Extensions of the Menchoff-Rademacher theorem with applications to ergodic theory

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Dedicated to Hillel Furstenberg upon his retirement

Abstract

We prove extensions of Menchoff's inequality and the Menchoff-Rademacher theorem for sequences $\{f_n\} \subset L_p$, based on the size of the norms of sums of sub-blocks of the first n functions.

The results are applied to the study of a.e. convergence of series $\sum_n \frac{a_n T^n g}{n^\alpha}$ when T is an L_2-contraction, $g \in L_2$, and $\{a_n\}$ is an appropriate sequence.

Given a sequence $\{f_n\} \subset L_p(\Omega, \mu)$, $1 < p \le 2$, of independent centered random variables, we study conditions for the existence of a set of x of μ-probability 1, such that for every contraction T on $L_2(\mathcal{Y}, \pi)$ and $g \in L_2(\pi)$, the random power series $\sum_n f_n(x) T^n g$ converges π-a.e. The conditions are used to show that for $\{f_n\}$ centered i.i.d. with $f_1 \in L \log^+ L$, there exists a set of x of full measure such that for every contraction T on $L_2(\mathcal{Y}, \pi)$ and $g \in L_2(\pi)$, the random series $\sum_n \frac{f_n(x) T^n g}{n}$ converges π-a.e.

1 Introduction

Motivated by the problem of almost everywhere convergence of Fourier series, Plancherel [37] studied the a.e. convergence of orthogonal series (for earlier work see the introduction of [38]). Rademacher [38] and Menchoff[1] [31] proved (independently) the following improvement of Plancherel's result (which for Fourier series had been observed by Hobson [22] to be equivalent to a result of Hardy [54, Theorem III.4.4]).

Theorem 1.1. *Let (Ω, μ) be a probability space, and let $\{f_n\} \subset L_2(\mu)$ be an orthogonal sequence. If $\sum_{n=1}^\infty ||f_n||_2^2 (\log n)^2 < \infty$, then the series $\sum_{n=1}^\infty f_n$ converges a.e.*

When (Ω, μ) is a σ-finite measure space, let $\tilde{\mu}$ be a probability equivalent to μ. The order preserving isometry between $L_p(\mu)$ and $L_p(\tilde{\mu})$ (e.g., [27, p. 189]) preserves pointwise convergence, so all statements concerning *one* L_p space proved for probability spaces, like Theorem 1.1, are valid also in σ-finite measure spaces. We therefore deal in this paper only with (Ω, μ) a probability space.

Menchoff [31, Part III] extended Theorem 1.1 to the following.

Theorem 1.2. *Let $\{f_n\}$ be a sequence in $L_2(\mu)$, and let $\{\rho_n\}$ be positive numbers such that*

$$\Big\| \sum_{k=j+1}^{l} f_k \Big\|_2^2 \le \sum_{k=j+1}^{l} \rho_k^2 \qquad \text{for every} \quad l > j \ge 0. \tag{1}$$

If $\sum_{n=1}^\infty \rho_n^2 (\log n)^2 < \infty$, then the series $\sum_{n=1}^\infty f_n$ converges a.e.

[1] We use Menchoff's own spelling of his name in the papers he wrote in French.

A sequence $\{f_n\} \subset L_2(\mu)$ satisfying (1), for $\{\rho_n\}$ with $\sum_{n=1}^\infty \rho_n^2 < \infty$, was called *quasi orthogonal* in [31, Part III]; this term is defined differently in [24], where an application of Theorem 1.2 is given.

An important step in the proof of Theorem 1.1 is the following inequality for n orthogonal functions (see Salem [41] for a different proof):

$$\Big\| \max_{1\le l\le n} |\sum_{k=1}^{l} f_k| \Big\|_2^2 \le (2+\log_2 n)^2 \sum_{k=1}^{n} ||f_k||_2^2 \ . \tag{2}$$

In the proof of Theorem 1.1 in Zygmund [54, §XIII.10] it is also proved that under the theorem's assumptions we have $\Big\| \sup_{n\ge1} |\sum_{k=1}^n f_k| \Big\|_2^2 \le A \sum_{k=1}^\infty (\log k)^2 ||f_k||_2^2$.

Khintchine and Kolmogorov [26] proved that if $\{f_n\} \subset L_2$ is a sequence of centered *independent* random variables, then convergence of $\sum_n ||f_n||_2^2$ implies a.e. convergence of $\sum_n f_n$. Marcinkiewicz and Zygmund [30] obtained results for series of centered independent random variables in L_p, $1 < p < 2$.

Remark. Note that for any sequence $\{f_n\} \subset L_p(\mu)$, $1 \le p < \infty$, convergence of $\sum_n ||f_n||_p$ implies a.e. absolute convergence of $\sum_n f_n$ (we may assume that μ is a probability, and obtain convergence of $\sum_n ||f_n||_1$).

Tandori [48] (for a detailed proof see also [49]) proved for $\{f_n\}$ orthogonal that if $\sum_{n=1}^\infty ||f_n||_2^2 \log n \log^+(\frac{1}{||f_n||_2^2}) < \infty$, then $\sum_n f_n$ converges a.e., which strictly improves Theorem 1.1 (see [34] and [49]). This condition is necessary in the following sense [46]: *If $\{a_n\}$ is a sequence such that $\sum_{n=1}^\infty a_n\phi_n$ converges a.e. for* every *orthonormal sequence $\{\phi_n\}$, then $\sum_{n=1}^\infty |a_n|^2 \log n \log_+(\frac{1}{|a_n|^2}) < \infty$.* If $\{|a_n|\}$ is non-increasing, then $\sum_{n=1}^\infty |a_n|^2 (\log n)^2 < \infty$ [47]. For additional information see [35]. Weber [51] extended Theorem 1.2 by the method of majorizing measures, replacing $\sum_n \rho_n^2 (\log n)^2 < \infty$ by $\sum_{n=1}^\infty \rho_n^2 (\log n)^{1-\delta} \Big(\log_+(\frac{1}{\rho_n^2}) \Big)^{1+\delta} < \infty$ for some $0 \le \delta < 1$.

In [31, Theorem 12] Menchoff proved that for $\{f_n\}$ orthogonal, convergence of the series $\sum_{n=1}^\infty ||f_n||_2^\alpha$ for some $\alpha < 2$ implies a.e convergence of $\sum_n f_n$. The inequality $||\cdot||_{\ell_2} \le ||\cdot||_{\ell_\alpha}$ for $1 \le \alpha < 2$ allows to deduce this result from the following *Billinglsey-Stechkin theorem* (see [6, p. 102, problem 6]; a proof for $p > 2$, based upon ideas of Stechkin, is given in Gaposhkin [15, Theorem 1.3.5]; Weber [53] has recently proved the theorem by the metric entropy method).

Theorem 1.3. *Let $\{f_n\}_{n=1}^\infty \subset L_p(\mu)$ with $1 < p < \infty$. Let $\{m_n\}_{n=1}^\infty$ be a sequence of* non-negative *numbers, such that for some $q > 1$ we have*

$$\Big\| \sum_{k=j+1}^{l} f_k \Big\|_p^p \le (\sum_{k=j+1}^{l} m_k)^q \quad \textit{for} \ \ l > j \ge 0.$$

If $\sum_{n=1}^\infty m_n < \infty$, then $\sum_{n=1}^\infty f_n$ converges a.e.

Remarks. 1. For $q = 1$ Theorem 1.3 is no longer true (e.g., Menchoff's example [31, Theorem 3]).

2. From [32, Theorem 1] we obtain $\Big\| \sup_{n\ge1} |\sum_{k=1}^n f_k| \Big\|_p^p \le C_{p,q} (\sum_{n=1}^\infty m_n)^q$.

2 Extensions of the Menchoff-Rademacher theorem

In this section we use strong maximal inequalities of Móricz [32] to obtain extensions of of Theorem 1.1 for sequences in L_p, and discuss their connection with previously known results.

Definition 2.1. A triangular sequence of real numbers $\{d(j,l) : 0 \le j < l \le n\}$, is said to be *super additive* if

$$d(j,k) + d(k,l) \le d(j,l) \quad \text{for any} \quad 0 \le j < k < l \le n. \tag{3}$$

Remark. If $\{d(j,l)\}$ is a non-negative super additive sequence and $q > 1$, then $\{(d(j,l))^q : 0 \le j < l \le n\}$, denoted by d^q, is also super additive, since $\alpha^q + \beta^q \le (\alpha+\beta)^q$ for every $\alpha, \beta \ge 0$ (in fact $\|\cdot\|_{\ell_q} \le \|\cdot\|_{\ell_1}$).

Example 2.1. Let $\{m_k\}_{k=1}^{\infty}$ be a sequence of non-negative numbers, and let $q \ge 1$. For any n and $0 \le j < l \le n$, define $d(j,l) = \sum_{k=j+1}^{l} m_k$, which is obviously super additive. By the previous remark, $\{(\sum_{k=j+1}^{l} m_k)^q\}$ is a super additive sequence.

Remark. [29] Let $d(j,l)$ be a non-negative super additive sequence defined for every $l > j \ge 0$. Then by (3) $d(0,n)$ is non-decreasing, and the sequence $\{m_n\}$ defined by $m_1 = d(0,1)$ and $m_n = d(0,n) - d(0,n-1)$ for $n > 1$ satisfies, by (3),

$$d(j,l) \le d(0,l) - d(0,j) = \sum_{k=j+1}^{l} m_k \quad \text{for } l > j \ge 0. \tag{4}$$

Definition. Let $\{d(j,l) :\ 0 \le j < l \le n\}$ be a *non-negative* super additive sequence, and for $\{f_k :\ 1 \le k \le n\} \subset L_p(\mu)$ put

$$A_n^{(d)} = \inf\left\{ A :\ \Big\| \sum_{k=j+1}^{l} f_k \Big\|_p^p \le A \cdot d(j,l) \quad \text{for every } 0 \le j < l \le n \right\}. \tag{5}$$

Clearly $A_{n_1}^{(d)} \le A_n^{(d)}$ for $n_1 < n$. Note that when $A_n^{(d)} < \infty$, we must have $\sum_{k=j+1}^{l} f_k = 0$ whenever $d(j,l) = 0$, so

$$A_n^{(d)} = \max\left\{ \frac{\|\sum_{k=j+1}^{l} f_k\|_p^p}{d(j,l)} :\ d(j,l) \ne 0,\ 0 \le j < l \le n \right\}.$$

If $d(j,l) > 0$ for $0 \le j < l \le n$, then $A_n^{(d)}$ is finite, by the above formula.

The following lemma (and proposition) can be deduced from Theorem 3 of Móricz [32], which was proved by the method of [42] (see [33, Theorem 3.1] for a more general form). For the sake of completeness we include a different proof, based on the proof of Menchoff's inequality as given in Doob [12, Ch. IV, Lemma 4.1, p. 156] (see also Zygmund [54, Ch. XIII, §10]).

Lemma 2.1. *Let $\{f_k\}_{k=1}^{n} \subset L_p(\mu)$, $1 < p < \infty$. Let $\{d(j,l) : 0 \le j < l \le n\}$ be a super additive sequence of non-negative numbers with $A_n^{(d)} < \infty$. Then*

$$\Big\| \max_{1 \le l \le n} \Big| \sum_{k=1}^{l} f_k \Big| \Big\|_p^p \le A_n^{(d)} (2 + \log_2 n)^p d(0,n).$$

Proof. Let $0 \le r$ be an integer with $2^r < n \le 2^{r+1}$. Put $g_k = f_k$ if $1 \le k \le n$; for $n < k \le 2^{r+1}$ put $g_k = 0$. Also, put $\tilde{d}(j,l) = d(j,l)$ if $0 \le j < l \le n$; put $\tilde{d}(j,l) = 0$ if $l > j \ge n$; for $j < n \le l$ put $\tilde{d}(j,l) = d(j,n)$. Clearly $\{\tilde{d}(j,l) : 0 \le j < l \le 2^{r+1}\}$ is a super additive sequence.

By the definitions of $\{g_k\}$ and $\tilde{d}$, for any $0 \le j < l \le 2^{r+1}$ we have

$$\Big\| \sum_{k=j+1}^{l} g_k \Big\|_p^p \le A_n^{(d)} \tilde{d}(j,l) \tag{$*$}$$

with the *same* $A_n^{(d)}$ as above.

For any $0 \le i \le r+1$ and $1 \le m \le 2^{r+1-i}$, define $S_{m,i} = \sum_{k=(m-1)2^i+1}^{m2^i} g_k$ and $S_i^* = \max_{1\le m\le 2^{r+1-i}} |S_{m,i}|$. Clearly, $|S_i^*|^p \le \sum_{m=1}^{2^{r+1-i}} |S_{m,i}|^p$. Integration, (*), and super additivity of $\tilde{d}$ yield

$$||S_i^*||_p^p \le \sum_{m=1}^{2^{r+1-i}} ||S_{m,i}||_p^p \le \sum_{m=1}^{2^{r+1-i}} A_n^{(d)}\tilde{d}((m-1)2^i, m2^i) \le A_n^{(d)}\tilde{d}(0,2^{r+1}) = A_n^{(d)}d(0,n).$$

Using the binary expansion of j, the sum $\sum_{k=1}^j g_k$ can be represented as a sum of disjoint blocks of different sizes $S_{m,i}$ for suitable m's and i's. By this we have that

$$\max_{1\le j\le 2^{r+1}} |\sum_{k=1}^j g_k| \le \sum_{i=0}^{r+1} S_i^*.$$

Hence

$$\Big\|\max_{1\le j\le n} |\sum_{k=1}^j f_k|\Big\|_p = \Big\|\max_{1\le j\le 2^{r+1}} |\sum_{k=1}^j g_k|\Big\|_p \le \sum_{i=0}^{r+1} \|S_i^*\|_p \le (r+2)[A_n^{(d)}d(0,n)]^{\frac{1}{p}}$$

and the result follows. □

Remarks. 1. For $p=2$ and $\{f_k\}_{k=1}^n$ orthogonal, Menchoff's inequality (2) follows by taking $d(j,l) = \sum_{k=j+1}^l ||f_k||_2^2$.

2. For $p=1$ we easily conclude

$$\Big\|\max_{1\le l\le n} |\sum_{k=1}^l f_k|\Big\|_1 \le \sum_{k=1}^n ||f_k||_1 \le \sum_{k=1}^n A_n^{(d)}d(k-1,k) \le A_n^{(d)}d(0,n).$$

3. Billingsley [6, p. 102, problem 5] outlines a proof of the lemma for the special case of $A_n^{(d)} = 1$ for $d(j,l)$ as in Example 2.1. In any case, defining $d'(i,j) = A_n^{(d)}d(i,j)$ we obtain $A_n^{(d')} = 1$, so for *fixed* n the assumption $A_n^{(d)} = 1$ is not a restriction.

4. The use of sequences satisfying (3) in the context of Menchoff's inequality is implicit in [42], and explicit in [44]. More general sequences were used in [33].

5. Also other authors, like Hannan [21, Lemma], Gaposhkin [16],[17, Theorem 3], and Houdré [23, e.g., Theorem 3.1], considered various extensions of Menchoff's inequality, or new applications of it, all beyond the scope of orthogonal functions.

6. An inspection of the proof of the lemma shows that the result is true for an arbitrary Banach lattice of functions.

Proposition 2.2. *Let $\{f_k\}_{k=1}^n \subset L_p(\mu)$ with $1 < p < \infty$. Let $\{d(j,l) : 0 \le j < l \le n\}$ be a super additive sequence of non-negative numbers with $A_n^{(d)}$ finite. Then for any $0 \le n_1 < n$, we have*

$$\Big|\Big|\max_{n_1<l\le n} |\sum_{k=n_1+1}^l f_k|\Big|\Big|_p^p \le A_n^{(d)}(2+\log_2(n-n_1))^p d(n_1,n).$$

Proof. For $n_1 = 0$ this is Lemma 2.1, so we assume $n_1 > 0$. Put $g_k = f_{k+n_1}$ for any $1 \le k \le n-n_1$, and put $\tilde{d}(j,l) = d(j+n_1, l+n_1)$ for any $0 \le j < l \le n-n_1$; clearly $\tilde{d}$ is a super additive sequence on $\{0 \le j < l \le n-n_1\}$.

