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Extensions of the Curzon Metric

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Abstract. By examining the behaviour of geodesics approaching the singularity of the Curzon solution, it is shown that the metric is capable of being extended in such a way that almost all such geodesics are complete. There are an infinite number of possible extensions. None are analytic, but all are C^∞.

1. Introduction

The general static axially symmetric vacuum solution of Einstein's field equations is given by the Weyl metric

$$ds^2 = -e^{2(\nu-\lambda)}(dr^2 + dz^2) - r^2 e^{-2\lambda} d\phi^2 + e^{2\lambda} dt^2 \tag{1}$$

where

$$\lambda_{rr} + \lambda_{zz} + r^{-1}\lambda_r = 0\,,$$

and

$$\nu_r = r(\lambda_r^2 - \lambda_z^2)\,, \qquad \nu_z = 2r\lambda_r\lambda_z\,.$$

We may pick for λ any finite multipole expansion in inverse powers of $R = \sqrt{r^2 + z^2}$, thus generating a whole range of "particle-like" solutions which have some interesting properties [1]. Israel's theorem [2] indicates that none of these solutions has a regular event horizon at $R = 0$, for the Schwarzschild solution appears as a rod of length $2m$ in Weyl coordinates and hence does not belong to the family. One should however be wary of the applicability of Israel's theorem even in this simple case for the equipotential surfaces are by no means always regular and may exhibit cusps or may even break up into two or more disjoint pieces. Furthermore it still remains an unsolved problem whether a realistic collapse of a spherically asymmetric system may or may not result in such a naked singularity [3].

It is also not at all clear that the $R = 0$ singularity is in general "point-like" [4]. Indeed if one considers the monopole solution (the so called Curzon metric),

$$\lambda = -\frac{m}{R}\,, \qquad \nu = -\frac{m^2 r^2}{2R^4}\,, \tag{2}$$

then the behaviour of invariants of the curvature tensor in the limit $R \to 0$ is strongly dependent on the direction of approach. Gautreau and Anderson [5] have shown for example that the limit of such invariants along the z-axis is in fact regular, while it is singular along other directions; which leads one to make the suggestion that possibly the Curzon metric "opens up" for particles approaching $R=0$ along the z-axis, allowing them to pass on into some new region. We propose to show in this paper that this is in fact so. The behaviour along the $r=0$ geodesics turns out to be general rather than peculiar for it appears to be the case that almost all geodesics approach the z-axis very rapidly (in Weyl's coordinates) as $R\to 0$, the only geodesics not exhibiting this behaviour being those which lie in the $z=0$ plane. By using a set of coordinates linked to infalling worldlines which approximate these geodesics, the metric becomes regular and C^∞ at $R=0$. However it is not analytic at $R=0$ and therefore possesses an infinite number of possible extensions.

2. Behaviour of Geodesics Near $R=0$

The geodesic equations for the Weyl metric are

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right)-\frac{\partial L}{\partial x^\mu}=0$$

where

$$L=-e^{2(\nu-\lambda)}(\dot{r}^2+\dot{z}^2)-r^2e^{-2\lambda}\dot{\phi}^2+e^{2\lambda}\dot{t}^2\,,$$

and $\cdot\equiv d/ds$. Two of these equations are immediately integrated to give

$$\dot{t}=Ke^{-2\lambda}$$
$$\dot{\phi}=He^{2\lambda}r^{-2}\,.$$

Using the other first integral $L=\varepsilon=\pm 1, 0$ according to whether the geodesic is timelike, spacelike or null, the remaining pair of geodesic equations may be expressed as a single differential equation

$$\begin{aligned} r''(K^2-H^2r^{-2}e^{4\lambda}-\varepsilon e^{2\lambda}) &= (1+r'^2)\,[K^2(\nu_r-2\lambda_r-r'(\nu_z-2\lambda_z)\\ &\quad +H^2e^{4\lambda}r^{-2}(r'\nu_z-\nu_r+r^{-1})+\varepsilon e^{2\lambda}(\lambda_r-\nu_r-r'(\lambda_z-\nu_z)]\,, \end{aligned} \tag{3}$$

where $'=d/dz$.

If we fix attention on null geodesics lying in a fixed plane $\phi=$ const (i.e. $\varepsilon=H=0$), and normalizing the affine parameter to make $K=1$, the equation reduces to

$$r''=(1+r'^2)(f_r-r'f_z) \tag{4}$$

where $f=\nu-2\lambda$. In doing this no generality should be lost as regards geodesics approaching $R=0$ because of the extreme smallness (at least in the case, as here, where the coefficient of the highest multipole moment in λ is negative) of the terms $H^2 e^{4\lambda} r^{-2}$ and $\varepsilon e^{2\lambda}$ which occur in the more general Eq. (3).

