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Exterior Powers of Lubin-Tate Groups

par S. Mohammad Hadi HEDAYATZADEH

Dédié à mon oncle Farzin

Résumé. Soient $\mathcal{O}$ l'anneau des entiers d'un corps local non-archimédien de charactéristique zéro et π une uniformisante de $\mathcal{O}$. On démontre que les puissances extérieures d'un schéma en $\mathcal{O}$-modules π-divisible de dimension au plus 1 sur un schéma de base localement noetherian existent et commutent avec changements de base arbitraires. De même, on calcule la hauteur et la dimension des puissances extérieures en termes de la hauteur du groupe p-divisible ou du schéma en $\mathcal{O}$-modules π-divisible donné. Dans le cas des groupes p-divisibles, on démontre l'existence des puissances extérieures sans aucune hypothèse sur le schéma de base.

Abstract. Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic zero and π a fixed uniformizer of $\mathcal{O}$. We prove that the exterior powers of a π-divisible module of dimension at most 1 over a locally Noetherian scheme exist and commute with arbitrary base change. We calculate the height and dimension of the exterior powers in terms of the height of the given π-divisible module. In the case of p-divisible groups, the existence of the exterior powers are proved without any condition on the basis.

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0. Introduction

The purpose of this paper is to prove the existence of "exterior powers" of p-divisible groups of dimension at most one or, more generally, of π-divisible modules of dimension at most one over locally Noetherian base schemes, where π is a uniformizer of a non-Archimedean local field of characteristic zero. This result is proved in [12] for π-divisible modules over fields and in this paper we will generalize it to locally Noetherian base schemes. Alternating morphisms of p-divisible groups are defined as compatible systems of alternating morphisms of their truncated Barsotti-Tate groups, as fppf sheaves (see Definition 1.16) and exterior powers of p-divisible groups are defined by the categorical universal property of exterior powers (see Definition 1.17). In this paper we prove the following theorems ($p > 2$):

Theorem 3.25. *Let S be a scheme and G a p-divisible group over S of height h and dimension at most 1. Then, there exists a p-divisible group $\bigwedge^r G$ over S of height $\binom{h}{r}$, and an alternating morphism $\lambda : G^r \to \bigwedge^r G$ such that for every morphism $f : S' \to S$ and every p-divisible group H over S', the following homomorphism is an isomorphism*

$$\mathrm{Hom}_{S'}(f^* \bigwedge^r G, H) \to \mathrm{Alt}^r_{S'}(f^* G, H), \quad \psi \mapsto \psi \circ f^* \lambda.$$

Moreover, the dimension of $\bigwedge^r G$ at $s \in S$ is $\binom{h-1}{r-1}$ (resp. 0) if the dimension of G at s is 1 (resp. 0).

Theorem 4.34. *Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic zero. Fix a uniformizer π of $\mathcal{O}$ and let S be a locally Notherian $\mathcal{O}$-scheme and $\mathcal{M}$ a π-divisible module over S of height h and dimension at most 1. Then, there exists a π-divisible module $\bigwedge^r_{\mathcal{O}} \mathcal{M}$ over S of height $\binom{h}{r}$, and an alternating morphism $\lambda : \mathcal{M}^r \to \bigwedge^r_{\mathcal{O}} \mathcal{M}$ such that for every morphism $f : S' \to S$ and every π-divisible module $\mathcal{N}$ over S', the homomorphism*

$$\mathrm{Hom}^{\mathcal{O}}_{S'}(f^* \bigwedge^r_{\mathcal{O}} \mathcal{M}, \mathcal{N}) \to \mathrm{Alt}^{\mathcal{O}}_{S'}((f^* \mathcal{M})^r, \mathcal{N})$$

induced by $f^*\lambda$ is as isomorphism. In other words, the r^{th}-exterior power of $\mathcal{M}$ exists and commutes with arbitrary base change. Moreover, the dimension of $\bigwedge_{\mathcal{O}}^{r}\mathcal{M}$ is the locally constant function

$$\dim : S \to \{0, \binom{h-1}{r-1}\}, \qquad s \mapsto \begin{cases} 0 & \textit{if} \quad \mathcal{M}_s \quad \textit{is étale} \\ \binom{h-1}{r-1} & \textit{otherwise.} \end{cases}$$

The results of this paper have many applications and in what follows, we mention a few.

The motivation for constructing such exterior powers comes from the (local) Langlands program, where one wants to have a rather explicit description of the deformation spaces of π-divisible modules. For example, over base fields, these exterior powers with their universal property are used to construct explicitly the Lubin-Tate tower at infinite level in [23] (equal characteristic) and the Rapoport-Zink spaces at infinite level in [21] (mixed characteristic).

The existence of the exterior powers of π-divisible modules will also explain why the exterior powers of the Galois representations attached to π-divisible modules are the Galois representation of a π-divisible module (the same way Anderson introduced t-motives in order to show that the exterior powers of Galois representations attached to Drinfel'd modules come from Drinfel'd modules [2]).

The universal property of the exterior powers of Barsotti-Tate groups, apart from being their "defining" property, is the most important one, and the construction of p-divisible groups with this property is what this paper is about. Note that this universal property makes them as functorial/categorical as one could ask. For instance, this allows one to induce a level structure on the exterior powers of a p-divisible group endowed with level structure. We will use the universal property extensively to prove the base change and descent property of these exterior powers. In section 5, we use the base change property to define a map on the Lubin-Tate space (see Example 5.1) and the descent property is used in the proof of the existence of the exterior powers. Alternating morphisms exist in "nature" (e.g. the Weil pairing of the group of p^n-torsion of an abelian variety induces an alternating morphism on the associated p-divisible group) and exterior powers "linearize" these morphisms. An immediate consequence of the existence of these exterior powers is the existence of exterior powers of "$1/g$" of the subgroup of p^n-torsion of an abelian scheme of signature $(1, g-1)$ (see Example 5.2 for more details). In section 5 we will give some detailed examples, where exterior powers can be used.

In a sequel, in preparation jointly with E. Mantovan, these results will be used to study the cohomology of Rapoport-Zink spaces from the perspective of the Langlands program. More specifically, we will compare the representations attached to the rigid fibers of the Lubin-Tate and Rapoport-Zink spaces, via the map between them, induced by taking the exterior powers. Note that this comparison, which is made possible thanks to the exterior powers of p-divisible groups, will give new insights into the cohomology of the Rapoport-Zink spaces and their representations, realizing the local Langlands correspondence for GL_n.

The formal Brauer group of a non-supersingular $K3$ surface has finite height and dimension one. The consequences of the existence of exterior powers applied to these formal groups will be the subject of another paper.

Finally, the results of this paper can be used in algebraic topology, where formal groups, or more generally p-divisible groups of dimension 1 abound. Here is an example from chromatic homotopy theory: Let $K(\mathbb{Z}/p^n, r)$ be the r^{th} Eilenberg-Mac Lane space attached to $\mathbb{Z}/p^n$. Let $\overline{K(h)}_* K(\mathbb{Z}/p^n, r)$ be the h^{th} ungraded Morava K-theory of $K(\mathbb{Z}/p^n, r)$ and $\overline{K(h)}^* K(\mathbb{Z}/p^n, r)$ its $\overline{K(h)}_*$-dual. These are finite Hopf algebras and denoting the finite group scheme $\mathbf{Spec}(\overline{K(h)}^* K(\mathbb{Z}/p^n, r))$ by $G_{n,r}$, the system $G(r) := (G_{n,r})_n$ is a Barsotti-Tate group of height $\binom{h}{r}$ and dimension $\binom{h-1}{r-1}$ (cf. [20] and [3]). The exterior powers of this p-divisible group can be used to understand the structure of the Morava K-theory. More precisely, one might ask if there is a canonical isomorphism (up to duality twists) $G(r) \cong \bigwedge^r G(1)$. Note that both sides have the same height and dimension. This is suggested by the Ravenel-Wilson theorem (cf. [20]) which says that there exists a canonical isomorphism between $\overline{K(h)}_* K(\mathbb{Z}/p^n, r)$ and the r^{th} exterior power of $\overline{K(h)}_* K(\mathbb{Z}/p^n, 1)$ via the $\circ$-product, and also by the computation of the Dieudonné module of these p-divisible groups in [3]. Similarly, one can expect the same result(s) for Morava E-theory of these Eilenberg-Mac Lane spaces.

Exterior powers of p-divisible groups have been studied also by Chen in [5]. Chen shows that if G is a p-divisible group of dimension 1 over a p-adic $W(\bar{\mathbb{F}}_p)$-algebra and if $\mathcal{P} = (P, Q, F, V^{-1})$ is the display attached to G, then the exterior power $\bigwedge^r P$ has a natural structure of a display, and therefore corresponds to a p-divisible group, called the r^{th}-exterior power of G. Using this and a comparison theorem between the Tate module and the filtered Dieudonné module of p-divisible groups (generalizing a result of

Faltings), she defines an exterior power map on the Lubin-Tate tower with nice properties. However, the exterior powers of G, constructed in [5], are not shown to satisfy the universal property of exterior powers as defined above. Also, the construction in [5] does not provide exterior powers of truncated Barsotti-Tate groups, as we do in this paper. Finally, our result is more general (as far as the exterior powers are concerned) in that it holds for π-divisible modules as well.

Let us present a brief sketch of the proof of these statements. Let us first sketch the proof for p-divisible groups. Let G be a p-divisible group of height h and dimension at most 1 over a base scheme S and denote by G_n the truncated Barsotti-Tate group of level n. When S is the spectrum of a field, the exterior powers of G are constructed in [12] and it is shown there that $\bigwedge^r G$ is a p-divisible group of height $\binom{h}{r}$ and dimension $\binom{h-1}{r-1}$ (resp. 0) if G has dimension 1 (resp. 0).

Let R be a local Artin ring of residual characteristic p and $S = \mathbf{Spec}(R)$. Here we use Zink's theory of displays (which is, in a suitable sense, a generalization of Dieudonné theory to more general base rings). Let $\mathcal{P}$ be the display attached to G. We construct the exterior powers of $\mathcal{P}$, which are again displays, and therefore correspond to certain p-divisible groups and we want to show that they are the exterior powers we are looking for.

We would like to emphasize that the construction of $\bigwedge^r \mathcal{P}$ as a display is not hard (albeit technical) and makes up only a small part of the paper; the core of the paper and what is truly difficult is that the p-divisible group attached to this display satisfies the universal property of the r^{th}-exterior power. It is worth mentioning here that if the dimension of G were bigger than 1, one could not define a Frobenius on $\bigwedge^r \mathcal{P}$, in other words, $\bigwedge^r \mathcal{P}$ does not exist as a display.

After defining alternating morphisms of displays, we construct a homomorphism

$$\beta : \mathrm{Alt}(\mathcal{P}^r, \mathcal{P}') \to \mathrm{Alt}(BT^r_{\mathcal{P}}, BT_{\mathcal{P}'}),$$

where $\mathcal{P}$ and $\mathcal{P}'$ are displays and $BT_{\mathcal{P}}$ and $BT_{\mathcal{P}'}$ are their corresponding p-divisible groups. The map β is one of the key ingredients of the proof of the main theorem. Put $\bigwedge^r \mathcal{P}$ for $\mathcal{P}'$ and denote by Λ the p-divisible group associated with $\bigwedge^r \mathcal{P}$. Under β, the universal alternating morphism $\mathcal{P}^r \to \bigwedge^r \mathcal{P}$ is mapped to an alternating morphism $\lambda : G^r \to \Lambda$, which we want to show is the universal one. One of the main results of the paper is that β is an isomorphism when R is a perfect field. This will allow us to reduce the proof of the universality of λ to the case over a perfect base field, where we already know the result [12].

When R is a complete local Noetherian ring of residue characteristic p, an approximation argument combined with the universal property of exterior powers will provide the exterior powers of G over R. In particular, the exterior powers of the universal deformation of a fixed connected p-divisible group of dimension 1 (defined over $\mathbb{F}_p$) over the universal deformation ring $\mathbb{Z}[\![x_1, \ldots, x_{h-1}]\!]$ exist.

Now, let S be a $\mathbb{Z}_{(p)}$-scheme. In order to construct the exterior powers of G, we construct the exterior powers of individual G_n and show that they form a Barsotti-Tate group. To show the existence of $\bigwedge^r G_n$, we prove faithfully flat descent results (descent of objects and morphisms) and use a result of Lau, which essentially says that under certain conditions, the pullback of a truncated Barsotti-Tate group defined over a $\mathbb{Z}_{(p)}$-scheme, via a faithfully flat and affine morphism is the pullback of the truncated universal Barsotti-Tate group defined over $\mathbb{Z}[\![x_1, \ldots, x_{h-1}]\!]$. We show that $\bigwedge^r G_n$ $(n \geq 1)$ sit in exact sequences, making them a Barsotti-Tate group.

Finally, let S be an arbitrary base scheme. We have a faithfully flat covering $S' := S[\frac{1}{p}] \coprod S_{(p)} \to S$, where $S[\frac{1}{p}]$ and $S_{(p)}$ are respectively the pullbacks of $S \to \mathbf{Spec}(\mathbb{Z})$ via the morphisms $\mathbf{Spec}(\mathbb{Z}[\frac{1}{p}]) \to \mathbf{Spec}(\mathbb{Z})$ and $\mathbf{Spec}(\mathbb{Z}_{(p)}) \to \mathbf{Spec}(\mathbb{Z})$. By base change, we then obtain p-divisible groups $G[\frac{1}{p}]$ over $S[\frac{1}{p}]$ and $G_{(p)}$ over $S_{(p)}$. Since p is invertible on $S[\frac{1}{p}]$, the p-divisible group $G[\frac{1}{p}]$ is étale, and thus, by Proposition 3.3 of [12], $\bigwedge^r G[\frac{1}{p}]$ exists. Over $S_{(p)}$, we also have constructed the exterior power $\bigwedge^r G_{(p)}$. These p-divisible groups glue together to produce the r^{th} exterior power of G' (the pullback of G) over S'. Again, using faithfully flat descent, we obtain the p-divisible group $\bigwedge^r G$ over S.

For π-divisible modules over complete local Noetherian rings, in order to prove Theorem 4.34, we use a generalization of the theory of displays, the so called *ramified displays*. We then develop a *ramified Dieudonné theory*, which is a finer version of the classical Dieudonné theory and is more consistent with the theory of ramified displays. Most of the results for p-divisible groups and displays hold for π-divisible modules and ramified displays as well. However, the result of Lau that was previously used (to pass from this case to arbitrary base scheme), does not hold and, in order to circumvent it, we need to assume that the base is locally Noetherian. Then, it is rather elementary, by devissage arguments, to reduce to the case of complete local

Noetherian rings where, again, we have already established the result.

Conventions. Throughout the article, unless otherwise specified, rings are commutative with 1. Group schemes are all commutative. Exact sequences of group schemes are exact as sequences of fppf sheaves.

Notations.

- p is a prime number, q is a power of p and $\mathbb{F}_q$ is the finite field with q elements.
- For natural numbers m and n, the binomial coefficient $\binom{n}{m}$ is defined to be zero when $m > n$.
- If R is a ring and r is an element of R, we denote by R/r the quotient ring R/rR.
- Let X be a scheme over a base scheme S. We identify X with the sheaf $\mathrm{Hom}_S(\_, X)$ on fppf site of S.
- Let X be a scheme over a base scheme S and $f : T \to S$ a morphism. We denote by X_T the fiber product $X \times_S T$. If $\mathcal{F}$ is a sheaf on a Grothendieck site over S, we denote by $f^*\mathcal{F}$ the pullback of $\mathcal{F}$ along f. So f^*X and X_T are identified as sheaves.
- Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a Grothendieck site. We denote by $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ the sheaf hom from $\mathcal{F}$ to $\mathcal{G}$.
- Let G be a finite flat group scheme over a base scheme S. Denote the Cartier dual $\underline{\mathrm{Hom}}_S(G, \mathbb{G}_{m,S})$ by G^*.
- Let R be a ring. Denote by $W(R)$ the ring of p-typical Witt vectors with coefficients in R.
- $\mathbb{Z}_q$ is the unramified extension $W(\mathbb{F}_q)$ of $\mathbb{Z}_p$ with residue field $\mathbb{F}_q$. Denote by $\mathbb{Q}_q$ the field of fractions of $\mathbb{Z}_q$.
- Let k be a perfect field of characteristic p. We denote by $\mathbb{E}_k$ the Dieudonné ring over k, i.e., the non-commutative polynomial ring in variables F, V and coefficients in $W(k)$ subject to relations $FV = VF = p$, $F\xi = \xi^\sigma F$ and $V\xi^\sigma = \xi V$ for all $\xi \in W(k)$, where ${}^\sigma : W(k) \to W(k)$ is the Frobenius morphism of $W(k)$.
- Let k be a perfect field of characteristic p and G a finite group scheme over k of p-power order. We denote the covariant Dieudonné module of G by $D_*(G)$. For details see [7].

1. Preliminaries

In this section, we gather the definitions and results from [12], that we need in the following sections. For proofs, please see *op. cit.* Fix a ring R, a perfect field k of characteristic $p > 2$ and a base scheme S.

Definition 1.1. A representable fppf-sheaf of R-modules over S will be called an R-module scheme. ▲

Remark 1.2. 1) Let $\mathcal{F}$ and $\mathcal{G}$ be fppf-sheaves of R-modules. We denote by $\mathrm{Hom}^R(\mathcal{F},\mathcal{G})$ and resp. $\underline{\mathrm{Hom}}^R(\mathcal{F},\mathcal{G})$ the R-module of R-linear morphisms and resp. the sheaf (of R-modules) of R-linear morphisms.

2) Let M be a finite locally free R-module scheme over S. The Cartier dual of M, i.e., the group scheme $\underline{\mathrm{Hom}}(M,\mathbb{G}_{m,S})$ has a natural R-module scheme structure given by the action of R on M. ♢

Definition 1.3. Let $M_1,\ldots,M_r,M$ and N be presheaves of R-modules on the fppf site of S.

(i) An *R-multilinear* or simply *multilinear* morphism from the product $\prod_{i=1}^r M_i$ to N is a morphism of presheaves of sets such that for every S-scheme T, the induced morphism $\prod M_i(T)\to N(T)$ is R-multilinear. The R-module of all such morphisms will be denoted by $\mathrm{Mult}_S^R(\prod_{i=1}^r M_i,N)$.

(ii) *Alternating R-multilinear morphisms* are defined similarly. The R-module of all alternating multilinear morphisms from M^r to N is denoted by $\mathrm{Alt}_S^R(M^r,N)$.

When no confusion is likely, we will drop the subscript S from the notation. ▲

Definition 1.4. Let $M_1\ldots,M_r$ and M be presheaves of R-modules and G a presheaf of Abelian groups. We denote by $\widetilde{\mathrm{Mult}}_S^R(\prod_{i=1}^r M_i,G)$ the group of morphisms $\varphi:\prod_{i=1}^r M_i\to G$ which are multilinear, when M_i are regarded as presheaves of Abelian groups and has the following weaker property than R-linearity: for every S-scheme T, every tuple $(m_1,\cdots,m_r)\in\prod_{i=1}^r M_i(T)$, every $a\in R$ and every i, we have $\varphi(a\cdot m_1,m_2,\cdots,m_r)=\varphi(m_1,\cdots,m_{i-1},a\cdot m_i,m_{i+1},\cdots,m_r)$. The elements of $\widetilde{\mathrm{Mult}}_S^R(\prod_{i=1}^r M_i,G)$ are called *pseudo-R-multilinear*. Similarly, we define *alternating pseudo-R-multilinear morphisms* and denote by $\widetilde{\mathrm{Alt}}_S^R(M^r,G)$ the group of such morphisms. When no confusion is likely, we will drop the subscript S from the notation. ▲

Remark 1.5. Note that the group $\widetilde{\mathrm{Mult}}^R(\prod_{i=1}^r M_i,G)$ has a natural structure of R-module through the action of R on one of the factors M_i, and this is independent of the factor we choose. Similarly, there is a natural R-module structure on the group $\widetilde{\mathrm{Alt}}^R(M^r,G)$. ♢

Definition 1.6. Let $M_1,\ldots,M_r,M$ and N be presheaves of R-modules and G a presheaf of Abelian groups. Define contravariant functors from the category of R-module schemes over S to the category of R-modules as follows:

(i)

$$T \mapsto \underline{\mathrm{Mult}}^R(M_1 \times \cdots \times M_r, N)(T) := \mathrm{Mult}_T^R(M_{1,T} \times \cdots \times M_{r,T}, N_T)$$

and resp.

$$T \mapsto \widetilde{\underline{\mathrm{Mult}}}^R(M_1 \times \cdots \times M_r, G)(T) := \widetilde{\mathrm{Mult}}_T^R(M_{1,T} \times \cdots \times M_{r,T}, G_T).$$

(ii)

$$T \mapsto \underline{\mathrm{Alt}}^R(M^r, N)(T) := \mathrm{Alt}_T^R(M_T^r, N_T)$$

and resp.

$$T \mapsto \widetilde{\underline{\mathrm{Alt}}}^R(M^r, G)(T) := \widetilde{\mathrm{Alt}}_T^R(M_T^r, G_T)).$$

▲

Proposition 1.7. *Let M, N and P and be sheaves of R-modules and G a sheaf of Abelian groups over S. We have natural isomorphisms*

$$\mathrm{Alt}^R(M^r, \underline{\mathrm{Alt}}^R(N^s, P)) \cong \mathrm{Alt}^R(M^r \times N^s, P) \quad \textit{and}$$

$$\mathrm{Alt}^R(M^r, \widetilde{\underline{\mathrm{Alt}}}^R(M, G)) \cong \widetilde{\mathrm{Alt}}^R(M^r \times N^s, G)$$

where elements on the right hand sides are morphisms that are alternating in M^r and N^s separately.

Proof. Standard. □

Remark 1.8. Let M be finite flat R-module scheme over S, and N (resp. G) be an affine R-module (resp. group) scheme over S, then using Weil restriction (cf. [8] Ch. I, §1, 6.6), one can show that the sheaf of morphisms of fppf-sheaves of sets, $\underline{\mathrm{Mor}}(M, N)$ (resp. $\underline{\mathrm{Mor}}(M, G)$), is representable by an affine scheme over S, which is of finite type resp. of finite presentation, if N (resp. G) is of finite type resp. of finite presentation, and therefore, $\underline{\mathrm{Hom}}^R(M, N)$ (resp. $\underline{\mathrm{Hom}}(M, G)$) being a closed subsheaf is representable by an affine R-module (resp. group) scheme. If in addition, N (resp. G) is of finite type or of finite presentation over S, then $\underline{\mathrm{Hom}}^R(M, N)$ (resp. $\underline{\mathrm{Hom}}(M, G)$) has the same property. Now, one can show by induction on r that $\underline{\mathrm{Mult}}^R(M^r, N)$ (resp. $\widetilde{\underline{\mathrm{Mult}}}^R(M^r, G)$) is representable by an affine R-module (group) scheme, and this scheme is of finite type resp. of finite presentation, if moreover, N (resp. G) is of finite type resp. of finite presentation. Finally, as $\underline{\mathrm{Alt}}^R(M^r, N)$ (resp. $\widetilde{\underline{\mathrm{Alt}}}^R(M^r, G)$) is a closed subfunctor, it is representable by an affine R-module (resp. group) scheme over S and is of finite type resp. of finite presentation, if moreover, N (resp. G) is of finite type resp. of finite presentation. ♢

Definition 1.9. Let $M_1, ..., M_r, M, N$ be left $\mathbb{E}_k \otimes_{\mathbb{Z}} R$-modules.

(i) Denote by $\mathrm{Mult}^R(\prod_{i=1}^r M_i, N)$ the group of $W(k) \otimes_{\mathbb{Z}} R$-multilinear morphisms $\ell : \prod_{i=1}^r M_i \to N$ satisfying the following conditions for all i and $m_i \in M_i$:

$$\ell(Vm_1, \dots, Vm_r) = V\ell(m_1, \dots, m_r),$$

$$\ell(m_1, \dots, m_{i-1}, Fm_i, m_{i+1}, \dots m_r) =$$

$$F\ell(Vm_1, \dots, Vm_{i-1}, m_i, Vm_{i+1}, \dots Vm_r).$$

(ii) Let $\mathrm{Alt}^R(M^r, N)$ denote the submodule of $\mathrm{Mult}^R(M^r, N)$ consisting of alternating morphisms. ▲

Proposition 1.10. *Let $M_1, \dots, M_r$ and M be finite local-local R-module schemes and N a finite local R-module scheme over k. Then there exists a natural isomorphism*

$$\Omega : \mathrm{Mult}^R(D_*(M_1) \times \cdots \times D_*(M_r), D_*(N)) \longrightarrow \mathrm{Mult}^R(M_1 \times \cdots \times M_r, N)$$

inducing an isomorphism

$$\Omega : \mathrm{Alt}^R(D_*(M)^r, D_*(N)) \to \mathrm{Alt}^R(M^r, N).$$

Proof. This is Corollary 1.23 and Remark 1.24 of [12]. □

Notations 1.11. In this paper, $\mathcal{O}$ will be the ring of integers of a non-Archimedean local field K of characteristic zero and residue field $\mathbb{F}_q$ of characteristic $p > 2$ ($q = p^f$). Fix a uniformizer π of $\mathcal{O}$.

Definition 1.12. Let $\mathcal{M}$ be an fppf sheaf of $\mathcal{O}$-modules over S. We call $\mathcal{M}$ a *π-Barsotti-Tate group* or *π-divisible module over S* if the following conditions are satisfied:

(i) $\mathcal{M}$ is π-divisible, i.e., the homomorphism $\pi : \mathcal{M} \to \mathcal{M}$ is an epimorphism.
(ii) $\mathcal{M}$ is π-torsion, i.e., the canonical homomorphism $\varinjlim_n \mathcal{M}[\pi^n] \to \mathcal{M}$ is an isomorphism.
(iii) $\mathcal{M}[\pi]$ is representable by a finite locally free $\mathcal{O}$-module scheme over S.

The order of $\mathcal{M}[\pi]$ is of the form q^h, where $h : S \to \mathbb{Q}_{\geq 0}$ is a locally constant function, called *the height* of $\mathcal{M}$. We denote by $\mathcal{M}_i$ the kernel of multiplication by π^i. ▲

Remark 1.13. The height of a π-divisible module is an integer (cf. Theorem B.14 of [12]). ♢

Remark 1.14. Let A be a Henselian local ring and $\mathcal{M}$ a π-divisible formal $\mathcal{O}$-module over A. The connected-étale sequence of $\mathcal{M}$, $0 \to \mathcal{M}^\circ \to \mathcal{M} \to \mathcal{M}^{\text{ét}} \to 0$, is in fact a sequence of π-divisible modules over A. We have $(\mathcal{M}^\circ)_n = (\mathcal{M}_n)^\circ$ and $(\mathcal{M}^{\text{ét}})_n = (\mathcal{M}_n)^{\text{ét}}$. ♢

Proposition 1.15. *Let $\mathcal{M}$ be an étale π-divisible module over S of height h. Then, for ever $n \geq 1$, there exists a finite étale cover $T \to S$ such that $\mathcal{M}_{n,T}$ is isomorphic to the constant group scheme $(\mathcal{O}/\pi^n)^h$. If S is connected, we can take T to be a connected finite étale cover.*

Proof. The proof is similar to the case of Barsotti-Tate groups. □

Definition 1.16. Let $\mathcal{M}_0, \ldots, \mathcal{M}_r, \mathcal{M}$ and $\mathcal{N}$ be π-divisible modules over S.

(i) An *$\mathcal{O}$-multilinear morphism* $\varphi : \mathcal{M}_1 \times \cdots \times \mathcal{M}_r \to \mathcal{M}_0$ is a system of $\mathcal{O}$-multilinear morphisms $\{\varphi_n : \mathcal{M}_{1,n} \times \cdots \times \mathcal{M}_{r,n} \to \mathcal{M}_{0,n}\}_n$ over S, compatible with the projections $\pi. : \mathcal{M}_{i,n+1} \twoheadrightarrow \mathcal{M}_{i,n}$ and $\pi. : \mathcal{M}_{0,n+1} \twoheadrightarrow \mathcal{M}_{0,n}$. In other words, it is an element of the inverse limit

$$\varprojlim_n \mathrm{Mult}_S^{\mathcal{O}}(\mathcal{M}_{1,n} \times \cdots \times \mathcal{M}_{r,n}, \mathcal{M}_{0,n}),$$

with transition homomorphisms induced by projections $\pi.: \mathcal{M}_{i,n+1} \twoheadrightarrow \mathcal{M}_{i,n}$. Denote the group of $\mathcal{O}$-multilinear morphisms from $\mathcal{M}_1 \times \cdots \times \mathcal{M}_r$ to $\mathcal{M}_0$ by $\mathrm{Mult}_S^{\mathcal{O}}(\mathcal{M}_1 \times \cdots \times \mathcal{M}_r, \mathcal{M}_0)$.

(ii) We define *alternating* $\mathcal{O}$-multilinear morphism $\varphi : \mathcal{N}^r \to \mathcal{M}$ similarly and denote the group of alternating $\mathcal{O}$-multilinear morphisms from $\mathcal{N}^r$ to $\mathcal{M}$ by $\mathrm{Alt}_S^{\mathcal{O}}(\mathcal{N}^r, \mathcal{M})$. ▲

Definition 1.17. (i) Let M be an R-module scheme and M' a group scheme over S. An alternating pseudo-R-multilinear morphism $\lambda : M^r \to M'$ or, by abuse of terminology, the group scheme M', is called an r^{th} *exterior power* of M over R, if for all group schemes N over S, the induced morphism

$$\lambda^* : \mathrm{Hom}(M', N) \to \widetilde{\mathrm{Alt}}^R(M^r, N), \quad \psi \mapsto \psi \circ \lambda,$$

is an isomorphism. If such M' and λ exist, we write $\bigwedge_R^r M$ (or $\bigwedge^r M$ if no confusion is likely) for M' and call λ *the universal alternating morphism* defining $\bigwedge_R^r M$.

(ii) Let $\mathcal{M}, \mathcal{M}'$ be π-divisible modules over S. An alternating $\mathcal{O}$-multilinear morphism $\lambda : \mathcal{M}^r \to \mathcal{M}'$ or, by abuse of terminology, the π-divisible module $\mathcal{M}'$, is called an r^{th} *exterior power* of $\mathcal{M}$ over $\mathcal{O}$, if for all π-divisible modules $\mathcal{N}$ over S, the induced morphism

$$\lambda^* : \mathrm{Hom}_S^{\mathcal{O}}(\mathcal{M}', \mathcal{N}) \to \mathrm{Alt}_S^{\mathcal{O}}(\mathcal{M}^r, \mathcal{N}), \quad \psi \mapsto \psi \circ \lambda,$$

is an isomorphism. If such $\mathcal{M}'$ and λ exist, we write $\bigwedge_{\mathcal{O}}^r \mathcal{M}$ (or $\bigwedge^r \mathcal{M}$ if no confusion is likely) for $\mathcal{M}'$ and call λ *the universal alternating morphism* defining $\bigwedge_{\mathcal{O}}^r \mathcal{M}$. ▲

Remark 1.18. 1) Exterior powers of finite R-module schemes over fields exist (Theorem 2.5 [12]).