For $0 \le j < l \le n-n_1$ we have

$$\|\sum_{k=j+1}^l g_k\|_p^p = \|\sum_{k=j+1+n_1}^{l+n_1} f_k\|_p^p \le A_n^{(d)}d(j+n_1, l+n_1) = A_n^{(d)}\tilde{d}(j,l),$$

which yields that $\tilde{A}^{(\tilde{d})}_{n-n_1}$, defined by (5) for $\tilde{d}$ and $\{g_k\}$, satisfies $\tilde{A}^{(\tilde{d})}_{n-n_1} \le A^{(d)}_n < \infty$. Using Lemma 2.1 we obtain

$$\Big\| \max_{n_1<j\le n} |\sum_{k=n_1+1}^{j} f_k| \Big\|_p^p = \Big\| \max_{1\le j\le n-n_1} |\sum_{k=1}^{j} g_k| \Big\|_p^p \le$$

$$\tilde{A}^{(\tilde{d})}_{n-n_1}(2+\log_2(n-n_1))^p \tilde{d}(0,n-n_1) \le A^{(d)}_n (2+\log_2(n-n_1))^p d(n_1,n).$$

□

Remark. A tighter inequality than that formulated in the proposition, which depends only on $n-n_1$, is given in the last line of the proof, using $\tilde{A}^{(\tilde{d})}_{n-n_1}$ instead of $A^{(d)}_n$.

Proposition 2.3. *Let $\{f_k\}_{k=1}^n \subset L_p(\mu)$, $1<p<\infty$, and let $\{d(j,l) : 0\le j<l\le n\}$ be a super additive sequence of non-negative numbers, such that $A^{(d^q)}_n < \infty$ for some $q>1$. Then for any $0 \le n_1 < n$ we have*

$$\Big\| \max_{n_1<l\le n} |\sum_{k=n_1+1}^{l} f_k| \Big\|_p^p \le C_{p,q} A^{(d^q)}_n d^q(n_1,n), \tag{6}$$

where $C_{p,q} = \big(1-2^{(1-q)/p}\big)^{-p}$.

Proof. We extend d to all pairs $l>j$, by putting $d(j,l)=d(j,n)$ for $l>n>j$ and $d(j,l)=0$ for $j\ge n$. It is easy to check that $d(j,l)$ is super additive for all $l>j\ge 0$. For $k>n$ define $f_k=0$, and let $\tilde{d}(j,l) = (A^{(d^q)}_n)^{1/q} d(j,l)$. It is easy to check that $\|\sum_{k=j+1}^{l} f_k\|_p^p \le (\tilde{d}(j,l))^q$ for $l>j\ge 0$. We can now apply Theorem 1 of Móricz [32]. □

Remark. A different value for $C_{p,q}$ was obtained by Longnecker and Serfling [29].

Notation. Unless otherwise specified, all logarithms in the sequel are to the base 2.

Theorem 2.4. *Let $\{f_n\}_{n=1}^\infty \subset L_p(\mu)$ with $1<p<\infty$. Let $\{d(j,l),\ l>j\ge 0\}$ be a non-negative super additive sequence. and let $1\le q<\infty$. Put $m_1 = d(0,1)$, $m_n = d(0,n)-d(0,n-1)$ for $n>1$, and define $A^{(d^q)}_n$ as in (5). If $\sum_{n=1}^\infty (A^{(d^q)}_{2n})^{1/q}(\log n)^{p/q} m_n$ converges, then $\sum_{n=1}^\infty f_n$ converges a.e. and in L_p-norm. Furthermore,*

$$\Big\| \sup_{n\ge 1} |\sum_{k=1}^{n} f_k| \Big\|_p \le 2\Big[\|f_1\|_p + \|f_2\|_p + p^{(p-1)/p}\Big(\sum_{n=1}^\infty (A^{(d^q)}_{2n})^{1/q} (\log n)^{p/q} m_n \Big)^{q/p} \Big]. \tag{7}$$

Proof. By the remark following Example 2.1, (4) yields $\hat{d}(j,l) := \sum_{k=j+1}^{l} m_k \ge d(j,l)$ for any $l>j\ge 0$. Hence $A^{(\hat{d}^q)}_n \le A^{(d^q)}_n$ for every n, so it is enough to prove the theorem for $\hat{d}$, i.e., we may assume $d(j,l) = \sum_{k=j+1}^{l} m_k$.

(a) Using the following facts: (i) the definitions of d^q and $A^{(d^q)}_n$; (ii) $A^{(d^q)}_n$ non-decreasing, so for $k \ge 2^v+1$, we have $A^{(d^q)}_{2^{v+1}} \le A^{(d^q)}_{2k}$; (iii) $\|\cdot\|_{\ell_q} \le \|\cdot\|_{\ell_1}$, we obtain

$$\sum_{v=1}^\infty \int v^p \Big| \sum_{k=2^v+1}^{2^{v+1}} f_k \Big|^p d\mu \le \sum_{v=1}^\infty v^p A^{(d^q)}_{2^{v+1}} \Big(\sum_{k=2^v+1}^{2^{v+1}} m_k \Big)^q \le$$

$$\sum_{v=1}^\infty \Big(\sum_{k=2^v+1}^{2^{v+1}} (A^{(d^q)}_{2k})^{1/q} (\log k)^{p/q} m_k \Big)^q \le \Big(\sum_{v=1}^\infty \sum_{k=2^v+1}^{2^{v+1}} (A^{(d^q)}_{2k})^{1/q} (\log k)^{p/q} m_k \Big)^q =$$

$$\Big(\sum_{n=1}^{\infty}(A_{2n}^{(d^q)})^{1/q}(\log n)^{p/q}m_n\Big)^q<\infty.$$

Hence (by Beppo Levi) the integrand $\sum_{v=1}^{\infty} v^p\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p$ converges a.e.

(b) For any naturals r and m we obtain, using Hölder's inequality,

$$\Big|\sum_{k=2^m+1}^{2^{m+r}} f_k\Big|^p=\Big|\sum_{v=m}^{m+r-1}\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p\le\Big(\sum_{v=m}^{m+r-1} v\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|\frac{1}{v}\Big)^p\le$$

$$\Big(\sum_{v=m}^{m+r-1} v^p\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p\Big)\Big(\sum_{v=m}^{m+r-1}\frac{1}{v^{p/(p-1)}}\Big)^{p-1}\le$$

$$\Big(\sum_{v=m}^{\infty}\frac{1}{v^{p/(p-1)}}\Big)^{p-1}\Big(\sum_{v=1}^{\infty} v^p\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p\Big).$$

The first factor in the last line converges to zero (as $m\to\infty$) as the tail of a convergent series (since $1<p<\infty$), while the last factor converges a.e. by (a), so $\{\sum_{k=1}^{2^m} f_k\}$ is a Cauchy sequence a.e., and hence converges a.e. By taking integrals of the above inequality, and considering the convergence proved in (a), $\{\sum_{k=1}^{2^m} f_k\}$ is a Cauchy sequence in L_p-norm, and hence converges in norm.

(c) Using Proposition 2.2, and the inequality $||\cdot||_{\ell_q}\le||\cdot||_{\ell_1}$, we have

$$\sum_{m=1}^{\infty}\int\max_{2^m<n\le 2^{m+1}}\Big|\sum_{k=2^m+1}^{n} f_k\Big|^p d\mu\le\sum_{m=1}^{\infty}A_{2^{m+1}}^{(d^q)}m^p\Big(\sum_{k=2^m+1}^{2^{m+1}} m_k\Big)^q\le$$

$$\sum_{m=1}^{\infty}\Big(\sum_{k=2^m+1}^{2^{m+1}}(A_{2k}^{(d^q)})^{1/q}(\log k)^{p/q}m_k\Big)^q\le\Big(\sum_{m=1}^{\infty}\sum_{k=2^m+1}^{2^{m+1}}(A_{2k}^{(d^q)})^{1/q}(\log k)^{p/q}m_k\Big)^q=$$

$$\Big(\sum_{n=1}^{\infty}(A_{2n}^{(d^q)})^{1/q}(\log n)^{p/q}m_n\Big)^q<\infty\,.$$

The above inequality clearly yields that $\max\limits_{2^m<n\le 2^{m+1}}|\sum_{k=2^m+1}^{n} f_k|\to 0$ as $m\to\infty$, almost evrywhere and in L_p-norm.

Now, (b) and (c) imply that $\sum_{n=1}^{\infty} f_n$ converges a.e. to $g:=\lim\limits_{m\to\infty}\sum_{n=1}^{2^m} f_n$, since for $2^m<n\le 2^{m+1}$, we have

$$|\sum_{k=1}^{n} f_k-g|\le|\sum_{k=1}^{2^m} f_k-g|+|\sum_{k=2^m+1}^{n} f_k|\le|\sum_{k=1}^{2^m} f_k-g|+\max_{2^m<n\le 2^{m+1}}|\sum_{k=2^m+1}^{n} f_k|\,.$$

By considering the norm convergence proved in (a) and (b), the L_p-norm convergence follows by taking the L_p-norm in the above inequality.

Proof that the maximal function is in L_p: The inequality in (b) with $m=1$ yields

$$\sup_{r\ge 1}\Big|\sum_{k=3}^{2^{r+1}} f_k\Big|^p\le\Big(\sum_{v=1}^{\infty}\frac{1}{v^{p/(p-1)}}\Big)^{p-1}\Big(\sum_{v=1}^{\infty} v^p\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p\Big).$$

Integration of the above inequality and application of (a) yield

$$\Big\|\sup_{r\ge 1}\Big|\sum_{k=3}^{2^{r+1}} f_k\Big|\Big\|_p^p\le p^{p-1}\Big(\sum_{n=1}^{\infty}(A_{2n}^{(d^q)})^{1/q}(\log n)^{p/q}m_n\Big)^q\qquad(*)$$

The inequality in (c) yields

$$\Big\|\sup_{m\ge1}\max_{2^m<n\le2^{m+1}}\Big|\sum_{k=2^m+1}^{n}f_k\Big|\Big\|_p^p\le\int\sum_{m=1}^{\infty}\max_{2^m<n\le2^{m+1}}\Big|\sum_{k=2^m+1}^{n}f_k\Big|^p d\mu\le$$

$$\Big(\sum_{n=1}^{\infty}(A_{2n}^{(d^q)})^{1/q}(\log n)^{p/q}m_n\Big)^q<\infty\,. \qquad (**)$$

Since

$$\sup_{n\ge1}|\sum_{k=1}^{n}f_k|\le|f_1|+|f_2|+\sup_{n\ge3}|\sum_{k=1}^{n}f_k|\ ,$$

combining (*) and (**) with

$$\Big\|\sup_{n\ge3}|\sum_{k=1}^{n}f_k|\Big\|_p\le||f_1||_p+||f_2||_p+\Big\|\sup_{m\ge2}|\sum_{k=3}^{2^m}f_k|\Big\|_p+\Big\|\sup_{m\ge1}\max_{2^m<n\le2^{m+1}}|\sum_{k=2^m+1}^{n}f_k|\Big\|_p$$

we obtain inequality (7) for the maximal function.

□

Remarks. 1. The proof of the a.e. convergence is based on the proof of the Menchoff-Rademacher theorem as given in Alexits [1, p. 80].

2. Let $p=2$ and $\{f_n\}_{n=1}^{\infty}$ orthogonal in L_2. By taking $q=1$ and $d(j,l)=\sum_{k=j+1}^{l}\|f_k\|_2^2$ in the above theorem, we simply get the Menchoff-Rademacher theorem.

3. Zygmund's proof of the Menchoff-Rademacher theorem [54, Theorem XIII.10.21] is different from Alexits's, and gives also the square-integrability of the maximal function. Our proof that the maximal function is in L_p is different from Zygmund's (which uses the Riesz-Fischer theorem).

4. Clearly $||f_n||_p^p\le A_n^{(d^q)}m_n^q$ for every n. For $p=1$ this yields $\sum_n||f_n||_1\le\sum_n A_{2n}^{(d^q)}m_n^q$. The condition of the theorem for $p=1$ implies $\sum_n(A_{2n}^{(d^q)})^{1/q}m_n<\infty$, which yields $\sum_n A_{2n}^{(d^q)}m_n^q<\infty$. Hence $\sum_n||f_n||_1<\infty$, so $\sum_n|f_n|<\infty$ a.e.

In the Menchoff-Rademacher theorem $q=1$, and $A_n^{(d^q)}=1$ for every n. When $q>1$ much more can be said, by using Proposition 2.3.

Theorem 2.5. *Let $\{f_n\}_{n=1}^{\infty}\subset L_p(\mu)$ with $1<p<\infty$. Let $\{d(j,l):\ l>j\ge0\}$ be a non-negative super additive sequence, and assume that $\{A_n^{(d^q)}\}$ is bounded for some $1<q<\infty$. If $\{d(0,n)\}$ is bounded (i.e., converges), then $\sum_{n=1}^{\infty}f_n$ converges a.e and in L_p-norm. Furthermore,*

$$\Big\|\sup_n|\sum_{k=1}^{n}f_k|\Big\|_p^p\le C_{p,q}\sup_n A_n^{(d^q)}\lim_{n\to\infty}d(0,n).$$

Proof. Let $A=\sup_n A_n^{(d^q)}$. Put $m_n=d(0,n)-d(0,n-1)$. Since $d(n_1,n)\le\sum_{k=n_1+1}^{n}m_k$ by (4), letting $n\to\infty$ in (6) yields

$$\Big\|\sup_{l>n_1}|\sum_{k=n_1+1}^{l}f_k|\Big\|_p^p\le AC_{p,q}\big(\sum_{k=n_1+1}^{\infty}m_k\big)^q\underset{n_1\to\infty}{\longrightarrow}0.$$

This shows all the assertions. □

Remark. Theorem 1.3 is the case where $d(j,l)=\sum_{k=j+1}^{l}m_k$ for $l>j\ge0$.

Notation. For a non-negative function $g(u)$ and α real we denote $[g(u)]^\alpha$ by $g^\alpha(u)$.

We will show that when $q>1$ Theorem 2.4 can be improved, even without boundedness of $\{A_n^{(d^q)}\}$, by assuming convergence of a smaller numerical series.

Theorem 2.6. *Let $\{f_n\}_{n=1}^\infty \subset L_p(\mu)$ with $1 < p < \infty$. Let $\{d(j,l),\ l > j \geq 0\}$ be a non-negative super additive sequence, and let $1 < q < \infty$. Put $m_1 = d(0,1)$, $m_n = d(0,n) - d(0,n-1)$ for $n > 1$, and define $A_n^{(d^q)}$ as in (5). Let $\psi(u)$ be a positive increasing function such that $\sum_{n=1}^\infty \frac{1}{\psi^{p/(p-1)}(n)}$ converges. If $\sum_{n=1}^\infty (A_{2n}^{(d^q)})^{1/q}\psi^{p/q}(\log n)m_n$ converges, then $\sum_{n=1}^\infty f_n$ converges a.e. and in L_p-norm. Furthermore, $\sup_{n\geq 1}|\sum_{k=1}^n f_k| \in L_p$, and if $\psi(0) \geq 1$ there exist $C > 0$ such that*

$$\Big\| \sup_{n\geq 1} |\sum_{k=1}^n f_k| \Big\|_p \leq 2\Big[||f_1||_p + ||f_2||_p + C^{(p-1)/p}\Big(\sum_{n=1}^\infty (A_{2n}^{(d^q)})^{1/q}\psi^{p/q}(\log n)m_n\Big)^{q/p}\Big] \quad (8)$$

Proof. The proof proceeds along the same lines as that of Theorem 2.4, Here we use Proposition 2.3 instead of Proposition 2.2. As in Theorem 2.4, it is enough to prove the theorem when $d(j,l) = \sum_{k=j+1}^l m_k$.

(a) Using the monotonicity of ψ, we obtain

$$\sum_{v=1}^\infty \int \psi^p(v)\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p d\mu \leq \sum_{v=1}^\infty \psi^p(v)A_{2^{v+1}}^{(d^q)}\Big(\sum_{k=2^v+1}^{2^{v+1}} m_k\Big)^q \leq$$

$$\sum_{v=1}^\infty \Big(\sum_{k=2^v+1}^{2^{v+1}} (A_{2k}^{(d^q)})^{1/q}\psi^{p/q}(\log k)m_k\Big)^q \leq \Big(\sum_{v=1}^\infty \sum_{k=2^v+1}^{2^{v+1}} (A_{2k}^{(d^q)})^{1/q}\psi^{p/q}(\log k)m_k\Big)^q =$$

$$\Big(\sum_{n=1}^\infty (A_{2n}^{(d^q)})^{1/q}\psi^{p/q}(\log n)m_n\Big)^q < \infty.$$

Hence (by Beppo Levi) the integrand $\sum_{v=1}^\infty \psi^p(v)\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p$ converges a.e.

(b) For any naturals r and m we obtain, using Hölder's inequality,

$$\Big|\sum_{k=2^m+1}^{2^{m+r}} f_k\Big|^p = \Big|\sum_{v=m}^{m+r-1}\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p \leq \Big(\sum_{v=m}^{m+r-1}\psi(v)\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|\frac{1}{\psi(v)}\Big)^p \leq$$

$$\Big(\sum_{v=m}^{m+r-1}\psi^p(v)\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p\Big)\Big(\sum_{v=m}^{m+r-1}\frac{1}{\psi^{p/(p-1)}(v)}\Big)^{p-1} \leq$$

$$\Big(\sum_{v=m}^\infty \frac{1}{\psi^{p/(p-1)}(v)}\Big)^{p-1}\Big(\sum_{v=1}^\infty \psi^p(v)\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p\Big).$$

Using (a) and (b) we conclude that $\{\sum_{k=1}^{2^m} f_k\}$ is a Cauchy sequence a.e. and in L_p-norm, hence converges a.e. and in norm.