If we now try for an asymptotic solution of the form

$$r \approx kz\,, \quad r' \approx k \quad \text{as} \quad z \to 0$$

we see, on substituting into (4) the explicit Curzon forms of λ and v given by (2), that

$$r'' \approx \frac{-m^2 k}{(1+k^2)\, z^3}$$

whence

$$r' \approx \frac{m^2 k}{2(1+k^2)\, z^2}$$

which contradicts the postulated behaviour for r' as $z \to 0$ unless $k=0$ (or possibly $k=\infty$, which is discussed below), i.e. the geodesics approach the z-axis asymptotically. The question remains how fast these geodesics approach the z-axis. An estimate may be obtained by neglecting terms in r'^2 and r^2/z^2 in Eqs. (4) and (2), resulting in

$$r'' \approx -\left(\frac{m^2}{z^4} + \frac{2m}{z^3}\right) r + \frac{2mr'}{z^2}\,.$$

The solution of this equation with correct behaviour as $z \to 0$ is

$$r \approx \varrho e^{-m/z}\,, \tag{5}$$

where $\varrho = \text{const}$. A similar treatment using r rather than z as the independent variable could be used to eliminate the case $k=\infty$, since $z=o(r)$, $z'=o(1)$ (as $r \to 0$), where $'$ now refers to d/dr, gives rise to the asymptotic equation

$$z'' = -\frac{m^2 z'}{r^3}$$

whose asymptotic solution is $z = O(e^{m^2/2r^2})$ (see Erdelyi [6]), which $\to \infty$ as $r \to 0$.

Thus the indications are that geodesics which approach $R=0$ either have the behaviour expressed by Eq. (5) or else they lie in the $z=0$ plane. [Note that in the above analysis we have not considered the possibility that some geodesics might conceivably spiral in towards $R=0$, a circumstance which, however unlikely, prevents us from being more categorical in our assertion.]

3. Extensions of the Curzon Metric

Treating Eq. (5) as though it were the exact equation of null geodesics entering $R=0$, we could use a "comoving" coordinate ϱ, and a "retarded" time u defined by

$$t = u + \int_{z_0}^{z} e^{2m/z} dz + \frac{\varrho^2 m}{2z^2}, \qquad z_0 = \text{const} > 0, \tag{6}$$

$$r = \varrho e^{-m/z}.$$

The terms "comoving" and "retarded" are of course not exactly applicable since (5) only represents the asymptotic form of the geodesic equations, but they do apply in a limiting sense as $z \to 0$. Transforming the Curzon metric to the new variable $x^0 = u$, $x^1 = \varrho$, $x^2 = z$, $x^3 = \phi$, we find

$$g_{00} = e^{-2m/R}.$$

Now if we approach $R = 0$ in such a way that $z \to 0+$ while ϱ remains bounded we see that

$$R = z(1 + \tfrac{1}{2}\varrho^2 z^{-2} e^{-2m/z} + O(z^{-4} e^{-4m/z})),$$

whence

$$g_{00} = O(e^{-2m/z}) \to 0 \quad \text{as} \quad z \to 0+.$$

Similarly

$$\begin{aligned}
g_{11} &= -e^{2m/R - m^2 r^2/R^4 - 2m/z} + e^{-2m/R} \varrho^2 m^2 z^{-4} \\
&= -1 + O(z^{-4} e^{-2m/z}), \\
g_{01} &= e^{-2m/R} \varrho m z^{-2} = O(z^{-2} e^{-2m/z}), \\
g_{22} &= -e^{2m/R - m^2 r^2/R^4}(1 + e^{-2m/z} m^2 \varrho^2 z^{-4}) + e^{-2m/R}(e^{2m/z} - \varrho^2 m z^{-3})^2 \\
&= O(z^{-8} e^{-2m/z}), \\
g_{02} &= e^{-2m/R}(e^{2m/z} - \varrho^2 m z^{-3}) = 1 + O(z^{-6} e^{-4m/z}), \\
g_{12} &= -e^{2m/R - m^2 r^2/R^4 - 2m/z} m \varrho z^{-2} + e^{-2m/R} \varrho m z^{-2}(e^{2m/z} - \varrho^2 m z^{-3}) \\
&= O(z^{-6} e^{-2m/z}), \\
g_{33} &= -\varrho^2 e^{2m/R - 2m/z} = -\varrho^2(1 + O(z^{-3} e^{-2m/z})), \\
g_{03} &= g_{13} = g_{23} = 0.
\end{aligned}$$

Thus for $z > 0$

$$ds^2 = 2du\,dz - d\varrho^2 - \varrho^2 d\phi^2 + O(z^{-8} e^{-2m/z})\, h_{\mu\nu} dx^\mu dx^\nu \tag{7}$$

where $h_{\mu\nu}$ is a tensor whose behaviour is regular at $z = 0$.