2) By the universal property, if $\bigwedge^r M$ exists, it has a natural R-module scheme structure. $\Diamond$

Proposition 1.19. *Let M be an R-module scheme over S, and Q the cokernel of multiplication by an element $x \in R$, i.e., we have an exact sequence $M \xrightarrow{x.} M \xrightarrow{p} Q \to 0$. Then, if $\bigwedge^r M$ and $\bigwedge^r Q$ exist, the sequence $\bigwedge^r M \xrightarrow{x.} \bigwedge^r M \xrightarrow{\bigwedge^r p} \bigwedge^r Q \to 0$ is exact.*

Proof. This is Proposition 2.25 of [12]. $\square$

Proposition 1.20. *Let M (resp. $\mathcal{M}$) be a finite étale $\mathcal{O}$-module scheme (resp. étale π-divisible module of height h) over S. Then $\bigwedge^r M$ (resp. $\bigwedge^r \mathcal{M}$) exists (resp. exists and has height $\binom{h}{r}$) and is étale. Moreover, if T is an S-scheme, the canonical homomorphism $\bigwedge^r(M_T) \to (\bigwedge^r M)_T$ (resp. $\bigwedge^r(\mathcal{M}_T) \to (\bigwedge^r \mathcal{M})_T$), induced by the universal property of $\bigwedge^r(M_T)$ (resp. $\bigwedge^r(\mathcal{M}_T)$) and λ_T, is an isomorphism.*

Proof. This is Proposition 3.3 of [12]. $\square$

Proposition 1.21. *Let $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_r$ and $\mathcal{M}$ be π-divisible modules over k. There exist natural functorial isomorphisms*

$$\mathrm{Mult}^{\mathcal{O}}(D_*(M_1) \times \cdots \times D_*(M_r), D_*(M_0)) \cong \mathrm{Mult}_k^{\mathcal{O}}(\mathcal{M}_1 \times \cdots \times \mathcal{M}_r, \mathcal{M}_0)$$

and

$$\mathrm{Alt}^{\mathcal{O}}(D_*(M)^r, D_*(M_0)) \cong \mathrm{Alt}_k^{\mathcal{O}}(\mathcal{M}_1^r, \mathcal{M}_0).$$

Proof. This is Corollary 3.7 of [12]. $\square$

Proposition 1.22. *Let $\mathcal{M}$ be a π-divisible module of height h and dimension 1 over k. The covariant Dieudonné module of $\bigwedge^r \mathcal{M}_i$ is canonically isomorphic to $\bigwedge^r D_*(\mathcal{M}_i)$ and the order of $\bigwedge^r \mathcal{M}_i$ is equal to $q^{i\binom{h}{r}}$. Furthermore, the universal alternating morphism $\mathcal{M}_i^r \to \bigwedge^r \mathcal{M}_i$ and the universal alternating morphism $D_*(\mathcal{M}_i)^r \to \bigwedge^r D_*(\mathcal{M}_i) \cong D_*(\bigwedge^r \mathcal{M}_i)$ correspond to each other under the isomorphism*

$$\mathrm{Alt}^{\mathcal{O}}(D_*(\mathcal{M}_i)^r, D_*(\bigwedge^r \mathcal{M}_i)) \cong \mathrm{Alt}^{\mathcal{O}}(\mathcal{M}_i^r, \bigwedge^r \mathcal{M}_i)$$

given by Proposition 1.10.

Proof. This is Corollary 3.21 and Remark 3.22 of [12]. $\square$

Theorem 1.23. *Let $\mathcal{M}$ be a π-divisible module of height h and dimension 1 over a field of characteristic $p > 2$. Then $\bigwedge_{\mathcal{O}}^r \mathcal{M}$ exists, and has height $\binom{h}{r}$. It has dimension $\binom{h-1}{r-1}$ (resp. 0) if $\mathcal{M}$ has dimension 1 (resp. 0). Moreover, for every n, we have $(\bigwedge_{\mathcal{O}}^r \mathcal{M})_n \cong \bigwedge_{\mathcal{O}}^r(\mathcal{M}_n)$.*

Proof. This is Theorem 3.36 of [12]. $\square$

2. Multilinear Theory of Displays

In this section, we assume some familiarity with the theory of displays (see [24] for details). We have included in the Appendix B the definitions, notations, constructions and results of the theory of displays that will be used in this section. We will develop a multilinear theory of displays, which, in the next section, will be related to the multilinear theory of group schemes developed by R. Pink. Fix a ring R.

2.1. Multilinear morphisms and the map β.

Definition 2.1. Let $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r$ and $\mathcal{P}$ be $3n$-displays over R.

(i) A *multilinear* morphism $\varphi : \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{P}_0$ is a $W(R)$-linear morphism $\varphi : P_1 \times \cdots \times P_r \to P_0$ satisfying the following conditions:
- φ restricts to a $W(R)$-multilinear morphism $\varphi : Q_1 \times \cdots \times Q_r \to Q_0$.
- *the V-condition*:

$$\forall q_i \in Q_i : \, V^{-1}\varphi(q_1, \dots, q_r) = \varphi(V^{-1}q_1, \dots, V^{-1}q_r).$$

The $W(R)$-module of multilinear morphisms $\mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{P}_0$ is denoted by $\mathrm{Mult}(\mathcal{P}_1 \times \cdots \times \mathcal{P}_r, \mathcal{P}_0)$.

(ii) *Alternating* multilinear morphisms $\varphi : \mathcal{P}^r \to \mathcal{P}_0$ are defined similarly. The $W(R)$-module of alternating morphisms $\mathcal{P}^r \to \mathcal{P}_0$ is denoted by $\mathrm{Alt}(\mathcal{P}^r, \mathcal{P}_0)$. ▲

Remark 2.2. Since the morphism F is uniquely determined by V^{-1}, the V-condition above implies that a multilinear morphism φ satisfies the *F-condition* as well: for all i, $x_i \in P_i$ and $q_j \in Q_j$ $(j \neq i)$ we have:

$$F\varphi(q_1, \dots, q_{i-1}, x_i, q_{i+1}, \dots, q_r) = \varphi(V^{-1}q_1, \dots, V^{-1}q_{i-1}, Fx_i, V^{-1}q_{i+1}, \dots, V^{-1}q_r). \quad \diamondsuit$$

Construction 2.3. Let $\mathcal{P}_1, \dots, \mathcal{P}_r, \mathcal{P}_0$ be $3n$-displays over R and $\varphi : \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{P}_0$ a multilinear morphism of $3n$-displays.

- Let S be an R-algebra. We extend φ to a multilinear morphism

$$\varphi_S : \mathcal{P}_{1,S} \times \cdots \times \mathcal{P}_{r,S} \to \mathcal{P}_{0,S}$$

as follows. For all $w_i \in W(S)$ and all $x_i \in P_i$, we set

$$\varphi_S(w_1 \otimes x_1, \dots, w_r \otimes x_r) := w_1 \dots w_r \otimes \varphi(x_1, \dots, x_r)$$

and extend $W(S)$-multilinearly to the whole product

$$\mathcal{P}_{1,S} \times \cdots \times \mathcal{P}_{r,S}.$$

- Given a nilpotent R-algebra $\mathcal{N}$ we extend φ to a $\widehat{W}(\mathcal{N})$-multilinear morphism

$$\widehat{\varphi} : \widehat{P}_1 \times \cdots \times \widehat{P}_r \to \widehat{P}_0$$

as follows. For all $\omega_i \in \widehat{W}(\mathcal{N})$ and all $x_i \in P_i$, we set:

$$\widehat{\varphi}(\omega_1 \otimes x_1, \dots, \omega_r \otimes x_r) := \omega_1 \dots \omega_r \otimes \varphi(x_1, \dots, x_r)$$

and extend $\widehat{W}(\mathcal{N})$-multilinearly to the whole product

$$\widehat{P}_1 \times \cdots \times \widehat{P}_r$$

. ▼

Lemma 2.4. *The multilinear morphisms φ_S and $\widehat{\varphi}$ constructed above satisfy the V-F conditions.*

Proof. The proof of the lemma for φ_S and $\widehat{\varphi}$ is similar and thus, we only prove the lemma for $\widehat{\varphi}$. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$-display over R. Take elements $w \in \widehat{W}(\mathcal{N})$ and $x \in P$. By construction of F and V on $\widehat{P}_{\mathcal{N}}$ (cf. Construction B.10), we have

$$F(w \otimes x) = F(w) \otimes F(x) = F(w) \otimes V^{-1}(V(1) \cdot x) =$$

$$V^{-1}(w \otimes V(1) \cdot x) = V^{-1}(V(1)w \otimes x).$$

So, it suffices to show that $\widehat{\varphi}$ satisfies the V-condition. For each i take an element $\hat{q}_i$ in $\widehat{Q}_{i,\mathcal{N}}$. As $\widehat{\varphi}$ is $\widehat{W}(\mathcal{N})$-multilinear, we can assume that either $\hat{q}_i \in \widehat{W}(\mathcal{N}) \otimes L_i$ or $\hat{q}_i \in \widehat{I}_{\mathcal{N}} \otimes T_i$, where for each i, we have fixed a normal decomposition $P_i = L_i \oplus T_i$, and that each $\hat{q}_i$ is a pure tensor (i.e., of the form $x \otimes y$). Since the construction of $\widehat{\varphi}$ is symmetric with respect to i and for the sake of simplicity, we can assume that $\hat{q}_j = w_j \otimes q_j \in \widehat{W}(\mathcal{N}) \otimes L_j$ for $1 \leq j \leq s$ and $\hat{q}_j = V(w_j) \otimes t_j$ for $s+1 \leq j \leq r$ for some $0 \leq s \leq r$. We divide the problem in two cases: $s < r$ and $s = r$. In the first case, we have:

$$\widehat{\varphi}(\hat{q}_1, \dots, \hat{q}_r) = \widehat{\varphi}(w_1 \otimes q_1 \dots, w_s \otimes q_s, V(w_{s+1}) \otimes t_{s+1}, \dots, V(w_r) \otimes t_r) =$$

$$w_1 \dots w_s V(w_{s+1}) \dots V(w_r) \otimes \varphi(q_1, \dots, q_s, t_{s+1}, \dots, t_r) =$$

$$V\big(F\big(w_1 \dots w_s V(w_{s+1}) \dots V(w_{r-1})\big) \cdot w_r\big) \otimes \varphi(q_1, \dots, q_s, t_{s+1}, \dots, t_r).$$

The element $V\big(F\big(w_1 \dots w_s V(w_{s+1}) \dots V(w_{r-1})\big) \cdot w_r\big)$ being in the ideal $\widehat{I}_{\mathcal{N}}$, it follows that $\widehat{\varphi}(\hat{q}_1, \dots, \hat{q}_r) \in \widehat{Q}_{0,\mathcal{N}}$. In the second case, we have $\widehat{\varphi}(\hat{q}_1, \dots, \hat{q}_r) = \widehat{\varphi}(w_1 \otimes q_1 \dots, w_r \otimes q_r) = w_1 \dots w_r \otimes \varphi(q_1, \dots, q_r)$. As by assumption $\varphi(q_1, \dots, q_r) \in Q_0$, we conclude again that $\widehat{\varphi}(\hat{q}_1, \dots, \hat{q}_r) \in \widehat{Q}_{0,\mathcal{N}}$. □

Construction 2.5. Given displays $\mathcal{P}_1, \dots, \mathcal{P}_r, \mathcal{P}_0$ over R and a multilinear morphism $\varphi : \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{P}_0$, we construct for all n, a map

$$\beta_{\varphi,n} : BT_{\mathcal{P}_1,n} \times \cdots \times BT_{\mathcal{P}_r,n} \to BT_{\mathcal{P}_0,n},$$

where $BT_{\mathcal{P}_i,n}$ is the kernel of multiplication by p^n on the p-divisible group $BT_{\mathcal{P}_i}$. Take a nilpotent R-algebra $\mathcal{N}$ and elements $[x_i]_n \in BT_{\mathcal{P}_i,n}(\mathcal{N})$ and set

$$\beta_{\varphi,n}([x_1]_n, \dots, [x_r]_n) :=$$
$$(-1)^{r-1} \sum_{i=1}^{r} [\widehat{\varphi}(V^{-1}g_1, \dots, V^{-1}g_{i-1}, x_i, g_{i+1}, \dots, g_r)],$$

where for all $1 \le j \le r$, we have abbreviated ${}_n g_{\mathcal{P}_j}(x_j)$ (from Notations B.12) to g_j. We show in the next lemma that this is a well-defined multilinear morphism. ▼

Remark 2.6. 1) Note that if $r = 1$, then $\beta_{\varphi,n} : BT_{\mathcal{P}_1,n} \to BT_{\mathcal{P}_0,n}$ is the restriction of the homomorphism $BT_\varphi : BT_{\mathcal{P}_1} \to BT_{\mathcal{P}_0}$ (using the functoriality of BT) to $BT_{\mathcal{P}_1,n}$.

2) The sign $(-1)^{r-1}$ in the above formula is there to make the diagram (3.4) commutative.

3) Let $\mathrm{Biext}^1(BT_{\mathcal{P}_1} \times BT_{\mathcal{P}_2}, \widehat{\mathbb{G}}_m)$ be the group of biextensions of the formal groups $BT_{\mathcal{P}_1}$ and $BT_{\mathcal{P}_2}$ in the sense of Mumford (cf. [19]). One can show that in the case, where $r = 2$ and $\mathcal{P}_0$ is the multiplicative display $\mathcal{G}_m$ (see Example B.9), homomorphism β induces a homomorphism

$$\mathrm{Mult}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{G}_m) \to \mathrm{Biext}^1(BT_{\mathcal{P}_1} \times BT_{\mathcal{P}_2}, \widehat{\mathbb{G}}_m)$$

that coincides (up to sign) with a similar homomorphism defined in [24] p. 110. ♢

Proposition 2.7. *The maps $\beta_{\varphi,n} : BT_{\mathcal{P}_1,n} \times \cdots \times BT_{\mathcal{P}_r,n} \to BT_{\mathcal{P}_0,n}$ satisfy the following properties:*

(i) *$\beta_{\varphi,n}$ are well-defined multilinear morphisms.*

(ii) *$\beta_{\varphi,n}$ are compatible with respect to projections $p : BT_{\mathcal{P}_i,n+1} \twoheadrightarrow BT_{\mathcal{P}_i,n}$ (induced by multiplication by p).*

(iii) *If the $3n$-displays $\mathcal{P}_1 \dots, \mathcal{P}_r$ are equal, then if φ is alternating, $\beta_{\varphi,n}$ is alternating as well.*

(iv) *The construction of $\beta_{\varphi,n}$ commutes with base change, i.e., if S is an R-algebra, and using the identification $BT_{\mathcal{P}_{i_S}} \cong (BT_{\mathcal{P}_i})_S$, $(\beta_{\varphi,n})_S$ and $\beta_{\varphi_S,n}$ are equal as multilinear morphisms*

$$\prod_{i=1}^{r} BT_{\mathcal{P}_{i_S},n} \to BT_{\mathcal{P}_{0_S},n}.$$

Proof. We fix a nilpotent R-algebra $\mathcal{N}$.

(i) For each i, take elements $[x_i]_n \in BT_{\mathcal{P}_i,n}(\mathcal{N})$. If we show that the element $\beta_{\varphi,n}([x_1]_n, \dots, [x_r]_n)$ does not depend on the representatives x_i of the class $[x_i]$ and that the map $\beta_{\varphi,n}$ is multilinear, then it follows that $\beta_{\varphi,n}([x_1]_n, \dots, [x_r]_n)$ lies in the kernel of multiplication by p^n (note that $p^n[x_i] = 0$). By multilinearity of $\widehat{\varphi}$, in order to show the independence of $\beta_{\varphi,n}([x_1]_n,\ldots,[x_r]_n)$ from the choice of the elements x_i, it is sufficient to show that if one x_j is in the subgroup $(V^{-1}-\mathrm{Id})G^{-1}_{\mathcal{P}_j}$ of $G^0_{\mathcal{P}_j}$, then the element $\beta_{\varphi,n}([x_1]_n,\ldots,[x_r]_n)$ is in the subgroup $(V^{-1}-\mathrm{Id})G^{-1}_{\mathcal{P}_0}$ of $G^0_{\mathcal{P}_0}$. So, assume that $x_j=(V^{-1}-\mathrm{Id})(z_j)$ for some $z_j\in G^{-1}_{\mathcal{P}_j}$ and for every i, set $g_i := {}_ng_{\mathcal{P}_j}(x_i)$. We then have

$$(V^{-1}-\mathrm{Id})(g_j)=p^nx_j=p^n(V^{-1}-\mathrm{Id})(z_j)=(V^{-1}-\mathrm{Id})(p^nz_j)$$

and since $V^{-1}-\mathrm{Id}$ is injective, it implies that $g_j=p^nz_j$. Set also:

- $A := (-1)^{r-1}\sum_{i=1}^{j-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{i-1},x_i,g_{i+1},\ldots,g_r)$,
- $B := (-1)^{r-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{j-1},x_j,g_{j+1},\ldots,g_r)$ and
- $C := (-1)^{r-1}\sum_{i=j+1}^{r}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{i-1},x_i,g_{i+1},\ldots,g_r)$.

We then have $\beta_{\varphi,n}([x_1]_n,\ldots,[x_r]_n)=[A+B+C]$. We develop each of the terms separately, by replacing x_j and g_j with $(V^{-1}-\mathrm{Id})(z_j)$ and respectively p^nz_j. An straightforward calculation will imply:

- $A=(-1)^{r-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{j-1},z_j,g_{j+1},\ldots,g_r)-$

$$(-1)^{r-1}\widehat{\varphi}(g_1,\ldots,g_{j-1},z_j,g_{j+1},\ldots,g_r). \tag{2.8}$$

- $B=(-1)^{r-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{j-1},V^{-1}z_j,g_{j+1},\ldots,g_r)-$

$$(-1)^{r-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{j-1},z_j,g_{j+1},\ldots,g_r). \tag{2.9}$$

- $C=(-1)^{r-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{j-1},V^{-1}z_j,$ $V^{-1}g_{j+1},\ldots,V^{-1}g_r)-$

$$(-1)^{r-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{j-1},V^{-1}z_j,g_{j+1},\ldots,g_r). \tag{2.10}$$

Now, adding up A, B and C and using equations (2.8), (2.9) and (2.10), we observe that four terms of the six terms cancel out and we obtain

$$\begin{gathered}\beta_{\varphi,n}([x_1]_n,\ldots,[x_r]_n)=[A+B+C]=\\ [(-1)^{r-1}\widehat{\varphi}(V^{-1}g_1,\ldots,V^{-1}g_{j-1},V^{-1}z_j,V^{-1}g_{j+1},\ldots,V^{-1}g_r)-\\ (-1)^{r-1}\widehat{\varphi}(g_1,\ldots,g_{j-1},z_j,g_{j+1},\ldots,g_r)]=\\ [(V^{-1}-\mathrm{Id})((-1)^{r-1}\widehat{\varphi}(g_1,\ldots,g_{j-1},z_j,g_{j+1},\ldots,g_r))].\end{gathered}$$

As the vector $(g_1,\ldots,g_{j-1},z_j,g_{j+1},\ldots,g_r)$ belongs to $\widehat{Q}_{1,\mathcal{N}}\times\cdots\times\widehat{Q}_{r,\mathcal{N}}$ and so by Lemma 2.4, $\widehat{\varphi}(g_1,\ldots,g_{j-1},z_j,g_{j+1},\ldots,g_r))$ belongs to $\widehat{Q}_{0,\mathcal{N}}=G^{-1}_{\mathcal{P}_0}$, we conclude that $\beta_{\varphi,n}([x_1]_n,\ldots,[x_r]_n)$ is the zero in the quotient $BT_{\mathcal{P}_0}$. This proves the independence from the choices of representatives. It remains to prove the multilinearity. Since $V^{-1}-\mathrm{Id}$ is an injective homomorphism and $\widehat{\varphi}$ is multilinear, a straightforward calculation shows that $\beta_{\varphi,n}$ is multilinear too. This proves part (i).

(ii) Take elements $[x_i]_{n+1} \in BT_{\mathcal{P}_i,n+1}(\mathcal{N})$. If we set $g_i := {}_{n+1}g_{\mathcal{P}_i}(x_{i+1})$, we have $p^n(px_i) = p^{n+1}x_i = (V^{-1} - \mathrm{Id})g_i$ and therefore ${}_n g_{\mathcal{P}_i}(px_i) = g_i$. Thus, we have

$$\beta_{\varphi,n}([px_1]_n, \dots, [px_r]_n) = (-1)^{r-1} \sum_{i=1}^{r} [\widehat{\varphi}(V^{-1}g_1, \dots,$$
$$V^{-1}g_{i-1}, px_i, g_{i+1}, \dots, g_r)]$$
$$= p(-1)^{r-1} \sum_{i=1}^{r} [\widehat{\varphi}(V^{-1}g_1, \dots,$$
$$V^{-1}g_{i-1}, x_i, g_{i+1}, \dots, g_r)]$$
$$= p\beta_{\varphi,n+1}([x_1]_{n+1}, \dots, [x_r]_{n+1})$$

where we have used the multilinearity of $\widehat{\varphi}$ for the second equality. This proves part (ii).

(iii) Set $\mathcal{P} := \mathcal{P}_1 = \cdots = \mathcal{P}_r$. Let $\sigma \in S_n$ be a permutation of n elements and define a new map $\psi : \mathcal{P}^r \to \mathcal{P}_0$ by setting $\psi(a_1, \dots, a_r) := \varepsilon \cdot \mathrm{sgn}(\sigma)\varphi(a_{\sigma(1)}, \dots, a_{\sigma(r)})$, where $\varepsilon \in \{1, -1\}$ is a fixed sign. Since φ is a multilinear morphism of $3n$-displays, it follows from the definition that the new map ψ is also a multilinear morphism of $3n$-displays. We claim that for any natural number n, any $1 \leq i \leq r$, any $[x_i] \in BT_{\mathcal{P},n}(\mathcal{N})$ and any permutation $\sigma \in S_n$ we have

$$\beta_{\psi,n}([x_1], \dots, [x_r]) = \varepsilon \cdot \mathrm{sgn}(\sigma)\beta_{\varphi,n}([x_{\sigma(1)}], \dots, [x_{\sigma(r)}]). \tag{2.11}$$

To prove the claim, it suffices to assume that σ is a transposition of the form $(t, t+1)$ with $1 \leq t \leq r-1$. Again, we set $g_i := {}_n g_{\mathcal{P}}(x_i)$. Using the multilinearity of $\widehat{\varphi}$, the formulae $(V^{-1} - \mathrm{Id})g_t = p^n x_t$ and $(V^{-1} - \mathrm{Id})g_{t+1} = p^n x_{t+1}$ and an straightforward calculation now implies that

$$\beta_{\psi,n}([x_1], \dots, [x_r]) - \varepsilon\,\mathrm{sgn}(\sigma)\beta_{\varphi,n}([x_{\sigma(1)}], \dots, [x_{\sigma(r)}]) =$$
$$\varepsilon(-1)^{r-1}\big[\widehat{\varphi}(V^{-1}g_1, \dots, V^{-1}g_{t-1}, V^{-1}g_{t+1} - g_{t+1}, x_t, g_{t+2}, g_{t+3}, \dots, g_r) -$$
$$\widehat{\varphi}(V^{-1}g_1, \dots, V^{-1}g_{t-1}, x_{t+1}, V^{-1}g_t - g_t, g_{t+2}, g_{t+3}, \dots, g_r)\big]_n =$$
$$\varepsilon(-1)^{r-1}\big[\widehat{\varphi}(V^{-1}g_1, \dots, V^{-1}g_{t-1}, p^n x_{t+1}, x_t, g_{t+2}, g_{t+3}, \dots, g_r) -$$
$$\widehat{\varphi}(V^{-1}g_1, \dots, V^{-1}g_{t-1}, x_{t+1}, p^n x_t, g_{t+2}, g_{t+3}, \dots, g_r)\big]_n =$$
$$\varepsilon(-1)^{r-1}\big[p^n\widehat{\varphi}(V^{-1}g_1, \dots, V^{-1}g_{t-1}, x_{t+1}, x_t, g_{t+2}, g_{t+3}, \dots, g_r) -$$
$$p^n\widehat{\varphi}(V^{-1}g_1, \dots, V^{-1}g_{t-1}, x_{t+1}, x_t, g_{t+2}, g_{t+3}, \dots, g_r)\big]_n = 0.$$

Now, assume that φ is alternating. We have to show that if two components of the vector $[\vec{x}] := ([x_1], \dots, [x_r]) \in BT_{\mathcal{P},n}(\mathcal{N})^r$ are equal, then $\beta_{\varphi,n}([\vec{x}]) = 0$. Using the identity (2.11) we can assume, without loss of generality, that the first two components of $[\vec{x}]$ are equal. Note also that φ being alternating, the extended multilinear morphism $\widehat{\varphi}$ is alternating as well. We have

$$\beta_{\varphi,n}([x_1],[x_1],[x_3],\dots,[x_r]) =$$

$$(-1)^{r-1}\big[\widehat{\varphi}(x_1,g_1,g_3,g_4,\dots,g_r)+\widehat{\varphi}(V^{-1}g_1,x_1,g_3,g_4,\dots,g_r)+$$

$$\sum_{i=3}^{r}\widehat{\varphi}(V^{-1}g_1,V^{-1}g_1,V^{-1}g_3,V^{-1}g_4,\dots,V^{-1}g_{i-1},x_i,g_{i+1},g_r)\big]_n,$$

where as before $g_i = {}_n g_{\mathcal{P}}(x_i)$. The last sum is zero, because $\widehat{\varphi}$ is alternating and if we use the fact that $\widehat{\varphi}$ is antisymmetric, the sum of the first two terms will be equal to

$$(-1)^{r-1}\big[\widehat{\varphi}(V^{-1}g_1,x_1,g_3,g_4,\dots,g_r)-\widehat{\varphi}(g_1,x_1,g_3,g_4,\dots,g_r)\big]_n =$$

$$(-1)^{r-1}\big[\widehat{\varphi}(V^{-1}g_1-g_1,x_1,g_3,g_4,\dots,g_r)\big]_n =$$

$$(-1)^{r-1}\big[\widehat{\varphi}(p^n x_1,x_1,g_3,g_4,\dots,g_r)\big]_n = (-1)^{r-1}\big[p^n\widehat{\varphi}(x_1,x_1,g_3,g_4,\dots,g_r)\big]_n,$$

which is zero, since $\widehat{\varphi}$ is alternating.

(iv) This follows from the fact that for every nilpotent S-algebra $\mathcal{M}$, the two groups $G^0_{\mathcal{P}}(\mathcal{M}) = \widehat{W}(\mathcal{M}) \otimes_{W(R)} P$ and $G^0_{\mathcal{P}_S}(\mathcal{M}) = \widehat{W}(\mathcal{M}) \otimes_{W(S)} P_S$ are canonically isomorphic and this isomorphism induces the isomorphism $(BT_{\mathcal{P}})_S \cong BT_{\mathcal{P}_S}$.

□

Remark 2.12. The detailed calculations of the proof can be found in [11]. ♢

Corollary 2.13. *The construction of β yields homomorphisms*

$$\beta : \mathrm{Mult}(\mathcal{P}_1 \times \cdots \times \mathcal{P}_r, \mathcal{P}_0) \to \mathrm{Mult}(BT_{\mathcal{P}_1} \times \cdots \times BT_{\mathcal{P}_0}, BT_{\mathcal{P}_0}),$$

and

$$\mathrm{Alt}(\mathcal{P}_1^r, \mathcal{P}_0) \to \mathrm{Alt}(BT^r_{\mathcal{P}_1}, BT_{\mathcal{P}_0}).$$

Question 2.14. *Are the morphisms β in the Corollary 2.13 isomorphisms?*

2.2. Exterior powers.

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$-display over R, with the tangent module, i.e, the R-module P/Q, of rank one. Recall that the rank of the projective R-module P/Q is equal to the rank of the projective $W(R)$-module T in any normal decomposition $P = L \oplus T$ (see Remark B.8).

Construction 2.15. We want to define a new $3n$-display denoted by

$$\bigwedge^r \mathcal{P} = (\Lambda^r P, \Lambda^r Q, \Lambda^r F, \Lambda^r V^{-1}).$$

Fix a normal decomposition $P = L \oplus T$ and $Q = L \oplus I_R T$. Although we use a normal decomposition for the construction, we will show in the next lemma, that this construction is independent from this choice.

- Define $\Lambda^r P$ to be the exterior power of P, $\bigwedge^r P$, over the ring $W(R)$.
- Define $\Lambda^r Q$ to be the image of $\bigwedge^r Q \xrightarrow{\bigwedge^r \iota} \bigwedge^r P$, where $\iota : Q \hookrightarrow P$ is the inclusion.
- Since by assumption, T is projective of rank one, we have

$$\Lambda^r P \cong \bigwedge^r L \oplus \bigwedge^{r-1} L \otimes T \quad \text{and} \quad \Lambda^r Q \cong \bigwedge^r L \oplus \bigwedge^{r-1} L \otimes I_R T.$$

  Define $\Lambda^r F : \Lambda^r P \to \Lambda^r P$ to be $\bigwedge^{r-1} V^{-1} \wedge F$.
- Define $\Lambda^r V^{-1} : \Lambda^r Q \to \Lambda^r P$ to be $\bigwedge^r V^{-1} : \bigwedge^r Q \to \bigwedge^r P$ restricted to $\Lambda^r Q$ (note that $\Lambda^r Q$ is a direct summand of $\bigwedge^r Q \cong \bigoplus_{i=0}^r (\bigwedge^{r-i} L \otimes \bigwedge^i I_R T)$). ▼

Lemma 2.16. *The construction of $\bigwedge^r \mathcal{P} = (\Lambda^r P, \Lambda^r Q, \Lambda^r F, \Lambda^r V^{-1})$ does not depend on the choice of a normal decomposition of P and defines a $3n$-display structure. The height and rank of $\bigwedge^r \mathcal{P}$ are respectively $\binom{h}{r}$ and $\binom{h-1}{r-1}$, where h is the height of $\mathcal{P}$. If $\mathcal{P}$ is a display, then $\bigwedge^r \mathcal{P}$ is a display as well. Furthermore, this construction commutes with the base change.*

Proof. Assume that we have shown that the morphism $\Lambda^r V^{-1}$ is independent from the choice of a normal decomposition and that this construction defines a $3n$-display structure. Then, as F is uniquely determined by V^{-1}, the morphism $\Lambda^r F$ will be independent from the choice of a normal decomposition as well. So, we first prove the canonicity of $\Lambda^r V^{-1}$ and then show that this construction yields a $3n$-display.

Set $N := \bigoplus_{i=2}^r (\bigwedge^{r-i} L \otimes \bigwedge^i I_R T)$. Since $\bigwedge^r Q \cong \Lambda^r Q \oplus N$ and since the morphism $\bigwedge^r V^{-1} : \bigwedge^r Q \to \bigwedge^r P$ is independent from the choice of a normal decomposition, if we show that the restriction of this morphism to the submodule N is zero, it follows that the canonical morphism $\bigwedge^r V^{-1}$ factors through the quotient $\bigwedge^r Q \twoheadrightarrow \Lambda^r Q$. Thus, the resulting morphism $\Lambda^r Q \to \Lambda^r P$, which is equal to $\Lambda^r V^{-1}$, is independent from the choice of a normal decomposition. So, it suffices to show that for every $i > 1$, the morphism

$$\bigwedge^r V^{-1} : \bigwedge^{r-i} L \otimes \bigwedge^i I_R T \to \bigwedge^r P$$

is trivial. For any $w \in W(R)$ and $x \in P$, we have $V^{-1}(V(w).x) = wF(x)$. It implies that this morphism factors through the image of the morphism

$$\bigwedge^{r-i} V^{-1} \wedge \bigwedge^i F : \bigwedge^{r-i} L \otimes \bigwedge^i T \to \bigwedge^r P.$$

The module $\bigwedge^i T$ being trivial for $i > 1$, we conclude that the morphism $\bigwedge^r V^{-1}$ restricted to $\bigwedge^{r-i} L \otimes \bigwedge^i I_R$ is zero, as desired.