(c) Using Proposition 2.3, we have

$$\sum_{m=1}^\infty \int \max_{2^m<n\leq 2^{m+1}}\Big|\sum_{k=2^m+1}^n f_k\Big|^p d\mu \leq C_{p,q}\sum_{m=1}^\infty A_{2^{m+1}}^{(d^q)}\Big(\sum_{k=2^m+1}^{2^{m+1}} m_k\Big)^q \leq$$

$$C_{p,q}\sum_{m=1}^\infty \Big(\sum_{k=2^m+1}^{2^{m+1}} (A_{2k}^{(d^q)})^{1/q}m_k\Big)^q \leq C_{p,q}\Big(\sum_{m=1}^\infty\sum_{k=2^m+1}^{2^{m+1}} (A_{2k}^{(d^q)})^{1/q}m_k\Big)^q =$$

$$C_{p,q}\Big(\sum_{n=1}^\infty (A_{2n}^{(d^q)})^{1/q}m_n\Big)^q \leq C_{p,q}\Big(\sum_{n=1}^\infty (A_{2n}^{(d^q)})^{1/q}\psi^{p/q}(\log n)m_n\Big)^q < \infty\,.$$

The above inequality yields that $\max_{2^m<n\leq 2^{m+1}}|\sum_{k=2^m+1}^n f_k| \to 0$ as $m \to \infty$, almost everywhere and in L_p-norm.

Now, (b) and (c) imply that $\sum_{n=1}^{\infty} f_n$ converges a.e. and in L_p-norm.

Proof that the maximal function is in L_p: The inequality in (b) with $m = 1$ yields

$$\sup_{r\geq 1}\Big|\sum_{k=3}^{2^{r+1}} f_k\Big|^p \leq \Big(\sum_{v=1}^{\infty}\frac{1}{\psi^{p/(p-1)}(v)}\Big)^{p-1}\Big(\sum_{v=1}^{\infty}\psi^p(v)\Big|\sum_{k=2^v+1}^{2^{v+1}} f_k\Big|^p\Big).$$

Integration of the above inequality and application of (a) yield

$$\Big\|\sup_{r\geq 1}\Big|\sum_{k=3}^{2^{r+1}} f_k\Big|\Big\|_p^p \leq K^{p-1}\Big(\sum_{n=1}^{\infty}(A_{2n}^{(d^q)})^{1/q}\psi^{p/q}(\log n)m_n\Big)^q \qquad (*)$$

By assumption $\psi(n) \to \infty$, so $\psi(x) \geq 1$ for $x \geq N$. The inequality in (c) yields

$$\Big\|\sup_{m\geq 1}\max_{2^m<n\leq 2^{m+1}}\Big|\sum_{k=2^m+1}^{n} f_k\Big|\Big\|_p^p \leq \int\sum_{m=1}^{\infty}\max_{2^m<n\leq 2^{m+1}}\Big|\sum_{k=2^m+1}^{n} f_k\Big|^p d\mu \leq$$

$$C_{p,q}\Big(\sum_{n=1}^{\infty}(A_{2n}^{(d^q)})^{1/q}m_n\Big)^q \leq C_{p,q}\Big(\sum_{n=1}^{\infty}(A_{2n}^{(d^q)})^{1/q}\psi^{p/q}(\log n)\, m_n\Big)^q + B < \infty\,, \qquad (**)$$

Where B is 0 if $\psi(0) \geq 1$, and is otherwise a finite sum without $\psi(\log n)$ of the terms having $\psi(\log n) < 1$. When $\psi(0) \geq 1$ we use (*) and (**), and obtain (8) for the maximal function with $C = K^{(p-1)/q} + (C_{p,q})^{1/q}$. □

Corollary 2.7. *Let $\{f_n\}_{n=1}^{\infty} \subset L_p(\mu)$ with $1 < p < \infty$. Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers, and put $d(j,l) = \sum_{k=j+1}^{l} m_k$ for $0 \leq j < l$. Fix $1 < q < \infty$, and define $A_n^{(d^q)}$ as in (5). If $\sum_{n=2}^{\infty}(A_{2n}^{(d^q)})^{1/q}(\log n)^{(p-1)/q}(\log\log n)^{(p-1)/q+\epsilon}m_n$ converges for some $\epsilon > 0$, then $\sum_{n=1}^{\infty} f_n$ converges a.e. and in L_p-norm. Furthermore,*

$$\Big\|\sup_{n\geq 1}|\sum_{k=1}^{n} f_k|\ \Big\|_p \leq$$

$$2\Big[||f_1||_p + ||f_2||_p + C\Big(\sum_{n=2}^{\infty}(A_{2n}^{(d)})^{1/q}(\log n)^{(p-1)/q}(\log\log n)^{(p-1)/q+\epsilon}m_n\Big)^{q/p}\Big]$$

for some $C > 0$.

When $q \geq p$, Theorem 2.4 (and Corollary 2.7) can be improved as follows.

Theorem 2.8. *Assume that in Theorem 2.4 $q \geq p > 1$. If $\sum_{n=1}^{\infty}\big(A_n^{(d^q)}\big)^{1/q}m_n < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges a.e. and in L_p-norm, with $\sup_{n\geq 1}|\sum_{k=1}^{n} f_k| \in L_p$.*

Proof. As in the proof of Theorem 2.6, we may assume $d(j,l) = \sum_{k=j+1}^{l} m_k$. For brevity denote $A_n^{(d^q)}$ by A_n. By definition, $||f_n||_p^p \leq A_n m_n^q$. By induction on l we prove $\|\sum_{k=j+1}^{l} f_k\|_p^p \leq (\sum_{k=j+1}^{l} A_k^{1/q}m_k)^q$ for $0 \leq j < l$, using $\alpha^{q/p} + \beta^{q/p} \leq (\alpha+\beta)^{q/p}$, as follows.

$$\Big\|\sum_{k=j+1}^{l+1} f_k\Big\|_p \leq \Big(\sum_{k=j+1}^{l} A_k^{1/q}m_k\Big)^{q/p} + \Big(A_{l+1}^{1/q}m_{l+1}\Big)^{q/p} \leq \Big(\sum_{k=j+1}^{l+1} A_k^{1/q}m_k\Big)^{q/p}.$$

Hence the assertions of the theorem follow from Theorem 2.5; see also Theorem 1.3. □

Theorem 2.9. *Let $\{f_n\}_{n=1}^\infty \subset L_p(\mu)$ with $1 < p < \infty$. Let $\{m_n\}_{n=1}^\infty$ be a sequence of non-negative numbers, and let $1 \le q < \infty$. Assume that for any $n > 0$ there is a constant $A_n < \infty$ such that*

$$\Big\| \sum_{k=j+1}^{l} f_k \Big\|_p^p \le A_n \Big(\sum_{k=j+1}^{l} m_k \Big)^q \quad \text{for } \ 0 \le j < l \le n. \tag{9}$$

Then $\sum_{n=1}^\infty f_n$ converges a.e. and in L_p-norm, with $\sup_{n\ge1} |\sum_{k=1}^n f_k| \in L_p(\mu)$, if one of the following sets of conditions holds:

(i) $q = 1$ and $\sum_{n=1}^\infty A_{2n} (\log n)^p m_n < \infty$.

(ii) $q > 1$, $\{A_n\}$ is bounded, and $\sum_{n=1}^\infty m_n < \infty$.

(iii) $p > q > 1$ and $\sum_{n=2}^\infty A_{2n}^{1/q} (\log n)^{(p-1)/q} (\log\log n)^{(p-1)/q+\epsilon} m_n < \infty$.

(iv) $q \ge p > 1$ and $\sum_{n=1}^\infty A_n^{1/q} m_n < \infty$.

Proof. The previous results apply to $d(j,l) = \sum_{k=j+1}^l m_k$, since $A_n^{(d^q)} \le A_n$. □

Proposition 2.10. *Let $\{a_n\}$ be a sequence of complex numbers, and let $1 < p < \infty$ and $1 \le t < \infty$. Let $\{f_n\} \subset L_p(\mu)$ such that for some constant $C > 0$ we have*

$$\Big\| \sum_{k=j+1}^{l} a_k f_k \Big\|_p \le C \Big(\sum_{k=j+1}^{l} |a_k|^t \Big)^{1/t} \qquad \text{for every} \quad l > j \ge 0 \tag{10}$$

If either

(i) $\quad p \le t$ *and* $\quad \sum_{n=1}^\infty |a_n|^p (\log n)^p < \infty$,

or

(ii) $\quad p > t$ *and* $\quad \sum_{n=1}^\infty |a_n|^t < \infty$,

then $\sum_{n=1}^\infty a_n f_n$ converges a.e. Furthermore, $\sup_{n\ge1} |\sum_{k=1}^n a_k f_k|$ is in $L_p(\mu)$.

Proof. (i) Since $t/p \ge 1$, condition (10) and the inequality $||\cdot||_{\ell_{t/p}} \le ||\cdot||_{\ell_1}$ yield

$$\Big\| \sum_{k=j+1}^{l} a_k f_k \Big\|_p^p \le C^p \Big(\sum_{k=j+1}^{l} |a_k|^{p\cdot t/p} \Big)^{p/t} \le C^p \sum_{k=j+1}^{l} |a_k|^p$$

so condition (9) is satisfied by the sequence $\{a_k f_k\}$, with $m_k = |a_k|^p$, $q = 1$, and $A_n = C^p$. Now Theorem 2.9(i) applies when $\sum_{n=1}^\infty |a_n|^p (\log n)^p < \infty$.

(ii) Condition (10) yields $\Big\| \sum_{k=j+1}^l a_k f_k \Big\|_p^p \le \Big(\sum_{k=j+1}^l |C a_k|^t \Big)^{p/t}$ for every $l > j \ge 0$, and since $p > t$ we obtain the a.e. convergence of $\sum_k a_k f_k$ from Theorem 1.3. Proposition 2.3 yields

$$\Big\| \max_{1\le l\le n} \Big| \sum_{k=1}^{l} a_k f_k \Big| \Big\|_p^p \le C_{p,p/t} \Big(\sum_{k=1}^{n} |C a_k|^t \Big)^{p/t}.$$

Letting $n \to \infty$ we obtain that $\sup_{n\ge1} |\sum_{k=1}^n a_k f_k|$ is in $L_p(\mu)$. □

Remarks. 1. When $p < t$, convergence of $\sum_n |a_n|^p (\log n)^p$ implies convergence of $\sum_n |a_n|^t$. When $p > t$, convergence of $\sum_n |a_n|^t$ implies that of $\sum_n |a_n|^p$.

2. Theorem 1.2 follows by applying Proposition 2.10(i) to $\{\frac{1}{\rho_k} f_k\}$, with $p = t = 2$ and $a_k = \rho_k$, since (10) follows from (1).

In the sequel we will denote the unit circle by Γ, and the normalized Haar (Lebesgue) measure by $d\lambda$.

Proposition 2.11. *Let $\{a_n\}$ be a sequence of complex numbers, and let $1 \le p < \infty$ and $q \ge 2$ with dual index $q' = q/(q-1)$. Let $\{f_n\} \subset L_p(\mu)$ such that for some constant $C > 0$ we have*

$$\Big\| \sum_{k=j+1}^{l} a_k f_k \Big\|_p \le C \Big\| \sum_{k=j+1}^{l} a_k \lambda^k \Big\|_{L_q(d\lambda)} \qquad \text{for any} \quad l > j \ge 0. \tag{11}$$

Then: (i) (10) *holds with $t = q'$. (ii) when $q' \le p$, for every $l > j \ge 0$ we have*

$$\Big\| \sum_{k=j+1}^{l} a_k f_k \Big\|_p \le C \Big(\sum_{k=j+1}^{l} |a_k|^{q'} \Big)^{1/q'} \le C (l-j)^{\frac{p-1}{p} - \frac{1}{q}} \Big(\sum_{k=j+1}^{l} |a_k|^p \Big)^{1/p}. \tag{12}$$

Proof. Note that (11) implies $||f_n||_p \le C$ when $a_n \ne 0$. For $q = \infty$ we have $q' = 1$, and we combine (11) with

$$\Big\| \sum_{k=j+1}^{l} a_k \lambda^k \Big\|_{L_\infty(d\lambda)} \le \sum_{k=j+1}^{l} |a_k| \le (l-j) \Big(\frac{1}{l-j} \sum_{k=j+1}^{l} |a_k|^p \Big)^{1/p}.$$

(i) Assume $2 \le q < \infty$. Then $1 < q' \le 2$, so by the Hausdorff-Young theorem we obtain

$$\Big(\int_\Gamma | \sum_{k=j+1}^{l} a_k \lambda^k |^q d\lambda \Big)^{1/q} \le \Big(\sum_{k=j+1}^{l} |a_k|^{q'} \Big)^{1/q'}.$$

(ii) When $1 < q' \le p$, the inequality $|| \cdot ||_{q'} \le || \cdot ||_p$ in probability spaces yields

$$\Big(\sum_{k=j+1}^{l} |a_k|^{q'} \Big)^{1/q'} = (l-j)^{1/q'} \Big(\frac{1}{l-j} \sum_{k=j+1}^{l} |a_k|^{q'} \Big)^{1/q'} \le (l-j)^{1/q'} \Big(\frac{1}{l-j} \sum_{k=j+1}^{l} |a_k|^p \Big)^{1/p}.$$

Using (11) the result follows. □

Remarks. 1. When $\sup_n \|f_n\|_p < \infty$ and all a_n are non-negative, (11) holds with $q = \infty$.

2. By Proposition 2.11, (11) implies (10) with $t = q'$, so Proposition 2.10 can be applied.

3. Recall the following definition (see [23] and the references therein): Let $1 \le p, q \le \infty$. A sequence of random variables $\{f_n\}$ is said to be (p,q)-*bounded*, if there is a universal constant $C > 0$ such that for *any finite sequence* of complex numbers $a_{j+1}, \dots, a_l$, $0 \le j < l$, (11) holds.

In Proposition 2.11 we assume that we are given only *one* sequence of complex numbers $\{a_n\}$ such that the pair $(\{a_n\}, \{f_n\})$ satisfies (11) for some $q \ge 2$ and obtain (10) with $t = q'$.

4. Houdré [23, Theorem 3.1] proved that if (11) holds for $q \ge p = 2$ and

$$\sum_{n=-\infty}^{\infty} |a_n|^2 |n|^{(q-2)/q} (\log(1+|n|))^2 < \infty \tag{13}$$

then $\sum_n a_n f_n$ converges a.e. (the proof in [23] does not need for $\{f_n\}$ to be $(2,q)$-bounded). When $q = 2$ this convergence follows from Proposition 2.10(i), and when $q > 2$ we can use Proposition 2.10(ii) with $t = q'$, since Hölder's inequality in $\ell_{2/q'}$ yields

$$\sum_{n=1}^{\infty} |a_n|^{q'} = \sum_{n=1}^{\infty} |a_n|^{q'} \frac{n^{(q-2)/(2q-2)} (\log(1+n))^{q'}}{n^{(q-2)/(2q-2)} (\log(1+n))^{q'}} \le$$

$$\Big(\sum_{n=1}^{\infty} |a_n|^2 n^{(q-2)/q} (\log(1+n))^2 \Big)^{q'/2} \Big(\sum_{n=1}^{\infty} \frac{1}{n (\log(1+n))^{2q'/(2-q')}} \Big)^{(2-q')/2} < \infty.$$

Note that convergence of $\sum_{n=1}^\infty |a_n|^{q'}$ does not imply (13). Specifically, for $q > 2$ define $a_n = n^{-(q-2)/2q}$ for $n = 2^k$, and $a_n = 2^{-n}$ otherwise. We then have

$$\sum_{n=1}^\infty |a_n|^2 n^{(q-2)/q} (\log n)^2 \geq \sum_{n\in\{2^k\}} |a_n|^2 n^{(q-2)/q} (\log n)^2 = \sum_{k=1}^\infty k^2 = \infty\,.$$

On the other hand, it is easy to check that $\sum_{n=1}^\infty |a_n|^{q'} < \infty$.

5. For a (p,q)-bounded sequence $\{f_n\}_{n\in\mathbb{Z}}$ with $p < 2 \leq q$, a.e. convergence of $\sum_{k=-\infty}^\infty a_k f_k$ is proved in [23] for any $\{a_n\}$ satisfying (13). The (p,q)-boundedness is used there to obtain that $\{f_n\}$ is a projection of a $(2,q)$-bounded sequence.