In these coordinates the Curzon metric is completely regular as $z \to 0+$ in the specified manner, and may for example be connected in a C^∞ manner across the plane $z = 0$ (noting that the functions $z^{-n} e^{-2m/z}$ have all derivatives $\to 0$ as $z \to 0+$) with flat space expressed in cylindrical polar coordinates

$$ds^2 = 2du\,dz - d\varrho^2 - \varrho^2 d\phi^2 \qquad (z < 0). \tag{8}$$

The spacetime resulting from such a conjunction is C^∞ at $z=0$, but not analytic. All outward going geodesics entering the upper half $z=0$ of the Curzon metric are now both past and future-complete. Inward going geodesics approaching $z=0$ from above do so asymptotically in these coordinates and hence are future incomplete. This situation may be remedied by using an "advanced" coordinate v, given by

$$t = v - \int^{z} e^{2m/z}\,dz - \tfrac{1}{2}\varrho_1{}^2 m/z^2$$
$$r = \varrho_1 e^{-m/z}\,.$$

The resulting metric can again be extended across $z=0$ in a similar manner. The lower half of the Curzon metric may be covered in entirely analogous fashion by making the coordinate transformation $z \to' -z$.

The extensions suggested here by various half-portions of Minkowski space lead to a multisheeted topology. However because of the lack of analyticity at $z=0$ this is by no means the only possible C^∞ extension. In fact an infinite number of possibilities present themselves. Perhaps the most natural extension is to connect the lower half to the upper half of the Curzon metric across $z=0$ in such a way that geodesics entering $z=0$ from below emerge into $z>0$ and geodesics entering from above emerge into $z<0$. From the above discussion this is clearly possible to achieve in a C^∞ manner, and would have the advantage that particles entering $R=0$ do not "disappear" from the external world.

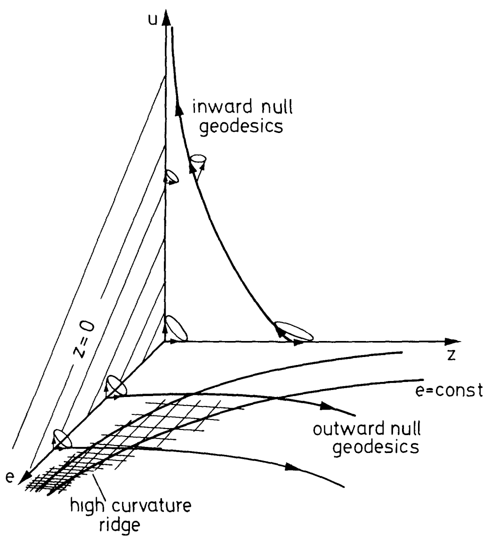

Fig. 1. The Curzon metric for $z>0$ near $R=0$, in u, z, ϱ coordinates

The plane $r=0$ ($r>0$) is not covered by any of the above coordinate patches and must be separately covered, for example by the original Weyl coordinates, in order to provide atlas of charts for the fully extended manifold. Approaching $R=0$ in this plane leads to infinite limits for invariants of the Riemann tensor [5], but in the patch covered by the u, ϱ, z, ϕ coordinates this singularity has been pushed out to $z=0$, $\varrho=\infty$, i.e. it appears as a ring at infinity with the null and timelike geodesics threading through it.

Thus the deceptively simple point-like appearance of $R=0$ in the Weyl coordinates must be abandoned. Indeed by using comoving coordinates it has more the appearance of an infinite plane ($z=0$) at which the space-time is momentarily flat. It is to be noted however that geodesics approaching this plane at high values of ϱ have to cross a ridge of high curvature close to $z=0$ before entering this flat region. This is clearly so because the curvature has an infinite limit along lines $r=kz$, $u=\text{const}$ ($k \neq 0$), i.e. along lines $\varrho = kze^{m/z}$ (see Fig. 1). It would be of interest to know whether these peculiarities exhibited by the Curzon metric are a special case, or whether other Weyl multipole "particles" have similar non-trivial extensions.

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