As P is a projective $W(R)$-module, its exterior powers are projective too. We have $\Lambda^r P = \bigwedge^r L \oplus \bigwedge^{r-1} L \otimes T$ and $\Lambda^r Q = \bigwedge^r L \oplus \bigwedge^{r-1} L \otimes I_R T$, and since $I_R(\bigwedge^{r-1} L \otimes T) = \bigwedge^{r-1} L \otimes I_R T$, we conclude that the direct sum $\bigwedge^r L \oplus \bigwedge^{r-1} L \otimes T$ is a normal decomposition of $\Lambda^r P$. We have to show that the morphism $\Lambda^r V^{-1}$ is an F^R-linear epimorphism. But we know that $V^{-1} : Q \to P$ is an F^R-linear epimorphism and therefore $\bigwedge^r V^{-1} : \bigwedge^r Q \to \bigwedge^r P$ is an F^R-linear epimorphism as well. As this morphism factors through the quotient $\bigwedge^r Q \twoheadrightarrow \Lambda^r Q$, the morphism $\Lambda^r V^{-1}$ is also an F^R-linear epimorphism.

Now, we show that the morphism $\Lambda^r F$ has the right properties, i.e., it is F^R-linear and satisfies the relation $w\Lambda^r F(x) = \Lambda^r V^{-1}(V(w).x)$ for every $w \in W(R)$ and every $x \in \Lambda^r P$. The fact that it is F^R-linear follows from its construction and the fact that V^{-1} and F are F^R-linear. Now take an element $w \in W(R)$ and $q_1 \wedge \cdots \wedge q_{r-1} \otimes t$ in the submodule $\bigwedge^{r-1} L \otimes T$, we have

$$w.\Lambda^r F(q_1 \wedge \cdots \wedge q_{r-1} \otimes t) = wV^{-1}q_1 \wedge \cdots \wedge V^{-1}q_{r-1} \wedge F(t) =$$
$$V^{-1}q_1 \wedge \cdots \wedge V^{-1}q_{r-1} \wedge wF(t) = V^{-1}q_1 \wedge \cdots \wedge V^{-1}q_{r-1} \wedge V^{-1}(V(1).x) =$$
$$\Lambda^r V^{-1}(V(1).q_1 \wedge \cdots \wedge q_{r-1} \otimes t).$$

A similar calculation shows that for any $w \in W(R)$ and any $x \in \bigwedge^r L$, we have $w.\Lambda^r F(x) = \Lambda^r V^{-1}(V(1).x)$ and therefore

$$\bigwedge^r \mathcal{P} = (\Lambda^r P, \Lambda^r Q, \Lambda^r F, \Lambda^r V^{-1})$$

is a $3n$-display.

By definition, the height of $\bigwedge^r \mathcal{P}$ is the rank of the projective $W(R)$-module $\Lambda^r P$, which is equal to $\binom{h}{r}$, with h the rank of P. The rank of $\bigwedge^r \mathcal{P}$ is equal to the rank of the projective $W(R)$-module $\bigwedge^{r-1} L \otimes T$, which is equal to $\binom{h-1}{r-1}$, since L has rank $h-1$ and T has rank one (cf. Remark B.8).

Since the construction of exterior powers of modules commutes with the base change, a straightforward calculation shows that, under the identification $W(R) \otimes_{F,W(R)} \Lambda^r P \cong \Lambda^r(W(R) \otimes_{F,W(R)} P)$, the morphism

$$(\Lambda^r V)^{N\sharp} : \Lambda^r P \to W(R) \otimes_{F,W(R)} \Lambda^r P$$

(cf. lemma 10, p.14 of [24]) is equal to the morphism

$$\Lambda^r(V^{N\sharp}) : \Lambda^r P \to \Lambda^r(W(R) \otimes_{F,W(R)} P).$$

Again, since $\bigwedge^r$ commutes with base change, $\Lambda^r(V^{N\sharp})$ is zero mod $I_R + pW(R)$, if $V^{N\sharp}$ is zero mod $I_R + pW(R)$. This shows that $\bigwedge^r \mathcal{P}$ is a display, if $\mathcal{P}$ is a display.

Finally, we have to show that for any R-algebra S, there exists a canonical isomorphism $\bigwedge^r(\mathcal{P}_S) \cong (\bigwedge^r \mathcal{P})_S$. This is straightforward. We explain why the pairs $((\Lambda^r P)_S, (\Lambda^r Q)_S)$ and $(\Lambda^r(P_S), \Lambda^r(Q_S))$ are canonically isomorphic, and leave the verification of the equality of the pairs

$$((\Lambda^r F)_S, (\Lambda^r V^{-1})_S)$$

and $(\Lambda^r(F_S), \Lambda^r(V_S^{-1}))$ to the reader. By definition, we have

$$(\Lambda^r P)_S = W(S) \otimes_{W(R)} \bigwedge^r P = \bigwedge^r (W(S) \otimes_{W(R)} P) = \Lambda^r(P_S)$$

and using a normal decomposition, we have

$$\begin{gathered}(\Lambda^r Q)_S = (W(S) \otimes_{W(R)} \bigwedge^r L) \oplus (I_S \otimes_{W(R)} \bigwedge^{r-1} L \otimes_{W(R)} T) \cong \\ (\bigwedge^r (W(S) \otimes_{W(R)} L)) \oplus ((I_S \otimes_{W(S)} W(S) \otimes_{W(R)} \bigwedge^{r-1} L \otimes_{W(R)} T)) \cong \\ \bigwedge^r L_S \oplus ((W(S) \otimes_{W(R)} \bigwedge^{r-1} L) \otimes_{W(S)} (I_S \otimes_{W(R)} T)) \cong \\ \bigwedge^r L_S \oplus (\bigwedge^{r-1} L_S) \otimes_{W(S)} I_S(W(S) \otimes_{W(R)} T) = \\ \bigwedge^r L_S \oplus (\bigwedge^{r-1} L_S \otimes_{W(S)} I_S T_S) = \Lambda^r(Q_S).\end{gathered}$$

The above isomorphisms are induced by the canonical isomorphism $(\bigwedge^r P)_S \cong \bigwedge^r(P_S)$, i.e., this isomorphism restricts to an isomorphism $(\Lambda^r Q)_S \cong \Lambda^r(Q_S)$. Thus, the latter isomorphism does not depend on the choice of a normal decomposition either. $\square$

Proposition 2.17. *Let $\mathcal{P}$ be a $3n$-display of rank one over R. The map*

$$\lambda : P^r \to \bigwedge^r P, \quad (x_1, \ldots, x_r) \mapsto x_1 \wedge \cdots \wedge x_r$$

defines an alternating morphism of $3n$-displays $\lambda : \mathcal{P}^r \to \bigwedge^r \mathcal{P}$ with the following universal property: For every $3n$-display $\mathcal{P}'$ over R, the following homomorphism is an isomorphism

$$\lambda^* \operatorname{Hom}(\bigwedge^r \mathcal{P}, \mathcal{P}') \to \operatorname{Alt}(\mathcal{P}^r, \mathcal{P}'), \quad \psi \mapsto \psi \circ \lambda.$$

Proof. It follows from the construction of $\bigwedge^r \mathcal{P} = (\Lambda^r P, \Lambda^r Q, \Lambda^r F, \Lambda^r V^{-1})$ that λ is an alternating morphism of $3n$-displays. We therefore only need to show the universal property. Let $\mathcal{P}' = (P', Q', F, V^{-1})$ be a $3n$-display over R and let $\varphi : \mathcal{P}^r \to \mathcal{P}'$ be an alternating morphism of $3n$-displays. We ought to show that there exists a unique morphism of $3n$-displays from $\bigwedge^r \mathcal{P}$ to $\mathcal{P}'$, whose composition with λ is φ. The morphism $\varphi : P^r \to P'$ is an alternating morphism of R-modules and the restriction of φ to Q^r is an alternating morphism $\varphi : Q^r \to Q'$. By the universal property of $\Lambda^r P =$

$\bigwedge^r P$, there exists a unique R-modules homomorphism $\bar{\varphi} : \bigwedge^r P \to P'$ such that $\bar{\varphi} \circ \lambda = \varphi$ and we claim that this morphism defines a morphism of $3n$-displays from $\bigwedge^r \mathcal{P}$ to $\mathcal{P}'$. Consider the following diagram:

$$\begin{tikzcd}
Q^r \arrow[rr, two heads, "\lambda"] \arrow[dr, hook] \arrow[dd, "\varphi"'] & & \bigwedge^r Q \arrow[dr] & \\
& P^r \arrow[rr, two heads, "\lambda"] \arrow[dd, "\varphi"'] & & \bigwedge^r P \arrow[dll, "\bar{\varphi}"] \\
Q' \arrow[dr, hook] & & & \\
& P'. & &
\end{tikzcd}$$

From what we said above and the definition, this diagram commutes. We want to show that the image of $\Lambda^r Q$ under $\bar{\varphi}$ lies inside Q'. Since by construction, $\Lambda^r Q$ is the image of the morphism $\bigwedge^r Q \to \bigwedge^r P$, and since the morphism $\lambda : Q^r \to \bigwedge^r Q$ is surjective, it is enough to show that the image of the composition

$$Q^r \xrightarrow{\lambda} \bigwedge{}^r Q \to \bigwedge{}^r P \xrightarrow{\bar{\varphi}} P'$$

lies inside Q'. This follows from the commutativity of the above diagram. Now, we have to show that $\bar{\varphi} \circ \Lambda^r V^{-1} = V^{-1} \circ \bar{\varphi}$. Take an element $q := q_1 \wedge q_2 \wedge \cdots \wedge q_r \in \bigwedge^r Q$. We have

$$\bar{\varphi} \circ \bigwedge{}^r V^{-1}(q) = \bar{\varphi}(V^{-1}(q_1) \wedge \cdots \wedge V^{-1}(q_r)) =$$

$$\varphi(V^{-1}(q_1), \ldots, V^{-1}(q_r)) = V^{-1}\varphi(q_1, \ldots, q_r) = V^{-1} \circ \bar{\varphi}(q),$$

where the third equality follows from the fact that φ satisfies the V-condition. This implies that for every $q \in \Lambda^r Q$, we have $\bar{\varphi} \circ \Lambda^r V^{-1}(q) = V^{-1} \circ \bar{\varphi}(q)$ and the claim is proved. By construction of $\bar{\varphi}$, we have $\bar{\varphi} \circ \lambda = \varphi$. It remains to show the uniqueness of $\bar{\varphi}$. Since the morphism $\lambda : P^r \to \bigwedge^r P$ is a surjective map (as sets), any morphism $\varphi_1 : \bigwedge^r \mathcal{P} \to \mathcal{P}'$ with $\varphi_1 \circ \lambda = \varphi$ is equal to $\bar{\varphi}$ as a morphism from $\bigwedge^r P$ to P' and therefore is equal to $\bar{\varphi}$ as a morphism of $3n$-displays. The proof is now achieved. $\square$

3. The Main Theorem for p-Divisible Groups

In this section, we prove the main theorem of the paper for p-divisible groups, namely, that the exterior powers of p-divisible groups of dimension at most 1 over arbitrary base exist and that the construction of the exterior power commutes with arbitrary base change.

3.1. Over fields β is an isomorphism.

Throughout this subsection, unless otherwise specified, k is a perfect field of characteristic p.

Construction 3.1. Let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ be Dieudonné displays over k, i.e., the display attached to a connected p-divisible group over k (see Example B.9) and $\varphi : \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{P}_0$ a multilinear morphism. For all $0 \leq i \leq r$, set $G_i := BT_{\mathcal{P}_i}$. The map φ induces a multilinear map $P_1 \times \cdots \times P_r \to P_0/p^n$ and since it is linear in each factor, we obtain a multilinear map

$$P_1/p^n \times \cdots \times P_r/p^n \to P_0/p^n.$$

As $P_i/p^n \cong D_*(G_{i,n})$, we have a V-F multilinear map

$$\tilde{\varphi_n} : D_*(G_{1,n}) \times \cdots \times D_*(G_{r,n}) \to D_*(G_{0,n})$$

i.e., an element of the group $\mathrm{Mult}(D_*(G_{1,n}) \times \cdots \times D_*(G_{r,n}), D_*(G_{0,n}))$ which is isomorphic to the group $\mathrm{Mult}(G_{1,n} \times \cdots \times G_{r,n}, G_{0,n})$ by Proposition 1.10. Hence, we obtain a multilinear map

$$\Omega(\tilde{\varphi_n}) : G_{1,n} \times \cdots \times G_{r,n} \to G_{0,n}.$$

▼

Theorem 3.2. *Let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ be (nilpotent) displays over k and*

$$\varphi : \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{P}_0$$

a multilinear morphism. Set $G_i := BT_{\mathcal{P}_i}$. Then the two morphisms $\Omega(\tilde{\varphi_n})$ and $\beta_{\varphi,n}$ are equal.

Proof. This theorem is one of the key results of this paper, and its proof consists of heavy, ad-hoc and tricky calculations, following carefully the isomorphism between the Dieudonné module of a connected p-divisible group over a perfect field of characteristic p, and its display. This isomorphism itself, is the composition of the isomorphism between the Dieudonné module of such a p-divisible group and its Cartier module on the one hand, and its Cartier module and its display (Proposition B.16) on the other hand. The former isomorphism is a well-known fact to experts and the author has written a detailed proof of it in [11] (such a proof seemed to be missing in the literature). We will therefore omit the proof for the sake of briefness and refer to *op. cit.* for the complete proof. □

Corollary 3.3. *Let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$ be (nilpotent) displays over k. The homomorphisms*

$$\beta : \mathrm{Mult}(\prod_{i=1}^{r} \mathcal{P}_i, \mathcal{P}_0) \to \mathrm{Mult}(\prod_{i=1}^{r} BT_{\mathcal{P}_i}, BT_{\mathcal{P}_0}) \quad \textit{and}$$

$$\beta : \mathrm{Alt}(\mathcal{P}_1^r, \mathcal{P}_0) \to \mathrm{Alt}(BT_{\mathcal{P}_1}^r, BT_{\mathcal{P}_0})$$

given in Corollary 2.13 are isomorphisms.

Proof. As usual, we only prove the first isomorphism, and omit the similar proof of the second one. For every i, set $G_i := BT_{\mathcal{P}_i}$ and denote by D_i the (covariant) Dieudonné module of G_i. Using the previous theorem, we obtain a commutative diagram

(3.4)
$$\begin{tikzcd}
\mathrm{Mult}(\mathcal{P}_1 \times \cdots \times \mathcal{P}_r, \mathcal{P}_0) \arrow[rr, "\beta"] \arrow[dr, "\cong"'] & & \mathrm{Mult}(G_1 \times \cdots \times G_r, G_0) \\
& \mathrm{Mult}(D_1 \times \cdots \times D_r, D_0) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where the left oblique isomorphism is given by the identifications of Dieudonné modules and displays and the right oblique isomorphism is given by Proposition 1.21. It follows at once that β is an isomorphism. $\square$

3.2. The affine base case.

In this subsection, we show the existence of exterior powers of p-divisible groups over complete local Noetherian rings with residue field of characteristic p, whose special fiber is connected of dimension 1. We also calculate the height of these exterior powers and the dimension of their special fiber. Moreover, we show that these exterior powers commute with arbitrary base change. We assume $p > 2$ and that p is nilpotent in R (unless otherwise specified).

Construction 3.5. Let $\mathcal{P}$ be a display over R, with tangent module of rank at most 1 and set $G := BT_{\mathcal{P}}$. Denote by Λ^r_R the p-divisible group associated with $\bigwedge^r \mathcal{P}$. The universal alternating morphism $\lambda : \mathcal{P}^r \to \bigwedge^r \mathcal{P}$ (cf. Proposition 2.17) induces an alternating morphism $\beta_{\lambda,n} : G^r_n \to \Lambda^r_{R,n}$ which, for every group scheme X, gives rise to a homomorphism $\lambda^*_n(X) : \mathrm{Hom}_R(\Lambda^r_{R,n}, X) \to \mathrm{Alt}_R(G^r_n, X)$. Sheafifying this morphism, we obtain a sheaf homomorphism

$$\underline{\lambda^*_n}(X) : \underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, X) \to \underline{\mathrm{Alt}}_R(G^r_n, X).$$

▼

Remark 3.6. Note that by Lemma 2.16 and Proposition B.11, the construction of $\bigwedge^r\mathcal{P}$, and therefore the formation of Λ^r_R commutes with the base change, i.e., if A is any R-algebra, then we have canonical isomorphisms $(\bigwedge^r\mathcal{P})_A \cong \bigwedge^r(\mathcal{P}_A)$ and $(\Lambda^r_R)_A \cong \Lambda^r_A$ (note that p is nilpotent in A as well, and therefore Λ^r_A is a p-divisible group). ♢

Theorem 3.7. *If R is a perfect field, then for every group scheme X over R, the morphism*

$$\lambda^*_n(X) : \mathrm{Hom}_R(\Lambda^r_{R,n}, X) \to \mathrm{Alt}_R(G^r_n, X)$$

is an isomorphism. Thus, for all n we have a canonical and functorial isomorphism $\Lambda^r_{R,n} \cong \bigwedge^r(G_n)$.

Proof. We know that for each n, the exterior power $\bigwedge^r G_n$ exists, is finite and its Dieudonné module is isomorphic to $\bigwedge^r D_*(G_n)$ (Proposition 1.22) which is isomorphic to $\left(\bigwedge^r D_*(G)\right)/p^n$. This shows that the canonical homomorphism $\bigwedge^r G_n \to \Lambda^r_{R,n}$ (induced by the universal property of $\bigwedge^r G_n$) is an isomorphism. Also, by Proposition 1.22, the universal alternating morphism $\tau_n : G_n^r \to \bigwedge^r G_n$ corresponds via the isomorphism $\mathrm{Mult}(D_*(G_n)^r, \bigwedge^r D_*(G_n)) \xrightarrow{\cong} \mathrm{Mult}(G_n^r, \bigwedge^r G_n)$ (Proposition 1.10), to the universal alternating morphism $D_*(G_n)^r \to \bigwedge^r D_*(G_n)$, which is the reduction of the morphism $\lambda : D_*(G)^r \to \bigwedge^r D_*(G)$ modulo p^n. It follows from the previous theorem, and after identifying the two group schemes $\bigwedge^r G_n$ and $\Lambda^r_{R,n}$, that the two alternating morphisms τ_n and $\beta_{\lambda,n}$ are equal, i.e., that the morphism $\beta_{\lambda,n}$ is the universal alternating morphism. This means that for every group scheme X, $\lambda_n^*(X)$ is an isomorphism. $\square$

Proposition 3.8. *For every group scheme X over R, and every ring homomorphism $R \to L$, with L a perfect field, the morphism $\underline{\lambda_n^*}(X)$ is an isomorphism on the L-rational points, i.e., the homomorphism $\lambda_n^*(X)(L)$: $\mathrm{Hom}_L(\Lambda^r_{L,n}, X_L) \to \mathrm{Alt}_L(G^r_{L,n}, X_L)$ is an isomorphism.*

Proof. This follows from Remark 3.6 and the previous theorem, noting that L has characteristic p. $\square$

Proposition 3.9. *Let R be a perfect field. Then, for every group scheme X over R, the morphism $\underline{\lambda_n^*}(X)$ is an isomorphism.*

Proof. Let I be a finite group scheme over R. Then the sheaf of Abelian groups $\underline{\mathrm{Hom}}_R(I, X)$ is representable and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(I, \underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, X)) \arrow[r, "\mathrm{Hom}(I, \underline{\lambda_n^*}(X))"] \arrow[d, "\cong"'] & \mathrm{Hom}_R(I, \underline{\mathrm{Alt}}_R(G_n^r, X)) \arrow[d, "\cong"] \\
\mathrm{Hom}_R(\Lambda^r_{R,n}, \underline{\mathrm{Hom}}_R(I, X)) \arrow[r, "\lambda_n^*(\underline{\mathrm{Hom}}_R(I,X))"'] & \mathrm{Alt}_R(G_n^r, \underline{\mathrm{Hom}}_R(I, X)).
\end{tikzcd}$$

The vertical isomorphisms are given by Proposition 1.7. The bottom homomorphism of this diagram is an isomorphism by Theorem 3.7 and therefore the top homomorphism is an isomorphism as well. We also know from the previous proposition that $\underline{\lambda_n^*}(X)$ is an isomorphism on the L-valued points, for every perfect field L, and in particular for the algebraic closure of R. It follows from the Proposition A.7, that $\underline{\lambda_n^*}(X)$ is an isomorphism. $\square$

Proposition 3.10. *The homomorphism $\underline{\lambda_n^*}(\mathbb{G}_m)$ is an isomorphism.*

Proof. Let L be a perfect field and s an L-valued point of the scheme $\mathbf{Spec}(R)$. By Remark 3.6, the group scheme $(\Lambda^r_{R,n})_L$ is canonically isomorphic to the group scheme $\Lambda^r_{L,n}$ and therefore, the fiber of the homomorphism

$\underline{\lambda_n^*}(\mathbb{G}_m)$ over s is the homomorphism

$$\underline{\lambda_n^*}(\mathbb{G}_m)_s : \underline{\mathrm{Hom}}_L(\Lambda^r_{L,n}, \mathbb{G}_{m,L}) \to \underline{\mathrm{Alt}}_L(G^r_{L,n}, \mathbb{G}_{m,L}),$$

which is an isomorphism by the previous proposition. As $\underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, \mathbb{G}_m)$ is a finite flat group scheme over R, and the group scheme $\underline{\mathrm{Alt}}_R(G^r_n, \mathbb{G}_m)$ is affine and of finite type over $\mathbf{Spec}(R)$ (cf. Remark 1.8), we can apply Remark A.6 and Proposition A.5 and conclude that $\underline{\lambda_n^*}(\mathbb{G}_m)$ is an isomorphism. $\square$

Proposition 3.11. *For every finite and flat group scheme X over R, the morphism $\underline{\lambda}_n^*(X)$ is an isomorphism. Consequently, $\beta_{\lambda,n} : G^r_n \to \Lambda^r_{R,n}$ is the r^{th}-exterior power of G_n in the category of finite and flat group schemes over R.*

Proof. As X is finite and flat over R, there exists a canonical isomorphism $X \cong \underline{\mathrm{Hom}}_R(X^*, \mathbb{G}_m)$, where X^* is the Cartier dual of X. We then obtain a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, X) \arrow[r, "\underline{\lambda}_n^*(X)"] \arrow[d, "\cong"'] & \underline{\mathrm{Alt}}_R(G^r_n, X) \arrow[d, "\cong"] \\
\underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, \underline{\mathrm{Hom}}_R(X^*, \mathbb{G}_m)) \arrow[r, "\underline{\lambda}_n^*(\underline{\mathrm{Hom}}_R(X^*,\mathbb{G}_m))"] \arrow[d, "\cong"'] & \underline{\mathrm{Alt}}_R(G^r_n, \underline{\mathrm{Hom}}_R(X^*, \mathbb{G}_m)) \arrow[d, "\cong"] \\
\underline{\mathrm{Hom}}_R(X^*, \underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, \mathbb{G}_m)) \arrow[r, "\underline{\mathrm{Hom}}_R(X^*, \underline{\lambda_n^*}(\mathbb{G}_m))"'] & \underline{\mathrm{Hom}}_R(X^*, \underline{\mathrm{Alt}}_R(G^r_n, \mathbb{G}_m)).
\end{tikzcd}$$

By the previous proposition, $\underline{\lambda_n^*}(\mathbb{G}_m)$ is an isomorphism, and thus so is $\underline{\lambda}_n^*(X)$. Taking the R-valued points of $\underline{\lambda}_n^*(X)$, we conclude that $\lambda_n^*(X)$ is an isomorphism. $\square$

Question 3.12. *Is the morphism $\lambda_n^*(X)$ an isomorphism for every group scheme X over R?*

From now on, $(R, \mathfrak{m})$ is a complete local Noetherian ring, with residue field k of characteristic p.

Proposition 3.13. *Let G a p-divisible group over R with connected special fiber of dimension 1. Then the r^{th}-exterior power, $\bigwedge^r G$, exists. Furthermore, for all n, the canonical homomorphism $\bigwedge^r(G_n) \to (\bigwedge^r G)_n$ induced by the universal property of $\bigwedge^r(G_n)$ is an isomorphism. Finally, the height of $\bigwedge^r G$ is equal to $\binom{h}{r}$ and its dimension at the closed point of R is equal to $\binom{h-1}{r-1}$.*

Proof. Let H be a p-divisible group over R. First assume that R is a local Artin ring. Then p is nilpotent in R and G is infinitesimal. Set $\bigwedge^r G := \Lambda^r_R$. By Proposition 3.11, the alternating morphism $\beta_{\lambda,n} : G^r_n \to (\bigwedge^r G)_n$ is the r^{th}-exterior power of G_n over R and therefore, the canonical homomorphism $(\bigwedge^r G)_n \to \bigwedge^r(G_n)$ is an isomorphism and the induced homomorphism $\mathrm{Hom}_R(\bigwedge^r G_n, H_n) \to \mathrm{Alt}_R(G_n^r, H_n)$ is an isomorphism. Taking the inverse limit of this isomorphism, we deduce that the canonical homomorphism $\mathrm{Hom}_R(G, H) \to \mathrm{Alt}_R(G^r, H)$ induced by the system $\{\beta_{\lambda,n} : G_n^r \to (\bigwedge^r G)_n\}_n$ is an isomorphism.

In the general case, set $X := \mathbf{Spec}(R)$, $\mathfrak{X} := \mathbf{Spf}(R)$ and for $i > 0$, $\mathfrak{X}_i := \mathbf{Spec}(R/\mathfrak{m}^i)$. Let $G(i)$ denote the base change of G to $\mathfrak{X}_i$. From above, we know that $\bigwedge^r G(i)$ exists for all i and we have a universal alternating morphism $\lambda(i) : G(i)^r \to \bigwedge^r G(i)$. We also know that the construction of the exterior power commutes with base change (note that $G(i)$ is infinitesimal), and thus the universal alternating morphism $\lambda(i+1) : G(i+1)^r \to \bigwedge^r G(i+1)$ restricts over $\mathfrak{X}_i$ to $\lambda(i) : G(i)^r \to \bigwedge^r G(i)$. By Proposition A.9 and Remark A.10, there exists a p-divisible group $\bigwedge^r G$ over X and an alternating morphism $\lambda : G^r \to \bigwedge^r G$ which restricts over each $\mathfrak{X}_i$ to $\lambda(i)$. It follows from the universal property of $\bigwedge^r G(i)$ and Remark A.10 that the alternating morphism λ is the universal alternating morphism making $\bigwedge^r G$ the r^{th}-exterior power of G over X.

The same arguments show that the truncated Barsotti-Tate groups $\bigwedge^r(G(i)_n)$ (of level n) form a compatible system and therefore define a truncated Barsotti-Tate group of level n over $\mathfrak{X}$. Using again Proposition A.9 and Remark A.10, we obtain a truncated Barsotti-Tate group of level n over X, denoted $\bigwedge^r(G_n)$ and an alternating morphism $\lambda_n : G_n^r \to \bigwedge^r(G_n)$. Like above, it is the universal alternating morphism making $\bigwedge^r(G_n)$ the r^{th}-exterior power of G_n over X. Since for all i the canonical homomorphism $\bigwedge^r(G(i)_n) \to (\bigwedge^r G(i))_n$ is an isomorphism, it follows from Proposition A.9 that the canonical homomorphism $\bigwedge^r(G_n) \to (\bigwedge^r G)_n$ is an isomorphism as well.

The dimension and height of a p-divisible group over a field is invariant under field extensions. We also know that the construction of the exterior powers of a p-divisible group (of dimension 1) over a field of characteristic p commutes with field extensions (cf. Remark 3.6). Thus, in order to determine the height of $\bigwedge^r G$, and its dimension at the closed point of R, we can assume that k is algebraically closed (note that $(\bigwedge^r G)_k \cong \bigwedge^r(G_k)$). By Remark B.8, the dimension and respectively the height of a p-divisible group is equal to the rank and respectively the height of the corresponding display. The result on the dimension and height of $\bigwedge^r G$ follows at once from Lemma 2.16. $\square$

Remark 3.14. Note that by construction of $\bigwedge^r G$ (respectively of $\bigwedge^r(G_n)$, the canonical homomorphism $\bigwedge^r(G_k) \to (\bigwedge^r G)_k$ (respectively $\bigwedge^r(G_{n,k}) \to (\bigwedge^r G_n)_k$) is an isomorphism. $\diamondsuit$

Now, we want to show that the exterior powers commute with arbitrary base change.

Notations 3.15. The ring R is a $\mathbb{Z}_{(p)}$-algebra and for all ring homomorphisms $R \to L$ with L a field, the characteristic of L is either p or zero. Let G be a p-divisible group over R, of height h, such that the dimension of G at points of $S := \mathbf{Spec}(R)$ of characteristic p is 1 and that the special fiber of G is connected. Fix n and set: $H := G_n$, $X := \underline{\mathrm{Hom}}_R(\bigwedge^r H, \mathbb{G}_m)$ and $Y := \underline{\mathrm{Alt}}_R(H^r, \mathbb{G}_m)$. Denote by α the canonical homomorphism $\alpha : X \to Y$ induced by the universal alternating morphism $\lambda : H^r \to \bigwedge^r H$.

Lemma 3.16. *Let $\bar{s}$ be a geometric point of S. Then $Y_{\bar{s}}$ is finite of order $p^{n\binom{h}{r}}$ over $\bar{s}$.*

Proof. Write $\bar{s} = \mathbf{Spec}(\Omega)$. We know that the exterior powers of H_Ω exist and we have a canonical homomorphism

$$f : \underline{\mathrm{Hom}}_\Omega(\bigwedge^r(H_\Omega), \mathbb{G}_{m,\Omega}) \to \underline{\mathrm{Alt}}_\Omega(H_\Omega^r, \mathbb{G}_\Omega) = Y_\Omega.$$

If Ω has characteristic p, then by assumption G_Ω has dimension 1 and therefore by Proposition 3.10, f is an isomorphism. By Proposition 3.13 we know that the order of $\bigwedge^r(H_\Omega)$ is equal to $p^{n\binom{h}{r}}$, and therefore its Cartier dual, which is isomorphic to Y_Ω has order $p^{n\binom{h}{r}}$ as well. Now assume that Ω has characteristic zero. Then $\underline{\mathrm{Hom}}_\Omega(\bigwedge^r(H_\Omega), \mathbb{G}_{m,\Omega})$ is a finite étale group scheme over Ω. By definition of $\bigwedge^r(H_\Omega)$ the homomorphism

$$f(\Omega) : \mathrm{Hom}_\Omega(\bigwedge^r(H_\Omega), \mathbb{G}_{m,\Omega}) \to \mathrm{Alt}_\Omega(H_\Omega^r, \mathbb{G}_{m,\Omega}) = Y_\Omega(\Omega)$$

is an isomorphism. The group $\mathrm{Hom}_\Omega(\bigwedge^r(H_\Omega), \mathbb{G}_{m,\Omega})$, being the Ω-valued points of a finite group scheme over Ω, is finite. Since the group scheme Y_Ω is of finite type over Ω and has finitely many Ω-valued points, it is finite over Ω. It is thus étale. As Ω is algebraically closed, the two finite étale group schemes X_Ω and Y_Ω are constant. So f is an isomorphism, because it is so on the Ω-valued points. Again, the order of Y_Ω is equal to the order of $\bigwedge^r(H_\Omega)$, which is equal to $p^{n\binom{h}{r}}$ by Proposition 1.20. $\square$

Proposition 3.17. *The homomorphism α is an isomorphism.*

Proof. Set $A := \mathcal{O}(X)$ and $B := \mathcal{O}(Y)$. We know that A is a finite flat R-module of rank $p^{n\binom{h}{r}}$ and B is a finitely generated R-algebra. Let $f : B \to A$ be the ring homomorphism corresponding to α and C the cokernel of f. By Remark 3.14 and Proposition 3.10, α_k is an isomorphism, which means that $C \otimes_R k = 0$. As A is finite over R, so is C and by Nakayama's lemma, we have $C = 0$. This implies that f is an epimorphism. Let $\bar{s} = \mathbf{Spec}(\Omega)$ be a geometric point of S. By previous lemma $\dim_\Omega B \otimes_R \Omega = p^{n\binom{h}{r}}$, which is equal to $\dim_\Omega A \otimes_R \Omega$. As f is an epimorphism, we conclude that $f \otimes_R \Omega$ is an isomorphism. Therefore $\alpha_{\bar{s}}$ is an isomorphism. It follows by Remark A.6 and Proposition A.5 that α is an isomorphism. □

Proposition 3.18. *Let T be an S-scheme and Z a finite flat group scheme over T. The homomorphism*

$$\underline{\lambda}_Z^* : \underline{\mathrm{Hom}}_T((\bigwedge^r H)_T, Z) \to \underline{\mathrm{Alt}}_T(H_T^r, Z)$$

induced by the alternating morphism $\lambda_T : H_T^r \to (\bigwedge^r H)_T$ is an isomorphism. In particular, $(\bigwedge^r H)_T$ is the r^{th}-exterior power of H_T in the category of finite flat group schemes over T and λ_T is the universal alternating morphism.