We deal here only with one pair $(\{a_n\},\{f_n\})$ that satisfies (11), rather than (p,q)-boundedness of $\{f_n\}$; for $q' < p$, condition (13) implies the a.e. convergence of $\sum_n a_n f_n$ by Proposition 2.10(ii) (see the previous remark). For $p \leq q' < 2$, we obtain the convergence from $\sum_{n=1}^\infty |a_n|^p (\log n)^p < \infty$, by Proposition 2.10(i). This last condition does not imply (13); for the sequence defined in remark 5 above,

$$\sum_{n=1}^\infty |a_n|^p (\log n)^p \leq \sum_{k=1}^\infty \frac{k^p}{2^{kp(q-2)/2q}} + \sum_{n=1}^\infty 2^{-np} (\log n)^p < \infty.$$

6. Let $\{f_n\} \subset L_p(\mu)$, $1 < p < \infty$, satisfy $\sup_n ||f_n||_p < \infty$, and let $\{a_n\}$ satisfy $\sum_{n=1}^\infty |a_n|^p n^{p-1} (\log n)^p < \infty$. Then $\sum_n |a_n| < \infty$, since putting $p' = \frac{p}{p-1}$ and using Hölder's inequality we have

$$\sum_{n=2}^N |a_n| = \sum_{n=2}^N \frac{1}{n^{\frac{p-1}{p}} \log n} |a_n| n^{\frac{p-1}{p}} \log n \leq \Big[\sum_{n=2}^N \frac{1}{n(\log n)^{p'}}\Big]^{\frac{1}{p'}} \Big[\sum_{n=2}^N |a_n|^p n^{p-1} (\log n)^p\Big]^{\frac{1}{p}}.$$

Hence $\sum_{n=1}^\infty |a_n f_n|$ converges a.e. For $p = 2$ this convergence was proved (using deeper results) by Houdré [23, Remark 3.4(iv)].

7. Let $p > 1$, define $a_n = \frac{1}{n(\log n)(\log\log n)^{(p+1)/2p}}$, and put $f_n \equiv 1$. Clearly, the series $\sum_{n=1}^\infty a_n f_n$ everywhere diverges, but since $p > 1$, for any "rate" $0 \leq b_n \leq Cn^{p-1} (\log n)^{p-1}$ we have

$$\frac{1}{C} \sum_{n=2}^\infty |a_n|^p b_n \leq \sum_{n=2}^\infty |a_n|^p n^{p-1} (\log n)^{p-1} = \sum_{n=2}^\infty \frac{1}{n(\log n)(\log\log n)^{(p+1)/2}} < \infty.$$

Thus the power of n in the condition of the previous remark is optimal, and the logarithm should be with power greater than $p-1$.

Theorem 2.12. *Let $\{f_n\}_{n=1}^\infty \subset L_p(\mu)$ with $1 \leq p < \infty$. Let $\{d(j,l) : 0 \leq j < l < \infty\}$ be a super additive sequence of non-negative numbers, and define $A_n^{(d)}$ as in (5).*

(i) If $\varphi(n)$ is a sequence decreasing to zero with $0 < \varphi(n) \leq C\varphi(2n)$, such that

$$\sum_{n=1}^\infty \frac{A_n^{(d)} \varphi^p(n) (\log n)^p d(0,n)}{n} < \infty,$$

then $\varphi(n) \sum_{k=1}^n f_k \to 0$ a.e. and in L_p-norm. Furthermore, $\sup_{n>0} |\varphi(n) \sum_{k=1}^n f_k| \in L_p(\mu)$.

(ii) If in addition $\sum_{n=1}^\infty [\varphi(n) - \varphi(n+1)] \Big(A_n^{(d)} d(0,n)\Big)^{\frac{1}{p}} < \infty$, then $\sum_{n=1}^\infty \varphi(n) f_n$ converges a.e. and in L_p-norm. Furthermore, $\sup_{n>0} |\sum_{k=1}^n \varphi(k) f_k| \in L_p(\mu)$.

Proof. We may and do assume that μ is a probability measure. Denote $S_n = \sum_{k=1}^n f_k$. By the definition of $A_n^{(d)}$, we have $||S_{2^m}||_p^p \le A_{2^m}^{(d)} d(0, 2^m)$.

By Proposition 2.2

$$\Big|\Big| \max_{2^m < n \le 2^{m+1}} | \sum_{k=2^m+1}^{n} f_k | \Big|\Big|_p^p \le 3^p A_{2^{m+1}}^{(d)} m^p d(2^m, 2^{m+1}).$$

Since d is a non-negative super additive sequence we have $d(2^m, 2^{m+1}) \le d(0, 2^{m+1})$ and $d(0, 2^m) \le d(0, 2^{m+1})$. By using this and the monotonicity of $A_n^{(d)}$, we have

$$\sum_{m=1}^{\infty} \left[\Big|\Big| \varphi(2^m) S_{2^m} \Big|\Big|_p^p + \Big|\Big| \varphi(2^m) \max_{2^m < n \le 2^{m+1}} | \sum_{k=2^m+1}^{n} f_k | \Big|\Big|_p^p \right] \le \qquad (*)$$

$$2 \cdot 3^p C^p \sum_{m=1}^{\infty} A_{2^{m+1}}^{(d)} m^p \varphi^p (2^{m+1}) d(0, 2^{m+1}).$$

The convergence of the right hand side above will imply the convergence to zero of $\{\varphi(n) \sum_{k=1}^n f_k\}_n$, a.e. and in L_p-norm. Indeed, by monotonicity of $A_n^{(d)}$, $\varphi(n)$, and $d(0, n)$, term by term estimation and the inequality $\varphi(n) \le C\varphi(2n)$ yield

$$\sum_{n=2^m+1}^{2^{m+1}} \frac{A_n^{(d)} \varphi^p(n) (\log n)^p d(0, n)}{n} \ge$$

$$2^m \cdot \frac{A_{2^m}^{(d)} \varphi^p(2^{m+1}) m^p}{2^{m+1}} d(0, 2^m) \ge \frac{1}{2C^p} A_{2^m}^{(d)} \varphi^p(2^m) (m-1)^p d(0, 2^m).$$

By summing on m, and considering the assumption of the theorem, the convergence of the right hand side of (*) follows, which implies convergence to zero (a.e. and in L_p-norm) of $\{\varphi(2^m) \sum_{k=1}^{2^m} f_k\}_m$ and $\{\varphi(2^m) \max_{2^m < n \le 2^{m+1}} | \sum_{k=2^m+1}^n f_k |\}_m$. The claimed convergence to zero (a.e. and in norm) of $\{\varphi(n) \sum_{k=1}^n f_k\}$ is now deduced as in Theorem 2.4.

Because the series in (*) converges, we can integrate the inequalities

$$\sup_{m \ge 1} |\varphi(2^m) S_{2^m}|^p \le \sum_{m=1}^{\infty} |\varphi(2^m) S_{2^m}|^p,$$

and

$$\sup_{m \ge 1} \left[\varphi(2^m) \max_{2^m < n \le 2^{m+1}} | \sum_{k=2^m+1}^{n} f_k | \right]^p \le \sum_{m=1}^{\infty} \left[\varphi(2^m) \max_{2^m < n \le 2^{m+1}} | \sum_{k=2^m+1}^{n} f_k | \right]^p.$$

This implies the integrability of the maximal function.

Using Abel's summation by parts,

$$\sum_{k=1}^{n} \varphi(k) f_k = \varphi(n) S_n + \sum_{k=1}^{n-1} [\varphi(k) - \varphi(k+1)] S_k. \qquad (**)$$

The first term on the right converges to zero, a.e. and in L_p-norm, as shown above. Since μ is a probability, the assumption yields

$$\sum_{k=1}^{\infty} [\varphi(k) - \varphi(k+1)] ||S_k||_1 \le \qquad (***)$$

$$\sum_{k=1}^{\infty} [\varphi(k) - \varphi(k+1)] ||S_k||_p \le \sum_{k=1}^{\infty} [\varphi(k) - \varphi(k+1)] \left(d(0, k) A_k^{(d)} \right)^{\frac{1}{p}} < \infty.$$

Hence the series on the right of (**) converges, absolutely a.e. by the convergence of the left term of (***), and in L_p-norm by the convergence of the middle term of (***). Hence $\{\sum_{k=1}^n \varphi(k) f_k\}$ converges a.e. and in L_p-norm.

For the maximal function we have

$$\sup_{n>0} \left|\sum_{k=1}^n \varphi(k) f_k\right| \le \sup_{n>0} |\varphi(n) S_n| + \sum_{k=1}^\infty [\varphi(k) - \varphi(k+1)] |S_k|.$$

The first term is in L_p as shown before. The second term is in L_p by (***). □

Remarks. 1. Let j be the first integer with $f_j \neq 0$, so $A_j^{(d)} > 0$. By definition, $A_n^{(d)}$ and $d(0,n)$ are non-decreasing, so we have

$$\sum_{n=j}^\infty \frac{\varphi^p(n)(\log n)^p A_j^{(d)} d(0,1)}{n} \le \sum_{n=j}^\infty \frac{A_n^{(d)} \varphi^p(n)(\log n)^p d(0,n)}{n}.$$

Hence for the convergence of the majorizing series, $\{\varphi(n)\}$ must decrease to zero faster than $\{\frac{1}{(\log n)^{1+1/p}}\}$. On the other hand, the condition $0 < \varphi(n) \le C\varphi(2n)$ does not allow φ to decrease to zero too fast: $\varphi(n) \ge \varphi(1)/n^{\log_2 C}$.

2. In contrast with Theorem 2.4, Theorem 2.12 gives conditions for a specific rate of convergence. It can happen that for given $\{f_n\}$ and d, the series $\sum_n f_n$ does not converge (so the condition of Theorem 2.4 does not hold); in that case Theorem 2.12 allows to evaluate the rate of growth of the partial sums.

3. In order to obtain the a.e. convergence of $\sum_{n=1}^\infty \varphi(n) f_n$ from Theorem 2.4, one must be able to compute (or estimate) the corresponding $A_n^{(d)}$.

The proof of the following theorem proceeds along the same lines as that of Theorem 2.12. Here we use Proposition 2.3 instead of Proposition 2.2.

Theorem 2.13. *Let $\{f_n\}_{n=1}^\infty \subset L_p(\mu)$ with $1 \le p < \infty$. Let $\{d(j,l) : 0 \le j < l < \infty\}$ be a super additive sequence of non-negative numbers, and let $q > 1$. Define $A_n^{(d^q)}$ as in (5).*

(i) If $\varphi(n)$ is a sequence decreasing to zero with $0 < \varphi(n) \le C\varphi(2n)$, such that

$$\sum_{n=1}^\infty \frac{A_n^{(d^q)} \varphi^p(n) d^q(0,n)}{n} < \infty,$$

then $\varphi(n) \sum_{k=1}^n f_k \to 0$ a.e. and in L_p-norm. Furthermore, $\sup_{n>0} |\varphi(n) \sum_{k=1}^n f_k| \in L_p(\mu)$.

(ii) If in addition $\sum_{n=1}^\infty [\varphi(n) - \varphi(n+1)] \left(A_n^{(d^q)} d^q(0,n)\right)^{\frac{1}{p}} < \infty$, then $\sum_{n=1}^\infty \varphi(n) f_n$ converges a.e. and in L_p-norm. Furthermore, $\sup_{n>0} |\sum_{k=1}^n \varphi(k) f_k| \in L_p(\mu)$.

Remarks. 1. The following condition was considered in Gaposhkin [17]:

There exists a positive non-decreasing sequence $\{\Psi(n)\}$, satisfying $\Psi(2n) \le C\Psi(n)$ for some positive constant C, such that for any nonnegative integers n and m

$$\Big\| \sum_{k=m+1}^{m+n} a_k f_k \Big\|_2^2 \le \Psi(n) \sum_{k=m+1}^{m+n} |a_k|^2. \tag{14}$$

If the pair $(\{a_n\}, \{f_n\})$ satisfies (14), then $A_n^{(2,1,2)} \le \Psi(n)$.

2. Condition (14) can fail even for orthogonal sequences. Take $a_n \equiv 1$ and $\{f_n\}$ orthogonal with $\{||f_n||_2\}$ unbounded; (14) does not hold, since $\Psi(1) < \infty$ implies $\sup_n ||f_n|| < \infty$.

Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the lower and the upper integral parts. For a given positive non-decreasing sequence $\{\Psi(n)\}_{n=1}^{\infty}$ define $\Lambda(1) = \Psi(1)$ and $\Lambda(n) = \sum_{k=0}^{\lfloor \log n \rfloor} \Psi(\lceil \frac{n}{2^{k+1}} \rceil)$. The following theorem is simply Theorem 4 of [32]. The above explicit formula for $\Lambda(n)$ is given in [33] with $Q = 1$ there.

Proposition 2.14. *Let $\{f_n\}_{n=1}^{\infty} \subset L_p(\mu)$, $1 < p < \infty$, and let $\{d(j,l)\}$ be a super additive sequence of non-negative numbers. Assume that there exists a positive non-decreasing sequence $\{\Psi(n)\}_{n=1}^{\infty}$ such that*

$$\Big\| \sum_{k=j+1}^{l} f_k \Big\|_p^p \leq \Psi^p(l-j) d(j,l) \qquad \text{for} \qquad l > j \geq 0. \tag{15}$$

Then for any $0 \leq n_1 < n$, we have

$$\Big\| \max_{n_1 < l \leq n} \Big| \sum_{k=n_1+1}^{l} f_k \Big| \Big\|_p^p \leq \Lambda^p(n-n_1) d(n_1, n).$$

Example 2.2. The following can be verified by the above formula for $\Lambda(n)$. If $\Psi(n) = (\log n)^{\beta}$ with $\beta \geq 0$, then $\Lambda(n) \leq (2 + \log n)^{\beta+1}$. If $\Psi(n) = n^{\alpha}(\log n)^{\beta}$ with $\alpha > 0$ and β any real, then $\Lambda(n) \leq K_{\alpha,\beta} n^{\alpha} (\log n)^{\beta}$.

Remarks. 1. Since $\{\Psi(n)\}$ is non-decreasing, condition (15) yields $A_n^{(d)} \leq \Psi^p(n)$.

2. The above example shows that when $\Psi(n) = (\log n)^{\beta}$ with $\beta \geq 0$, Proposition 2.14 gives no more than Proposition 2.2, although the assumption in Proposition 2.14 is stronger.

Theorem 2.15. *Let $\{f_n\}_{n=1}^{\infty} \subset L_p(\mu)$ with $1 < p < \infty$. Let $\{d(j,l),\ l > j \geq 0\}$ be a non-negative super additive sequence, and put $m_1 = d(0,1)$, $m_n = d(0,n) - d(0,n-1)$ for $n > 1$. Assume that for some $\alpha > 0$ and β real, condition (15) holds with $\Psi(n) = n^{\alpha}(\log n)^{\beta}$. If $\sum_{n=2}^{\infty} n^{\alpha p} (\log n)^{p(\beta+1)-1} (\log \log n)^{p-1+\epsilon} m_n$ converges for some $\epsilon > 0$, then $\sum_{n=1}^{\infty} f_n$ converges a.e. and in L_p-norm, with $\sup_{n \geq 1} |\sum_{k=1}^{n} f_k| \in L_p(\mu)$.*

Proof. The proof proceeds along the same lines as that of Theorem 2.6, with $q = 1$ and $\psi(u) = u^{(p-1)/p} (\log u)^{(p-1)/p+\epsilon}$. We use Proposition 2.14 instead of Proposition 2.3, where the estimation of $\Lambda(n)$ is taken from Example 2.2. ☐

3 Applications to ergodic theory

In this section we look at the problem of a.e. convergence of series $\sum_n n^{-\alpha} a_n T^n f$, $\alpha < 1$, for power-bounded operators on L_p. We apply the previous results in order to obtain conditions on $\{a_n\}$ and on the function $f \in L_p$, which ensure the a.e. convergence for an appropriate α. For contractions on L_2 we obtain conditions on f in terms of $\{\langle T^n f, f \rangle\}$.

Theorem 3.1. *Let T be a power bounded operator on $L_p(\mu)$, $1 < p < \infty$, and $f \in L_p(\mu)$ such that for some $0 < \beta \leq 1$, we have*

$$K := \sup_{n>0} || \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} T^k f ||_p < \infty \ . \tag{16}$$

Let $\{b_n\}$ be a sequence of complex numbers such $\sum_{n=1}^{\infty} |b_n - b_{n+1}| < \infty$.

(i) When $0<\beta<\frac{p-1}{p}$*, for every* $\epsilon>0$ *the series* $\displaystyle\sum_{n=2}^{\infty}\frac{b_nT^nf}{n^{1-\beta}(\log n)^{1+\epsilon}}$ *converges a.e. and in* L_p*-norm; moreover,* $\displaystyle\frac{1}{n^{1-\beta}(\log n)^{1/p+\epsilon}}\sum_{k=1}^{n}b_kT^kf\to 0$ *a.e. and in* L_p*-norm.*

(ii) When $\frac{p-1}{p}\le\beta\le 1$*, for every* $\epsilon>0$ *the series* $\displaystyle\sum_{n=2}^{\infty}\frac{b_nT^nf}{n^{1/p}(\log n)^{1+\frac{1}{p}+\epsilon}}$ *converges a.e. and in* L_p*-norm.*

In each of the above cases, the corresponding maximal function is in L_p*.*

Proof. Since $\{b_n\}$ is of bounded variation, it converges. Put $V=\sum_{n=1}^{\infty}|b_n-b_{n+1}|$. Inspection of the proof of Lemma 1 in [10] shows that if (16) holds, then also

$$\sup_{n>0}||\frac{1}{n^{1-\beta}}\sum_{k=1}^{n}b_kT^kf||_p\le K':=KV+K\sup_n|b_n|<\infty. \qquad (*)$$

For any $j\ge 0$ the sequence $\{b_{j+n}\}_{n=1}^{\infty}$ is also of bounded variation, and clearly $\sum_{n=1}^{\infty}|b_{j+n}-b_{j+n+1}|\le V$. Applying (*) to the sequence $\{b_{j+n}\}_{n=1}^{\infty}$, and noting that K and V, hence K', are independent of j, we obtain

$$||\sum_{k=j+1}^{l}b_kT^kf||_p^p=||T^j\sum_{k=1}^{l-j}b_{j+k}T^kf||_p^p\le(\sup_{k\ge 0}||T^k||^p)(K')^p(l-j)^{p(1-\beta)}. \qquad (**)$$

For positive α and γ let $\varphi(u)=\frac{1}{u^{\alpha}(\log u)^{\gamma}}$. Using the derivative we obtain that $\varphi(n)-\varphi(n+1)\le\frac{\alpha+\gamma/\log_e 2}{n^{\alpha+1}(\log n)^{\gamma}}$ for $n\ge 2$. Put $f_n=b_nT^nf$, and for $l>j\ge 0$ define $d(j,l)=l-j$.

(i) Put $q=p(1-\beta)>1$. From (**) we obtain that $A_n^{(d^q)}\le(K')^p\sup_{k\ge 0}||T^k||^p$. Theorem 2.13 applies, with the appropriate α and γ.

(ii) Since $p(1-\beta)\le 1$, using (**) we have that $A_n^{(d)}\le(K')^p\sup_{k\ge 0}||T^k||^p$. So, Theorem 2.12 applies. □

Remarks. 1. The estimate (**) in the proof allows us to use the results of Gál and Koksma [14], which yield the same "strong laws of large numbers with rates" as in the above theorem; in case (i) we use [14, Theorem 5], and in case (ii) we use [14, Theorem 3].

2. The case $b_n\equiv 1$ was treated in Gaposhkin [18, Theorem 3] when $p=2$ and T is unitary on L_2, in Derriennic and Lin [11, Corollary 3.7] when T is a Dunford-Schwartz operator, and in Weber [52, Proposition 1.6] in the general case treated here. Applying Kronecker's lemma to the series in (i) (with $b_n\equiv 1$) yields the same "strong law with rate" as Weber for $\beta>(p-1)/p$, but our rate obtained directly in (i) is better; the rate in the "strong law of large numbers" obtained from (ii) by Kronecker's lemma is the same as Weber's when $\beta=(p-1)/p$, but worse than Weber's (in the power of the logarithm) when $\beta<(p-1)/p$. For T Dunford-Schwartz, our result is better than [11] when $\beta\le(p-1)/p$. For T unitary, Gaposhkin's results are better than ours.

3. Sublinear growth conditions on the norms $\{||\sum_{k=1}^{n}T^kf||\}$ were used also in [10] and [9] to obtain for f the pointwise ergodic theorem with rate, as well as a.e. convergence of the one-sided ergodic Hilbert transform. Our present results are more precise.

For an $L_2(\mu)$-bounded sequence $\{f_j\}_{j=1}^{\infty}$ and any integer $n\ge 0$ we define

$$\Phi(n):=\sup_{j\ge 1}|\int f_j\overline{f_{j+n}}d\mu|<\infty.$$

Clearly, $\Phi(n)\le\sup_{j\ge 1}||f_j||_2^2$.

Remarks. 1. $L_2(\mu)$-bounded sequences $\{f_j\}$, with $\int f_j d\mu = 0$ and $\sum_{n=0}^{\infty} \Phi(n) < \infty$, were considered in Gaposhkin [19], and were called *weakly correlated* sequences.

3. For an isometry operator V, $f \in L_2(\mu)$, and $f_n := V^n f$ (i.e., $\{f_n\}$ is wide sense stationary) we have $\Phi(n) = |\int f_n \overline{f_0} d\mu|$.

The following lemma appears in Gaposhkin [16, Lemma 1] (see also Serfling [43, Lemma 2.1], Weber [51, Lemma 14]).

Lemma 3.2. *Let $\{a_n\}$ be a sequence of complex numbers, and let $\{f_n\}$ be an $L_2(\mu)$-bounded sequence. Then for any $n, m \geq 1$*

$$\Big\| \sum_{k=m+1}^{m+n} a_k f_k \Big\|_2^2 \leq [\Phi(0) + 2 \sum_{k=1}^{n-1} \Phi(k)] \sum_{k=m+1}^{m+n} |a_k|^2. \tag{17}$$

Corollary 3.3. *Let $\{f_n\}$ be an $L_2(\mu)$-bounded sequence, and put $\sigma_n = \Phi(0) + 2\sum_{k=1}^{n} \Phi(k)$. Let $\{a_n\}$ be a sequence of complex numbers. The series $\sum_{n=1}^{\infty} a_n f_n$ converges a.e. and in L_2-norm, with $\sup_{n>0} |\sum_{k=1}^{n} a_k f_k|$ in $L_2(\mu)$, in either of the following cases:*

(i) $\sigma_n = O(n^{\alpha} (\log n)^{\beta})$ *for some $\alpha > 0$ and β real, and the series $\sum_{n=2}^{\infty} |a_n|^2 n^{\alpha} (\log n)^{\beta+1} (\log\log n)^{1+\epsilon}$ converges for some $\epsilon > 0$.*

(ii) $\sigma_n = O((\log n)^{\beta})$ *with $\beta \geq 0$, and $\sum_{n=1}^{\infty} |a_n|^2 (\log n)^{\beta+2}$ converges,*

Proof. Put $d(j,l) = \sum_{k=j+1}^{l} |a_k|^2$ and $p = 2$.

(i) By Lemma 3.2 we obtain (15) for $\{a_n f_n\}$, with $\Psi^2(n) = \sigma_n = O(n^{\alpha} (\log n)^{\beta})$. The result follows by applying Theorem 2.15.

(ii) By Lemma 3.2 we obtain (9) for $\{a_n f_n\}$, with $q = 1$, $m_n = |a_n|^2$, and $A_n = \sigma_n$. Theorem 2.9(i) yields the result. □

Remarks. 1. Obviously $\sigma_n = O(n)$, but convergence of $\sum_n |a_n|^2 n \log n (\log\log n)^{1+\epsilon}$ implies $\sum_n |a_n| < \infty$ by Cauchy's inequality, so the interest in (i) is when $\alpha < 1$.

2. Without refering to the order of σ_n, Theorem 2.9(i) yields the desired convergence when $\sum_n |a_n|^2 \sigma_{2n} (\log n)^2 < \infty$. This, with σ_{2n} replaced by σ_n, was obtained by Gaposhkin [16, Theorem 1]. Note that this is not important for the classes of $\{\Phi(n)\}$ considered there.

3. Part (ii) was proved in [16, Corollaries 1 and 2]. A better result (smaller power of $\log n$) for part (i) was obtained in [16, Corollary 3], under a mild additional additional condition on $\Phi(n)$.

4. In the stationary case, Gaposhkin [16] proved that under a given rate of decay to zero of $\{\Phi(n)\}$, the convergence of $\sum_{n=1}^{\infty} |a_n|^2 \sigma_n^2 (\log n)^2$ is an optimal condition for the a.e. convergence of $\sum_{n=1}^{\infty} a_n f_n$. Note that for $\{f_n\}$ orthonormal, Corollary 3.3(ii) becomes the Menchoff-Rademacher theorem.

Let T be a contraction of a Hilbert space $\mathcal{H}$. Define $T_n := T^n$ for $n \geq 0$ and $T_n := (T^*)^{|n|}$ for $n < 0$. Then $\{\langle T_n f, f \rangle\}$ is a positive semi-definite sequence [39, Appendix, §9] (see also [27, Proposition 3.1, p. 94]), so by Herglotz's theorem it is the Fourier coefficients of a positive measure ν_f on the unit circle Γ. By the unitary dilation theorem of B. Sz. Nagy [39, Theorem III, p.469] (the proof of which uses the positive semi-definiteness of $\{\langle T_n f, f \rangle\}$), there exist a larger Hilbert space $\mathcal{H}'$, an orthogonal projection $P_{\mathcal{H}}$ on $\mathcal{H}$, and unitary operator U on $\mathcal{H}'$ such that for every $g \in \mathcal{H}'$ and every integer n we have $T_n P_{\mathcal{H}} g = P_{\mathcal{H}} U^n g$. For $f \in \mathcal{H}$, the above identity yields

$$\langle T_n f, f \rangle = \langle P_{\mathcal{H}} U^n f, f \rangle = \langle U^n f, P_{\mathcal{H}}^* f \rangle = \langle U^n f, P_{\mathcal{H}} f \rangle = \langle U^n f, f \rangle.$$

By the spectral representation theorem for unitary operators, ν_f is the spectral measure of f with respect to U, with Fourier coefficients $\{\langle T_n f, f \rangle\}$.

Definition 3.1. For a contraction T on $\mathcal{H}$ and $f \in \mathcal{H}$, we call ν_f the *unitary spectral measure* of f (with respect to T). When ν_f is absolutely continuous, we say that f has *spectral density*, which is $\frac{d\nu_f}{d\lambda}$.

Remark. There are cases where all that is needed is to extend T to an isometry, i.e., we need an *isometry dilation*. If V is an isometry dilation of T, then we still have $\langle T^n f, f\rangle = \langle V^n f, f\rangle$ for all nonnegative n and $f \in \mathcal{H}$.

Proposition 3.4. *Let $\{a_n\}$ be a sequence of complex numbers. Let T be a contraction of $L_2(\mu)$ and $f \in L_2(\mu)$. For any integers $m, n \geq 1$ we have the following:*

(i) $\Big|\Big|\sum_{k=m+1}^{m+n} a_k T^k f\Big|\Big|_2^2 \leq [||f||_2^2 + 2\sum_{k=1}^{n-1}|\langle T^k f, f\rangle|]\sum_{k=m+1}^{m+n}|a_k|^2.$

(ii) For $1 < u < \infty$ and $v := \frac{u}{u-1}$,

$$\Big|\Big|\sum_{k=m+1}^{m+n} a_k T^k f\Big|\Big|_2^2 \leq \left[||f||_2^2 + 2n^{\frac{1}{u}}(\sum_{k=1}^{n-1}|\langle T^k f, f\rangle|^v)^{\frac{1}{v}}\right]\sum_{k=m+1}^{m+n}|a_k|^2 \ .$$

(iii) If f has spectral density in $L_u(d\lambda)$, $1 < u < \infty$, then

$$\Big\|\sum_{k=m+1}^{m+n} a_k T^k f\Big\|_2^2 \leq \Big\|\frac{d\nu_f}{d\lambda}\Big\|_{L_u(d\lambda)}\Big(\sum_{k=m+1}^{m+n}|a_k|^{\frac{2u}{u+1}}\Big)^{\frac{u+1}{u}} \leq n^{1/u}\Big\|\frac{d\nu_f}{d\lambda}\Big\|_{L_u(d\lambda)}\sum_{k=m+1}^{m+n}|a_k|^2 \ .$$

(iv) If f has bounded spectral density, then

$$\Big|\Big|\sum_{k=m+1}^{m+n} a_k T^k f\Big|\Big|_2^2 \leq \Big\|\frac{d\nu_f}{d\lambda}\Big\|_{L_\infty(d\lambda)}\sum_{k=m+1}^{m+n}|a_k|^2.$$

Proof. (i) We first prove it when $T = V$ is isometry. We take $\Phi(n) = |\langle V^n f, f\rangle|$ and $f_n = V^n f$, hence (i) follows (for V) by Lemma 3.2.

Now for T a contraction, let V be the isometry dilation of T, and let $P_{\mathcal{H}}$ be the corresponding projection. By the discussion preceeding the proposition, $\langle T^n f, f\rangle = \langle V^n f, f\rangle$, so we have

$$\Big|\Big|\sum_{k=m+1}^{m+n} a_k T^k f\Big|\Big|_2^2 = \Big|\Big|P_{\mathcal{H}}\Big(\sum_{k=m+1}^{m+n} a_k V^k f\Big)\Big|\Big|_2^2 \leq \Big|\Big|\sum_{k=m+1}^{m+n} a_k V^k f\Big|\Big|_2^2 \leq$$

$$[||f||_2^2 + 2\sum_{k=1}^{n-1}|\langle V^k f, f\rangle|]\sum_{k=m+1}^{m+n}|a_k|^2 = [||f||_2^2 + 2\sum_{k=1}^{n-1}|\langle T^k f, f\rangle|]\sum_{k=m+1}^{m+n}|a_k|^2.$$

(ii) Using Hölder's inequalty

$$\sum_{k=1}^{n-1}|\langle T^k f, f\rangle| \leq n^{\frac{1}{u}}(\sum_{k=1}^{n-1}|\langle T^k f, f\rangle|^v)^{\frac{1}{v}},$$

hence (ii) follows from (i).

(iii) In the proof of (i) we could use the unitary dilation U of T instead of using the isometry dilation, so it suffices to prove for U. Denote the spectral density $h = \frac{d\nu_f}{d\lambda} \in L_u(d\lambda)$. When $u < \infty$, the spectral theorem and Hölder's inequality yield

$$\Big\|\sum_{k=m+1}^{m+n} a_k U^k f\Big\|_2^2 = \int_\Gamma\Big|\sum_{k=m+1}^{m+n} a_k\lambda^k\Big|^2 h(\lambda)d\lambda \leq ||h||_{L_u(d\lambda)}\Big(\int_\Gamma\Big|\sum_{k=m+1}^{m+n} a_k\lambda^k\Big|^{\frac{2u}{u-1}}\Big)^{\frac{u-1}{u}}d\lambda.$$

Hence $\Big\|\sum_{k=m+1}^{m+n} a_k U^k f\Big\|_2 \le ||h||_{L_u(d\lambda)}^{1/2} \Big\|\sum_{k=m+1}^{m+n} a_k \lambda^k\Big\|_{L_q(d\lambda)}$ with $q = 2u/(u-1)$. We now apply Proposition 2.11(i-ii) with $p = 2$.

(iv) Again we prove only for U unitary. Put $h(\lambda) = \frac{d\nu_f}{d\lambda}$. The spectral theorem now yields

$$\Big\|\sum_{k=m+1}^{m+n} a_k U^k f\Big\|_2^2 = \int \Big|\sum_{k=m+1}^{m+n} a_k \lambda^k\Big|^2 h(\lambda)d\lambda \le ||h||_\infty \sum_{k=m+1}^{m+n} |a_k|^2.$$

□

Remark. For T unitary, (i) and (iv) appear (without proof) in Gaposkin [17].