Proof. The proof is similar to the proof of Proposition 3.11. □

Corollary 3.19. *Let T be an S-scheme. The base change to T of the alternating morphism $\tau : G^r \to \bigwedge^r G$ given by Proposition 3.13 is the universal alternating morphism, i.e., for every p-divisible group G' over T, the following homomorphism is an isomorphism*

$$\tau_T^* : \mathrm{Hom}_T((\bigwedge^r G)_T, G') \to \mathrm{Alt}_T(G_T^r, G') \quad \psi \mapsto \psi \circ \tau_T.$$

Proof. We have

$$\mathrm{Hom}_T((\bigwedge^r G)_T, G') = \varprojlim_n \mathrm{Hom}_T((\bigwedge^r G_n)_T, G'_n)$$

and $\mathrm{Alt}_T(G_T^r, G') = \varprojlim_n \mathrm{Alt}_T(G_{n,T}^r, G'_n)$ and the homomorphism

$$\tau_{n,T}^* : \mathrm{Hom}_T((\bigwedge^r G_n)_T, G'_n) \to \mathrm{Alt}_T(G_{n,T}^r, G'_n)$$

is induced by the alternating morphisms $\tau_{n,T} : G_{n,T}^r \to (\bigwedge^r G_n)_T$. By previous proposition, $\tau_{n,T}$ is the universal alternating morphism, and therefore the homomorphisms $\tau_{n,T}^*$ are isomorphisms. Hence τ_T^* is an isomorphism as well. □

Remark 3.20. In virtue of the previous corollary, the r^{th}-exterior power of G_T exists and we can write $\bigwedge^r G_T$ instead of $(\bigwedge^r G)_T$ and $\bigwedge^r (G_T)$. The same holds (by Proposition 3.18) for the groups $G_{n,T}$. ◊

3.3. The general case.

In this subsection we prove the main theorem over any base scheme. We first show a result which will serve as a tool to transfer the question of the existence of exterior powers over an (almost) arbitrary base, to the question over a special complete local Noetherian base, where we know the answer. We then prove some faithfully flat descent properties, which together with the mentioned proposition and the results from the last sections, will provide a proof of the main theorem. The prime number p is not 2.

The following proposition and its proof are due to E. Lau. We include the proof for the sake of completeness. We refer to [15], [14] and [22] for more details.

Proposition 3.21. *Let G_0 over $\mathbb{F}_p$ be a connected p-divisible group of dimension 1 and height h, and $\mathcal{G}$ over $R := \mathbb{Z}_p[\![x_1, \ldots, x_{h-1}]\!]$ be the universal deformation of G_0. Let H be a truncated Barsotti-Tate group of level $n \geq 1$ and of height h over a $\mathbb{Z}_{(p)}$-scheme S. We assume that the fibers of H in points of characteristic p of S have dimension* 1*. Then there exist morphisms*

$$S \xleftarrow{\varphi} Y \xrightarrow{\psi} \mathbf{Spec}\, R$$

with φ faithfully flat and affine, such that $\varphi^* H \cong \psi^* \mathcal{G}_n$.

Proof. Let $\mathcal{Y}$ over $\mathbb{Z}_{(p)}$ be the algebraic stack of truncated Barsotti-Tate groups of level n and height h and let $\mathcal{Y}_0 \subset \mathcal{Y}$ be the substack of truncated Barsotti-Tate groups as in the proposition. The inclusion $\mathcal{Y}_0 \hookrightarrow \mathcal{Y}$ is an open immersion because for a morphism $S \to \mathcal{Y}$ corresponding to a truncated Barsotti-Tate group H over S, the points of characteristic p of S in which the dimension of H is not equal to 1 form an open and closed subscheme S_1 of $S \times \mathbf{Spec}\, \mathbb{F}_p$, and we have $S \times_{\mathcal{Y}} \mathcal{Y}_0 = S \smallsetminus S_1$. The group scheme $\mathcal{G}_n$ is a truncated Barsotti-Tate group over R of height h and level n, and has the property that all the fibers in points of characteristic p of R have dimension 1. Therefore, by definition of $\mathcal{Y}_0$, it defines a unique morphism $\alpha : \mathbf{Spec}\, R \to \mathcal{Y}_0$. We claim that α is faithfully flat and affine. Then for H over S as in the proposition, which corresponds to a morphism $S \to \mathcal{Y}_0$, we can take $Y = S \times_{\mathcal{Y}_0} \mathbf{Spec}\, R$ (note that since α is a representable morphism, Y is indeed a scheme). Indeed, since α is faithfully flat and affine, the projection $\varphi : Y \to S$, which is the base change of α, is faithfully flat and affine. Also, by the universal property of $\mathcal{G}$ and the definition of $\mathcal{Y}_0$, we have $\varphi^* H \cong \psi^* \mathcal{G}_n$, where $\psi : Y \to \mathbf{Spec}\, R$ is the projection on $\mathbf{Spec}\, R$.

The morphism α is affine, because $\mathbf{Spec}\, R$ is affine and the diagonal of $\mathcal{Y}$ is affine. It is easy to see that α is surjective on geometric points. Indeed, let k be an algebraically closed field. If k has characteristic zero, then $\mathcal{Y}(k) = \mathcal{Y}_0(k)$ has only only one isomorphism class (corresponding to the constant truncated Barsotti-Tate group of height h and level n). If k has characteristic p, then $\mathcal{Y}_0(k)$ has precisely $h - 1$ isomorphism classes corresponding to the étale rank. These isomorphism classes all occur in the fibers of $\mathcal{G}_n$ over R. It remains to show that α is flat.

Let T be the following functor on $\mathbb{Z}_{(p)}$-schemes: $T(S)$ is the set of isomorphism classes of pairs (H, a) where $\pi : H \to S$ is an open object of $\mathcal{Y}_0$ and where $a : \mathcal{O}_S^{p^{nh}} \cong \pi_* \mathcal{O}_H$ is an isomorphism of $\mathcal{O}_S$-modules. Then T is representable by a quasi-affine scheme of finite type over $\mathbb{Z}_{(p)}$ (see Proposition (1.8) of [22]). The morphism $T \to \mathcal{Y}_0$ defined by forgetting a is a $GL_{p^{nh}}$-torsor and thus smooth. By [14], the algebraic stack $\mathcal{Y}$ is smooth over $\mathbb{Z}_{(p)}$. Hence the same is true for $\mathcal{Y}_0$ and T. Let $t \in T$ be a closed point with residue field $\mathbb{F}_p$ such that the associated group over $\mathbb{F}_p$ is $G_{0,n}$. The image of t in $\mathcal{Y}(\mathbb{F}_p)$ is also denoted by t. The homomorphism of tangent spaces $T_{T,t} \to T_{\mathcal{Y},t}$ is surjective, because $T \to \mathcal{Y}$ is smooth. Let $Z \to T$ be a regular immersion (thus Z is smooth) with $t \in Z$ such that $T_{Z,t} \to T_{\mathcal{Y},t}$ is bijective. After shrinking Z we can assume that $Z \to \mathcal{Y}$ is smooth. Indeed, let $U = Z \times_{\mathcal{Y}} T$ and let $u \in U$ be a closed point with residue field $\mathbb{F}_p$ lying over $(t, t) \in Z \times T$. Then the second projection $T_{U,u} \to T_{T,t}$ is surjective. Since U and T are smooth, the projection $U \to T$ is smooth in u. Thus there is an open subscheme U_0 of U containing u such that $U_0 \to T$ is smooth. If we replace Z by the image of $U_0 \to Z$, which is open, because this map is smooth, then $Z \to \mathcal{Y}$ is smooth. Let $S = \hat{\mathcal{O}}_{Z,t}$ and let H over S be the truncated Barsotti-Tate group corresponding to the given morphism $\mathbf{Spec}\, S \to \mathcal{Y}$. The special fiber of H is isomorphic to $G_{0,n}$. By [14] there is a p-divisible group G' over S with special fiber G_0 such that G'_n is isomorphic to H. Since the first-order deformations of G_0 and of $G_{0,n}$ coincide, it follows that G' is a universal deformation of G_0. Thus the morphism α can be written as a composition $\mathbf{Spec}\, R \cong \mathbf{Spec}\, S \to Z \to \mathcal{Y}$. Here $Z \to \mathcal{Y}$ is smooth and thus flat, and $\mathbf{Spec}\, S \to Z$ is flat, because it is a completion of a Noetherian ring. Thus the composition is flat as well. $\square$

Now we prove the faithfully flat descent of the universal alternating morphism and the exterior powers.

Lemma 3.22. *Let $f : T \to S$ be a faithfully flat morphism of schemes and let H (respectively G) be a finite flat group scheme (respectively a p-divisible group) over S.*

1) *Assume that we are given an alternating morphism $\tau : H^r \to \Lambda$ (respectively $\tau : G^r \to \Lambda$), with Λ a finite flat group scheme (respectively a p-divisible group) over S, such that for all morphisms $g : T' \to T$, the pullback $g^* f^* \tau : g^* f^* H^r \to g^* f^* \Lambda$ (respectively $g^* f^* \tau : g^* f^* G^r \to g^* f^* \Lambda$) is the universal alternating morphism in the category of finite flat group schemes (respectively p-divisible groups) over T'. Then τ is the universal alternating morphism in the category of finite flat group schemes (respectively p-divisible groups) over S, i.e., $\Lambda = \bigwedge^r H$ (respectively $\Lambda = \bigwedge^r G$).*

2) *Assume that we have an alternating morphism $\tau' : f^*H^r \to \Lambda'$ (respectively $\tau' : f^*G^r \to \Lambda'$), with Λ' a finite and flat group scheme (respectively a p-divisible group) over T, such that for all morphisms $g : T' \to T$, the pullback $g^*\tau' : g^*f^*H^r \to g^*\Lambda'$ (respectively $g^*\tau' : g^*f^*G^r \to g^*\Lambda'$) is the universal alternating morphism in the category of finite flat group schemes (respectively p-divisible groups) over T'. Then there exists a finite flat group scheme (respectively a p-divisible group) Λ over S and an alternating morphism $\tau : H^r \to \Lambda$ (respectively $\tau : G^r \to \Lambda$), such that for every morphism $h : S' \to S$, the pullback $h^*\tau$ is the universal alternating morphism in the category of finite flat group schemes (respectively p-divisible groups) over S', i.e., $h^*\Lambda = \bigwedge^r(h^*H)$ (respectively $h^*\Lambda = \bigwedge^r(h^*G)$). In particular, $\Lambda = \bigwedge^r H$ (respectively $\Lambda = \bigwedge^r G$).*

Proof. Since alternating morphisms and homomorphisms of p-divisible groups are defined as compatible systems of alternating morphisms and homomorphisms of their truncated Barsotti-Tate groups, we will only prove the lemma for truncated Barsotti-Tate groups, and the result for the p-divisible groups follows.

For the proof of both parts of the lemma, we use faithfully flat descent.

1) Take a finite flat group scheme X over S. We have to show that the canonical homomorphism

$$\tau^* : \mathrm{Hom}_S(\Lambda, X) \to \mathrm{Alt}^r_S(H, X)$$

induced by τ is an isomorphism. So, take an alternating morphism $\varphi : H^r \to X$. Letting g be the identity morphism of T, we see that in particular, $f^*\tau : f^*H^r \to f^*\Lambda$ is the universal alternating morphism, and therefore there exists a unique group scheme homomorphism $a' : f^*\Lambda \to f^*X$ such that $a' \circ f^*\tau = f^*\varphi$. We want to descend the homomorphism a' to a homomorphism $a : \Lambda \to X$. Then since f is faithfully flat, we have $a \circ \tau = \varphi$ and a with this property is unique. Set $T' := T \times_S T$ and let $p_i : T' \to T$ $(i = 1, 2)$ be the two projections. We have to show that $p_1^*a' = p_2^*a'$. But this is true, since $p_1^*f^* = p_2^*f^*$ and by assumption, for $i = 1, 2$, we know that $p_i^*f^*\tau : p_i^*f^*H^r \to p_i^*f^*\Lambda$ is the universal alternating morphism and we have $p_i^*a' \circ p_i^*f^*\tau = p_i^*f^*\varphi$.

2) We want to descend the finite flat group scheme Λ' to a group scheme over S. Set $T' := T \times_S T$ and let $p_i : T' \to T$ $(i = 1, 2)$ be the two projections. We should prove that the base changes of Λ' via p_1 and p_2 are canonically isomorphic. By assumption, we know that $p_i^*\Lambda' = \bigwedge^r(p_i^*f^*H)$ $(i = 1, 2)$. The two compositions $p_1^*f^*$ and $p_2^*f^*$

are equal and thus, there exists a unique isomorphism $p_1^*\Lambda' \cong p_2^*\Lambda'$ by the uniqueness of the exterior powers. This shows that the group scheme Λ' descends to a group scheme over S. Since f is faithfully flat and Λ' is finite flat over T, we conclude that Λ is finite flat over S. The same arguments show that the alternating morphism τ' descends to a morphism $\tau : H^r \to \Lambda$, and again by the faithfully flatness of f, it should be alternating. Now let T' be the base change $T \times_S S'$ and denote by h' and respectively f' the projection $T' \to T$ and respectively $T' \to S'$. The morphism f', being the base change of f, is faithfully flat. By construction of τ ($f^*\tau = \tau'$) and the assumptions on τ', we observe that all the hypotheses of the first part of the lemma are satisfied for the alternating morphism $h^*\tau : h^*H^r \to h^*\Lambda$ (note that we are considering the lemma for the faithfully flat morphism $f' : T' \to S'$). Consequently, the alternating morphism $h^*\tau$ is the universal alternating morphism in the category of the finite flat group schemes over S'.

□

Now, we show the existence of the exterior powers over arbitrary base. We need two lemmas.

Lemma 3.23. *Let S be a scheme over $\mathbb{Z}_{(p)}$ and H a truncated Barsotti-Tate group of level n and height h over S, such that the dimension of the fibers at points of S of characteristic p is 1. Then there exists a truncated Barsotti-Tate group Λ_n over S of level n and height $\binom{h}{r}$, and an alternating morphism $\lambda_n : H^r \to \Lambda_n$ such that for all morphisms $\psi : S' \to S$ and all finite flat group schemes X over S', the induced homomorphism*

$$\mathrm{Hom}_{S'}(\psi^*\Lambda_n, X) \to \mathrm{Alt}^r_{S'}(\psi^*H, X)$$

is an isomorphism. Moreover fibers of Λ_n at points of S of characteristic p have dimension $\binom{h-1}{r-1}$.

Proof. Let G_0, R and $\mathcal{G}$ be as in the statement of the Proposition 3.21. The assumptions of Proposition 3.13 and Corollary 3.19 are satisfied (cf. Notations 3.15). Thus, the p-divisible group $\bigwedge^r\mathcal{G}$ exists over R and we have the universal alternating morphism $\tau : \mathcal{G}^r \to \bigwedge^r\mathcal{G}$. Furthermore, there exists a canonical isomorphism $\bigwedge^r(\mathcal{G}_n) \cong (\bigwedge^r\mathcal{G})_n$, induced by the alternating morphism $\tau_n : \mathcal{G}_n^r \to (\bigwedge^r\mathcal{G})_n$ (which is then the universal one). Also, for every morphism $\varphi : T' \to \mathbf{Spec}\, R$, we have $\varphi^*\bigwedge^r\mathcal{G}_n \cong \bigwedge^r\varphi^*\mathcal{G}_n$ (cf. Proposition 3.18). By Proposition 3.13, the height of $\bigwedge^r\mathcal{G}$ is equal to $p^{\binom{h}{r}}$, and its dimension at the closed point of R is equal to $\binom{h-1}{r-1}$. So, the order of $\bigwedge^r\mathcal{G}_n$ over R is equal to $p^{n\binom{h}{r}}$, in other words, $\bigwedge^r\mathcal{G}_n$ is a truncated Barsotti-Tate group of level n and height $\binom{h}{r}$, and its dimension at the closed point of R is $\binom{h-1}{r-1}$.

The group scheme H satisfies the hypotheses of the Proposition 3.21 and therefore, there exists a faithfully flat and affine morphism $f : T \to S$ and a morphism $g : T \to \mathbf{Spec}\, R$ such that $f^*H \cong g^*\mathcal{G}_n$. By the above discussions, we have an alternating morphism $g^*\tau_n : f^*H^r \cong g^*\mathcal{G}_n^r \to g^*\bigwedge^r\mathcal{G}_n$ such that for all morphisms $g' : T' \to T$, the pullback $g'^*g^*\tau_n$ is the universal alternating morphism

$$g'^*f^*H^r \cong g'^*g^*\mathcal{G}_n^r \to g'^*g^*\bigwedge^r\mathcal{G}_n.$$

By Lemma 3.22 2), there exists a finite flat group scheme Λ_n over S and an alternating morphism $\lambda_n : H^r \to \Lambda_n$, which has the desired properties stated in the lemma. As f is faithfully flat and $f^*\Lambda_n \cong g^*\bigwedge^r\mathcal{G}_n$ is a truncated Barsotti-Tate group of level n and height $\binom{h}{r}$, the group scheme Λ_n is also a truncated Barsotti-Tate group of level n and height $\binom{h}{r}$. The dimension of Λ_n at points of S of characteristic p is $\binom{h-1}{r-1}$. $\square$

Lemma 3.24. *Let S be a scheme over $\mathbb{Z}_{(p)}$ and G a p-divisible group over S of height h, such that the dimension of G at points of characteristic p of S is 1. Let $\bigwedge^r G_n$ be the truncated Barsotti-Tate group of level n over S provided by the previous lemma (applied to G_n). Then there exist natural monomorphisms $i_n : \bigwedge^r G_n \hookrightarrow \bigwedge^r G_{n+1}$, which make the inductive system $(\bigwedge^r G_n)_{n\geq 1}$ a Barsotti-Tate group over S of height $\binom{h}{r}$ and dimension $\binom{h-1}{r-1}$ at points of S of characteristic p.*

Proof. For every n we have an exact sequence $G_{n+1} \xrightarrow{p^n} G_{n+1} \xrightarrow{\xi_n} G_n \to 0$. By Proposition 1.19, the induced sequence

$$\bigwedge^r G_{n+1} \xrightarrow{p^n} \bigwedge^r G_{n+1} \xrightarrow{\bigwedge^r \xi_n} \bigwedge^r G_n \to 0$$

is exact as well. Since by previous lemma $\bigwedge^r G_{n+1}$ is a truncated Barsotti-Tate group of level $n+1$, we have $p^{n+1}\bigwedge^r G_{n+1} = 0$. It implies that there exists a unique homomorphism $i_n : \bigwedge^r G_n \to \bigwedge^r G_{n+1}$ making the following diagram commutative:

$$\begin{tikzcd}
\bigwedge^r G_{n+1} \arrow[r, "p^n"] & \bigwedge^r G_{n+1} \arrow[r, "\bigwedge^r \xi_n"] \arrow[d, "p"'] & \bigwedge^r G_n \arrow[r] \arrow[dl, "i_n"] & 0 \\
& \bigwedge^r G_{n+1}. & &
\end{tikzcd}$$

We want to show that i_n is a monomorphism and it identifies $\bigwedge^r G_n$ with $(\bigwedge^r G_{n+1})[p^n]$. Since $p^n\bigwedge^r G_n = 0$, there exists a homomorphism $j_n : \bigwedge^r G_n \to (\bigwedge^r G_{n+1})[p^n]$ whose composition with $\iota : (\bigwedge^r G_{n+1})[p^n] \hookrightarrow \bigwedge^r G_{n+1}$ is equal to i_n. Also, as $\bigwedge^r G_{n+1}$ is a truncated Barsotti-Tate group of level $n+1$, the image of multiplication by p is equal to the kernel of multiplication by p^n and therefore the homomorphism $p : \bigwedge^r G_{n+1} \to \bigwedge^r G_{n+1}$ factors through the inclusion $\iota : (\bigwedge^r G_{n+1})[p^n] \hookrightarrow \bigwedge^r G_{n+1}$ and induces an epimorphism $q_n : \bigwedge^r G_{n+1} \to (\bigwedge^r G_{n+1})[p^n]$:

$$\begin{tikzcd}
0 \arrow[r] & (\bigwedge^r G_{n+1})[p^n] \arrow[rr, "\iota"] & & \bigwedge^r G_{n+1} \arrow[r, "p^n"] & \bigwedge^r G_{n+1} \\
& & \bigwedge^r G_n \arrow[ul, "j_n"] \arrow[ur, "i_n"'] & & \\
& & \bigwedge^r G_{n+1}. \arrow[u, two heads, "\bigwedge^r \xi_n"] \arrow[uul, bend left, "q_n"] \arrow[uur, bend right, "p"'] & &
\end{tikzcd}$$

The composition of $\bigwedge^r \xi_n \circ j_n$ and q_n with ι are equal and since ι is a monomorphism, we have $\bigwedge^r \xi_n \circ j_n = q_n$. So, $j_n : \bigwedge^r G_n \to (\bigwedge^r G_{n+1})[p^n]$ is an epimorphism (q_n is an epimorphism). The two group schemes $\bigwedge^r G_n$ and $(\bigwedge^r G_{n+1})[p^n]$ have the same order over S (note that $(\bigwedge^r G_{n+1})[p^n]$ is a truncated Barsotti-Tate group of level n) and thus the epimorphism j_n is in fact an isomorphism. Hence i_n is a monomorphism identifying $\bigwedge^r G_n$ with $(\bigwedge^r G_{n+1})[p^n]$. By previous lemma, the order of $\bigwedge^r G_n$ is equal to $p^{n\binom{h}{r}}$. This proves that the inductive system $\bigwedge^r G_1 \xhookrightarrow{i_1} \bigwedge^r G_2 \xhookrightarrow{i_2} \bigwedge^r G_3 \xhookrightarrow{i_3} \dots$ is a Barsotti-Tate group over S of height $\binom{h}{r}$. The statement on the dimension follows from Theorem 1.23. $\square$

Theorem 3.25 (The Main Theorem)**.** *Let S be a scheme and G a p-divisible group over S of height h, and dimension at most* 1*. Then, there exists a p-divisible group $\bigwedge^r G$ over S of height $\binom{h}{r}$, and an alternating morphism $\lambda : G^r \to \bigwedge^r G$ such that for every morphism $f : S' \to S$ and every p-divisible group H over S', the following homomorphism is an isomorphism*

$$\mathrm{Hom}_{S'}(f^* \bigwedge^r G, H) \to \mathrm{Alt}^r_{S'}(f^* G, H), \quad \psi \mapsto \psi \circ f^* \lambda.$$

Moreover, the dimension of $\bigwedge^r G$ at $s \in S$ is $\binom{h-1}{r-1}$ (resp. 0*) if the dimension of G at s is* 1 *(resp.* 0*).*

Proof. We have flat morphisms $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{p}] \xhookrightarrow{\iota} \mathbf{Spec}\,\mathbb{Z} \xleftarrow{\rho} \mathbf{Spec}\,\mathbb{Z}_{(p)}$ which define a faithfully flat morphism from the disjoint union

$$\iota \sqcup \rho : \mathbf{Spec}\,\mathbb{Z}[\frac{1}{p}] \sqcup \mathbf{Spec}\,\mathbb{Z}_{(p)} \to \mathbf{Spec}\,\mathbb{Z}.$$

Let $S[\frac{1}{p}]$ and respectively $S_{(p)}$ be the pullbacks of $S \to \mathbf{Spec}\,\mathbb{Z}$, via ι and respectively ρ. The two morphisms $S[\frac{1}{p}] \to S$ and $S_{(p)} \to S$ induce a morphism $\pi : S[\frac{1}{p}] \sqcup S_{(p)} \to S$, which is the base change of $\iota \sqcup \rho$ via $S \to \mathbb{Z}$. It is therefore faithfully flat. Let S_0 and S_1 be respectively subsets of $S_{(p)}$ where G has dimension 0 and respectively 1. Then $S_{(p)} = S_0 \sqcup S_1$ and G is étale over S_0. Since p is invertible on $S[\frac{1}{p}]$, all p-divisible groups over it are étale. Set $X := S_0 \sqcup S[\frac{1}{p}]$ and $Y := S_1$. We have a faithfully flat morphism $\pi : X \sqcup Y \to S$, with G_X étale and G_Y of dimension 1.

The Barsotti-Tate group $(\bigwedge^r G_{Y,n})_{n\geq 1}$ provided by Lemma 3.24 gives rise to a p-divisible group $\bigwedge^r G_Y$ over S of height $\binom{h}{r}$, such that $(\bigwedge^r G_Y)_n = \bigwedge^r G_{Y,n}$. The universal alternating morphisms $\lambda_{(p),n} : G^r_{Y,n} \to \bigwedge^r G_{Y,n}$ provided by Lemma 3.23 are compatible with the projections $\bigwedge^r G_{Y,n+1} \twoheadrightarrow \bigwedge^r G_{Y,n}$. Indeed, the projections $\bigwedge^r \xi_n : \bigwedge^r G_{Y,n+1} \twoheadrightarrow \bigwedge^r G_{Y,n}$ are induced by the universal property of $\bigwedge^r G_{Y,n+1}$ applied to the alternating morphism

$$G^r_{Y,n+1} \to G^r_{Y,n} \xrightarrow{\lambda_{Y,n}} \bigwedge^r G_{Y,n}.$$

Therefore, the system $(\lambda_{Y,n})_{n\geq 1}$ gives rise to an alternating morphism $\lambda_Y : G^r_Y \to \bigwedge^r G_Y$. It follows from Lemma 3.23 that for every morphism $f : S' \to S_Y$ and every p-divisible group H over S', the induced homomorphism $\mathrm{Hom}_{S'}(f^*\bigwedge^r G_Y, H) \to \mathrm{Alt}_{S'}(f^*G_Y, H)$ is an isomorphism.

We can apply Proposition 1.20 and obtain a p-divisible group $\bigwedge^r G_X$ over S_X of height $\binom{h}{r}$, and an alternating morphism $\lambda_X : G^r_X \to \bigwedge^r G_X$ such that for every morphism $f : S' \to S_X$ and every p-divisible group H over S', the induced homomorphism $\mathrm{Hom}_{S'}(f^*\bigwedge^r G_X, H) \to \mathrm{Alt}_{S'}(f^*G_X, H)$ is an isomorphism.

It is easy to see that the disjoint union of the two universal alternating morphisms λ_X and λ_Y glue to an alternating morphism

$$\lambda_X \sqcup \lambda_Y : G^r_X \sqcup G^r_Y \cong (G_X \sqcup G_Y)^r \longrightarrow \bigwedge^r G_X \sqcup \bigwedge^r G_Y$$

and it is the universal alternating morphism over $S_X \sqcup S_Y$, i.e., $\bigwedge^r G_X \sqcup \bigwedge^r G_Y \cong \bigwedge^r (G_X \sqcup G_Y)$. Since both λ_X and λ_Y stay the universal alternating morphism after any base change, their disjoint union has the same property. The pullback of G via the faithfully flat morphism π is $G_X \sqcup G_Y$. It follows from Lemma 3.22 that the p-divisible group $\bigwedge^r (G_X \sqcup G_Y)$ and respectively the disjoint union $\lambda_X \sqcup \lambda_Y$ descend to a p-divisible group $\bigwedge^r G$ over S and respectively to an alternating morphism $\lambda : G^r \to \bigwedge^r G$ over S, and this is the universal alternating morphism over S and after any base change of S.

The statement on the dimension follows from the previous lemma. □

Remark 3.26. When the base scheme is locally Noetherian, there is a rather elementary way to bypass Lau's result (Proposition 3.21). We will pursue this way in the next section, when dealing with π-divisible modules, where we don't possess a generalization of Lau's result. For more details, we refer to the results following Corollary 4.23 to the end of section 4. ◊

4. The Main Theorem for π-Divisible Modules

In this section, we generalize the main theorem of the previous section to π-divisible modules. In fact, we are going to explain how the results from sections 2 and 3, that led to the main theorem, can be generalized so as to imply the generalized version of the theorem. Recall that a main ingredient of these results was displays. We used displays in order to find potential candidates for the exterior powers of a p-divisible group. Therefore, if we want to use the same methods, we should have a variant of displays for π-divisible modules. These are *ramified displays*. We refer to Appendix B for their definition and properties. They have been studied in the Ph.D. thesis of T. Ahsendorf (cf. [1]). We also need a refined version of Dieudonné theory adapted to π-divisible modules. The rest of the generalization can be carried on quite easily. We only consider π-divisible modules that are defined over $\mathcal{O}$-schemes and assume that the action of $\mathcal{O}$ on their tangent space is given by scalar multiplication. In this section k denotes an algebraically closed field of characteristic p.

Remark 4.1.