Proposition 3.5. *Let $\{a_n\}$ be a sequence of complex numbers, and let $1 < u < \infty$ with dual index $v := u/(u-1)$. Let T be a contraction of $L_2(\mu)$ and $f \in L_2(\mu)$. For any integers $m, n \ge 1$ we have the following:*

(i) $\Big\|\sum_{k=m+1}^{m+n} a_k T^k f\Big\|_2^2 \le n^{1/v}(\sum_{k=m+1}^{m+n} |a_k|^u)^{2/u}\big(||f||_2^{2v} + 2\sum_{k=1}^{n-1} |\langle T^k f, f\rangle|^v\big)^{1/v}.$

(ii) If $1 < u \le 2$, and f has spectral density in $L_u(d\lambda)$, then

$$\Big\|\sum_{k=m+1}^{m+n} a_k T^k f\Big\|_2^2 \le 2^{1/v}\Big\|\frac{d\nu_f}{d\lambda}\Big\|_{L_u(d\lambda)} n^{1/v}(\sum_{k=m+1}^{m+n} |a_k|^u)^{2/u}. \tag{18}$$

Proof. (i) We first prove the proposition when $T = U$ is a unitary operator. Using Hölder's inequality

$$\|\sum_{k=m+1}^{m+n} a_k U^k f\|_2^2 = \sum_{k,i=m+1}^{m+n} a_k \bar{a}_i \langle U^k f, U^i f\rangle \le (\sum_{k=m+1}^{m+n} |a_k|^u)^{2/u}(\sum_{k,i=m+1}^{m+n} |\langle U^{k-i} f, f\rangle|^v)^{1/v} =$$

$$(\sum_{k=m+1}^{m+n} |a_k|^u)^{2/u}(\sum_{k,i=1}^{n} |\langle U^{k-i} f, f|^v)^{1/v} \le n^{1/v}(\sum_{k=m+1}^{m+n} |a_k|^u)^{2/u}(\sum_{k=-(n-1)}^{n-1} |\langle U^k f, f\rangle|^v)^{1/v} =$$

$$n^{1/v}(\sum_{k=m+1}^{m+n} |a_k|^u)^{2/u}(||f||_2^{2v} + 2\sum_{k=1}^{n-1} |\langle U^k f, f\rangle|^v)^{1/v}.$$

Now for T a contraction, let U be the unitary dilation of T, and let $P_{\mathcal{H}}$ be the corresponding projection. By the discussion preceeding Proposition 3.4 $\langle T^n f, f\rangle = \langle U^n f, f\rangle$ for $n \ge 0$, so using the previous calculation we have

$$\Big\|\sum_{k=m+1}^{m+n} a_k T^k f\Big\|_2^2 = \Big\|P_{\mathcal{H}}\Big(\sum_{k=m+1}^{m+n} a_k U^k f\Big)\Big\|_2^2 \le \Big\|\sum_{k=m+1}^{m+n} a_k U^k f\Big\|_2^2 \le$$

$$n^{1/v}(\sum_{k=m+1}^{m+n} |a_k|^u)^{2/u}(||f||_2^{2v} + 2\sum_{k=1}^{n-1} |\langle T^k f, f\rangle|^v)^{1/v}.$$

(ii) Follows from (i) by the Hausdorff-Young theorem. □

Example 3.1. For f in $L_2(\Gamma, d\lambda)$, where $d\lambda$ is the Lebesgue measure on the unit circle Γ, define $Uf(\lambda) := \lambda f(\lambda)$. Then U is unitary on $L_2(\Gamma, d\lambda)$. Fix $1 \le p \le \infty$ and $0 \le g \in L_p(d\lambda)$, and let $f = \sqrt{g} \in L_2$; clearly $\langle U^k f, f\rangle = \int \lambda^k g(\lambda)d\lambda$, so the unitary spectral measure of f with respect to U is absolutely continuous with density g.

Example 3.2. Let $\{g_k : -\infty < k < \infty\} \subset \mathcal{H}$ be an orthonormal sequence; then $g_n = U^n g_0$, where U is the bilateral shift on the closed subspace generated by $\{g_k\}$. For any $\{c_k\} \in \ell_2(\mathbb{Z})$ define the *moving average* sequence $f_n := \sum_{k=-\infty}^{\infty} c_k g_{n+k}$, where the series is convergent by the Riesz-Fischer theorem. Clearly $f_n = U^n f_0$ (so $\{f_n\}$ is a well defined wide sense stationary process). Denote by ν the spectral measure of f_0 with respect to U; then $\frac{d\nu}{d\lambda} = |a(\lambda)|^2$, where $a(\lambda) := \sum_{k=-\infty}^{\infty} c_k \lambda^k$ is defined in $L_2(d\lambda)$-norm by Riesz-Fischer, and hence $\frac{d\nu}{d\lambda} \in L_1(d\lambda)$. If we impose $\{c_k\} \in \ell_1$, then $a(\lambda)$ is a continuous function on Γ, so $\frac{d\nu}{d\lambda} \in L_2(d\lambda)$. When $\{c_k\} \in \ell_p$, $1 < p < 2$, then $a(\lambda) \in L_q(d\lambda)$ (where $q = \frac{p}{p-1} > 2$), by the Hausdorff-Young theorem, so $\frac{d\nu}{d\lambda} \in L_{q/2}(d\lambda)$, and with $u = \min\{\frac{q}{2}, 2\}$, Proposition 3.5(ii) applies to U and $f = f_0$.

Corollary 3.6. *Let $\{a_n\}$ be a sequence of complex numbers, and let $1 < u < \infty$ with dual index v. Let T be a contraction of $L_2(\mu)$ and $f \in L_2(\mu)$. The series $\sum_{n=1}^{\infty} a_n T^n f$ converges a.e. and in L_2-norm, and $\sup_{n\geq 1} |\sum_{k=1}^{n} a_k T^k f|$ is in $L_2(\mu)$, if for some $\epsilon > 0$ any of the following sets of conditions is satisfied:*

(i) $\sum_{n=1}^{\infty} |a_n|^2 n^{[1+\gamma(u-1)]/u} \log n (\log\log n)^{1+\epsilon} < \infty$ *and* $\sum_{k=1}^{n} |\langle T^k f, f\rangle|^v \leq C n^\gamma$, *for some* $0 \leq \gamma < 1$.

(ii) $\sum_{n=1}^{\infty} |a_n|^u n^{(1+\gamma)(u-1)/2} (\log n)^{u/2} (\log\log n)^{u/2+\epsilon} < \infty$ *and* $\sum_{k=1}^{n} |\langle T^k f, f\rangle|^v \leq C n^\gamma$, *for some* $0 \leq \gamma < 1$ *and* $1 < u < 2$.

(iii) $\sum_{n=1}^{\infty} |a_n|^{2u/(u+1)} < \infty$ *and f has spectral density in $L_u(d\lambda)$.*

Proof. (i) Put $d(j,l) = \sum_{k=j+1}^{l} |a_k|^2$ and $p = 2$. By Proposition 3.4(ii), (15) holds for $\{a_n T^n f\}$ with $\Psi^2(n) \leq C' n^{\frac{1}{u}+\frac{\gamma}{v}}$. Theorem 2.15 yields the result.

(ii) By Proposition 3.5(i), (9) holds with $A_n = C' n^{(\gamma+1)/v}$, $p = 2$, $m_n = |a_n|^u$, and $q = 2/u$. Since $1 < u < 2$, we have $q > 1$ and Theorem 2.9(iii) applies.

(iii) By the first inequality in Proposition 3.4(iii) we obtain (10) with $p = 2$ and $t = 2u/(u+1) < 2$. Hence Proposition 2.10(ii) applies.

□

Remarks. 1. For any $u > 1$ and $\gamma = 0$, the condition on $\{a_n\}$ given in (i) or in (ii) implies the condition in (iii). Indeed, by Hölder's inequality

$$\sum_{n=1}^{\infty} |a_n|^{\frac{2u}{u+1}} \leq \Big(\sum_{n=1}^{\infty} |a_n|^2 n^{\frac{1}{u}} \log n (\log\log n)^{1+\epsilon}\Big)^{\frac{u}{u+1}} \Big(\sum_{n=1}^{\infty} \frac{1}{n(\log n)^u (\log\log n)^{u(1+\epsilon)}}\Big)^{\frac{1}{u+1}}.$$

Similarly, the condition in (ii) with $\gamma = 0$ implies (iii).

2. If $1 < u \leq 2$ and f has spectral density in $L_u(d\lambda)$, then, as mentioned before, $\sum_{k=1}^{\infty} |\langle T^k f, f\rangle|^v$ converges. The previous remark shows that in this case (iii) yields a better result (weaker assumptions on $\{a_n\}$) than (i) or (ii).

3. If $2 \leq u < \infty$ and $\sum_{k=1}^{\infty} |\langle T^k f, f\rangle|^v$ converges, then by the Hausdorff-Young theorem f has spectral density in $L_u(d\lambda)$. By Remark 1 above (iii) yields a better result than (i). Thus, for $u \geq 2$ (i) is relevant only for $\gamma > 0$.

4. By the computation in Remark 1 above, the condition on $\{a_n\}$ given in [23, Corollary 3.3(i)] (for unitary operators) when f has spectral density in L_u implies the condition in (iii).

5. (i) and (ii) are equivalent for $u = 2$, but for $1 < u < 2$ and $\gamma = 0$, (ii) does not imply (i). Specifically, for *any* $1 < u < 2$ there exists a positive sequence $\{a_n\}$ such that the series $\sum_{n=1}^{\infty} |a_n|^u n^{(u-1)/2} (\log n)^{u/2} (\log\log n)^{u/2+\epsilon}$ converges, but $\sum_{n=1}^{\infty} |a_n|^2 n^{1/u} \log n (\log n \log n)^{1+\epsilon}$ diverges.

Define $a_n = (2^k)^{-1/2u}$ for $n = 2^k$, and $a_n = 2^{-n}$ otherwise. We have

$$\sum_{n\in\{2^k\}} |a_n|^2 n^{1/u} \log n (\log\log n)^{1+\epsilon} = \sum_{k=1}^{\infty} k (\log k)^{1+\epsilon} = \infty\,,$$

so (i) does not hold. On the other hand,

$$\sum_{n\in\{2^k\}} |a_n|^u n^{(u-1)/2}(\log n)^{u/2}(\log\log n)^{u/2+\epsilon} =$$

$$\sum_{k=1}^{\infty}(2^k)^{-1/2}(2^k)^{(u-1)/2}k^{u/2}(\log k)^{u/2+\epsilon} = \sum_{k=1}^{\infty}\frac{k^{u/2}(\log k)^{u/2+\epsilon}}{(2^{(2-u)/2})^k}\,.$$

The last sum converges since for $u < 2$ the denominator has exponential growth. The convergence of the series is not affected by adding the convergent series $\sum_{n\notin\{2^k\}}\cdot$, so (ii) holds.

6. Recall that for any T *power-bounded* on L_2, convergence of $\sum_{n=1}^{\infty}|a_n|^2 n(\log n)^2$ implies a.e. convergence of $\sum a_n T^n f$, by Remark 6 to Proposition 2.11. In each case of Corollary 3.6 the power of n in the series is less than 1.

We next show that when $u = \infty$, Corollary 3.6(i) remains true (only) when $\gamma > 0$.

Corollary 3.7. *Let $\{a_n\}$ be a sequence of complex numbers. Let T be a contraction of $L_2(\mu)$ and $f \in L_2(\mu)$. The series $\sum_{n=1}^{\infty} a_n T^n f$ converges a.e. and in L_2-norm, and $\sup_{n\geq 1}|\sum_{k=1}^{n} a_k T^k f|$ is in $L_2(\mu)$, if any of the following sets of conditions is satisfied:*

(i) $\sum_{n=1}^{\infty}|a_n|^2 n^{\gamma}\log n(\log\log n)^{1+\epsilon} < \infty$ and $\sum_{k=1}^{n}|\langle T^k f, f\rangle| \leq Cn^{\gamma}$, for some $\epsilon > 0$ and $0 < \gamma < 1$.

(ii) $\sum_{n=1}^{\infty}|a_n|^2(\log n)^2 < \infty$ and f has bounded spectral density.

Proof. (i) Put $d(j,l) = \sum_{k=j+1}^{l}|a_k|^2$ and $p = 2$. By Proposition 3.4(i), (15) holds for $\{a_n T^n f\}$ with $\Psi^2(n) \leq C' n^{\gamma}$. Theorem 2.15 yields the result.

(ii) By Proposition 3.4(iv), (9) holds with $A_n = C'$, $p = 2$, $q = 1$, and $m_n = |a_n|^2$, so Theorem 2.9(i) applies. □

Remarks. 1. If $\sum_{n=1}^{\infty}|\langle T^n f, f\rangle| < \infty$, then the unitary spectral measure of f is absolutely continuous with *continuous* Radon-Nikodym derivative. Hence the spectral density of f is in $L_u(d\lambda)$ for any $1 < u \leq \infty$, and we can use either Corollary 3.7(ii), or Corollary 3.6(iii) with some $u < \infty$ large. These two results are not comparable. When $a_n = 1/(\sqrt{n}\log^2 n)$, only Corollary 3.7(ii) applies; if we define $a_{2^k} = 1/k$ and $a_n = 0$ for n not a power of 2, then Corollary 3.6(iii) applies with any $u > 1$, while $\sum_n |a_n|^2\log^2 n = \infty$.

2. Let $\{f_n\} \subset L_2(\mu)$ be orthonormal, and let T be induced on L_2 by the shift, i.e., $Tg = 0$ for $g \in \{f_n\}^{\perp}$ and $Tf_n = f_{n+1}$ for $n \geq 1$. Applying part (ii) to T with $f = f_1$ yields the Menchoff-Rademacher theorem. Menchoff's example in this context shows that when $\gamma = 0$ (i) is no longer sufficient for a.e. convergence of $\sum_n a_n f_n$. Applying Corollary 3.6(iii) yields Menchoff's [31, Theorem 12].

Example 3.3. On $L_2(\Gamma,\nu)$ define $Uf(\lambda) = \lambda f(\lambda)$. Then $\int|Uf|^2 d\nu = \int|f|^2 d\nu$, and hence U is a unitary operator (with $U^*f(\lambda) = \bar{\lambda}f(\lambda)$). The sequence $X_n(\lambda) := U^n\mathbf{1} = \lambda^n$ is wide sense stationary with $\langle X_n, X_0\rangle = \int \lambda^n d\nu$, so its spectral measure is ν.

This example exhibits a wide sense stationary process with *any* pre-assigned spectral measure. It is a concretization of (the general) Example 4 in Doob [12, p. 479].

Definition 3.2. Let $\{a_n\}$ be a sequence of (complex) numbers, and let $1 \leq t < \infty$; we say that $\{a_n\} \in W_t$ if $\sup_{n>0}\frac{1}{n}\sum_{k=1}^{n}|a_k|^t < \infty$. If $\{a_n\}$ is bounded we say that $\{a_n\} \in W_{\infty}$. For $t > s \geq 1$ we have $W_t \subset W_s \subset W_1$.

Corollary 3.8. *Let $\{a_n\} \in W_t$, $1 < t \leq 2$ be a sequence of complex numbers, and let $1 < u < \infty$ with dual index v. Let T be a contraction of $L_2(\mu)$ and $f \in L_2(\mu)$. The* *series* $\sum_{n=2}^{\infty} \frac{a_n T^n f}{n^{\alpha}(\log n)^{\beta}(\log\log n)^{\delta}}$ *converges a.e. and in L_2-norm, with* $\sup_{n\geq 2}\Big|\sum_{k=2}^{n} \frac{a_k T^k f}{k^{\alpha}(\log k)^{\beta}(\log\log k)^{\delta}}\Big|$ *in $L_2(\mu)$, for α, β, and δ determined according to the following conditions:*

(i) If $\sum_{k=1}^{n} |\langle T^k f, f\rangle|^v \leq Cn^{\gamma}$ for some $0 \leq \gamma < 1$, then $\alpha = [1+\gamma(u-1)]/2u + 1/t$, $\beta = 1$, and $\delta > 1$.

(ii) If $\sum_{k=1}^{n} |\langle T^k f, f\rangle| \leq Cn^{\gamma}$ for some $0 < \gamma < 1$, then $\alpha = \gamma/2 + 1/t$, $\beta = 1$, and $\delta > 1$.

(iii) If $\sum_{k=1}^{n} |\langle T^k f, f\rangle| \leq C(\log n)^{\eta}$ for some $\eta > 0$, then $\alpha = 1/t$, $\beta = (3+\eta)/2$, and $\delta > 1/2$.

(iv) If f has bounded spectral density, then $\alpha = 1/t$, $\beta = 3/2$, and $\delta > 1/2$.

Proof. The method of proof of [10, Lemma 2], can be used to show that if $\{a_n\} \in W_t$ then $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2/t}\log n(\log\log n)^{1+\epsilon}} < \infty$ for every $\epsilon > 0$. Put $b_n = \frac{a_n}{n^{\alpha}(\log n)^{\beta}(\log\log n)^{\delta}}$, and obtain the values of α, β, and δ by applying to $\{b_n\}$ Corollary 3.6(i) for (i), Corollary 3.7(i) for (ii), Corollary 3.3(ii) for (iii), and Corollary 3.7(ii) for (iv). □

Remarks. 1. Under the assumption $\sum_n |\langle T^n f, f\rangle|/(\log n)^{\eta} < \infty$, Gaposhkin [16, Theorem 5] showed, for $a_n \equiv 1$, that $\sum_n \frac{T^n f}{\sqrt{n}(\log n)^{(3+\eta)/2}}$ converges a.e. The assumption is stronger than our assumption in (iii), and the convergence statement is better.

2. In Example 3.1 take $0 \leq g \in L_2(d\lambda)$ unbounded. Then $f := \sqrt{g}$ satisfies $\sum_{k=1}^{\infty} |\langle T^k f, f\rangle|^2 \leq ||g||^2_{L_2(d\lambda)}$, but since g is unbounded, $\sum_{k=1}^{\infty} |\langle T^k f, f\rangle| = \infty$.