1) $\mathcal{O}$-Multilinear and alternating morphisms of ramified $3n$-displays are defined as in Definition 2.1 with the obvious additional requirement due to the presence of $\mathcal{O}$. Let $R \to S$ be a ring homomorphism and $\mathcal{N}$ a nilpotent R-algebra. Let $\varphi : \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{P}_0$ be an $\mathcal{O}$-multilinear morphisms of ramified $3n$-displays over R. The base change,
$$\varphi_S : \mathcal{P}_{1,S} \times \cdots \times \mathcal{P}_{r,S} \to \mathcal{P}_{0,S}$$
and the $\mathcal{O}$-multilinear morphism $\widehat{\varphi} : \widehat{P}_1 \times \cdots \times \widehat{P}_r \to \widehat{P}_0$ are constructed as in Construction 2.3.
2) The $\mathcal{O}$-linear morphism
$$\beta : \mathrm{Mult}^{\mathcal{O}}(\mathcal{P}_1 \times \cdots \times \mathcal{P}_r, \mathcal{P}_0) \to \mathrm{Mult}^{\mathcal{O}}(BT_{\mathcal{P}_1} \times \cdots \times BT_{\mathcal{P}_r}, BT_{\mathcal{P}_0})$$
is constructed as in Construction 2.5. It maps alternating morphisms to alternating morphisms and commutes with base change (cf. Proposition 2.7). Note that the $\mathcal{O}$-linearity follows from the construction of β and the fact that we are considering $\mathcal{O}$-multilinear morphisms.
3) Now let $\mathcal{P}$ be of rank 1. Exterior powers of $\mathcal{P}$ are constructed as in Construction 2.15. They enjoy the same properties as the exterior powers of a $3n$-display, stated in Lemma 2.16, namely, they have the universal property of exterior powers, their construction is independent of the choice of a normal decomposition and commutes with base change. If $\mathcal{P}$ is nilpotent, then its exterior powers are

also nilpotent. If $\mathcal{P}$ has height h, then $\bigwedge_{\mathcal{O}}^{r}\mathcal{P}$ has height $\binom{h}{r}$ and rank $\binom{h-1}{r-1}$. $\diamondsuit$

Construction 4.2. We have an isomorphism

$$W(k)\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_q \xrightarrow{\cong} \prod_{i\in\mathbb{Z}/f\mathbb{Z}} W(k)$$

sending an element $(w\otimes a)$ to the element $(w\otimes a)_i = (wa^{\sigma^{-i}})_i$. The automorphism $\sigma\otimes\mathrm{Id}$ on the left hand side induces an automorphism on the right hand side, permuting the factors. Tensoring with $\mathcal{O}$ over $\mathbb{Z}_q$, we obtain an isomorphism $W(k)\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O} \cong \prod W(k)\widehat{\otimes}_{\mathbb{Z}_q}\mathcal{O} \cong \prod W_{\mathcal{O}}(k)$ (cf. Proposition B.24), with $\sigma\otimes\mathrm{Id}$ permuting the factors. Let D be a Dieudonné module over k, endowed with an action of $\mathcal{O}$, which acts on the tangent space of D (i.e. on D/VD) through the scalar multiplication. Let us call such an action *a scalar action*. The Dieudonné module D is a $W(k)\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$-module and therefore decomposes into a product $M_0\times M_1\times\cdots\times M_{f-1}$, where each M_i is a $W_{\mathcal{O}}(k)$-module. The Verschiebung of D is a $\sigma^{-1}\otimes\mathrm{Id}$-linear monomorphism (cf. Lemma B.13 of [12]) and so, for every $i\in\mathbb{Z}/f\mathbb{Z}$, induces a semi-linear monomorphism $V: M_i\to M_{i+1}$. It then follows that each M_i is a free $W_{\mathcal{O}}(k)$-module of rank h[1]. Let us denote M_0 by $\mathbb{H}(D)$. It is a free $W_{\mathcal{O}}(k)$-module of rank h and is endowed with an injective ${}^{F_\pi^{-1}}$-linear $\mathcal{O}$-module homomorphism $V_\pi := V^f$. If D is the Dieudonné module of a π-divisible module, we call $\mathbb{H}(D)$ the *ramified Dieudonné module* of $\mathcal{M}$ and denote it by $\mathbb{H}(\mathcal{M})$. ▼

Lemma 4.3. *Let D be a Dieudonné module over k, with a scalar $\mathcal{O}$-action. There exists an ${}^{F_\pi}$-linear $\mathcal{O}$-module homomorphism $F_\pi:\mathbb{H}(D)\to\mathbb{H}(D)$ such that $F_\pi\circ V_\pi = V_\pi\circ F_\pi = \pi$.*

Proof. We show at first that the tangent space of D is canonically isomorphic to $\mathbb{H}(D)/V_\pi\mathbb{H}(D)$. With the notations of the above construction, we have $D\cong M_0\times\cdots\times M_{f-1}$ and therefore,

$$VD \cong VM_{f-1}\times VM_0\times VM_1\times\cdots\times VM_{f-2}.$$

Thus, the tangent space of D is isomorphic to the product $M_0/VM_{f-1}\times M_1/VM_0\times\cdots\times M_{f-1}/VM_{f-2}$. Since by assumption, the action of $\mathcal{O}$ on this k-vector space is via scalar multiplication, for every $i\in\mathbb{Z}/f\mathbb{Z}\smallsetminus\{0\}$, the quotient M_i/VM_{i-1} is trivial and so $VM_{i-1}=M_i$. Consequently, $V^fM_0 = VM_{f-1}$ and so, the tangent space of D is isomorphic to M_0/V^fM_0, which is by definition equal to $\mathbb{H}(D)/V_\pi\mathbb{H}(D)$. Since π goes to zero in k and the

[1] mod π, D is free of rank fh and by Nakayama lemma, and the facts that $W_{\mathcal{O}}(k)$ is a dvr (Proposition B.24) and D has no π-torsion, it is free over $W_{\mathcal{O}}(k)$ of rank fh. As $V: M_i\to M_{i+1}$ is injective, each M_i is a free $W_{\mathcal{O}}(k)$-module of rank h.

the action of $\mathcal{O}$ on the tangent space of D is by scalar multiplication, we have $\pi\big(\mathbb{H}(D)/V_\pi\mathbb{H}(D)\big) = 0$, i.e., $\pi\mathbb{H}(D) \subseteq V_\pi\mathbb{H}(D)$. Set

$$F_\pi := V_\pi^{-1}\pi : \mathbb{H}(D) \to \mathbb{H}(D).$$

It is a well defined ${}^{F_\pi}$-linear $\mathcal{O}$-module homomorphism and by definition we have $F_\pi \circ V_\pi = V_\pi \circ F_\pi = \pi$. □

Remark 4.4. We have seen in the proof of the previous lemma that $VM_{i-1} = M_i$ for every $i \in \mathbb{Z}/f\mathbb{Z} \smallsetminus \{0\}$. Since V is injective (cf. Lemma B.13 of [12]), we conclude that $V : M_{i-1} \to M_i$ is a semilinear isomorphism, for every $i \in \mathbb{Z}/f\mathbb{Z} \smallsetminus \{0\}$ and we have

$$D \cong \mathbb{H}(D) \times V\mathbb{H}(D) \times V^2\mathbb{H}(D) \times \cdots \times V^{f-1}\mathbb{H}(D).$$

◊

Corollary 4.5. *The quadruple $\mathcal{P} = (\mathbb{H}(D), V_\pi\mathbb{H}(D), F_\pi, V_\pi^{-1})$ is a ramified $3n$-display over k of height h and its rank is equal to the dimension of the k-vector space D/VD.*

Proof. It follows from the fact that $\mathbb{H}(D)$ is a ramified Dieudonné module over k and the equivalence of ramified Dieudonné modules and ramified $3n$-displays over k (cf. point 5 in the list in subsection B.2). □

Construction 4.6. In Construction 4.2, we have constructed a functor, $\mathbb{H}$, from the category of Dieudonné modules over k with a scalar $\mathcal{O}$-action to the category of ramified Dieudonné modules over k with scalar $\mathcal{O}$-action. Now, we construct a functor in the other direction, which will be a quasi-inverse to $\mathbb{H}$. Let H be a ramified Dieudonné module over k such that the action of $\mathcal{O}$ on it is scalar. For every $i = 0, \ldots, f-1$, set $H_i := W(k) \otimes_{\sigma^{-i}, W(k)} H$ and let $\mathbb{D}(H)$ be the finite free $W(k)\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}$-module $H_0 \times H_1 \times \cdots \times H_{f-1}$. ▼

Lemma 4.7. *There are operators V and F on $\mathbb{D}(H)$ that make it a Dieudonné module with scalar $\mathcal{O}$-action.*

Proof. Define operators V and F on $\mathbb{D}(H)$ as follows:

$$V(x_0, 1 \otimes x_1, \ldots, 1 \otimes x_{f-1}) := (V_\pi(x_{f-1}), 1 \otimes x_0 \ldots, 1 \otimes x_{f-2})$$

and

$$F(x_0, 1 \otimes x_1, \ldots, 1 \otimes x_{f-1}) := (px_1, 1 \otimes px_2, \ldots, 1 \otimes px_{f-1}, 1 \otimes V_\pi^{-1}(px_0)).$$

These are well-defined maps for the following reasons. Every element of H_i can be written as $1 \otimes x_i$ with $x_i \in H$, because $\sigma : W(k) \to W(k)$ is an automorphism. As $pH \subseteq \pi H \subseteq V_\pi H$ and V_π is injective on H (this follows from the fact that H is a free $W_{\mathcal{O}}(k)$-module, that $F_\pi \circ V_\pi = \pi$ and that π is a non-zero divisor of $W_{\mathcal{O}}(k)$), the element $V_\pi^{-1}(px_0)$ is well-defined. It is now straightforward to check that F and V are respectively ${}^\sigma$ and ${}^{\sigma^{-1}}$-linear and that $F \circ V = p = V \circ F$. Therefore, we have a Dieudonné module endowed with an $\mathcal{O}$-action. By definition, the tangent space of $\mathbb{D}(H)$ is isomorphic to $\mathbb{D}(H)/V\mathbb{D}(H) \cong H/V_\pi H$, since by assumption, the action of $\mathcal{O}$ on this vector space is by scalar multiplication, we conclude that the action of $\mathcal{O}$ on $\mathbb{D}(H)$ is scalar. $\square$

Lemma 4.8. *The functors $\mathbb{D}$ and $\mathbb{H}$ are quasi-inverse one to the other.*

Proof. Let D be a Dieudonné module over k with scalar $\mathcal{O}$-action. Then, as we explained before, we have $D \cong M_0 \times M_1 \times \cdots \times M_{f-1}$ with $V : M_{i-1} \to M_i$ a ${}^{\sigma^{-1}}$-linear isomorphism for every $i \in \mathbb{Z}/f\mathbb{Z} \smallsetminus \{0\}$. Therefore, we have linear isomorphisms $W(k) \otimes_{\sigma^{-1},W(k)} M_{i-1} \to M_i$ induced by V and so isomorphisms $W(k) \otimes_{\sigma^{-i},W(k)} M_0 \cong M_i$ for every i. Since by definition, we have $\mathbb{H}(D) = M_0$, we conclude that $M_i \cong W(k) \otimes_{\sigma^{-i},W(k)} \mathbb{H}(D)$. This shows that $\mathbb{D}(\mathbb{H}(D)) \cong D$. Now, let H be a ramified Dieudonné module over k with scalar $\mathcal{O}$-action. Then, by construction, $\mathbb{D}(H) = H_0 \times H_1 \times \cdots \times H_{f-1}$, where $H_i = W(k) \otimes_{\sigma^{-i},W(k)} H$. If we decompose the Dieudonné module $\mathbb{D}(H)$ as the product $M_0 \times \cdots \times M_{f-1}$, as in Construction 4.2, then by construction, we have $M_i = H_i$. In particular, $\mathbb{H}(\mathbb{D}(H)) = M_0 = H_0 = H$. $\square$

Remark 4.9. The above arguments and the fact that the category of $3n$-displays (respectively ramified $3n$-displays) over k is equivalent to the category of Dieudonné modules (respectively ramified Dieudonné modules) over k, show that we have functors $\mathbb{H}$ and $\mathbb{D}$ which define an equivalence of categories between the category of $3n$-displays over k with a scalar $\mathcal{O}$-action and the category of ramified $3n$-displays over k with a scalar $\mathcal{O}$-action. $\diamondsuit$

Lemma 4.10. *Let $D_0, \dots, D_r$ (respectively $\mathcal{P}_0, \dots, \mathcal{P}_r$) be Dieudonné modules (respectively $3n$-displays) over k with scalar $\mathcal{O}$-action. There exist canonical and functorial isomorphisms*

$$\mathrm{Mult}^{\mathcal{O}}(\mathbb{H}(D_1) \times \cdots \times \mathbb{H}(D_r), \mathbb{H}(D_0)) \cong \mathrm{Mult}^{\mathcal{O}}(D_1 \times \cdots \times D_r, D_0)$$

and

$$\mathrm{Mult}^{\mathcal{O}}(\mathbb{H}(\mathcal{P}_1) \times \cdots \times \mathbb{H}(\mathcal{P}_r), \mathbb{H}(\mathcal{P}_0)) \cong \mathrm{Mult}^{\mathcal{O}}(\mathcal{P}_1 \times \cdots \times \mathcal{P}_r, \mathcal{P}_0).$$

Proof. Since the category of $3n$-displays (respectively ramified $3n$-displays) over k is equivalent to the category of Dieudonné modules (respectively ramified Dieudonné modules) over k, it is enough to show the result for Dieudonné modules. Let us denote $\mathbb{H}(D_i)$ by H_i and let $\varphi : H_1 \times \cdots \times H_r \to H_0$ be a $W_{\mathcal{O}}(k)$-multilinear morphism satisfying the V_π-condition. Define $\chi(\varphi) : D_1 \times \cdots \times D_r \to D_0$ as follows. For every $i = 1, \dots, r$, take an element $x_i \in D_i$. We know that every element of D_i can be written in a unique way as a sum $a_0 + Va_1 + \cdots + V^{f-1}a_{f-1}$ with $a_j \in \mathbb{H}(D_i)$ (cf. Remark 4.4). Therefore, we may assume that each x_i is of the form $V^{\alpha_i} y_i$ with $y_i \in \mathbb{H}(D_i)$

and $0 \leq \alpha_i \leq f-1$. Set

$$\chi(\varphi)(V^{\alpha_1}y_1,\ldots,V^{\alpha_r}y_r) := \begin{cases} V^{\alpha}\varphi(y_1,\ldots,y_r) & \text{if } \alpha_1=\alpha_2=\cdots=\alpha_r=\alpha \\ 0 & \text{otherwise} \end{cases}$$

It follows from the construction that $\chi(\varphi)$ is $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}$-multilinear. We have to check that it satisfies the V-condition. If α is strictly smaller that $f-1$, then $\alpha+1 \leq f-1$, and by definition, we have

$$\begin{aligned} V\chi(\varphi)(V^{\alpha}y_1,\ldots,V^{\alpha}y_r) &= V^{\alpha+1}(\varphi)(y_1,\ldots,y_r) \\ &= \chi(\varphi)(V^{\alpha+1}y_1,\ldots,V^{\alpha+1}y_r). \end{aligned}$$

If $\alpha = f-1$, then we have $V\chi(\varphi)(V^{f-1}y_1,\ldots,V^{f-1}y_r) = V^f\varphi(y_1,\ldots,y_r)$. Since φ satisfies the V_π-condition, and on $\mathbb{H}(D_0)$, V_π is equal to V^f, we conclude that

$$V^f\varphi(y_1,\ldots,y_r) = \varphi(V^f y_1,\ldots,V^f y_r) = \chi(\varphi)(V^f y_1,\ldots,V^f y_r).$$

This shows that $\chi(\varphi)$ belongs to $\mathrm{Mult}^{\mathcal{O}}(D_1\times\cdots\times D_r, D_0)$ and that we have an $\mathcal{O}$-linear homomorphism

$$\chi : \mathrm{Mult}^{\mathcal{O}}(\mathbb{H}(D_1)\times\cdots\times\mathbb{H}(D_r),\mathbb{H}(D_0)) \to \mathrm{Mult}^{\mathcal{O}}(D_1\times\cdots\times D_r, D_0).$$

Now, we want to define a homomorphism, Ξ, in the other direction, which will be the inverse of χ. Let $\psi : D_1\times\cdots\times D_r \to D_0$ be a $W\otimes_{\mathbb{Z}_p}\mathcal{O}$-multilinear morphism satisfying the V-condition. We then obtain a $W_{\mathcal{O}}(k)$-multilinear morphism $\Xi(\psi) : \mathbb{H}(D_1)\times\cdots\times\mathbb{H}(D_r) \to \mathbb{H}(D_0)$ by restricting ψ on the first components of D_i. As ψ satisfies the V-condition, it also satisfies V^f-condition, and thus, $\Xi(\psi)$ satisfies the V_π-condition. Consequently, we have a $\mathcal{O}$-linear homomorphism

$$\Xi : \mathrm{Mult}^{\mathcal{O}}(D_1\times\cdots\times D_r, D_0) \to \mathrm{Mult}^{\mathcal{O}}(\mathbb{H}(D_1)\times\cdots\times\mathbb{H}(D_r),\mathbb{H}(D_0)).$$

By construction, the composition $\Xi\circ\chi$ is the identity. As for the composition $\chi\circ\Xi$, we have

$$\begin{gathered} \chi\circ\Xi(\psi)(V^{\alpha}y_1,\ldots,V^{\alpha}y_r) = V^{\alpha}\Xi(\psi)(y_1,\ldots,y_r) = \\ V^{\alpha}\psi(y_1,\ldots,y_r) = \psi(V^{\alpha}y_1,\ldots,V^{\alpha}y_r) \end{gathered}$$

where the last equality follows from the fact that ψ satisfies the V-condition. Thus $\chi\circ\Xi$ is the identity. $\square$

Remark 4.11. 1) Let $\mathcal{P}_0,\ldots,\mathcal{P}_r$ be $3n$-displays over k with scalar $\mathcal{O}$-action. Then the $\mathcal{O}$-linear homomorphism χ given in the above Lemma is given by the following formula. Take elements $\varphi \in \mathrm{Mult}^{\mathcal{O}}(\mathbb{H}(\mathcal{P}_1)\times\cdots\times\mathbb{H}(\mathcal{P}_r),\mathbb{H}(\mathcal{P}_0))$ and $\mathbf{x} := (\vec{x}_1,\ldots,\vec{x}_r) \in P_1\times\cdots\times P_r$. Write $\vec{x}_i = (x_{i,0},\ldots,x_{i,f-1})$ according to the decomposition $P_i = P_{i,0}\times\cdots\times P_{i,f-1}$. Then

$$\chi(\varphi)(\mathbf{x}) = (\varphi(x_{1,0},\ldots,x_{r,0}), V\varphi(V^{-1}x_{1,1},\ldots,V^{-1}x_{r,1}),\ldots$$

$$\ldots, V^{f-1}\varphi(V^{-f+1}x_{1,f-1}, \ldots, V^{-f+1}x_{r,f-1}))$$

note that for every $j > 0$, we have $P_{i,j} = V^j P_{i,0}$ and the formula makes sense.

2) Let D be a nilpotent Dieudonné module over k with scalar $\mathcal{O}$-action and such that the dimension of its tangent space is 1. Then, by Proposition 1.22 and Theorem 1.23, the exterior power $\bigwedge^r_{W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}} D$ is again a nilpotent Dieudonné module with scalar $\mathcal{O}$-action (we are using the equivalence of p-divisible groups with $\mathcal{O}$-actions and π-divisible modules). We have a canonical isomorphism

$$\mathbb{H}(\bigwedge^r_{W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}} D) \cong \bigwedge^r_{W_{\mathcal{O}}(k)} \mathbb{H}(D).$$

Indeed, if $D = M_0 \times \cdots \times M_{f-1}$, then

$$\bigwedge^r_{W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}} D = \bigwedge^r_{W_{\mathcal{O}}(k)} M_0 \times \cdots \times \bigwedge^r_{W_{\mathcal{O}}(k)} M_{f-1}.$$

It follows that under the isomorphism of the previous lemma, the universal alternating morphism $\lambda : D^r \to \bigwedge^r_{W\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}} D$ corresponds to the universal alternating morphism

$$\lambda : \mathbb{H}(D)^r \to \bigwedge^r_{W_{\mathcal{O}}(k)} \mathbb{H}(D).$$

$\Diamond$

Construction 4.12. Let D be a Dieudonné module over k with a scalar $\mathcal{O}$-action. Define a map $\mathrm{Tr} : D \to \mathbb{H}(D)$ as follows. Take an element x of D. It can be written uniquely as a sum $x_0 + Vx_1 + \cdots + V^{f-1}x_{f-1}$ with $x_i \in \mathbb{H}(D)$. Now set $\mathrm{Tr}(x) := x_0 + x_1 + \cdots + x_{f-1}$. Since the Verschiebung is an $\mathcal{O}$-linear homomorphism, the map Tr is also an $\mathcal{O}$-linear homomorphism. In the same fashion, if $\mathcal{P} = (P, Q, F, V^{-1})$ is a $3n$-display over k, we obtain an $\mathcal{O}$-linear homomorphism $\mathrm{Tr} : \mathcal{P} \to \mathbb{H}(\mathcal{P})$ (cf. Remark 4.9). Now, let $\mathcal{N}$ be a nilpotent k-algebra, we want to define similarly, an $\mathcal{O}$-linear homomorphism $\mathrm{Tr} : \widehat{P}(\mathcal{N}) \to \widehat{\mathbb{H}(\mathcal{P})}(\mathcal{N})$. We have decompositions $P = P_0 \times \ldots P_{f-1}$ and $Q = Q_0 \times \cdots \times Q_{f-1}$ and we know that in fact, for every $i = 1, \ldots, f-1$, the two $W_{\mathcal{O}}(k)$-modules P_i and Q_i are equal (and are equal to $V^i P_0$). We also know that $Q_0 = V^f P_0$. Thus, we have a decomposition

$$\widehat{W}(\mathcal{N}) \otimes_{W(k)} P = \widehat{W}(\mathcal{N}) \otimes_{W(k)} P_0 \times \widehat{W}(\mathcal{N}) \otimes_{W(k)} Q_1 \times \cdots \times \widehat{W}(\mathcal{N}) \otimes_{W(k)} Q_{f-1}.$$

Note that each component $\widehat{W}(\mathcal{N}) \otimes_{W(k)} Q_i$ ($i = 1, \ldots, f-1$) is a submodule of $\widehat{Q}_{\mathcal{N}}$ and on it the morphism V^{-i} is defined. So, we can define an $\mathcal{O}$-linear homomorphism $\mathrm{Tr} : \widehat{P}_{\mathcal{N}} \to \widehat{\mathbb{H}(P)}_{\mathcal{N}}$ by sending an element $(x_0, q_1, \ldots, q_{f-1})$

to the element $(\mu\otimes\mathrm{Id})(x_0+V^{-1}q_1+\cdots+V^{-f+1}q_{f-1})$, where $u:\widehat{W}(\mathcal{N})\to\widehat{W}_{\mathcal{O}}(\mathcal{N})$ is the canonical morphism given in Proposition B.22. ▼

Notations. Let $\mathcal{P}$ be a ramified $3n$-display over k with an $\mathcal{O}$-action. In the following proposition and the theorem that follows it, we denote by $BT^{\mathcal{O}}_{\mathcal{P}}$ the formal group associated with $\mathcal{P}$. So, if $\mathcal{P}$ is a $3n$-displays in the classical sense, we denote its formal group by $BT^{\mathbb{Z}_p}_{\mathcal{P}}$.

Proposition 4.13. *Let $\mathcal{P}$ be a $3n$-display over k with a scalar $\mathcal{O}$-action. Then* Tr *induces an isomorphism*

$$\mathrm{Tr}:BT^{\mathbb{Z}_p}_{\mathcal{P}}\xrightarrow{\cong}BT^{\mathcal{O}}_{\mathbb{H}(\mathcal{P})}$$

of formal $\mathcal{O}$-modules.

Proof. We have to show that $\mathrm{Tr}((V^{-1}-\mathrm{Id})\widehat{Q}_{\mathcal{N}})\subseteq(V_\pi^{-1}-\mathrm{Id})(\widehat{\mathbb{H}(Q)}_{\mathcal{N}})$. Take an element $(q_0,\dots,q_{f-1})\in\widehat{Q}_{\mathcal{N}}$. We have

$$\mathrm{Tr}((V^{-1}-\mathrm{Id})(q_0,\dots,q_{f-1}))=$$

$$Tr(V^{-1}q_1-q_0,V^{-1}q_2-q_1,\dots,V^{-1}q_{f-1}-q_{f-2},V^{-1}q_0-q_{f-1})$$

$$(\mu\otimes\mathrm{Id})(V^{-1}q_1-q_0+V^{-2}q_2-V^{-1}q_1+\cdots+V^{-f+1}q_{f-1}$$

$$-V^{-f+2}q_{f-2}+V^{-f}q_0-V^{-f+1}q_{f-1})$$

$$=(\mu\otimes\mathrm{Id})(V^{-f}q_0-q_0)=(\mu\otimes\mathrm{Id})(V^{-f}q_0)-(\mu\otimes\mathrm{Id})(q_0).$$

It is thus sufficient to show that

$$(\mu\otimes\mathrm{Id})(V^{-f}q_0)=V_\pi^{-1}(\mu\otimes\mathrm{Id})(q_0). \tag{4.14}$$

We can assume that either $q_0=\xi\otimes y_0$ with $\xi\in\widehat{W}(\mathcal{N}),y_0\in Q_0$ or $q_0=V^f\xi\otimes x_0$ with $\xi\in\widehat{W}(\mathcal{N}),y_0\in P_0$. In the first case, we have

$$\begin{aligned}(\mu\otimes\mathrm{Id})(V^{-f}q_0)&=\mu\otimes\mathrm{Id})(F^f\xi\otimes V^{-f}y_0)\\&=\mu(F^f\xi)\otimes V_\pi^{-1}y_0\\&=F_\pi\mu(\xi)\otimes V_\pi^{-1}y_0\\&=V_\pi^{-1}(\mu\otimes\mathrm{Id})(\xi\otimes y_0)\\&=V_\pi^{-1}(\mu\otimes\mathrm{Id})(q_0)\end{aligned}$$

where we have used Proposition B.22 for the third equality. In the second case, we have

$$\begin{aligned}(\mu\otimes\mathrm{Id})(V^{-f}q_0) &= (\mu\otimes\mathrm{Id})(\xi\otimes F^f x_0)\\ &= \mu(\xi)\otimes V^{-f}(V^f F^f x_0)\\ &= \mu(\xi)\otimes V_\pi^{-1}(p^f x_0)\\ &= \mu(p^{f-1}\xi)\otimes V_\pi^{-1}(px_0)\\ &= \mu(F^{f-1}V^{f-1}\xi)\otimes V_\pi^{-1}(px_0)\\ &= \mu(F^{f-1}V^{f-1}\xi)\otimes V_\pi^{-1}\pi(\frac{p}{\pi}x_0)\\ &= \mu(F^{f-1}V^{f-1}\xi)\otimes F_\pi(\frac{p}{\pi}x_0)\\ &= V_\pi^{-1}(V_\pi\mu(F^{f-1}V^{f-1}\xi)\otimes\frac{p}{\pi}x_0)\\ &= V_\pi^{-1}(\frac{p}{\pi}V_\pi\mu(F^{f-1}V^{f-1}\xi)\otimes x_0) = V_\pi^{-1}(\mu(V^f\xi)\otimes x_0)\\ &= V_\pi^{-1}(\mu\otimes\mathrm{Id})(q_0)\end{aligned}$$

where for the penultimate equality we have used Proposition B.22. This proves the claim (equality (4.14)) and it follows that Tr induces an $\mathcal{O}$-linear homomorphism $\mathrm{Tr} : BT_{\mathcal{P}}^{\mathbb{Z}_p} \to BT_{\mathbb{H}(\mathcal{P})}^{\mathcal{O}}$ and we have to show that it is an isomorphism. Let us construct $\widehat{P}_{0,\mathcal{N}}, \widehat{Q}_{0,\mathcal{N}}$ and the monomorphism $V^{-f}-\mathrm{Id} : \widehat{Q}_{0,\mathcal{N}} \to \widehat{P}_{0,\mathcal{N}}$ in the same way that we constructed $\widehat{P}_{\mathcal{N}}, \widehat{Q}_{\mathcal{N}}$ and $V^{-1}-\mathrm{Id} : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}$. Let us denote by $BT_{\mathcal{P}_0}^{\mathbb{Z}_p}(\mathcal{N})$ the cokernel of $V^{-f}-\mathrm{Id} : \widehat{Q}_{0,\mathcal{N}} \to \widehat{P}_{0,\mathcal{N}}$. It follows that the homomorphism Tr is equal to the composition $\widehat{P}_{\mathcal{N}} \xrightarrow{\mathrm{Tr}'} \widehat{P}_{0,\mathcal{N}} \xrightarrow{\mu\otimes\mathrm{Id}} \widehat{\mathbb{H}(P)}_{\mathcal{N}}$ where Tr' is the homomorphism sending $(x_0, q_1, \ldots, q_{f-1})$ to the sum $x_0+V^{-1}q_1+\cdots+V^{-f+1}q_{f-1}$. It also follows from the above calculations that $\mathrm{Tr}'((V^{-1}-\mathrm{Id})\widehat{Q}_{\mathcal{N}}) \subseteq (V^{-f}-\mathrm{Id})(\widehat{Q}_{0,\mathcal{N}})$ and $(\mu\otimes\mathrm{Id})((V^{-f}-\mathrm{Id})(\widehat{Q}_{0,\mathcal{N}})) \subseteq (V_\pi^{-1}-\mathrm{Id})(\widehat{\mathbb{H}(Q)}_{\mathcal{N}})$. In other words, we have a commutative diagram:

$$\begin{tikzcd} \widehat{P}_{\mathcal{N}} \arrow[r, "\mathrm{Tr}'"] \arrow[d, two heads] \arrow[rr, bend left, "\mathrm{Tr}"] & \widehat{P}_{0,\mathcal{N}} \arrow[r, "\mu\otimes\mathrm{Id}"] \arrow[d, two heads] & \widehat{\mathbb{H}(P)}_{\mathcal{N}} \arrow[d, two heads] \\ BT_{\mathcal{P}}^{\mathbb{Z}_p}(\mathcal{N}) \arrow[r, "\mathrm{Tr}'"'] \arrow[rr, bend right, "\mathrm{Tr}"'] & BT_{\mathcal{P}_0}^{\mathbb{Z}_p}(\mathcal{N}) \arrow[r, "\overline{\mu\otimes\mathrm{Id}}"'] & BT_{\mathbb{H}(\mathcal{P})}^{\mathcal{O}}(\mathcal{N}). \end{tikzcd}$$

It therefore suffices to show that Tr' and $\overline{\mu\otimes\mathrm{Id}}$ are isomorphisms. The homomorphism $\mathrm{Tr}' : \widehat{P}_{\mathcal{N}} \to \widehat{P}_{0,\mathcal{N}}$ is surjective, since $\mathrm{Tr}'(x_0, 0, \ldots, 0) = x_0$

for every $x_0 \in \widehat{P}_{0,\mathcal{N}}$. So, Tr' is surjective as well. Let $(x_0, , \dots, x_{f-1})$ be an element in the kernel of Tr'. It means that there is an element $q_0 \in \widehat{Q}_{0,\mathcal{N}}$ such that

$$\mathrm{Tr}'(x_0, x_1, \dots, x_{f-1}) = x_0 + V^{-1}x_1 + \cdots + V^{-f+1}x_{f-1} = (V^{-f} - \mathrm{Id})(q_0).$$

Define, recursively, elements $y_i \in \widehat{Q}_{\mathcal{N}}$, with $i = 0, \dots, f-1$, as follows: $y_0 := q_0$ and for $0 < i < f-1$, set $y_i := V^{-1}y_{i+1} - x_i$ where we calculate $i+1$ modulo f, e.g., $y_{f-1} = V^{-1}y_0 - x_{f-1}$ and so on. It follows that the element $(y_0, y_1, \dots, y_{f-1})$ belongs to $\widehat{Q}_{\mathcal{N}}$ and we have

$$(V^{-1} - \mathrm{Id})(y_0, y_1, \dots, y_{f-1}) =$$
$$(V^{-1}y_1 - y_0, \dots, V^{-1}y_{f-1} - y_{f-2}, V^{-1}y_0 - y_{f-1}) = (x_0, x_1, \dots, x_{f-1})$$

and so $(x_0, \dots, x_{f-1})$ is zero in $BT_{\mathcal{P}}^{\mathbb{Z}_p}(\mathcal{N})$. It implies that Tr' is injective too, and hence an isomorphism. It remains to prove that $\overline{\mu \otimes \mathrm{Id}} : BT_{\mathcal{P}_0}^{\mathbb{Z}_p}(\mathcal{N}) \to BT_{\mathbb{H}(\mathcal{P})}^{\mathcal{O}}(\mathcal{N})$ is an isomorphism. We can reduce to the case, where $\mathcal{N}^2 = 0$. Under this condition, there exist commutative diagrams (for details, we refer [24]):

$$\begin{tikzcd}
0 \arrow[r] & \widehat{Q}_{0,\mathcal{N}} \arrow[r, hook] \arrow[d, equal] & \widehat{P}_{0,\mathcal{N}} \arrow[r] \arrow[d, "V^{-1}-\mathrm{Id}"'] & \mathcal{N} \otimes_k \mathcal{T}(\mathcal{P}_0) \arrow[r] \arrow[d, "\exp"'] & 0 \\
0 \arrow[r] & \widehat{Q}_{0,\mathcal{N}} \arrow[r, "V^{-1}-\mathrm{Id}"'] & \widehat{P}_{0,\mathcal{N}} \arrow[r] & BT_{\mathcal{P}_0}^{\mathbb{Z}_p}(\mathcal{N}) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathbb{H}(Q)}_{\mathcal{N}} \arrow[r, hook] \arrow[d, equal] & \widehat{\mathbb{H}(P)}_{\mathcal{N}} \arrow[r] \arrow[d, "V^{-1}-\mathrm{Id}"'] & \mathcal{N} \otimes_k \mathcal{T}(\mathbb{H}(\mathcal{P})) \arrow[r] \arrow[d, "\exp"'] & 0 \\
0 \arrow[r] & \widehat{\mathbb{H}(Q)}_{\mathcal{N}} \arrow[r, "V^{-1}-\mathrm{Id}"'] & \widehat{\mathbb{H}(P)}_{\mathcal{N}} \arrow[r] & BT_{\mathbb{H}(\mathcal{P})}^{\mathcal{O}}(\mathcal{N}) \arrow[r] & 0
\end{tikzcd}$$

where the vertical morphisms $V^{-1} - \mathrm{Id}$ in the middle are extensions of the usual (horizontal) $V^{-1} - \mathrm{Id}$ and the exponential morphism, which is $\mathcal{O}$-linear, is given by these diagrams. What is proved in [1] and [24] is that the vertical $V^{-1} - \mathrm{Id}$ are isomorphisms and so are the exponential morphisms. Note that the tangent spaces $\mathcal{T}(\mathcal{P}_0)$ and $\mathcal{T}(\mathbb{H}(\mathcal{P}))$ are equal ($P_0 = \mathbb{H}(P)$ and $Q_0 = \mathbb{H}(Q)$) and the equality (4.14) implies that we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{N} \otimes_k \mathcal{T}(\mathcal{P}_0) & \xrightarrow{\exp} & BT_{\mathcal{P}_0}^{\mathbb{Z}_p}(\mathcal{N}) \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \overline{\mu\otimes\mathrm{Id}}} \\ \mathcal{N} \otimes_k \mathcal{T}(\mathbb{H}(\mathcal{P})) & \xrightarrow[\exp]{} & BT_{\mathbb{H}(\mathcal{P})}^{\mathcal{O}}(\mathcal{N}). \end{array}$$

Since the exponential morphisms are isomorphisms, we conclude that $\overline{\mu \otimes \mathrm{Id}}$ is also an isomorphism. □

Remark 4.15. Let $\mathcal{M}_0, \ldots, \mathcal{M}_r$ be π-divisible modules over a base scheme S. Let $\varphi : \prod_{i=1}^r \mathcal{M}_i \to \mathcal{M}_0$ be an $\mathcal{O}$-multilinear morphism, when $\mathcal{M}_i$ are considered as p-divisible groups with $\mathcal{O}$-action. Then φ is not an $\mathcal{O}$-multilinear morphism of π-divisible modules, but an appropriate "twist" of it will be. Indeed, if we set $\varphi^\sharp_{ne} := \frac{1}{u^{n(r-1)}}\varphi_n$, then a straightforward calculation shows that

$$\pi^e \varphi^\sharp_{e(n+1)}(g_1, \ldots, g_r) = \varphi^\sharp_{ne}(\pi^e g_1, \ldots, \pi^e g_r).$$

Thus, the system $\{\varphi^\sharp_{ne}\}_n$ belongs to the inverse limit

$$\varprojlim_n \mathrm{Mult}^{\mathcal{O}}(\mathcal{M}_{1,ne} \times \cdots \times \mathcal{M}_{r,ne}, \mathcal{M}_{0,ne}) \cong$$
$$\varprojlim_n \mathrm{Mult}^{\mathcal{O}}(\mathcal{M}_{1,n} \times \cdots \times \mathcal{M}_{r,n}, \mathcal{M}_{0,n}) = \mathrm{Mult}^{\mathcal{O}}(\mathcal{M}_1 \times \cdots \times \mathcal{M}_r, \mathcal{M}_0).$$

So, in the sequel, if we identify the $\mathcal{O}$-module of multilinear morphisms of π-divisible modules with that of the corresponding p-divisible groups with $\mathcal{O}$-action, we are implicitly using the above twist. ◊

The following lemma will be used in the proof of the next proposition.

Lemma 4.16. *Let A be a ring and $\alpha \in A$ a non-zero divisor. Let $M_0, \ldots, M_r$ be A-modules and $\varphi : \prod_{i=1}^r M_i \to M_0$ an A-multilinear morphism. Let $y_{i,0}, \ldots, y_{i,n-1}$ and $w_{i,0}, \ldots, w_{i,n}$ be elements of M_i ($n \geq 1$ and $1 \leq i \leq r$) subject to the following relations: $\forall i, j$, $w_{i,j+1} = w_{i,j} + \alpha y_{i,j}$. Then we have*

$$\sum_{i=1}^r \sum_{j=0}^{n-1} \varphi(w_{1,j+1}, \ldots, w_{i-1,j+1}, y_{i,j}, w_{i+1,j}, \ldots, w_{r,j}) =$$

$$\sum_{i=1}^r \sum_{j=0}^{n-1} \varphi(w_{1,n}, \ldots, w_{i-1,n}, y_{i,j}, w_{i+1,0}, \ldots, w_{r,0}).$$

Proof. We refer to [11] for the (rather technical but elementary) proof of the lemma. □

Proposition 4.17. *Let $\mathcal{P}_0, \dots, \mathcal{P}_r$ be $3n$-displays over k with scalar $\mathcal{O}$-action. Let us denote by G_i (respectively $\tilde{G}_i$) the formal $\mathcal{O}$-module $BT^{\mathbb{Z}_p}_{\mathcal{P}_i}$ (respectively $BT^{\mathcal{O}}_{\mathbb{H}(\mathcal{P}_i)}$). Then the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Mult}^{\mathcal{O}}(\mathcal{P}_1 \times \cdots \times \mathcal{P}_r, \mathcal{P}_0) & \xrightarrow{\Xi} & \mathrm{Mult}^{\mathcal{O}}(\mathbb{H}(\mathcal{P}_1) \times \cdots \times \mathbb{H}(\mathcal{P}_r), \mathbb{H}(\mathcal{P}_0)) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \mathrm{Mult}^{\mathcal{O}}(G_1 \times \cdots \times G_r, G_0) & \xrightarrow[\mathrm{Tr}^*]{} & \mathrm{Mult}^{\mathcal{O}}(\tilde{G}_1 \times \cdots \times \tilde{G}_r, \tilde{G}_0) \end{array}$$

where Ξ and Tr^* are respectively the $\mathcal{O}$-linear homomorphism given in Lemma 4.10 and the $\mathcal{O}$-linear homomorphism induced by the isomorphisms $\mathrm{Tr} : G_i \to \tilde{G}_i$ given in Proposition 4.13.

Proof. Take an $\mathcal{O}$-multilinear morphism $\varphi \in \mathrm{Mult}^{\mathcal{O}}(\mathbb{H}(\mathcal{P}_1) \times \cdots \times \mathbb{H}(\mathcal{P}_r), \mathbb{H}(\mathcal{P}_0))$, a nilpotent k-algebra $\mathcal{N}$, and a vector $\mathbf{x} = (\vec{x}_1, \dots, \vec{x}_r) \in G_{1,n}(\mathcal{N}) \times \cdots \times G_{r,n}(\mathcal{N})$. Using Remark 4.15, we have to show that

$$\frac{1}{u^{n(r-1)}} \mathrm{Tr}\big(\beta(\chi(\varphi))(\mathbf{x})\big) = \beta(\varphi)(\mathrm{Tr}(\vec{x}_1), \dots, \mathrm{Tr}(\vec{x}_r))$$

where χ is the isomorphism given in the proof of Lemma 4.10 and the equality should hold in the $\mathcal{O}$-module $G_{0,ne}(\mathcal{N})$. By definition (cf. Construction 2.5), we have

$$\beta(\chi(\varphi))(\mathbf{x}) = (-1)^{r-1} \sum_{i=1}^{r} [\widehat{\chi(\varphi)}(V^{-1}\vec{g}_1, \dots, V^{-1}\vec{g}_{i-1}, \vec{x}_i, \vec{g}_{i+1}, \dots, \vec{g}_r)]$$

where elements $\vec{g}_i$ are given by the formula $p^n \vec{x}_i = (V^{-1} - \mathrm{Id})(\vec{g}_i)$. By definition of χ (cf. Lemma 4.10 and Remark 4.11), we have

$$\begin{aligned} \mathrm{Tr}\big([\widehat{\chi(\varphi)}(V^{-1}\vec{g}_1, \dots, V^{-1}\vec{g}_{i-1}, \vec{x}_i, \vec{g}_{i+1}, \dots, \vec{g}_r)]\big) = \\ \overline{\mu \otimes \mathrm{Id}}[\sum_{j=0}^{f-1} (\widehat{\varphi}(V^{-j-1}g_{1,j+1}, \dots, \\ V^{-j-1}g_{i-1,j+1}, V^{-j}x_{i,j}, V^{-j}g_{i+1,j}, \dots, V^{-j}g_{r,j}))] \end{aligned}$$

where $\vec{x}_i = (x_{i,0}, \dots, x_{i,f-1})$ and $\vec{g}_i = (g_{i,0}, \dots, g_{i,f-1})$. Therefore, we have

$$\frac{1}{u^{n(r-1)}} \mathrm{Tr}\big(\beta(\chi(\varphi))(\mathbf{x})\big) =$$

$$\begin{aligned} \frac{(-1)^{r-1}}{u^{n(r-1)}} \overline{\mu \otimes \mathrm{Id}}\Big[\sum_{i=1}^{r}\sum_{j=0}^{f-1} (\widehat{\varphi}(V^{-j-1}g_{1,j+1}, \dots, \\ \dots, V^{-j-1}g_{i-1,j+1}, V^{-j}x_{i,j}, V^{-j}g_{i+1,j}, \dots, V^{-j}g_{r,j}))\Big]. \end{aligned} \tag{4.18}$$

Again, by the definition of β, we have $\beta(\varphi)(\mathrm{Tr}(\vec{x}_1), \ldots, \mathrm{Tr}(\vec{x}_r)) = (-1)^{r-1} \sum_{i=1}^{r}[\widehat{\varphi}(V_\pi^{-1}h_1, \ldots, V_\pi^{-1}h_{i-1}, \mathrm{Tr}(\vec{x}_i), h_{i+1}, \ldots, h_r)]$ where $\pi^{ne}\mathrm{Tr}(\vec{x}_i) = (V_\pi^{-1} - \mathrm{Id})(h_i)$. Since $p^n\vec{x}_i = (V^{-1} - \mathrm{Id})(\vec{g}_i)$, for every $j = 0, \ldots, f-1$, we have

$$\pi^{ne}x_{i,j} = \frac{1}{u^n}(V^{-1}g_{i,j+1} - g_{i,j}) \tag{4.19}$$

and therefore $\pi^{ne}\mathrm{Tr}(\vec{x}_i) = \frac{1}{u^n}(\mu\otimes\mathrm{Id})(V^{-f}g_{i,0} - g_{i,0}) = \frac{1}{u^n}(V_\pi^{-1} - \mathrm{Id})(\mu\otimes\mathrm{Id})g_{i,0}$ which implies that $h_i = (\mu\otimes\mathrm{Id})\frac{1}{u^n}g_{i,0}$. Thus,

$$\widehat{\varphi}(V_\pi^{-1}h_1, \ldots, V_\pi^{-1}h_{i-1}, \mathrm{Tr}(\vec{x}_i), h_{i+1}, \ldots, h_r)] =$$

$$\widehat{\varphi}(V_\pi^{-1}h_1, \ldots, V_\pi^{-1}h_{i-1}, (\mu\otimes\mathrm{Id})\sum_{j=0}^{f-1}V^{-j}x_{i,j}, h_{i+1}, \ldots, h_r)]$$

$$= \frac{1}{u^{n(r-1)}}\overline{\mu\otimes\mathrm{Id}}\sum_{j=0}^{f-1}[\widehat{\varphi}(V^{-f}g_{1,0}, \ldots, V^{-f}g_{i-1,0}, V^{-j}x_{i,j}, g_{i+1,0}, \ldots, g_{r,0})].$$

Consequently, we have

$$\beta(\varphi)(\mathrm{Tr}(\vec{x}_1), \ldots, \mathrm{Tr}(\vec{x}_r)) =$$

$$\frac{(-1)^r}{u^{n(r-1)}}\overline{\mu\otimes\mathrm{Id}}\sum_{i=1}^{r}\sum_{j=0}^{f-1}[\widehat{\varphi}(V^{-f}g_{1,0}, \ldots, V^{-f}g_{i-1,0}, V^{-j}x_{i,j}, g_{i+1,0}, \ldots, g_{r,0})].$$

So, it is sufficient to show the following equality

$$\sum_{i=1}^{r}\sum_{j=0}^{f-1}\widehat{\varphi}(V^{-j-1}g_{1,j+1}, \ldots, V^{-j-1}g_{i-1,j+1},$$
$$V^{-j}x_{i,j}, V^{-j}g_{i+1,j}, \ldots, V^{-j}g_{r,j}) =$$

$$\sum_{i=1}^{r}\sum_{j=0}^{f-1}\widehat{\varphi}(V^{-f}g_{1,0}, \ldots, V^{-f}g_{i-1,0}, V^{-j}x_{i,j}, g_{i+1,0}, \ldots, g_{r,0}). \tag{4.20}$$

For every $i = 1, \ldots, r$ and $j = 0, \ldots, f$, set $w_{i,j} := V^{-j}g_{i,j}$ and $y_{i,j} := V^{-j}x_{i,j}$. Since V^{-1} is an $\mathcal{O}$-linear homomorphism, it follows from the relations (4.19) that

$$u^n\pi^{ne}y_{i,j} = V^{-j}x_{i,j} = V^{-j-1}g_{i,j+1} - V^{-j}g_{i,j} = w_{i,j+1} - w_{i,j}.$$

If we define $\alpha := u^n\pi^{ne}$, then α is a non-zero divisor of $\mathcal{O}$ and we have $w_{i,j+1} = \alpha y_{i,j} + w_{i,j}$. With these new notations, the equality (4.20) becomes the following equality and holds thanks to Lemma 4.16

$$\sum_{i=1}^{r}\sum_{j=0}^{f-1}\widehat{\varphi}(w_{1,j+1}, \ldots, w_{i-1,j+1}, y_{i,j}, w_{i+1,j}, \ldots, w_{r,j}) =$$

$$\sum_{i=1}^{r}\sum_{j=0}^{f-1}\widehat{\varphi}(w_{1,f},\ldots,w_{i-1,f},y_{i,j},w_{i+1,0},\ldots,w_{r,0}).$$

$\square$

Corollary 4.21. *Let $\mathcal{P}_0,\ldots,\mathcal{P}_r$ be ramified displays over k. The $\mathcal{O}$-linear homomorphism*

$$\beta:\mathrm{Mult}^{\mathcal{O}}(\mathcal{P}_1\times\cdots\times\mathcal{P}_r,\mathcal{P}_0)\to\mathrm{Mult}^{\mathcal{O}}(BT_{\mathcal{P}_1}\times\cdots\times BT_{\mathcal{P}_r},BT_{\mathcal{P}_0})$$

is an isomorphism.

Proof. Let $\tilde{\mathcal{P}}_i$ be a display over k such that $\mathbb{H}(\tilde{\mathcal{P}}_i)\cong\mathcal{P}_i$ (cf. Remark 4.9). By Corollary 3.3

$$\beta:\mathrm{Mult}(\tilde{\mathcal{P}}_1\times\cdots\times\tilde{\mathcal{P}}_r,\tilde{\mathcal{P}}_0)\to\mathrm{Mult}(BT_{\tilde{\mathcal{P}}_1}\times\cdots\times BT_{\tilde{\mathcal{P}}_r},BT_{\tilde{\mathcal{P}}_0})$$

is an isomorphism. Since β preserves the $\mathcal{O}$-module structure, it induces an isomorphism

$$\mathrm{Mult}^{\mathcal{O}}(\tilde{\mathcal{P}}_1\times\cdots\times\tilde{\mathcal{P}}_r,\tilde{\mathcal{P}}_0)\xrightarrow{\cong}\mathrm{Mult}^{\mathcal{O}}(BT_{\tilde{\mathcal{P}}_1}\times\cdots\times BT_{\tilde{\mathcal{P}}_r},BT_{\tilde{\mathcal{P}}_0}).$$

The result now follows from Proposition 4.17 (Ξ, Tr^* and β on the left hand side are isomorphisms). $\square$

One of the main results of subsection 3.2 is the existence of the exterior powers of truncated Barsotti-Tate groups of dimension 1 over local Artin rings with residue characteristic p (cf. Propositions 3.10 and 3.11). The proof of this result is based on the fact that β is an isomorphism when the base is a perfect field of characteristic p. Now that we have the above isomorphism, we can prove in the same way, the similar statement (Proposition 4.22 below). We therefore omit its proof and the auxiliary statements leading to it.

Let R be a local Artin $\mathcal{O}$-algebra with residue characteristic p and $\mathcal{M}$ a π-divisible module over R of height h, whose special fiber is connected of dimension 1. Let us denote by $\mathcal{P}$ the ramified display associated with $\mathcal{M}$. The exterior power $\bigwedge^r\mathcal{P}$ exists (cf. Remark 4.1 3)) and we denote by Λ^r_R the π-divisible module associated with $\bigwedge^r\mathcal{P}$. The universal alternating morphism $\lambda:\mathcal{P}^r\to\bigwedge^r\mathcal{P}$ induces an alternating morphism $\beta_\lambda:\mathcal{M}^r\to\Lambda^r_R$. So, we obtain an alternating morphism $\beta_{\lambda,n}:\mathcal{M}^r_n\to\Lambda^r_{R,n}$. As in Construction 3.5, for every group scheme X, we obtain the morphism

$$\underline{\lambda_n^*}(X):\underline{\mathrm{Hom}}_R(\Lambda^r_{R,n},X)\to\widetilde{\underline{\mathrm{Alt}}}^{\mathcal{O}}_R(\mathcal{M}^r_n,X).$$

Note that $\underline{\mathrm{Hom}}_R(\Lambda^r_{R,n},X)$ is the $\mathcal{O}$-module of group scheme homomorphisms. If R is a perfect field, then by Remark 4.11 2), Corollary 4.21, Propositions 1.21 and 1.22, $\Lambda^r_{R,n}$ is the r^{th}-exterior power of $\mathcal{M}_n$ and $\beta_{\lambda,n}$ is the universal alternating morphism.

Proposition 4.22. *For every finite and flat group scheme X over R, the morphisms*

$$\underline{\lambda}_n^*(X) : \underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, X) \to \widetilde{\underline{\mathrm{Alt}}}^{\mathcal{O}}_R(\mathcal{M}_n^r, X)$$

and

$$\underline{\lambda}_n^*(\mathbb{G}_m) : \underline{\mathrm{Hom}}_R(\Lambda^r_{R,n}, \mathbb{G}_m) \to \widetilde{\underline{\mathrm{Alt}}}^{\mathcal{O}}_R(\mathcal{M}_n^r, \mathbb{G}_m)$$

are isomorphisms. Consequently, $\beta_{\lambda,n} : \mathcal{M}_n^r \to \Lambda^r_{R,n}$ is the r^{th}-exterior power of $\mathcal{M}_n$ in the category of finite and flat group schemes over R (in the sense of Definition 1.17).

Corollary 4.23. *The $\mathcal{O}$-module scheme $\widetilde{\underline{\mathrm{Alt}}}^{\mathcal{O}}_R(\mathcal{M}_n^r, \mathbb{G}_m)$ is finite and flat over R and has order $q^{n\binom{h}{r}}$.*

Using theses results, we will reduce the general case to the case over a local Artin ring. We need few technical lemmas:

Lemma 4.24. *Let S be a base scheme, A a ring and b an element of A such that A/b is a finite set. Let*

$$0 \to N \to M \xrightarrow{pr} \underline{A/b} \to 0$$

be a short exact sequence of finite A-module schemes over S, where $\underline{A/b}$ denotes the constant A-module scheme over S corresponding to the A-module A/b. Assume that b annihilates M and there exists an scheme-theoretic section $s : S \to M$ such that its composition with the projection $pr : M \twoheadrightarrow \underline{A/b}$ is the closed embedding $S \hookrightarrow \underline{A/b}$ corresponding to the element $1 \in \underline{A/b}$. Then there exists an A-module scheme section, $\underline{A/b} \hookrightarrow M$, of the projection $pr : M \twoheadrightarrow \underline{A/b}$ and so the above short exact sequence is split.

Proof. We have

$$\mathrm{Hom}^A_S(\underline{A/b}, M) \cong \mathrm{Hom}^A(A/b, M(S)) \cong M(S)[b] = M(S)$$

where for the last equality, we used the fact that M is annihilated by b. Let $g : \underline{A/b} \to M$ be the A-linear homomorphism mapping to the section s under this isomorphism. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^A_S(\underline{A/b}, M) \arrow[r, "pr^*"] \arrow[d, "\cong"'] & \mathrm{Hom}^A_S(\underline{A/b}, \underline{A/b}) \arrow[d, "\cong"] \\
M(S) \arrow[r, "pr_S"'] & \underline{A/b}(S)
\end{tikzcd}$$

where the horizontal morphisms are induced by the projection pr, the left vertical isomorphism is the one given above and the right vertical isomorphism is given similarly. By assumption, $pr_S(s)$ is the closed embedding

$S \hookrightarrow \underline{A/b}$ corresponding to the element $1 \in \underline{A/b}$ and thus, under the isomorphism $\underline{A/b}(S) \cong \mathrm{Hom}_S^A(\underline{A/b}, \underline{A/b})$, it is mapped to the identity morphism. It follows from the definition of g and the commutativity of the above diagram, that $pr \circ g$ is the identity of $\underline{A/b}$. $\square$

Lemma 4.25. *Let S be a scheme whose underlying set consists of one point. Let A be a ring and $b_1, \dots, b_n$ elements of A such that the quotient ring A/b_i is a finite set (for every $i = 1, \dots, n$). Let*

$$(\xi) \qquad 0 \to N \to M \xrightarrow{pr} \bigoplus_{i=1}^{n} \underline{A/b_i} \to 0$$

be a short exact sequence of finite flat A-module schemes, such that for every i, b_i annihilates M. Then there exists a finite and faithfully flat S-scheme T such that the above short exact sequence splits over T.

Proof. For every $j = 1, \dots, n$, let us denote by N_j the kernel of the projection

$$M \xrightarrow{pr} \!\!\!\!\!\rightarrow \bigoplus_{i=1}^{n} \underline{A/b_i} \xrightarrow{pr_j} \!\!\!\!\!\rightarrow \bigoplus_{i=1}^{j} \underline{A/b_i}$$

and set $N_0 := M$. Then, for every $j = 0, \dots, n-1$, we have a short exact sequences

$$(\xi_j): \qquad 0 \to N_{j+1} \to N_j \to \underline{A/b_{j+1}} \to 0.$$

Note that it suffices to find for every $j = 0, \dots, n-1$, a finite and faithfully flat morphism $T_i \to S$ such that (ξ_j) is split over T_j. For, then (ξ) will be split over $T = T_0 \times_S T_1 \times \dots \times_S T_{n-1}$. We prove the following assertion: let $0 \to P \to Q \xrightarrow{pr} \underline{A/b} \to 0$ be a short exact sequence of finite flat A-module schemes over S, where $b \in A$ is an element annihilating Q and such that A/b is a finite set, then there exists a finite and faithfully flat morphism $T \to S$ such that this short exact sequence splits over T. We can write Q as a disjoint union $\coprod_{x \in A/b} Q_x$ of closed submodule schemes, with $Q_0 = P$. Set $T := Q_1$. Since Q is finite and flat over S, all Q_x and in particular T are also finite and flat over S. By assumption, S has a single point and so, $T \to S$ is surjective. This shows that T is faithfully flat and finite over S. The embedding $T \hookrightarrow Q$ induces a morphism $T \to Q_T$ over T, i.e., a section of the structural morphism $Q_T \to T$. By definition, this section has the property that its composition with the projection $Q_T \twoheadrightarrow \underline{A/b}_T = \coprod_{x \in A/b} T$ is the embedding corresponding to the element $1 \in A/b$. Now, we are in the situation of the previous lemma, and applying it, we deduce that the short exact sequence

$$0 \to P_T \to Q_T \xrightarrow{pr} \underline{A/b}_T \to 0$$

is split. This proves the above assertion. By assumption, b_{j+1} annihilates M and since N_j is a submodule scheme of M, it annihilates N_j too. We apply the above assertion to (ξ_j) and obtain the desired T_j. $\square$

Lemma 4.26. *Let $\mathcal{M}$ be a π-divisible module over a local Artin ring R. Then, for every n, there exists a finite and faithfully flat R-scheme T such that $\mathcal{M}_{n,T}$ becomes isomorphic to the direct sum $(\mathcal{M}_n^0)_T \oplus (\underline{\mathcal{O}/\pi^n})^{h^{\acute{e}t}}$, where $h^{\acute{e}t}$ is the height of $\mathcal{M}^{\acute{e}t}$.*

Proof. Fix n and set $S := \mathbf{Spec}(R)$. By Proposition 1.15, there is a connected finite étale cover $S' \to S$ such that $\mathcal{M}^{\text{ét}}_{n,S'}$ is the constant $\mathcal{O}$-module scheme $(\underline{\mathcal{O}/\pi^n})^{h^{\text{ét}}}$ over S'. Consider the short exact sequence

$$0 \to (\mathcal{M}_n^0)_{S'} \to \mathcal{M}_{n,S'} \to (\underline{\mathcal{O}/\pi^n})^{h^{\text{ét}}} \to 0$$

induced by the connected-étale sequence of $\mathcal{M}_n$ over S (cf. Remark 1.14). The $\mathcal{O}$-module $\mathcal{M}_{n,S'}$ is annihilated by π^n. Since S' is finite over S, it is the spectrum of a finite R-algebra R'. Thus, the Krull dimension of R' is equal to the Krull dimension of R, which is zero. Since R is Noetherian and R' is finite over it, R' is also Noetherian. This shows that R' is an Artin ring. Further, $S' = \mathbf{Spec}(R')$ is connected, which means that R' is local. Consequently, S' is a scheme with 1 point. We can now apply the previous lemma and find a finite faithfully flat morphism $T \to S'$ that splits the above short exact sequence. Hence $\mathcal{M}_{n,T} \cong (\mathcal{M}_n^0)_T \oplus (\underline{\mathcal{O}/\pi^n})^{h^{\text{ét}}}$. Since $T \to S'$ is finite and faithfully flat and $S' \to S$ is finite, surjective (it is a cover) and étale, the composition $T \to S' \to S$ is finite and faithfully flat. $\square$

From now on, we assume that $\mathcal{M}$ is a π-divisible module over a base scheme S (defined over $\mathbf{Spec}(\mathcal{O})$), of height h and dimension at most 1.

Lemma 4.27. *Let S be the spectrum of a local Artin $\mathcal{O}$-algebra. Then, the $\mathcal{O}$-module scheme $\widetilde{\underline{\mathrm{Alt}}}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m)$ is finite and flat over S. Moreover, its order is the constant function $q^{n\binom{h}{r}}$.*

Proof. If the dimension of the special fiber of $\mathcal{M}$ is zero, then it is an étale π-divisible module and by Proposition 1.20 that the exterior power $\bigwedge^r \mathcal{M}_n$ exists and is a finite étale $\mathcal{O}$-module scheme of order $q^{n\binom{h}{r}}$. We also know that the construction of this exterior power commutes with base change. Thus

$$\widetilde{\underline{\mathrm{Alt}}}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m) \cong \underline{\mathrm{Hom}}_S(\bigwedge^r \mathcal{M}_n, \mathbb{G}_m)$$

The latter, being the Cartier dual of $\bigwedge^r \mathcal{M}_n$, is a finite flat $\mathcal{O}$-module scheme of order $q^{n\binom{h}{r}}$. So we may assume that the dimension of the special fiber of $\mathcal{M}$ is 1 and so the closed point of S has characteristic p. If we can find a finite and faithfully flat S-scheme T such that the base change of $\underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m)$ to T is finite and flat over T, then

$$\underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m)$$

is finite and flat over S [2]. So, it is enough to find such a T. Let T be a finite faithfully flat S-scheme (provided by the last lemma) such that

$$(\star) \qquad \mathcal{M}_{n,T} \cong (\mathcal{M}_n^0)_T \oplus (\underline{\mathcal{O}/\pi^n})^{h^{\text{ét}}}$$

and to simplify the notations, set $\Gamma := (\underline{\mathcal{O}/\pi^n})^{h^{\text{ét}}}$and $Y := \underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_S((\mathcal{M}_n^0)^r, \mathbb{G}_m)$. Since $\mathcal{M}^0$ is a connected π-divisible module of dimension 1, by Corollary 4.23, Y is finite and flat over S. Thus $Y_T = \underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_S((\mathcal{M}_n^0)^r, \mathbb{G}_m)_T \cong \underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_T((\mathcal{M}_n^0)^r_T, \mathbb{G}_m)$ is finite and flat over T. By isomorphism $(\star)$ and adjunction formulas given in Proposition 1.7, we have

$$\underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_T(\mathcal{M}_{n,T}^r, \mathbb{G}_m) \cong \underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_T((\mathcal{M}_n^0)^r_T \oplus \Gamma^r, \mathbb{G}_m) \cong$$

$$\underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_T(\Gamma^r, \underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_T((\mathcal{M}_n^0)^r_T, \mathbb{G}_m)) \cong \underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_T(\Gamma^r, Y_T).$$

By Proposition 1.20, the exterior power $\bigwedge^r \Gamma$ exists, and is isomorphic to $(\underline{\mathcal{O}/\pi^n})^{\binom{h^{\text{ét}}}{r}}$. We therefore have

$$\underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_T(\Gamma^r, Y_T) \cong \underline{\mathrm{Hom}}^{\mathcal{O}}_T(\bigwedge^r \Gamma, Y_T) \cong \bigoplus \underline{\mathrm{Hom}}^{\mathcal{O}}_T(\underline{\mathcal{O}/\pi^n}, Y_T).$$

So, it suffices to show that $\underline{\mathrm{Hom}}^{\mathcal{O}}_T(\underline{\mathcal{O}/\pi^n}, Y_T)$ is finite and flat over T. We claim that this is isomorphic to Y_T. Indeed, since $\mathcal{M}_n$ is annihilated by π^n, its connected component $\mathcal{M}_n^0$ is annihilated by π^n too, and therefore Y_T is annihilated by π^n as well. It follows that for every T-scheme T', we have isomorphisms

$$\underline{\mathrm{Hom}}^{\mathcal{O}}_T(\underline{\mathcal{O}/\pi^n}, Y_T)(T') \cong \mathrm{Hom}^{\mathcal{O}}_{T'}(\underline{\mathcal{O}/\pi^n}, Y_{T'}) \cong$$

$$\mathrm{Hom}^{\mathcal{O}}(\mathcal{O}/\pi^n, Y(T')) \cong \mathrm{Hom}^{\mathcal{O}}(\mathcal{O}, Y(T')) \cong Y_T(T').$$

This proves the claim. Hence the finiteness and flatness of $\underline{\mathrm{Hom}}^{\mathcal{O}}_T(\underline{\mathcal{O}/\pi^n}, Y_T)$.