3. Let T be a symmetric (i.e., $T^* = T$) contraction on L_2. If $f \in L_2$ satisfies $\sum_{n=1}^{\infty} |\langle T^n f, f\rangle| < \infty$, then $\sum_n \|T^n f\|^2 < \infty$, so $\sum_n |T^n f(x)|^2 < \infty$ a.e. For $\{a_n\} \in W_t$, $1 < t \leq 2$, and any $\delta > 1/2$ Cauchy's inequality yields

$$\sum_{n=2}^{\infty} \frac{|a_n T^n f(x)|}{n^{1/t}(\log n)^{1/2}(\log\log n)^{\delta}} \leq \Big(\sum_{n=2}^{\infty} \frac{|a_n|^2}{n^{2/t}\log n(\log\log n)^{1+\epsilon}}\Big)^{\frac{1}{2}} \Big(\sum_{n=2}^{\infty} |T^n f(x)|^2\Big)^{\frac{1}{2}} < \infty,$$

which gives a better rate than the general result in (iv).

Lemma 3.9. *Let θ be one-sided shift of an ergodic Markov chain $\{\xi_n\}$ with invariant initial distribution μ and Markov operator T. For $f \in L_2$ define $X_n = f(\xi_n)$. Then $\{X_n\}$ is strictly stationary and* $\mathrm{E}(X_n X_0) = \langle T^n f, f\rangle$.

Proof. Since $X_n = f(\xi_n) = X_0 \circ \theta^n$, the sequence $\{X_n\}$ is strictly stationary. Since $f \in L_2(\mu)$, we have $\{X_n\} \subset L_2(\mathbb{P}_{\mu})$. We have

$$\mathrm{E}(X_n X_0) = \int f(\xi_n) f(\xi_0) d\mathbb{P}_{\mu} = \int \Big[\int f(\xi_n) f(\xi_0) d\mathbb{P}_x\Big] d\mu =$$

$$\int T^n f(x) f(x) \mu(dx) = \langle T^n f, f\rangle,$$

and the result follows. □

Remarks. 1. With the help of the lemma, we can make assumptions on $\{\langle T^n f, f\rangle\}$ and apply the previous results to the operator induced by the shift.

2. The Lemma applies also to the 2-sided shift.

Example 3.4. On $(-\pi, \pi]$ there exists a finite measure ν, singular with respect to the Lebesgue measure , such that its Fourier coefficients $\{\hat{\nu}(k)\}$ tend to zero Zygmund [54, Theorem 10.12, vol. II, p. 146]. It is not hard to modify ν to be defined on the unit circle Γ, and concentrated on $\frac{\pi}{2} \leq \arg\lambda \leq \pi$.

On $L_2(\Gamma, \nu)$ we define $Uf(\lambda) = \lambda f(\lambda)$. For ν-almost every λ we have $|1-\lambda| \geq \sqrt{2}$, so clearly $I - U$ is invertible on $L_2(\nu)$, and $g(\lambda) := \frac{1}{1-\lambda} \in L_2(\nu)$. As in Example 3.3, we take the stationary sequence $U^n\mathbf{1} = \lambda^n$, which has spectral measure ν. Since $\langle U^n\mathbf{1}, \mathbf{1}\rangle = \hat{\nu}(n) \to 0$, we have $U^n\mathbf{1} \to 0$ weakly by Foguel [13]. Since $(I - U)g = \mathbf{1}$, and g is in the closed subspace generated by $\{U^n\mathbf{1}\}$ (see Lin-Sine [28]), also $U^n g \to 0$ weakly. Hence $\sum_{k=0}^n U^k\mathbf{1} \to g$ weakly, so $\sum_{k=0}^\infty \langle U^k\mathbf{1}, \mathbf{1}\rangle$ converges, but since ν is singular, $\sum_{k=0}^\infty |\langle U^k\mathbf{1}, \mathbf{1}\rangle|^2 = \infty$. The example shows that the following conditions can live together:
(i) $f \in (I - U)L_2(\nu)$.
(ii) The spectral measure of f is singular, so $\sum_{k=0}^\infty |\langle U^k f, f\rangle|^2 = \infty$.
(iii) $\sum_{k=0}^\infty \langle U^k f, f\rangle$ converges, but only *conditionally*, and in particular $\langle U^k f, f\rangle \to 0$.

4 Random power series of L_2-contractions

In this section we treat a.e. convergence of random power series of contractions in L_2 spaces. Norm convergence of such series was considered in [36]. Let $\{f_n\}$ be independent random variables on the probability space (Ω, μ). For a contraction T on $L_2(\mathcal{Y}, \pi)$ of some measure space, we define the (formal) random power series of T by $\sum_{k=1}^\infty f_k(x)T^k g$, $g \in L_2(\pi)$. We are interested in having for a.e. x the a.e. convergence of all random power series of L_2-contractions. To be more precise, we want a *universal null set* in Ω, such that when $x \in \Omega$ is outside this null set, for every contraction T on $L_2(\pi)$ and $g \in L_2(\pi)$ the series $\sum_{k=1}^\infty f_k(x)T^k g$ converges π-a.e. and, in particular, for every orthonormal sequence $\{g_k\} \subset L_2(\pi)$ the series $\sum_{k=1}^\infty f_k(x)g_k$ converges π-a.e. By [46] we must have $\sum_{n=1}^\infty |f_n(x)|^2 \log n \log^+(1/|f_n(x)|) < \infty$ a.e., and if $|f_n(x)|$ is a.e. non-increasing (e.g., $f_n(x) = c_n e_n(x)$ with $|e_n(x)| \equiv 1$ a.e. and $|c_n|$ decreasing) then [47] necessarily $\sum_{n=1}^\infty |f_n(x)|^2 (\log n)^2 < \infty$ a.e.

Given complex numbers $a_0, a_1, \ldots, a_n$ and a unitary operator U on a Hilbert space, the spectral theorem yields that $||\sum_{k=0}^n a_k U^k|| \leq \max_{|\lambda|=1} \left|\sum_{k=0}^n a_k \lambda^k\right|$. The unitary dialtion theorem yields that for every contraction T on a complex Hilbert space we have

$$\Big\|\sum_{k=0}^n a_k T^k\Big\| \leq \max_{|\lambda|=1}\Big|\sum_{k=0}^n a_k \lambda^k\Big| \tag{19}$$

As (19) suggests, application of the previous methods requires good estimates on $C(\Gamma)$-norms of blocks of the generating random Fourier series $\sum_n f_n(x)\lambda^k$. Throughout this section our (complex valued) random coefficients $\{f_n\}$ will be independent.

Proposition 4.1. *Let $\{f_n\}$ be symmetric independent complex valued random variables on (Ω, μ). Then (with 0/0 interpreted as 1)*

$$\left\|\sup_{j\geq 0}\sup_{l>j}\exp\left\{\frac{\max_{|\lambda|=1}|\sum_{k=j+1}^l f_k\lambda^k|^2}{\log(l+1)(\sum_{k=j+1}^l |f_k|^2)}\right\}\right\|_{L_1(\mu)} < \infty\,.$$

Hence for a.e. $x \in \Omega$ we have

$$\sup_{l>j\geq 0} \frac{\max_{|\lambda|=1}|\sum_{k=j+1}^l f_k(x)\lambda^k|}{\sqrt{\log(l+1)}(\sum_{k=j+1}^l |f_k(x)|^2)^{1/2}} < \infty. \tag{20}$$

The proposition was proved by Weber [50] (using the metric entropy method).

Theorem 4.2. *Let $1 < p \leq 2$, and let $\{f_n\} \subset L_p(\Omega, \mu)$ be a sequence of independent centered random variables. If*

$$\sum_{n=1}^\infty \|f_n\|_2^2 (\log n)^3 < \infty \qquad (p = 2), \tag{21}$$

or

$$\sum_{n=2}^{\infty} \|f_n\|_p^p (\log n)^p (\log\log n)^{p/2+\epsilon} < \infty \quad \textit{for some} \quad \epsilon > 0 \qquad (1 < p < 2), \tag{22}$$

then there exists a subset $\Omega^* \subset \Omega$ with $\mu(\Omega^*) = 1$, such that when $x \in \Omega^*$, for every contraction T on a space $L_2(\pi)$ and any $g \in L_2(\pi)$, the series

$$\sum_{n=1}^{\infty} f_n(x) T^n g \quad \textit{converges } \pi \textit{ a.e.} \tag{23}$$

When $\{f_n\}$ are symmetric, for $x \in \Omega^*$ there is a constant $K_x < \infty$, determined only by $\{f_n(x)\}$ (and p), such that

$$\Big\| \sup_{n\geq 1} \Big| \sum_{k=1}^{n} f_k(x) T^k g \Big| \Big\|_2 \leq K_x \|g\|_2 \tag{24}$$

In the general case, if π is a probability, then for $x \in \Omega^*$ we have $sup_{n\geq 1} |\sum_{k=1}^n f_k(x)T^k g|$ in $L_p(\pi)$.

Proof. We first prove the case that each f_n is symmetric. By Beppo Levi's theorem, conditions (21) or (22) imply, respectively, that for μ a.e. $x \in \Omega$ we have

$$\sum_{n=1}^{\infty} |f_n(x)|^2 (\log n)^3 < \infty \qquad (*)$$

or

$$\sum_{n=1}^{\infty} |f_n(x)|^p (\log n)^p (\log\log n)^{p/2+\epsilon} < \infty \qquad (**)$$

By symmetry of $\{f_n\}$, Proposition 4.1 applies. We define Ω^* as the set of x for which either (*) or (**) (according to $p = 2$ or $1 < p < 2$), together with (20), hold. Fix $x \in \Omega^*$. Given a contraction T on $L_2(\pi)$ and $g \in L_2(\pi)$, (19), (20), and $\|\cdot\|_{\ell_2} \leq \|\cdot\|_{\ell_p}$ yield

$$\|\sum_{k=j+1}^{l} f_k(x) T^k g\|_2 \leq \|g\|_2 \max_{|\lambda|=1} \Big| \sum_{k=j+1}^{l} f_k(x) \lambda^k \Big| \leq \|g\|_2 C_x \sqrt{\log(l+1)} (\sum_{k=j+1}^{l} |f_k(x)|^p)^{1/p}.$$

Hence for $x \in \Omega^*$, (9) is satisfied by $\{f_n(x)T^n g\} \subset L_2(\pi)$, with $m_n = |f_n(x)|^p$, $q = \frac{2}{p} \geq 1$, and $A_n = \|g\|_2^2 C_x^2 \log(n+1)$. When $p = 2$, using (*) we have

$$\sum_{n=1}^{\infty} A_{2n} (\log n)^2 m_n \leq C \|g\|_2^2 C_x^2 \sum_{n=1}^{\infty} (\log n)^3 |f_n(x)|^2 < \infty.$$

Hence Theorem 2.9(i), applied to $\{f_n(x)T^n g\} \subset L_2(\pi)$, yields the π-a.e. convergence of the series $\sum_{n=1}^\infty f_n(x) T^n g$ and the estimate (24) for the maximal function of the partial sums.

When $1 < p < 2$, using (**) and $q = 2/p$ we obtain

$$\sum_{n=1}^{\infty} A_{2n}^{1/q} (\log n)^{1/q} (\log\log n)^{1/q+\epsilon} m_n \leq C \|g\|_2^p C_x^p \sum_{n=1}^{\infty} (\log n)^p (\log\log n)^{p/2+\epsilon} |f_n(x)|^p < \infty,$$

and now Theorem 2.9(iii) applies. This concludes the proof when $\{f_n\}$ are symmetric.

We now prove the general case of $\{f_n\}$ centered. Let $\{f'_n\}$ defined on (Ω', μ') be an independent copy of $\{f_n\}$, and put $h_n(x, x') = f_n(x) - f'_n(x')$ on $(\Omega \times \Omega', \mu \times \mu')$. Then

h_n is symmetric with $\|h_n\|_p \le 2\|f_n\|_p$, so applying to $\{h_n\}$ the result for the symmetric case proved above, we obtain a set $E \subset \Omega \times \Omega'$ with $\mu \times \mu'(E) = 1$, such that for fixed $(x, x') \in E$ and any contraction T of $L_2(\pi)$ and $g \in L_2(\pi)$, the series $\sum_{n=1}^\infty h_n(x,x')T^n g$ converges π a.e. Define $E_x = \{x' \in \Omega' : (x,x') \in E\}$ and put $\Omega^* = \{x \in \Omega : \mu'(E_x) = 1\}$. By Fubini's theorem, for μ a.e. x we have $\mu'(E_x) = 1$, so $\mu(\Omega^*) = 1$.

Now fix $x \in \Omega^*$. Let T be a contraction on $L_2(\mathcal{Y}, \pi)$ and $g \in L_2(\pi)$. In order to show that $\sum_{n=1}^\infty f_n(x)T^n g$ converges π-a.e., take any $x' \in E_x$ and consider the identity

$$\sum_{n=1}^{N} f_n(x)T^n g = \sum_{n=1}^{N} h_n(x,x')T^n g + \sum_{n=1}^{N} f'_n(x')T^n g \qquad (***)$$

As $N \to \infty$, the first sum on the right hand side converges π-a.e., since $(x, x') \in E$. We show that $x' \in E_x$ can be chosen such that the second sum is also π-a.e. convergent.

As mentioned in the introduction, we may assume that π is a probability, so for $p \le 2$ we have $\|T^n g\|_{L_p(\pi)} \le \|T^n g\|_{L_2(\pi)} \le \|g\|_{L_2(\pi)}$. The appropriate condition (21) or (22) yields

$$\int \sum_{n=1}^\infty ||f'_n||_p^p |T^n g|^p d\pi \le \|g\|_{L_2(\pi)}^p \sum_{n=1}^\infty ||f'_n||_p^p < \infty\,,$$

so by Beppo Levi's theorem we have that the series $\sum_{n=1}^\infty ||f'_n||_p^p |T^n g(y)|^p$ converges π-a.e. Hence for π almost every fixed $y \in \mathcal{Y}$, the Marcinkiewicz-Zygmund theorem [30, Theorem 5'] (see [8, p. 114]), applied to the independent centered sequence $\{f'_n T^n g(y)\} \subset L_p(\mu')$, yields that the series $\sum_{n=1}^\infty f'_n T^n g(y)$ converges μ'-a.e. By Fubini's theorem, we have that for μ'-a.e. x' the series $\sum_{n=1}^\infty f'_n(x')T^n g$ converges π a.e. Since $\mu'(E_x) = 1$, this shows that we can find $x' \in E_x$ for which also the second term on the right hand side of (***) converges π-a.e., and the π-a.e. convergence of $\sum_{n=1}^\infty f_n(x)T^n g$ when $x \in \Omega^*$ is proved.

Fix $x \in \Omega^*$. Let $(\mathcal{Y}, \pi)$ be a probability space, T a contraction on $L_2(\pi)$, $g \in L_2(\pi)$, and $y \in \mathcal{Y}$. Since $\{T^n g(y) f'_n\}$ are centered independent in $L_p(\mu')$, the inequality of [5] yields

$$\| \sum_{k=j+1}^{l} T^k g(y) f'_k\|_p^p \le 2 \sum_{k=j+1}^{l} |T^k g(y)|^p \|f'_k\|_p^p \quad \text{ for } \; l > j \ge 0.$$

Hence for $d(j,l) := \sum_{k=j+1}^{l} |T^k g(y)|^p \|f'_k\|_p^p$ we have $A_n^{(d)} \le 2$ for every n. By Theorem 2.4, with $q = 1$, we have

$$\int \sup_{n\ge1} \Big| \sum_{k=1}^{n} f'_k(x')T^k g(y)\Big|^p d\mu'(x') \le C \sum_{k=1}^{\infty} (\log k)^p \|f'_k\|_p^p\, |T^k g(y)|^p.$$

Integrating the above with respect to π and using Fubini's theroem we obtain

$$\int \Big[\int \sup_{n\ge1} \Big| \sum_{k=1}^{n} f'_k(x')T^k g(y)\Big|^p d\pi(y)\Big] d\mu'(x') \le C \sum_{k=1}^{\infty} \|f'_k\|_p^p\, \|T^k g\|_p^p (\log k)^p$$

Since $\|T^k g\|_p \le \|g\|_2$ (for $p < 2$ because π is a probability), the appropriate condition (21) or (22) now implies convergence of the last series. Hence for a.e. x' we have $\int \sup_{n\ge1} |\sum_{k=1}^n f'_k(x')T^k g(y)|^p\, d\pi(y) < \infty$, and x' can be chosen in E_x since $\mu'(E_x) = 1$. With this x' the suprema of the sums on the right hand side of (***) are both in $L_p(\pi)$, which proves the assertion. □

Remarks. 1. When $p = 2$, we have in the general case $\sup_n |\sum_{k=1}^n f_k(x)T^k g|$ in $L_2(\pi)$ even if π is not finite.

2. By considering for each $\lambda \in \Gamma$ the "rotation" it induces and applying the theorem to $g(z) = z$, we obtain that (21) or (22) implies that for a.e. $x \in \Omega$ the random Fourier series $\sum_n f_n(x)\lambda^n$ converges for every λ. When $\{f_n\}$ are symmetric, Billard's theorem

[25, Theorem 3, p. 58] yields a.e. uniform convergence of the series. In this case, (19) yields that *for a.e. x the series $\sum_{n=1}^{\infty} f_n(x)T^n$ converges in operator norm for any contraction T on L_2 (uniformly in all contractions).*

3. For $p = 2$ and $f_n = c_n\epsilon_n$ (throughout the paper $\{\epsilon_n\}$ is a Rademacher sequence defined on the unit interval), (21) becomes $\sum_{n=1}^{\infty} |c_n|^2(\log n)^3 < \infty$. Rosenblatt [40, Theorem 11] used a stronger assumption, namely $\sum_{n=1}^{\infty} |c_n|^2\sqrt{n}(\log n)^{2+\delta} < \infty$ for some $\delta > 0$, in order to prove the assertion of the theorem.

4. For $1 < p < 2$ and $f_n = c_n\epsilon_n$, (22) becomes $\sum_{n=1}^{\infty} |c_n|^p(\log n)^p(\log\log n)^{p/2+\epsilon} < \infty$. This condition and $\sum_{n=1}^{\infty} |c_n|^2(\log n)^3 < \infty$ are not comparable. The sequence $c_n = 1/(\sqrt{n}\log^3 n)$ satisfies only the second condition, while $c_n = 1/(k^2)$ for $n = 2^k$ and $c_n = 0$ otherwise satisfies only the first one.

Corollary 4.3. *Let $1 < p \le 2$, and let $\{f_n\} \subset L_p(\Omega,\mu)$ be a sequence of independent centered random variables with $\sup_n ||f_n||_p < \infty$. Then there exists a subset $\Omega^* \subset \Omega$, with $\mu(\Omega^*) = 1$, such that when $x \in \Omega^*$, for every contraction T on a space $L_2(\pi)$ and any $g \in L_2(\pi)$, the series*

$$\sum_{n=2}^{\infty} \frac{f_n(x)T^n g}{n^{1/p}(\log n)^{\beta}(\log\log n)^{\gamma}} \quad \textit{converges } \pi \textit{ a.e.,} \tag{25}$$

with $\beta = 2$ and $\gamma > 1/2$ when $p = 2$, and with $\beta = 1 + 1/p$ and $\gamma > 1/2 + 1/p$ when $1 < p < 2$.

Proof. Apply the previous theorem to $\{\frac{f_n}{n^{1/p}(\log n)^{\beta}(\log\log n)^{\gamma}}\}$. □

Remarks. 1. The convergence (25) implies (e.g., [11, Lemma 2.19]) that for any $\alpha > 1/p$

$$\sum_{n=2}^{\infty} \frac{f_n(x)T^n g}{n^{\alpha}} \quad \text{converges } \pi \text{ a.e.} \tag{26}$$

2. For $p = 2$ and under the additional assumption that $\{f_n\}$ are *symmetric identically distributed*, Boukhari and Weber [7, Corollary 3.3] proved (26), and also (25) with $\beta > 2$ and $\gamma = 0$. M. Weber has informed us that when $p = 2$ the general symmetric case can be deduced also from the main result of [7].

Theorem 4.4. *Let $\{f_n\}$ be a sequence of i.i.d. centered random variables on (Ω,μ). If $\int |f_1|\log^+|f_1|d\mu < \infty$, then there exists a subset $\Omega^* \subset \Omega$ with $\mu(\Omega^*) = 1$, such that when $x \in \Omega^*$, for every contraction T on a space $L_2(\pi)$ and any $g \in L_2(\pi)$, the series*

$$\sum_{n=1}^{\infty} \frac{f_n(x)T^n g}{n} \quad \textit{converges } \pi \textit{ a.e..} \tag{27}$$

with $\sup_{n\ge1} |\sum_{k=1}^{n} \frac{f_k(x)T^k g}{k}| \in L_2(\pi)$.

When $\{f_n\}$ are symmetric, then the above assertions are true if we assume only $\int |f_1|\log^+\log^+|f_1|d\mu < \infty$.

Proof. We start with the general centered case. Since $\{f_n\}$ are assumed identically distributed with $f_1 \in L_1(\mu)$, we have $\sum_{n=1}^{\infty} \mu\{|f_n| > n\} < \infty$, so for a.e. $x \in \Omega$ we have $|f_n(x)| > n$ only for finitely many n. Hence it is sufficient to prove the assertions for $\{f_n\mathbf{1}_{\{|f_n|\le n\}}\}$ instead of $\{f_n\}$.

Put $h_1 := f_1\mathbf{1}_{\{|f_1|\le 1\}}$ and $h_n := f_n\mathbf{1}_{\{|f_n|\le n/\log^3 n\}}$ for $n \ge 2$. Throughout this proof, the logarithm is the natural one, and $\log^3 t$ denotes $(\log t)^3$. By definition, $\{h_n\}$ is a sequence of independent *bounded* random variables. Put $\mathbf{E}h_n = \int h_n d\mu$.

For a contraction T on $L_2(\mathcal{Y},\pi)$ and $g\in L_2(\pi)$ we have the identity

$$\sum_{n=1}^N \frac{f_n(x)\mathbf{1}_{\{|f_n|\le n\}}(x)T^ng}{n} =$$

$$\sum_{n=1}^N \frac{(h_n(x)-\mathbf{E}h_n)T^ng}{n} + \sum_{n=1}^N \frac{\mathbf{E}h_nT^ng}{n} + \sum_{n=1}^N \frac{f_n(x)\mathbf{1}_{\{n/\log^3 n<|f_n|\le n\}}(x)T^ng}{n}. \qquad (*)$$

We have to find a universal set of x (independent of T and g) for which the assertion of the theorem holds. Note that the second sum does not depend on x.

For the first sum on the right hand side of (*), we want to apply Theorem 4.2 to $\{\frac{h_n-\mathbf{E}h_n}{n}\}$ with $p=2$, so we show that $\{\frac{h_n-\mathbf{E}h_n}{n}\}$ satisfies (21). Denoting $f:=f_1$ and using $\|h_n-\mathbf{E}h_n\|_2\le 2\|h_n\|_2$, we obtain, via Fubini's theorem on $(\Omega\times\mathbb{N})$,

$$\sum_{n=2}^\infty \Big\|\frac{h_n}{n}\Big\|_2^2\log^3 n = \sum_{n=2}^\infty \int \frac{1}{n^2}|f_n|^2\mathbf{1}_{\{|f_n|\le n/\log^3 n\}}\log^3 n d\mu =$$

$$\sum_{n=2}^\infty \int \frac{1}{n^2}|f|^2\mathbf{1}_{\{|f|\le n/\log^3 n\}}\log^3 n d\mu = \int |f(x)|^2 \sum_{\{n\ge 2:\frac{n}{\log^3 n}\ge|f(x)|\}} \frac{\log^3 n}{n^2}d\mu \le$$

$$\int |f(x)|^2 \sum_{\{n\ge \max\{2,|f(x)|\log^3|f(x)|\}\}} \frac{\log^3 n}{n^2}d\mu\,,$$

since $0<a\le n/\log^3 n \implies a\log^3 a\le n$. We now estimate the tail of the convergent series $\sum_{n=2}^\infty n^{-2}\log^3 n$, which has eventually decreasing terms, by the integral test. Computing $\frac{d}{dt}[t^{-1}\log^3 t]$ we see that there is a constant C such that for t large (i.e., $t>K$) we have $t^{-2}\log^3 t\le C\frac{d}{dt}[-t^{-1}\log^3 t]$. Since for large values of $|f|$ we have also $\log(|f|\log^3|f|)\le 2\log|f|$, the last integral is bounded by

$$C_1+\int \mathbf{1}_{\{|f|\ge K\}}|f|^2 \sum_{\{n:\, n\ge[|f|\log^3|f|]\}} \frac{\log^3 n}{n^2}d\mu \le$$

$$C_1+C_2\int \mathbf{1}_{\{|f|\ge K\}}|f|^2\frac{(\log(|f|\log^3|f|))^3}{|f|(\log|f|)^3}d\mu =$$

$$C_1+8C_2\int \mathbf{1}_{\{|f|\ge K\}}|f|d\mu \le C_1+8C_2\int|f|d\mu<\infty.$$

Thus $\sum_{n=2}^\infty \big\|\frac{h_n-\mathbf{E}h_n}{n}\big\|_2^2\log^3 n<\infty$. Let Ω^{**} be the set given by Theorem 4.2, so for fixed $x\in\Omega^{**}$, for any contraction T on $L_2(\pi)$ and $g\in L_2(\pi)$ we have π a.e. convergence of the first sum of (*). Note that only *integrability* of f was needed.

For the second sum in (*), we show that $\sum_{n=1}^\infty\frac{|\mathbf{E}h_n|}{n}<\infty$. Since f_n is centered, $\mathbf{E}h_n=-\mathbf{E}(f_n\mathbf{1}_{\{|f_n|>n/\log^3 n\}})$ for $n\ge 2$.

Claim. *There exists N such that if $n>N$ and $a>n/\log^3 n$, then $n<2a\log^3 a$.*

Proof. Fix N with $\frac{\log n}{\log(\log^3 n)}>10$ for $n>N$. For $n>N$ and $a>\frac{n}{\log^3 n}$ we have

$$a\log^3 a>\frac{n}{\log^3 n}[\log n-\log(\log^3 n)]^3>\frac{n}{\log^3 n}(0.9\log n)^3>\frac12 n.$$

We return to the second sum in (*). For N given by the claim large enough we have

$$\sum_{n=2}^\infty\frac{|\mathbf{E}h_n|}{n}\le\sum_{n=2}^\infty\frac{\mathbf{E}(|f_n|\mathbf{1}_{\{|f_n|\ge n/\log^3 n\}})}{n}=\sum_{n=2}^\infty\int\frac{|f|\mathbf{1}_{\{|f|>n/\log^3 n\}}}{n}d\mu\le$$

$$\sum_{n=2}^{N}\int \frac{|f|\mathbf{1}_{\{|f|>n/\log^3 n\}}}{n}d\mu + \int \mathbf{1}_{\{|f|\ge e\}}|f| \sum_{\{N<n\le 2|f|\log^3|f|\}} \frac{1}{n}d\mu \le$$

$$C + \int \mathbf{1}_{\{|f|\ge e\}}|f|(\log 2 + \log|f| + 3\log\log|f|)d\mu < \infty,$$

since the last integral is finite by assumption. Hence $\sum_{n=1}^\infty \frac{|\mathbf{E}h_n|}{n}$ converges; thus, as remarked in the introduction, for any contraction T and $g \in L_2(\pi)$, the series $\sum_{n=1}^\infty \frac{|\mathbf{E}h_n T^n g|}{n}$ converges π-a.e.

For the third sum in (*) we use the previous computation to obtain

$$\int \sum_{n=2}^{\infty} \frac{|f_n|\mathbf{1}_{\{n/\log^3 n<|f_n|\le n\}}}{n}d\mu \le \int \sum_{n=2}^{\infty} \frac{|f|\mathbf{1}_{\{n/\log^3 n<|f|\}}}{n}d\mu < \infty.$$

Hence by Beppo Levi $\sum_{n=1}^\infty \frac{|f_n(x)|\mathbf{1}_{\{n/\log^3 n<|f_n|\le n\}}(x)}{n} < \infty$ converges on a set Ω' with $\mu(\Omega')=1$. Now it is clear that for $x\in\Omega'$ the series $\sum_{n=1}^\infty \frac{|f_n(x)\mathbf{1}_{\{n/\log^3 n<|f_n|\le n\}}(x)\,T^n g|}{n}$ converges π a.e. We define $\Omega^* = \Omega' \cap \Omega^{**}$, so for $x \in \Omega^*$ we have π a.e. convergence in (*).

By Theorem 4.2 the maximal function of the first term in (*) is square integrable. For $x \in \Omega^*$ the suprema of the last two terms in (*) are bounded by the corresponding π-a.e. absolutely convergent series; each series is square integrable by the triangle inequality and the absolute convergence of the series of coefficients. This yields the desired square integrability of the maximal function.

When $\{f_n\}$ are symmetric, so are $\{h_n\}$, and $\mathbf{E}h_n = 0$. Hence the second term in (*) vanishes identically. To treat the third sum in this case, we give a direct proof of the a.e. convergence of $\sum_{n=1}^\infty \frac{|f_n(x)|\mathbf{1}_{\{n/\log^3 n<|f_n|\le n\}}(x)}{n} < \infty$, which uses only the condition $\int |f|\log^+\log^+|f|d\mu < \infty$. Indeed, using the claim as before we obtain

$$\int \sum_{n=2}^{\infty} \frac{|f_n|\mathbf{1}_{\{n/\log^3 n<|f_n|\le n\}}}{n}d\mu = \int \sum_{n=2}^{\infty} \frac{|f|\mathbf{1}_{\{n/\log^3 n<|f|\le n\}}}{n}d\mu \le$$

$$C + \int \mathbf{1}_{\{|f|\ge e\}}|f| \sum_{\{|f|\le n\le 2|f|\log^3|f|\}} \frac{1}{n}d\mu \le$$

$$C + \int \mathbf{1}_{\{|f|\ge e\}}|f|\big(\log 2 + \log|f| + \log\log^3|f| - \log(|f|-1)\big)d\mu \le$$

$$C + \int \mathbf{1}_{\{|f|\ge e\}}|f|\big(\log 2 + 3\log(\log|f|) + \frac{2}{|f|}\big)d\mu < \infty.$$

Since the application of Theorem 4.2 required only integrability of f, we finish the proof of the assertions as above. □

Remarks. 1. When $\{f_n\}$ are centered i.i.d. and we take T the identity, we obtain μ almost sure convergence of $\sum_{n=1}^\infty \frac{f_n(x)}{n}$. By the discussion following Theorem 6 of [30], in general there is no weaker integrability condition on f_1 that ensures this convergence.

2. When $\{f_n\}$ are symmetric i.i.d. which satisfy the assertion of the theorem, taking all multiplications by λ (with $|\lambda|=1$) we obtain pointwise convergence of the random Fourier series $\sum_{n=1}^\infty \frac{f_n(x)\lambda^n}{n}$ (which is in fact uniform in λ [25, p. 58]). By [45] we must have $f_1 \in L\log^+\log^+ L$.

An inspection of the proof of Theorem 4.2 shows that in fact we prove the following.

Theorem 4.5. *Let $\{a_n\}$ be a sequence which satisfies*

(i) $\displaystyle\sum_{n=1}^{\infty}|a_n|^2(\log n)^3 < \infty \qquad \text{for} \quad p=2,$

or

$$\sum_{n=2}^{\infty}|a_n|^p(\log n)^p(\log\log n)^{p/2+\epsilon} < \infty \qquad \text{for} \quad 1<p<2 \ \text{and} \ \epsilon>0.$$

(ii) $\displaystyle\max_{|\lambda|=1}|\sum_{k=j+1}^{l} a_k\lambda^k| \le C\sqrt{\log(l+1)}\Big(\sum_{k=j+1}^{l}|a_k|^p\Big)^{1/p}$ *for every* $l>j\ge 0$.

Then for every contraction T on $L_2(\pi)$ and $g \in L_2(\pi)$, the series $\sum_{n=1}^{\infty} a_nT^ng$ converges a.e. and in L_2-norm. In particular, the Fourier series $\sum_{n=1}^{\infty} a_n\lambda^n$ converges for every $|\lambda| = 1$.

Remark. By Proposition 4.1, for a.e. $x \in [0,1]$ the sequence $a_n := \frac{\epsilon_n(x)}{\sqrt{n}\log^3(n+1)}$ satisfies both conditions of the theorem with $p = 2$. However, we have no specific example of the appropriate "choice of signs".

Recall that a Dunford-Schwartz operator on $L_1(\mathcal{Y},\pi)$ is a contraction T which is also a contraction of $L_\infty(\mathcal{Y},\pi)$, and therefore is also a contraction of each $L_s(\mathcal{Y},\pi)$, $1<s<\infty$, by the Riesz-Thorin theorem (for a simple proof for Markov operators, see [27, p. 65]). The Dunford-Schwartz theorem gives a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} T^kg$ for every $g \in L_1(\pi)$.

Our results obviously apply to T Dunford-Schwartz and $g \in L_2(\pi)$, and this raises the question about what happens for $g \in L_s$, $1 \le s < 2$.

Recently, Assani [4] proved the π-a.e. convergence of $\sum_{k=1}^{\infty}\frac{f_k(x)T^kg}{k}$ for $g \in L_s(\pi)$, $s>1$, when $\{f_n\} \subset L_p$, $p>1$, are centered i.i.d. This extends previous results of Rosenblatt [40, Theorem 18] (for $\{f_n\}$ a Rademacher sequence), Boukhari and Weber [7] ($p=2$ and $\{f_n\}$ symmetric), and Assani [2],[3] (convergence of $\frac{1}{n}\sum_{k=1}^{n} f_k(x)T^kg$). Note that for $g \in L_2(\pi)$ Theorem 4.4 yields Assani's result under the weaker requirement that the i.i.d. variables $\{f_n\}$ are in $L\log^+ L$.

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