To prove the statement on the order of $\underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m)$, it suffices to assume that S is the spectrum of an algebraically closed field. The statement then follows from the fact that the exterior power $\bigwedge^r \mathcal{M}_n$ has order $q^{n\binom{h}{r}}$ (cf. Theorem 1.23) and so does its Cartier dual $\underline{\mathrm{Hom}}_S(\bigwedge^r \mathcal{M}_n, \mathbb{G}_m) \cong \underline{\widetilde{\mathrm{Alt}}}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m)$. $\square$

[2]a module over a ring is finite and flat if and only if it is so after base change to a faithfully flat ring extension

Lemma 4.28. *Let S be the spectrum of a complete local Noetherian $\mathcal{O}$-algebra. Then, the $\mathcal{O}$-module scheme $\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)$ is finite and flat over S.*

Proof. By Remark 1.8, we know that $\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)$ is an affine scheme of finite type over S, and so is separated over S. Let R be a local Artin $\mathcal{O}$-algebra and $f : \mathbf{Spec}(R) \to S$ an $\mathcal{O}$-scheme morphism. We have

$$\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)_R \cong \widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_R(\mathcal{M}^r_{R,n}, \mathbb{G}_m)$$

and by last lemma, $\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_R(\mathcal{M}^r_{R,n}, \mathbb{G}_m)$ is finite flat over R. We can now apply Lemma A.1 and conclude that $\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)$ is finite and flat over S. □

Lemma 4.29. *Let S be a locally Noetherian $\mathcal{O}$-scheme. Then, the $\mathcal{O}$-module scheme $\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)$ is finite and flat over S. Moreover, its order is the constant function $q^{n\binom{h}{r}}$.*

Proof. Set $X := \widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)$. We show at first that X is flat over S. We can assume that S is the spectrum of a local ring R. By assumption, R is a Noetherian ring and therefore, the completion $R \to \widehat{R}$ is a faithfully flat morphism. Thus, X is flat over R if and only if $X_{\widehat{R}}$ is flat over $\widehat{R}$. But

$$X_{\widehat{R}} = \widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)_{\widehat{R}} \cong \widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_{\widehat{R}}(\mathcal{M}^r_{\widehat{R},n}, \mathbb{G}_m)$$

which is finite and flat over $\widehat{R}$ by previous lemma. This shows the flatness of X over S. We now prove that X is finite over S. We can assume that S is affine, say $S = \mathbf{Spec}(R)$ and then X is affine too, say $X = \mathbf{Spec}(A)$. Let L be a field and $f : \mathbf{Spec}(L) \to S$ be a morphism. Again, we have

$$X_L \cong \widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}^r_n, \mathbb{G}_m)_L \cong \widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_L(\mathcal{M}^r_{L,n}, \mathbb{G}_m)$$

which is finite over L by Lemma 4.27. It follows that X is quasi-finite over S. If we show that X is proper over S, then it will follow that it is finite over S. We use the valuative criterion of properness. So, let D be a valuation ring and E its fraction field. Denote also by $\widehat{D}$ and respectively $\widehat{E}$ the completions of D and E. Assume that we have a commutative "solid" diagram

(4.30)
$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d, hook] \\ A \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] & E \end{tikzcd}$$

where the vertical ring homomorphisms are the obvious ones, and we want to lift g to a homomorphism $\tilde{g} : A \to D$ filling the diagram (note that X is separated over S and therefore if such a morphism exists, it is unique). We can complete this diagram to the following diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[r, hook] \arrow[d, hook] & \widehat{D} \arrow[d, hook] \\ A \arrow[r, "g"'] & E \arrow[r, hook] & \widehat{E}. \end{tikzcd}$$

If we can find a homomorphism $\tilde{g} : A \to \widehat{D}$ making the above diagram commutative, then since $E \cap \widehat{D} = D$, the image of $\tilde{g}$ will be inside D and we are done. So, we may replace D and respectively E by $\widehat{D}$ and $\widehat{E}$ and assume that D is a complete valuation ring. Then, by assumption $A \otimes_R D$ is finite over D and therefore X_D is proper over D. Consider the following diagram

$$\begin{tikzcd} R \arrow[rr] \arrow[d] & & D \arrow[dd, hook] \\ A \arrow[urr, dashed, "\tilde{g}"] \arrow[drr, "g"] \arrow[d] & & \\ A \otimes_R D \arrow[rr, "g_D"'] \arrow[uurr, dashed, "\tilde{g}_D"] & & E \end{tikzcd}$$

induced by base change. The valuative criterion of properness, applied to $A \otimes_R D$, implies that there exists a unique $\tilde{g}_D : A \otimes_R D \to D$ making the above diagram commutative. Let $\tilde{g}$ be the composition $A \to A \otimes_R D \xrightarrow{\tilde{g}_D} D$, then $\tilde{g}$ fills the diagram (4.30) and this proves that X is proper over S.

In order to calculate the order of X, we may assume that S is the spectrum of a field. The statement on the order now follows from Lemma 4.27. $\square$

Proposition 4.31. *Let S be a locally Noetherian $\mathcal{O}$-scheme. The exterior power $\bigwedge^r \mathcal{M}_n$ exists in the category of finite flat group schemes over S, and commutes with arbitrary base change. Moreover, its order is the constant function $q^{n\binom{h}{r}}$. Furthermore, for every S-scheme T, the canonical base change homomorphism $\bigwedge^r(\mathcal{M}_{n,T}) \to (\bigwedge^r \mathcal{M}_n)_T$ is an isomorphism.*

Proof. Set $\Lambda_n^r := \underline{\mathrm{Hom}}_S(\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m), \mathbb{G}_m)$. According to the previous lemma, being the Cartier dual of $\widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m)$, it is finite and flat over S. By Cartier duality, we have a canonical isomorphism

$$\alpha : \underline{\mathrm{Hom}}_S(\Lambda_n^r, \mathbb{G}_m) \xrightarrow{\cong} \widetilde{\underline{\mathrm{Alt}}}{}^{\mathcal{O}}_S(\mathcal{M}_n^r, \mathbb{G}_m)$$

and by adjunction formulas (cf. Proposition 1.7), we have an alternating morphism $\lambda_n : \mathcal{M}_n^r \to \Lambda_n^r$. More precisely, we have isomorphisms

$$\mathrm{Alt}_S^{\mathcal{O}}(\mathcal{M}_n^r, \Lambda_n^r) \cong \widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \underline{\mathrm{Hom}}_S(\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m), \mathbb{G}_m)) \cong$$

$$\mathrm{Hom}_S^{\mathcal{O}}(\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m), \widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m))$$

and λ_n corresponds, via this isomorphism, to the identity morphism. Let G be a finite flat group scheme over S. The same argument as in the proof of Proposition 3.11 shows that $\lambda_n^*(G)$ is an isomorphism. Thus, Λ_n^r is the r^{th}-exterior power of $\mathcal{M}_n$ in the category of finite flat group schemes over S, and we can write $\bigwedge^r \mathcal{M}_n \cong \Lambda_n^r$. As $\bigwedge^r \mathcal{M}_n$ is the Cartier dual of $\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m)$ and by previous lemma, $\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m)$ has order equal to $q^{n\binom{h}{r}}$, we deduce that $\bigwedge^r \mathcal{M}_n$ has order equal to the constant function $q^{n\binom{h}{r}}$. What we have proved above is that if $\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m)$ is finite and flat over S (for any base $\mathcal{O}$-scheme S), then $\bigwedge^r \mathcal{M}_n$ exists and it is isomorphic to the Cartier dual of $\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m)$. Now, let T be an S-scheme. Since by assumption, S is locally Noetherian, $\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m)$ is finite and flat over S by previous lemma and therefore, the base change $\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m)_T \cong \widetilde{\underline{\mathrm{Alt}}}{}_T^{\mathcal{O}}(\mathcal{M}_{n,T}^r, \mathbb{G}_m)$ is finite and flat over T. It follows that $\bigwedge^r(\mathcal{M}_{n,T})$ exists and since the Cartier duality commutes with base change, the two $\mathcal{O}$-module schemes $\bigwedge^r(\mathcal{M}_{n,T})$ and $(\bigwedge^r \mathcal{M}_n)_T$ are canonically isomorphic. $\square$

Proposition 4.32. *Let S be a locally Notherian $\mathcal{O}$-scheme. There exist natural monomorphisms $i_n : \bigwedge^r \mathcal{M}_n \hookrightarrow \bigwedge^r \mathcal{M}_{n+1}$, which make the inductive system $(\bigwedge^r \mathcal{M}_n)_{n \geq 1}$ a π-Barsotti-Tate group over S of height $\binom{h}{r}$ and dimension a locally constant function*

$$\dim : S \to \{0, \binom{h-1}{r-1}\}, \qquad s \mapsto \begin{cases} 0 & \textit{if} \quad \mathcal{M}_s \quad \textit{is étale} \\ \binom{h-1}{r-1} & \textit{otherwise.} \end{cases}$$

Proof. By previous proposition, $\bigwedge^r \mathcal{M}_n$ are finite flat $\mathcal{O}$-module schemes over S. Since $\mathcal{M}_n$ is annihilated by π^n, the $\mathcal{O}$-module scheme $\widetilde{\underline{\mathrm{Alt}}}{}_S^{\mathcal{O}}(\mathcal{M}_n^r, \mathbb{G}_m)$ is also annihilated by π^n. Thus, its Cartier dual, $\bigwedge^r \mathcal{M}_n$ is annihilated by π^n as well. We also know by previous proposition that $\bigwedge^r \mathcal{M}_n$ has order equal to the constant function $q^{n\binom{h}{r}}$. Now, the similar arguments as in the proof of Lemma 3.24 provide monomorphisms $i_n : \bigwedge^r \mathcal{M}_n \hookrightarrow \bigwedge^r \mathcal{M}_{n+1}$ and imply this proposition. We therefore omit the proof. $\square$

Remark 4.33. Since λ_n are the universal alternating morphisms, they are compatible with the projections $\mathcal{M}_{n+1}^r \to \mathcal{M}_n^r$ and $\bigwedge^r \mathcal{M}_{n+1} \to \bigwedge^r \mathcal{M}_n$ and so yield an alternating morphism $\lambda : \mathcal{M}^r \to \bigwedge^r \mathcal{M}$. $\lozenge$

Theorem 4.34 (The Main Theorem for π-divisible modules)**.** *Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic zero. Fix a uniformizer π of $\mathcal{O}$ and let S be a locally Notherian $\mathcal{O}$-scheme and $\mathcal{M}$ a π-divisible module over S of height h and dimension at most 1. Assume that the action of $\mathcal{O}$ on the tangent space of $\mathcal{M}$ is given by scalar multiplication. Then, there exists a π-divisible module $\bigwedge_{\mathcal{O}}^r \mathcal{M}$ over S of height $\binom{h}{r}$, and an alternating morphism $\lambda : \mathcal{M}^r \to \bigwedge_{\mathcal{O}}^r \mathcal{M}$ such that for every morphism $f : S' \to S$ and every π-divisible module $\mathcal{N}$ over S', the homomorphism*

$$\mathrm{Hom}_{S'}^{\mathcal{O}}(f^* \bigwedge_{\mathcal{O}}^r \mathcal{M}, \mathcal{N}) \to \mathrm{Alt}_{S'}^{\mathcal{O}}((f^*\mathcal{M})^r, \mathcal{N})$$

induced by $f^*\lambda$ is as isomorphism. In other words, the r^{th}-exterior power of $\mathcal{M}$ exists and commutes with arbitrary base change. Moreover, the dimension of $\bigwedge_{\mathcal{O}}^r \mathcal{M}$ is the locally constant function

$$\dim : S \to \{0, \binom{h-1}{r-1}\}, \qquad s \mapsto \begin{cases} 0 & \text{if } \ \mathcal{M}_s \ \text{ is étale} \\ \binom{h-1}{r-1} & \text{otherwise.} \end{cases}$$

Proof. By Proposition 4.31, $f^*\bigwedge^r \mathcal{M}_n$ is isomorphic to $\bigwedge^r (f^*\mathcal{M}_n)$. Therefore, $f^*\lambda_n : (f^*\mathcal{M}_n)^r \to f^*\bigwedge^r \mathcal{M}_n$ induces an isomorphism

$$\mathrm{Hom}_{S'}(f^* \bigwedge^r \mathcal{M}_n, \mathcal{N}_n) \to \mathrm{Alt}_{S'}^{\mathcal{O}}((f^*\mathcal{M}_n)^r, \mathcal{N}_n).$$

Taking the inverse limit of these isomorphisms, we conclude that $\mathrm{Hom}_{S'}$ $(f^*\bigwedge^r \mathcal{M}, \mathcal{N}) \to \mathrm{Alt}_{S'}^{\mathcal{O}}((f^*\mathcal{M})^r, \mathcal{N})$ is an isomorphism. The statement on height and dimension follows from the previous proposition. ☐

5. Examples

In this section, p is an odd prime number, f is a positive natural number and $q = p^f$.

Example 5.1. Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field of characteristic zero and residue field $\mathbb{F}_q$ and S an $\mathcal{O}$-scheme. Let F be a π-divisible module of height h and dimension 1 over a locally Noetherian scheme S. By Theorem 4.34, the exterior power $\bigwedge_{\mathcal{O}}^r F$ exists and is a π-divisible module of height $\binom{h}{r}$ and dimension $\binom{h-1}{r-1}$. Fix π-divisible modules F_0 and G_0 of height h and dimensions respectively 1 and d over $\mathbb{F}_q$. Let $k/\mathbb{F}_q$ be a perfect field. Define the *deformation functor* that assigns to any local Artin $\mathcal{O}$-algebra R whose residue field is an overfield of k, the set $Def(G_0)(R)$ of the pairs (G, α) up to isomorphisms, where G is a d dimensional formal $\mathcal{O}$-module over R and

$$\alpha : G \times_R R/\mathfrak{m}_R \xrightarrow{\cong} G_0 \times_{\mathbb{F}_q} R/\mathfrak{m}_R$$

is an isomorphism of formal $\mathcal{O}$-modules. Then the functor $Def(G_0)$ is represented by the formal scheme $\mathbf{Spf}(W_{\mathcal{O}}(k)[\![t_1,\dots,t_{d(h-d)}]\!])$, called the Lubin-Tate (moduli) space of G_0 (cf. [4] or [13]). Taking the r^{th} exterior power induces in a natural way a morphism

$$\det : \mathbf{Spf}(W_{\mathcal{O}}(k)[\![t_1,\dots,t_{(h-1)}]\!]) \to \mathbf{Spf}\left(W_{\mathcal{O}}(k)[\![T_1,\dots,T_{\binom{h-1}{r-1}\binom{h-1}{r}}]\!]\right)$$

of formal $\mathcal{O}$-schemes. Indeed, let R be as above and let (F,α) be a deformation of F_0 over R. Then $(\bigwedge_{\mathcal{O}}^h F, \bigwedge_{\mathcal{O}}^h \alpha)$ is a deformation of $\bigwedge_{\mathcal{O}}^h F_0$ over R, of dimension $\binom{h-1}{r-1}$. This is so, because the construction of exterior powers commutes with base change:

$$\begin{aligned}
(\bigwedge_{\mathcal{O}}^h F) \times_R R/\mathfrak{m}_R &\cong \bigwedge_{\mathcal{O}}^h (F \times_R R/\mathfrak{m}_R) \\
&\cong \bigwedge_{\mathcal{O}}^h (F_0 \times_{\mathbb{F}_q} R/\mathfrak{m}_R) \\
&\cong (\bigwedge_{\mathcal{O}}^h F) \times_{\mathbb{F}_q} R/\mathfrak{m}_R.
\end{aligned}$$

■

Example 5.2. Let $L/\mathbb{Q}$ be an imaginary quadratic field extension and denote by $\mathcal{O}_L$ its ring of integers and assume that p splits in $\mathcal{O}_L$, say $p\mathcal{O}_L = \mathfrak{q}\cdot\mathfrak{p}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are different primes of $\mathcal{O}_L$. Let S be an $\mathcal{O}_L$-scheme defined over $\mathbf{Spf}(\mathbb{Z}_p)$ and $\mathcal{A}$ an Abelian scheme over S of relative dimension g. Assume that $\mathcal{O}_L$ acts on $\mathcal{A}$, i.e., we have a ring homomorphism $\mathcal{O}_L \to \mathrm{End}_S(\mathcal{A})$ and the induced action on the relative Lie algebra of $\mathcal{A}$ has signature $(g-1,1)$. More precisely, the decomposition $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{O}_S \cong \mathcal{O}_S \times \mathcal{O}_S$, according to the splitting of p, induces a decomposition of the relative Lie algebra of $\mathcal{A}$ into a product of two locally free $\mathcal{O}_S$-modules and the component corresponding to $\mathfrak{p}$ has rank 1 and the other component, corresponding to $\mathfrak{q}$, has rank $g-1$. The p-divisible group, $\mathcal{A}[p^\infty]$, associated with $\mathcal{A}$ has a natural structure of $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module. We can decompose $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ into a product $\mathcal{O}_{L,\mathfrak{q}} \times \mathcal{O}_{L,\mathfrak{p}}$, where $\mathcal{O}_{L,\mathfrak{q}}$ and $\mathcal{O}_{L,\mathfrak{p}}$ are respectively the completions of $\mathcal{O}_L$ with respect to $\mathfrak{q}$ and $\mathfrak{p}$. Since p is split in $\mathcal{O}_L$, they are both isomorphic to $\mathbb{Z}_p$. So, we have $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$. This decomposition induces a decomposition $\mathcal{A}[p^\infty] \cong \mathcal{A}[\mathfrak{q}^\infty] \times \mathcal{A}[\mathfrak{p}^\infty]$ of $\mathcal{A}[p^\infty]$ into p-divisible groups of height g over S. By assumption on the action of $\mathcal{O}_L$ on $\mathcal{A}$, $\mathcal{A}[\mathfrak{p}^\infty]$ has dimension 1 and $\mathcal{A}[\mathfrak{q}^\infty]$ has dimension $g-1$. So, for every $r > 0$, the exterior power $\bigwedge^r \mathcal{A}[\mathfrak{p}^\infty]$ exists and its height and dimension are respectively $\binom{g}{r}$ and $\binom{g-1}{r-1}$. Note that there exists a natural number $m \le n$ such that $\mathcal{A}[\mathfrak{p}^\infty]$ has slopes zero and $\frac{1}{n-m}$ with multiplicities respectively m and $n-m$, and $\mathcal{A}[\mathfrak{q}^\infty]$ has slopes zero and $\frac{n-1}{n-m}$ with multiplicities respectively m and

$n-m$. Note that, by construction, and for all i, we also obtain the exterior powers $\mathcal{A}[\mathfrak{p}^i]$. ■

Example 5.3. Let $\mathcal{E}$ be an elliptic scheme (i.e., Abelian scheme of relative dimension 1) over a base scheme S. Then, the associated p-divisible group $\mathcal{E}[p^\infty]$ has rank 2 and dimension 1 at points of characteristic p. Thus, the second exterior power $\bigwedge^2\mathcal{E}[p^\infty]$ is a p-divisible group of height 1 and dimension 1 at points of characteristic p. This means that at these points, $\bigwedge^2\mathcal{E}[p^\infty]$ is a multiplicative group of height and dimension 1 and so is isomorphic to μ_{p^∞}. At points of characteristic zero, $\bigwedge^2\mathcal{E}[p^\infty]$ is an étale p-divisible group of height 1 and if we pass to an algebraic closure, we obtain the constant p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$, which is again isomorphic to μ_{p^∞}. So, at all geometric points, we have $\bigwedge^2\mathcal{E}[p^\infty] \cong \mu_{p^\infty}$.

The Weil pairing $\omega_n : \mathcal{E}[p^n] \times \mathcal{E}[p^n] \to \mu_{p^n}$ is perfect and induces an alternating morphism $\omega : \mathcal{E}[p^\infty] \times \mathcal{E}[p^\infty] \to \mu_{p^\infty}$. We want to show that ω is in fact the universal alternating morphism and $\bigwedge^2\mathcal{E}[p^\infty]$ is canonically isomorphic to μ_{p^∞}. Indeed, by the universal property of $\bigwedge^2\mathcal{E}[p^\infty]$, we have a homomorphism $\tilde{\omega} : \bigwedge^2\mathcal{E}[p^\infty] \to \mu_{p^\infty}$ such that $\omega = \tilde{\omega} \circ \lambda$ (where λ is the universal alternating morphism of $\bigwedge^2\mathcal{E}[p^\infty]$) and we have to show that $\tilde{\omega}$ is an isomorphism. It is an isomorphism, if and only if it is so over every geometric point, and therefore, we may assume that S is the spectrum of an algebraically closed field. Then, as explained above, $\bigwedge^2\mathcal{E}[p^\infty]$ is isomorphic to μ_{p^∞}. So we have a homomorphism $\tilde{\omega} : \mu_{p^\infty} \to \mu_{p^\infty}$ and we want to show that it is an isomorphism. By Cartier duality, we can consider the dual of this homomorphism, namely $\tilde{\omega}^* : \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p$. If this is not an isomorphism, it factors through multiplication by p. Thus, $\tilde{\omega}$ factors through multiplication by p. It follows that on p-torsion points, $\omega_1 = \tilde{\omega}_1 \circ \lambda_1$ is the zero morphism, which is in contradiction with the fact that the Weil pairing is perfect. ■

Appendix A. Algebraic Geometry Results

In this section we prove some auxiliary results from algebraic geometry that are used in the paper.

Lemma A.1. *Let $X = \mathbf{Spec}(A)$ with A a complete local Noetherian ring and let $f : Y \to X$ be a separated morphism with the following property: for every local Artin ring R and every morphism $\mathbf{Spec}(R) \to X$, the base change of f to R, $f_R : Y_R \to \mathbf{Spec}(R)$, is a finite and flat morphism. Then f is a finite and flat morphism.*

Proof. The hypothesis on f implies that it is a quasi-finite morphism. Since A is a local Henselian ring, by Theorem 4.2, p. 32 of [18], f is a finite morphism. Thus, in particular, f is affine and we can write $Y = \mathbf{Spec}(B)$

with B a finite A-algebra. Denote by $\mathfrak{m}$ be the maximal ideal of A, by A_n the local Artin ring $A/\mathfrak{m}^{n+1}$ and by B_n, the finite A_n-algebra $B \otimes_A A_n$. Then, by assumption, for every n, B_n is a finite flat A_n-algebra. By the local flatness criterion (cf. Theorem 22.3 of [16]) B is then flat over A. □

Lemma A.2. *Let $\varphi : X \to Y$ be a surjective morphism of schemes over a base scheme S. If X is finite over S and Y is separated and of finite type over S, then Y is finite over S.*

Proof. Let $f : X \to S$ and $Y \to S$ be the structural morphisms. As φ is surjective, the fibers of f surject onto the fibers of g and since f is finite, the fibers of g are finite sets and therefore, g is a quasi-finite morphism. If we show that g is proper, then it will be finite. Since g is already by assumption separated, we only need to show the universal closedness. Note that since f is finite, it is proper as well. As the properties of being proper and surjective are preserved under base change, in order to show that g is a universally closed morphism, it is sufficient to show that it is a closed map of topological spaces. Let $F \subseteq Y$ be closed. Since φ is surjective, we have $F = \varphi(\varphi^{-1}(F))$ and thus $g(F) = f(\varphi^{-1}(F))$, which is closed in S, because f is proper and therefore a closed map and $\varphi^{-1}(F)$ is a closed subset of X. □

Definition A.3. Let X be a scheme over a scheme S and let $s \in S(L)$ be an L-valued point, with L a field. Denote by X_s the base extension of X via $s : \mathbf{Spec}(L) \to S$. By the *order* of X_s over s, we mean $\dim_L \Gamma(X_s, \mathcal{O}_{X_s})$. In particular, if $s \in S$, the order of the fiber X_s is $\dim_{\kappa(s)} \Gamma(X_s, \mathcal{O}_{X_s})$. ▲

Lemma A.4. *Let X, Y be affine schemes over a local ring R. Assume that X is finite and flat and Y is of finite type over $S := \mathbf{Spec}(R)$, that the fibers of X and Y have the same order over every point of S, and that we have an S-morphism $\varphi : X \to Y$ which is an isomorphism on the special fiber. Then φ is an isomorphism.*

Proof. We show at first that φ is a closed embedding. Set $A := \Gamma(X, \mathcal{O}_X)$ and $B := \Gamma(Y, \mathcal{O}_Y)$. By assumption, we have $X = \mathbf{Spec}(A)$ and $Y = \mathbf{Spec}(B)$, with A a flat and finite R-algebra (i.e., finite as R-module). The morphism $\varphi : X \to Y$ corresponds to a ring homomorphism $f : B \to A$. We want to show that f is surjective. Write C for the cokernel of f and denote by k the residue field of R. Tensoring the exact sequence of R-modules

$$B \xrightarrow{f} A \to C \to 0$$

with k over R, we obtain the exact sequence

$$B \otimes_R k \xrightarrow{f \otimes_R \mathrm{Id}_k} A \otimes_R k \to C \otimes_R k \to 0.$$

By hypothesis, $B \otimes_R k \xrightarrow{f \otimes_R \mathrm{Id}_k} A \otimes_R k$ is an isomorphism, and therefore, $C \otimes_R k$ is the zero k-vector space. As A is a finitely generated R-module and C is a quotient, we can apply the Nakayama's lemma to C and deduce that $C = 0$. This shows that f is surjective. Write K for the kernel of f, i.e., we have a short exact sequence $0 \to K \to B \xrightarrow{f} A \to 0$. As A is flat and finitely generated and R is local, it is free. This implies that the above short exact sequence is split (as R-modules) and we can write $B \cong K \oplus A$, and so $B \otimes_R k \cong (K \otimes_R k) \oplus (A \otimes_R k)$. Again, since by assumption f is an isomorphism after tensoring with k over R, we have $K \otimes_R k = 0$. Assume for the moment that B is a finitely generated R-module. Then, K being a quotient of B, is also finitely generated and we can apply once again Nakayama's lemma and conclude that $K = 0$, which achieves the proof of the proposition. So, we have to show that B is a finitely generated R-module or equivalently, that Y is a finite S-scheme. Fix a point $s \in S$. As $\varphi : X \to Y$ is a closed embedding, the induced morphism $\varphi_s : X_s \to Y_s$ is a closed embedding as well. By assumption, X_s is finite over s and the fibers X_s and Y_s have the same order over s, which should be then finite. This shows that the embedding φ_s is in fact an isomorphism (a surjective map of vector spaces of the same finite dimension is an isomorphism). Consequently, the morphism φ is surjective as a map between topological spaces. We can now apply Lemma A.2 and conclude that Y is a finite scheme over S. □

Proposition A.5. *Let S be a base scheme and $\varphi : X \to Y$ a morphism over S with X finite and flat and Y of finite type and separated over S. Assume that for every geometric point s of S, X_s and Y_s have the same order over s and that φ is an isomorphism over every closed point of S. Then φ is an isomorphism.*

Proof. Denote by $f : X \to S$, respectively $g : Y \to S$ the structural morphisms of X, respectively of Y. Assume that we have shown the proposition for S, X and Y affine. Let $S = \bigcup_{\alpha\in\Lambda} S_\alpha$, and $Y = \bigcup_{\alpha\in\Lambda} Y_\alpha$ be open affine coverings, such that $g(Y_\alpha) \subseteq S_\alpha$. Set $X_\alpha := \varphi^{-1}(Y_\alpha)$, and therefore we have also $f(X_\alpha) \subseteq S_\alpha$. Since by hypothesis, f is finite and thus affine and g is separated, by [10] I.6.2 (v), we know that φ is an affine morphism and therefore $\bigcup_{\alpha\in\Lambda} X_\alpha$ is an open affine covering of X. Denote by $\varphi_\alpha : X_\alpha \to Y_\alpha$ the restriction of φ. We know that for all $\alpha \in \Lambda$, the morphism φ_α is an isomorphism, and so, it follows that φ is as isomorphism. So, it is enough to show the statement in the affine case. Set $A := \Gamma(X, \mathcal{O}_X)$, $B := \Gamma(Y, \mathcal{O}_Y)$ and $R := \Gamma(S, \mathcal{O}_S)$ and denote by $h : B \to A$ the ring homomorphism corresponding to $\varphi : X \to Y$. We want to show that for every maximal ideal $\mathfrak{m}$ of R, the localization $h_\mathfrak{m} : B_\mathfrak{m} \to A_\mathfrak{m}$ of h is an isomorphism. It follows then that h is an isomorphism. We can therefore assume further that R is a local ring. Let s be a point of S, and $\bar{\kappa}$ an algebraic closure of $\kappa(s)$. Since by assumption the $\bar{\kappa}$-vector spaces $B \otimes_R \kappa(s) \otimes_{\kappa(s)} \bar{\kappa}$ and $A \otimes_R \kappa(s) \otimes_{\kappa(s)} \bar{\kappa}$ have the same finite dimension, the $\kappa(s)$-vector spaces $B \otimes_R \kappa(s)$ and $A \otimes_R \kappa(s)$

have the same dimension too. This shows that the fibers of X and Y have the same order. We also know by assumption that φ is an isomorphism over the special fiber. We can now apply the previous lemma, and conclude that φ is an isomorphism. □

Remark A.6. Let X, Y, S and φ be as in the last proposition with φ an isomorphism at every geometric point of S, then the conditions of the proposition are satisfied and we can draw the same conclusion. ◊

Proposition A.7. *Let $\psi : G \to H$ be a homomorphism of affine group schemes of finite type over $\mathbf{Spec}(k)$, where k is a field. Assume that for every finite group scheme I over k, the induced homomorphism of groups $\psi_*(I) : \mathrm{Hom}(I, G) \to \mathrm{Hom}(I, H)$ is an isomorphism and also the induced homomorphism on $\bar{k}$-valued points, $\psi(\overline{k}) : G(\overline{k}) \to H(\overline{k})$, is an isomorphism. Then ψ is an isomorphism.*

Proof. This is Lemma 3.37 of [12]. □

Definition A.8. Let $(A, \mathfrak{m})$ be a complete local Noetherian ring and denote by $\mathfrak{X}$ and $\mathfrak{X}_n$ the formal scheme $\mathbf{Spf}(A)$ respectively the affine scheme $\mathbf{Spec}(A/\mathfrak{m}^n)$. We also set $X := \mathbf{Spec}(A)$.

(i) A *truncated Barsotti-Tate group of level* i over $\mathfrak{X}$ is a system $\mathfrak{G} = (G(n))_{n\geq 1}$ of truncated Barsotti-Tate groups of level i over $\mathfrak{X}_n$ endowed with isomorphisms $G(n+1)|_{\mathfrak{X}_n} \cong G(n)$, where $G(n+1)|_{\mathfrak{X}_n}$ is the base change of $G(n+1)$ to $\mathfrak{X}_n$. A *homomorphism* $\varphi : \mathfrak{G} \to \mathfrak{H}$ between two truncated Barsotti-Tate groups of level i over $\mathfrak{X}$ is a system $(\varphi(n))_{n\geq 1}$ of homomorphisms $\varphi(n) : G_n \to H_n$ over $\mathfrak{X}_n$, such that for all n, $\varphi(n+1)|_{\mathfrak{X}_n} = \varphi(n)$. Denote by $\mathfrak{BT}_i\,/\mathfrak{X}$ (resp. by $\mathfrak{BT}_i\,/X$) the category of truncated Barsotti-Tate groups of level i over $\mathfrak{X}$ (resp. over X). Multilinear and alternating morphisms of truncated Barsotti-Tate groups of level i over $\mathfrak{X}$ are defined similarly.

(ii) *p-Divisible groups* over $\mathfrak{X}$ and their homomorphisms, multilinear and alternating morphisms are defined similarly. Denote by p-$\mathfrak{Div}\,/\mathfrak{X}$ (resp. by p-$\mathfrak{Div}\,/X$) the category of p-divisible groups over $\mathfrak{X}$ (resp. over X). ▲

Let G be an object of p-$\mathfrak{Div}\,/X$ (respectively of $\mathfrak{BT}_i\,/X$) and denote by $G(n)$ the pullback of G to $\mathfrak{X}_n$. We have canonical isomorphisms $G(n+1)|_{\mathfrak{X}_n} \cong G(n)$ and therefore, the system $(G(n))_{n\geq 1}$ defines a p-divisible group (respectively a truncated Barsotti-Tate groups of level i) over $\mathfrak{X}$ that we denote by $\mathfrak{F}(G)$.

For the proof of the following proposition, we refer to [17], Ch. II, lemma 4.16, p. 75 or [6], lemma 2.4.4, p. 17.

Proposition A.9. *The functors $\mathfrak{F} : \mathfrak{BT}_i/X \longrightarrow \mathfrak{BT}_i/\mathfrak{X}$ and $\mathfrak{F} : p\text{-}\mathfrak{Div}/X \longrightarrow p\text{-}\mathfrak{Div}/\mathfrak{X}$ are equivalences of categories.*

Remark A.10. The same arguments as in the (omitted) proof of the above proposition show that under these functors, the group of multilinear and alternating morphisms of p-divisible groups (resp. truncated Barsotti-Tate groups of level i) over X is isomorphic to the group of multilinear and respectively alternating morphisms of p-divisible groups (resp. truncated Barsotti-Tate groups of level i) over $\mathfrak{X}$. $\Diamond$

Appendix B. Displays and Ramified Displays

In the first subsection, we recall the main definitions and results of the theory of displays (see [24] for details), make some definitions and notations and prove a result that will be used in this paper. In the second subsection, we define ramified displays and state their main properties, for proof of which, we refer to [1]. Unless otherwise specified, R is a ring, and F^R and V^R are respectively the Frobenius and Verschiebung of the ring of Witt vectors $W(R)$. We denote by I_R the image of the Verschiebung. We sometimes denote the Frobenius and Verschiebung without the superscript R, when no confusion is likely.

B.1. Displays.

Definition B.1. A $3n$-display over R is a quadruple $\mathcal{P} = (P, Q, F, V^{-1})$, where P is a finitely generated $W(R)$-module, $Q \subseteq P$ is a submodule and $F : P \to P$ and $V^{-1} : Q \to P$ are F^R-linear morphisms, subject to the following axioms:

(i) $I_R P \subseteq Q \subseteq P$ and there is a decomposition of P into the direct sum of $W(R)$-modules $P = L \oplus T$, called *a normal decomposition*, such that $Q = L \oplus I_R T$.

(ii) $V^{-1} : Q \to P$ is an F^R-linear epimorphism (i.e., the $W(R)$-linearization $(V^{-1})^\sharp : W(R) \otimes_{F^R, W(R)} Q \to P$ is surjective).

(iii) For any $x \in P$ and $w \in W(R)$ we have

$$V^{-1}(V^R(w)x) = wF(x).$$

▲

Remark B.2. Note that from the last axiom, it follows that F is uniquely determined by V^{-1}. Indeed, we have for every $x \in P$:

$$F(x) = V^{-1}(V^R(1)x).$$

It follows also from this relation and F^R-linearity of V^{-1}, that for every $y \in Q$, we have

$$F(y) = V^{-1}(V^R(1)y) = F^R V^R(1) V^{-1}(y) = pV^{-1}(y).$$

$\Diamond$

Construction B.3. According to Lemma 10, p.14 of [24], there exists a unique $W(R)$-linear map $V^\sharp : P \to W(R) \otimes_{F,W(R)} P$, satisfying the following equations:

$$V^\sharp(wF(x)) = pw \otimes x, \quad w \in W(R), x \in P$$

and

$$V^\sharp(wV^{-1}(y)) = w \otimes y, \quad w \in W(R), y \in Q.$$

If we denote by $F^\sharp : W(R) \otimes_{F,W(R)} P \to P$ the $W(R)$-linearization of $F : P \to P$, we have the properties:

$$F^\sharp \circ V^\sharp = p.\,\mathrm{Id}_P \quad \text{and} \quad V^\sharp \circ F^\sharp = p.\,\mathrm{Id}_{W(R)\otimes_{F,W(R)}P}\,. \tag{2.4}$$

Denote by $V^{n\sharp}$ the composition

$$P \xrightarrow{V^\sharp} W(R)\otimes_F P \xrightarrow{\mathrm{Id}\otimes_F V^\sharp} W(R)\otimes_{F^2} P \to \dots \xrightarrow{\mathrm{Id}\otimes_{F^n} V^\sharp} W(R)\otimes_{F^n} P.$$

$\blacktriangledown$

Construction B.5. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$-display over a ring R and let $\varphi : R \to S$ be a ring homomorphism. We are going to construct a $3n$-display, which will be the $3n$-display obtained from $\mathcal{P}$ by base change with respect to $\varphi : R \to S$. Set $\mathcal{P}_S := (P_S, Q_S, F_S, V_S^{-1})$, where:

- P_S is $W(S) \otimes_{W(R)} P$,
- Q_S is the kernel of the morphism $W(S) \otimes_{W(R)} P \to S \otimes_R P/Q$,
- $F_S : P_S \to P_S$ is the morphism $F^S \otimes F$ and
- $V_S^{-1} : Q_S \to P_S$ is the unique $W(S)$-linear homomorphism, which satisfies the following properties:

$$V_S^{-1}(w \otimes y) = F^S(w) \otimes V^{-1}(y), \quad w \in W(S), y \in Q$$

and

$$V_S^{-1}(V^S(w) \otimes x) = x \otimes F(x), \quad w \in W(S), x \in P.$$

If $P = L \oplus T$ is a normal decomposition of P, then one can show that $P_S = L_S \oplus T_S$ is a normal decomposition of P_S, where $L_S := W(S) \otimes_{W(R)} L$ and $T_S = W(S) \otimes_{W(R)} T$ and that we have $Q_S = L_S \oplus I_S \otimes_{W(R)} T$ (note that $I_S T_S = I_S \otimes_{W(R)} T$). For the details of this construction, in particular to see why this construction produces a $3n$-display, refer to Definition 20 and the discussions following it, p.20 of [24]. $\blacktriangledown$

Definition B.6. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$-display over R. Assume that p is nilpotent in R. Then $\mathcal{P}$ is called *display* or *nilpotent* if it satisfies the *nilpotence or V-nilpotence condition*, i.e., if there exists a natural number N such that the morphism

$$V^{N\sharp} : P \to W(R) \otimes_{F^N,W(R)} P$$

is zero modulo $I_R + pW(R)$. ▲

Definition B.7. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$-display over R.

(i) The *tangent module* of $\mathcal{P}$, denoted by $\mathcal{T}(\mathcal{P})$, is the R-module P/Q.

(ii) The *rank* of $\mathcal{P}$, is the rank of $\mathcal{T}(\mathcal{P})$ over R and the *height* of $\mathcal{P}$, is the rank of P over $W(R)$. ▲

Remark B.8. 1) Using the normal decomposition $P = L \oplus T$ and $Q = L \oplus I_R T$, we observe that $\mathcal{T}(\mathcal{P})$ is isomorphic to $T/I_R T$, which is a projective R-module and therefore the rank of $\mathcal{P}$ is equal to the rank of T over $W(R)$. The height of $\mathcal{P}$ is equal to the sum of ranks of L and T over $W(R)$.

2) If R is a perfect field of characteristic $p > 0$ and $\mathcal{P}$ is the display attached to a connected p-divisible group G, then $\mathcal{T}(\mathcal{P})$, which is an R-vector space, is canonically isomorphic to the tangent space of G. The rank and height of $\mathcal{P}$ are respectively equal to the dimension and height of G. ♢

Example B.9. (1) Let k be a perfect field of characteristic p and let M be a Dieudonné module over k. This means that M is a finite free $W(k)$-module, equipped with semi-linear operators F and V satisfying $FV = p = VF$. Then, the quadruple $\mathcal{P}_M := (M, VM, F, V^{-1})$ is a $3n$-display over k (note that $V : M \to M$ is injective, and so $V^{-1} : VM \to M$ is well-defined). Such a $3n$-display is called a *Dieudonné $3n$-display*. In other words, a Dieudonné $3n$-display, is the $3n$-display attached to a p-divisible group over k. Note that $\mathcal{P}_M$ is nilpotent, if and only if $V : M/pM \to M/pM$ is nilpotent.

(2) The *multiplicative display* $\mathcal{G}_m = (P, Q, F, V^{-1})$ over a ring R is defined as follows: We set $P := W(R)$, $Q = I_R$ and define the maps $F : P \to P$ and $V^{-1} : Q \to P$ by: $\forall w \in W(R)$,

$$Fw = F^R w, \quad V^{-1}(V^R w) = w.$$

If p is nilpotent in R, then $\mathcal{G}_m$ is a nilpotent display. ■

For the next construction, we fix a $3n$-display $\mathcal{P} = (P, Q, F, V^{-1})$ over R, where p is nilpotent. Let $\mathcal{N}$ be a nilpotent R-algebra. This means that $\mathcal{N}$ is an associative commutative R-algebra, where every element is nilpotent (in particular, $\mathcal{N}$ is not a unital ring). Set $S := R \oplus \mathcal{N}$. This has a natural structure of an associative commutative unital R-algebra. The following construction is a recapitulation of some of the constructions and results in subsection 3 of [24].

Construction B.10. Let $\widehat{W}(\mathcal{N})$ be the subset of $W(\mathcal{N})$ consisting of Witt vectors with finitely many non-zero elements. Denote by $\widehat{I}_{\mathcal{N}}$ the $W(R)$-submodule $V\widehat{W}(\mathcal{N})$ of $\widehat{W}(\mathcal{N})$ (consisting of Witt vectors, whose first component is zero). Set $\widehat{P}_{\mathcal{N}} := \widehat{W}(\mathcal{N}) \otimes_{W(R)} P$ and $\widehat{Q}_{\mathcal{N}} := \widehat{P}_{\mathcal{N}} \cap Q_S$, where Q_S

is the base change of Q (as in $3n$-display), i.e., $Q_S = \mathrm{Ker}(W(S) \otimes_{W(R)} P \to S \otimes_R P/Q)$. We extend the maps $F : P \to P$ and $V^{-1} : Q \to P$ to maps $F : \widehat{P}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}$ and $V^{-1} : \widehat{Q}_{\mathcal{N}} \to \widehat{P}_{\mathcal{N}}$ as follows. We set $F := F^{\mathcal{N}} \otimes F$, where $F^{\mathcal{N}} : \widehat{W}(\mathcal{N}) \to \widehat{W}(\mathcal{N})$ is the Frobenius. We let V^{-1} act on $\widehat{W}(\mathcal{N}) \otimes_{W(R)} L$ as $F \otimes V^{-1}$ and on $\widehat{I}_{\mathcal{N}} \otimes_{W(R)} T$ as $V^{-1} \otimes F$ (since the action of V on Witt vectors is injective, the map $V^{-1} : \widehat{I}_{\mathcal{N}} \to \widehat{W}(\mathcal{N})$ is well-defined). If we want to look at $\widehat{P}_{\mathcal{N}}$ and $\widehat{Q}_{\mathcal{N}}$ as functors on Nil_R, we denote $\widehat{P}_{\mathcal{N}}$ by $G^0_{\mathcal{P}}(\mathcal{N})$ and $\widehat{Q}_{\mathcal{N}}$ by $G^{-1}_{\mathcal{P}}(\mathcal{N})$. Denote by $BT_{\mathcal{P}}(\mathcal{N})$ the cokernel of $V^{-1} - \mathrm{Id} : G^{-1}_{\mathcal{P}}(\mathcal{N}) \to G^0_{\mathcal{P}}(\mathcal{N})$. ▼

Here is a summary of the results that we will need in the paper (cf. [24], pp. 22, 77, 81 and 94).

Proposition B.11. *Let $\mathcal{P}$ be a $3n$-display over R, then:*

- *The sequence*
$$0 \longrightarrow G^{-1}_{\mathcal{P}}(\mathcal{N}) \xrightarrow{V^{-1}-\mathrm{Id}} G^0_{\mathcal{P}}(\mathcal{N}) \longrightarrow BT_{\mathcal{P}}(\mathcal{N}) \longrightarrow 0$$
is exact.
- *$BT_{\mathcal{P}}$ is a finite dimensional formal group and the construction $\mathcal{P} \rightsquigarrow BT_{\mathcal{P}}$ commutes with base change, i.e., if $R \to S$ is a ring homomorphism, then there exists an canonical isomorphism $(BT_{\mathcal{P}})_S \cong BT_{\mathcal{P}_S}$.*
- *If p is nilpotent in R and $\mathcal{P}$ is nilpotent, then $BT_{\mathcal{P}}$ is an infinitesimal p-divisible group.*
- *If $pR = 0$, and $\mathcal{P}$ is nilpotent, then the Frobenius and respectively Verschiebung of the p-divisible group $BT_{\mathcal{P}}$ are $BT_{\mathcal{P}}(Fr_{\mathcal{P}})$ and respectively $BT_{\mathcal{P}}(Ver_{\mathcal{P}})$, where $Fr_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}^{(p)}$ and $Ver_{\mathcal{P}} : \mathcal{P}^{(p)} \to \mathcal{P}$ are the Frobenius and Verschiebung of $\mathcal{P}$ (with $\mathcal{P}^{(p)}$ denoting the base change of $\mathcal{P}$ with respect to the morphism $Frob_p : R \to R, r \mapsto r^p$).*
- *If R is an excellent local ring, then the functor $\mathcal{P} \mapsto BT_{\mathcal{P}}$ gives an equivalence of categories between the category of (nilpotent) displays over R and infinitesimal p-divisible groups over R.*

Notations B.12. For a nilpotent R-algebra $\mathcal{N}$, we denote by $[b]$ the class of an element $b \in G^0_{\mathcal{P}}(\mathcal{N})$ modulo $(V^{-1} - \mathrm{Id})G^{-1}_{\mathcal{P}}(\mathcal{N})$. If $[b]$ is annihilated by p^n, we write $[b]_n$ to emphasize the fact that this element belongs to the kernel of p^n. In this case, $p^n b$ belongs to the subgroup $(V^{-1} - \mathrm{Id})G^{-1}_{\mathcal{P}}(\mathcal{N})$ of $G^0_{\mathcal{P}}(\mathcal{N})$, and therefore, since $V^{-1} - \mathrm{Id} : G^{-1}_{\mathcal{P}}(\mathcal{N}) \to G^0_{\mathcal{P}}(\mathcal{N})$ is injective, there exists a unique element ${}_n g_{\mathcal{P}}(b) \in G^{-1}_{\mathcal{P}}(\mathcal{N})$ with $(V^{-1} - \mathrm{Id})({}_n g_{\mathcal{P}}(b)) = p^n b$.

Remark B.13. It follows from the construction of $BT_{\mathcal{P}}$ that for any nilpotent R-algebra $\mathcal{N}$, any $w \in \widehat{W}(\mathcal{N})$ and any $x \in P$, we have $[Fw \otimes x] =$

$[w \otimes Vx]$ and $[Vw \otimes x] = [w \otimes Fx]$. Indeed, by Construction B.10, we know that

$$(V^{-1} - \mathrm{Id})(w \otimes Vx) = Fw \otimes x - w \otimes Vx$$

and that

$$(V^{-1} - \mathrm{Id})(Vw \otimes x) = w \otimes Fx - Vw \otimes x.$$

$\diamondsuit$

Example B.14. In the case of the multiplicative display

$$\mathcal{G}_m = (P, Q, F, V^{-1})$$

over a ring R, we have $G^{-1}_{\mathcal{G}_m}(\mathcal{N}) = \widehat{I}_{\mathcal{N}}$ and $G^{0}_{\mathcal{G}_m}(\mathcal{N}) = \widehat{W}(\mathcal{N})$. Note that the image of the morphism $V^{-1} - \mathrm{Id} : \widehat{I}_{\mathcal{N}} \to \widehat{W}(\mathcal{N})$ is equal to the image of the morphism $V - \mathrm{Id} : \widehat{W}(\mathcal{N}) \to \widehat{W}(\mathcal{N})$, and therefore, $BT_{\mathcal{G}_m}(\mathcal{N})$ is canonically isomorphic to the cokernel of the morphism $V - \mathrm{Id} : \widehat{W}(\mathcal{N}) \to \widehat{W}(\mathcal{N})$. The Artin-Hasse exponential $E(\_, 1)$ gives rise to a short exact sequence (see [7], Proposition 117 of [24]):

$$0 \to \widehat{W}(\mathcal{N}) \xrightarrow{V - \mathrm{Id}} \widehat{W}(\mathcal{N}) \xrightarrow{E(\_,1)} \widehat{\mathbb{G}}_m(\mathcal{N}) \to 0$$

where $\widehat{\mathbb{G}}_m$ is the multiplicative formal group. Hence a canonical isomorphism $BT_{\mathcal{G}_m} \cong \widehat{\mathbb{G}}_m$. ■

The following proposition is proposition 90, p. 84 of [24]:

Proposition B.15. *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-Display over a ring R. There is a canonical surjection*

$$\mathbb{E}_R \otimes_{W(R)} P \twoheadrightarrow M(BT_{\mathcal{P}}), \quad e \otimes x \mapsto (u \mapsto [ue \otimes x])$$

where $\mathbb{E}_R$ is the Cartier ring, the ring opposite to the ring $\mathrm{End}(\widehat{W})$. *The kernel of this morphism is the $\mathbb{E}_R$-submodule generated by the elements $F \otimes x - 1 \otimes Fx$, for $x \in P$, and $V \otimes V^{-1}y - 1 \otimes y$, for $y \in Q$.*

Proposition B.16. *Let k be a perfect field of characteristic p and $\mathcal{P}$ a Dieudonné display over k. We have an $\mathbb{E}_k$-linear isomorphism:*

$$\mu : P \longrightarrow M(BT_{\mathcal{P}}), \qquad x \mapsto (\xi \mapsto [\xi \otimes x]).$$

Proof. Denote by I the kernel of the morphism in the last proposition ($R = k$). Then the morphism $(\mathbb{E}_k \otimes_{W(k)} P)/I \to M(BT_{\mathcal{P}})$ sending the class of $e \otimes x$ modulo I to the morphism $(u \mapsto [ue \otimes x])$, is an isomorphism. Now, we have a canonical morphism $\varphi : P \to (\mathbb{E}_k \otimes_{W(k)} P)/I$ sending an element x to $[1 \otimes x]$. The composition of this morphism with the above isomorphism is the morphism given in the statement of the proposition. So, it suffices to show that φ is an isomorphism. Since by assumption $\mathcal{P}$ is the $3n$-display of a p-divisible group, it has a Verschiebung and so we can define a map $\psi : (\mathbb{E}_k \otimes_{W(k)} P)/I \to P$ by sending $F^i \otimes x$ to $F^i x$ and $V^j \otimes y$ to $V^j y$ with $i, j \geq 0$ and $x, y \in P$. We claim that this is a well-defined $\mathbb{E}_k$-linear homomorphism and is the inverse to the morphism φ. It follows from the definition that ψ is an $\mathbb{E}_k$-linear homomorphism. Again, by definition, elements of the form $F \otimes x - 1 \otimes Fx$ or $V \otimes V^{-1}y - 1 \otimes y$ map to zero and since they generate the ideal I, we see that ψ is well-defined. It is clear that the composition $\psi \circ \varphi$ is the identity of P. As in the quotient, elements $F^i \otimes x$ and $1 \otimes F^i x$, respectively $V^j \otimes y$ and $1 \otimes V^j y$, are identified, it follows that $\varphi \circ \psi$ is the identity of $(\mathbb{E}_k \otimes_{W(k)} P)/I$. $\square$

B.2. Ramified displays. Fix an $\mathcal{O}$-algebra R.

Definition B.17. The set of *ramified Witt vectors*, denoted by $W_{\mathcal{O}}(R)$, is the set

$$W_{\mathcal{O}}(R) := \{(x_0, x_1, \dots) \mid x_i \in R\} = R^{\mathbb{N}}.$$

The map

$$\mathrm{w}_n : W_{\mathcal{O}}(R) \to R, \qquad \underline{x} := (x_0, x_1, \dots) \mapsto x_0^{q^n} + \pi x_1^{q^{n-1}} + \cdots + \pi^n x_n$$

is called the n^{th} Witt polynomial. ▲

Remark B.18. The association $R \mapsto W_{\mathcal{O}}(R)$ is functorial on the category of $\mathcal{O}$-algebras. ♢

Theorem B.19. *There exists a unique $\mathcal{O}$-algebra structure on $W_{\mathcal{O}}(R)$ with following properties:*

a) *The Witt polynomials $\mathrm{w}_n : W_{\mathcal{O}}(R) \to R$ are $\mathcal{O}$-algebra homomorphisms.*
b) *If $R \to S$ is an $\mathcal{O}$-algebra morphism, the induced map $W_{\mathcal{O}}(R) \to W_{\mathcal{O}}(S)$ is an $\mathcal{O}$-algebra morphism.*

Remark B.20. It follows from the theorem that $W_{\mathcal{O}}$ is a functor from the category of $\mathcal{O}$-algebras to itself. Also, if we denote by $\mathrm{Id}_{\mathcal{O}\text{-alg}}$ the identity functor on the category of $\mathcal{O}$-algebras, the Witt polynomials define natural transformations of functors $\mathrm{w}_n : W_{\mathcal{O}} \to \mathrm{Id}_{\mathcal{O}\text{-alg}}$. ♢

Proposition B.21. *There are $\mathcal{O}$-linear endomorphisms F_π and V_π on $W_{\mathcal{O}}(R)$, with the properties:*

1) *for every $\underline{x} \in W_{\mathcal{O}}(R)$, we have $\mathrm{w}_0(V_\pi \underline{x}) = 0, \mathrm{w}_n(V_\pi \underline{x}) = \pi \mathrm{w}_{n-1}(\underline{x})$ and $\mathrm{w}_n(F_\pi \underline{x}) = \mathrm{w}_{n+1}(\underline{x})$.*
2) *F_π is an $\mathcal{O}$-algebra homomorphism.*
3) *$F_\pi V_\pi = V_\pi F_\pi = \pi$ and for every $\underline{x}, \underline{y} \in W_{\mathcal{O}}(R)$, we have*

$$V_\pi(F_\pi(\underline{x})\underline{y}) = \underline{x} V_\pi(\underline{y}).$$

Proposition B.22. *There exists a unique natural transformation of functors $\mu : W \to W_{\mathcal{O}}$ such that $\mathrm{w}_n \circ \mu = \mathrm{w}_{fn}$ for all n. For all $a \in R$ and all $w \in W(R)$, we have:*

- $\mu([a]) = [a]$
- $\mu(F^f w) = F_\pi(\mu(w))$
- $\mu(Vw) = (\frac{p}{\pi})V_\pi\mu(F^{f-1}w)$.

Remark B.23. 1) The first property determines uniquely F_π and V_π and the other properties follow from the first one. It is easy to see that $V_\pi(x_0, \dots) = (0, x_0, \dots)$, and so $I_R := \mathrm{Im}(V_\pi) = \mathrm{Ker}(\mathrm{w}_0)$.

2) If $\mathcal{O}$ is the ring of p-adic integers, then we obtain the usual ring of Witt vectors. ♢

Proposition B.24. *Let k be a perfect field of characteristic p.*

a) *$W_{\mathcal{O}}(k)$ is a complete discrete valuation ring with residue field k and maximal ideal generated by π.*

b) *If k contains $\mathbb{F}_q$, then there exists a canonical $\mathcal{O}$-algebra isomorphism*

$$W(k)\widehat{\otimes}_{\mathbb{Z}_q}\mathcal{O} \cong W_{\mathcal{O}}(k).$$

Proof. The first statement is a standard one, stated e.g. in [9]. For the second statement, note that $W_{\mathcal{O}}(k)$ is an $\mathcal{O}$-algebra and contains also $W(k)$ as subring. There exists therefore a canonical $\mathcal{O}$-algebra homomorphism $W(k) \otimes_{\mathbb{Z}_q} \mathcal{O} \to W_{\mathcal{O}}(k)$. Denote by L the fraction field of $W(k)$. Since $\mathbb{F}_q \subseteq k$, we have $\mathbb{Z}_q \subseteq W(k)$. The extension $K/\mathbb{Q}_q$ is totally ramified and $L/\mathbb{Q}_q$ is unramified. Therefore, $L \otimes_{\mathbb{Q}_q} K$ is a field with ring of integers $W(k) \otimes_{\mathbb{Z}_q} \mathcal{O}$, uniformizer π and residue field k. It follows that the completed tensor product $W(k)\widehat{\otimes}_{\mathbb{Z}_q}\mathcal{O}$ is a π-adically complete and therefore, the homomorphism $W(k) \otimes_{\mathbb{Z}_q} \mathcal{O} \to W_{\mathcal{O}}(k)$ extends to an $\mathcal{O}$-algebra homomorphism $W(k)\widehat{\otimes}_{\mathbb{Z}_q}\mathcal{O} \to W_{\mathcal{O}}(k)$. As both sides are complete discrete valuation rings with the same residue field and uniformizer, this homomorphism is an isomorphism. □

Now we define the ramified displays.

Definition B.25. A ramified $3n$-display over R is a quadruple

$$\mathcal{P} = (P, Q, F, V^{-1}),$$

where P is a finitely generated $W_{\mathcal{O}}(R)$-module, $Q \subseteq P$ is a submodule and F, V^{-1} are F_π-linear morphisms $F : P \to P$ and $V^{-1} : Q \to P$, subject to the following axioms:

(i) $I_R P \subseteq Q \subseteq P$ and there is a decomposition of P into the direct sum of $W(R)$-modules $P = L \oplus T$, called *a normal decomposition*, such that $Q = L \oplus I_R T$.

(ii) $V^{-1} : Q \to P$ is an F_π-linear epimorphism (i.e., the $W_{\mathcal{O}}(R)$-linearization $(V^{-1})^\sharp : W_{\mathcal{O}}(R) \otimes_{F_\pi, W_{\mathcal{O}}(R)} Q \to P$ is surjective).

(iii) For any $x \in P$ and $w \in W_{\mathcal{O}}(R)$ we have $V^{-1}(V_\pi(w)x) = wF(x)$. ▲

Remark B.26. 1) Note that from the last axiom, it follows that F is uniquely determined by V^{-1}. Indeed, for every $x \in P$, we have $F(x) = V^{-1}(V_\pi(1)x)$. It follows also from this relation and F_π-linearity of V^{-1}, that for every $y \in Q$, we have

$$F(y) = V^{-1}(V_\pi(1)y) = F_\pi V_\pi(1)V^{-1}(y) = \pi V^{-1}(y).$$

2) Since $W_\mathcal{O}(R)$ is an $\mathcal{O}$-algebra, P and Q have a natural $\mathcal{O}$-module structure and the morphisms F and V^{-1} are $\mathcal{O}$-linear (note that F_π is $\mathcal{O}$-linear). ♢

Definition B.27. Let k be a perfect field of characteristic p, which is an $\mathcal{O}$-algebra. A ramified Dieudonné module over k is a finite free $W_\mathcal{O}(k)$-module M endowed with an F_π-linear morphism $F : M \to M$ and an F_π^{-1}-linear morphism $V : M \to M$, such that $FV = \pi = VF$. ▲

Similar constructions, remarks and propositions, as in section 2, hold for ramified $3n$-displays. Because of these similarities, we will only list them, without giving details. Fix a ramified $3n$-display $\mathcal{P} = (P, Q, F, V^{-1})$ over R .

(1) The tangent module, rank and height of $\mathcal{P}$ are defined analogously (cf. Definitions B.7.)
(2) Nilpotent ramified displays are defined similarly, when π is nilpotent in R.
(3) We have the base change of a $3n$-display, with respect to a ring homomorphism $R \to S$.
(4) Assume that $pR = 0$. Denote by $\mathcal{P}^{(q)}$ the base change of $\mathcal{P}$ with respect to the ring homomorphism $\mathrm{Frob}^f : R \to R$, sending r to r^q. Frobenius $Fr_\mathcal{P} : \mathcal{P} \to \mathcal{P}^{(q)}$ and Verschiebung $Ver_\mathcal{P} : \mathcal{P}^{(q)} \to \mathcal{P}$ are defined similarly. Note that we have $Fr_\mathcal{P} \circ Ver_\mathcal{P} = \pi.\,\mathrm{Id}_{\mathcal{P}^{(p)}}$ and $Ver_\mathcal{P} \circ Fr_\mathcal{P} = \pi.\,\mathrm{Id}_\mathcal{P}$.
(5) There is an equivalence of categories between the category of ramified $3n$-displays and the category of ramified Dieudonné modules. Under this equivalence, nilpotent displays correspond to ramified Dieudonné modules on which V is topologically (in the π-adic topology) nilpotent.
(6) Let $\mathcal{N}$ be a nilpotent R-algebra. We construct $\widehat{P}(\mathcal{N}), \widehat{Q}(\mathcal{N}), G^0_\mathcal{P}(\mathcal{N})$ and $G^{-1}_\mathcal{P}(\mathcal{N})$ as in Construction B.10. Also, we define

$$V^{-1} - \mathrm{Id} : G^{-1}_\mathcal{P}(\mathcal{N}) \to G^0_\mathcal{P}(\mathcal{N})$$

and set $BT_\mathcal{P}(\mathcal{N})$ to be its cokernel.
(7) For every nilpotent R-algebra $\mathcal{N}$, we have an exact sequence

$$0 \longrightarrow G^{-1}_\mathcal{P}(\mathcal{N}) \xrightarrow{V^{-1}-\mathrm{Id}} G^0_\mathcal{P}(\mathcal{N}) \longrightarrow BT_\mathcal{P}(\mathcal{N}) \longrightarrow 0.$$

(8) The functor $BT_{\mathcal{P}}$ from the category of nilpotent R-algebras to the category of $\mathcal{O}$-modules is a finite dimensional formal $\mathcal{O}$-module. The construction $\mathcal{P} \rightsquigarrow BT_{\mathcal{P}}$ commutes with base change.

(9) If π is nilpotent in R and $\mathcal{P}$ is nilpotent, then $BT_{\mathcal{P}}$ is an infinitesimal π-divisible module.

(10) If $pR = 0$, then the Frobenius and Verschiebung morphisms of the π-divisible module $BT_{\mathcal{P}}$ are $BT_{\mathcal{P}}(Fr_{\mathcal{P}})$ and respectively $BT_{\mathcal{P}}(Ver_{\mathcal{P}})$.

(11) If R is a Noetherian ring and π is nilpotent, then the functor BT, from the category of (nilpotent) displays over R to the category of infinitesimal π-divisible modules is an equivalence of categories.

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