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THE EXTERIOR THREE-PARTICLE WAVE FUNCTION*

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ABSTRACT

For three particles interacting via forces of finite range, it is shown that the wave function interior to the finite volume where all three force ranges overlap completely determines the exterior wave function, provided only the wave functions (half off-shell t matrices) of the isolated two-particle subsystems are also known inside their own range of force. The determination is provided by the solution of a one-variable integral equation with a compact kernel, whose resolvent applied to any parametrization of the interior wave function supplies the equivalent of a phase shift analysis for three-particle final states (exact description of overlapping resonances throughout the Dalitz plot), and (if the interior three-particle forces are also known) a matrix equation for the interior wave function.

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For (central, spin-independent) forces of finite range the exterior wave function for two particles can be uniquely specified by a single phase shift in each angular momentum state, which parameters also specify the scattering cross section. So far, the corresponding description for three particle final states has not been constructed. At first sight, the Faddeev[1] decomposition into the three channels which asymptotically contain an interacting pair plus a free particle would seem to solve the problem. However, so long as the interacting pair are within the range of the two-particle force, their energy and momentum are not necessarily connected as they will be in the three free particle final state, and they can pick up momentum from the outgoing wave in one of the other two channels (cf. Figure 1). This produces a non-local[2] interaction which falls off only with the inverse distance to the third particle, even if the two particle forces are of finite range. This in turn produces a singularity in the Faddeev equations which must be removed (e.g. by contour rotation or iteration) before they can be solved. In addition, any strongly interacting system can be expected to have three-body forces where all three particle ranges overlap, and these must be specified, in addition to the two particle forces, before the physical prediction of three-particle cross sections can be achieved. Thus the Faddeev equations provide a dynamical description, but do not provide a means of separating the exterior from the interior wave function. We show below that by reformulating the problem in in configuration space, it <u>is</u> possible to make such a separation. Because of the long-range effect described above, the formalism <u>necessarily</u> requires a complete description of the wave functions of the isolated two-particle systems inside the range of the two-particle forces (half off-shell two-particle t matrix), but still can be made whether or not there are three-particle forces inside the finite volume where the force ranges of all three pairs overlap. Hence, to the extent that one believes a covariant description of the half off-shell two-particle t matrices (e. g. via the Blankenbecler-Sugar equation), the description could also be extended to the relativistic case. If the forces are known in the interior region, the method also provides non-singular dynamical equations for the three-body problem.

The wave function $\Psi^M(\underline{r}_1, \underline{r}_2, \underline{r}_3)$ for three particles of masses m_1, m_2, m_3 can be re-expressed in terms of the new coordinates

$$
\begin{aligned}
\underline{R} &= (m_1\underline{r}_1 + m_2\underline{r}_2 + m_3\underline{r}_3)/(m_1 + m_2 + m_3) \\
\underline{x}_i &= [2m_j m_k/(m_j + m_k)]^{\frac{1}{2}} (\underline{r}_j - \underline{r}_k) \\
\underline{y}_i &= [2m_i(m_j + m_k)/(m_1 + m_2 + m_3)]^{\frac{1}{2}} \\
&\quad [-\underline{r}_i + (m_j\underline{r}_j + m_k\underline{r}_k)/(m_j + m_k)]
\end{aligned}
\qquad (1)
$$

i,j,k cyclic on 1,2,3, and decomposed into Faddeev channels and radial and angular parts according to

$$
\Psi_J^M = e^{i\underline{P}\cdot\underline{R}} \sum_{s=1}^{3} \sum_{\ell\lambda} \frac{U_{\ell\lambda}^{Js}(x_s, y_s)}{x_s y_s} Y_{J\ell\lambda}^M (\theta_{\underline{x}_s}, \phi_{\underline{x}_s}, \theta_{y_s}, \phi_{y_s}) \qquad (2)
$$

where $\underline{P}$ is the total momentum and the $Y_{J\ell\lambda}^M$ are the two-direction spherical harmonics as defined by Blatt and Weisskopf[3]. If the interactions are due to central, spin-independent potentials $V^i(|\underline{r}_j - \underline{r}_k|)$ and $W^i(x_i) = V_i([(m_j + m_k)/2m_j m_k]^{\frac{1}{2}} x_i)$, the radial wave functions $U_{\ell\lambda}^{Ji}(x, y)$ are uniquely[4] specified for each value of J (which index we now drop) by the dynamical equations

$$\left[\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} - \frac{\ell(\ell+1)}{x_i^2} - \frac{\lambda(\lambda+1)}{y_i^2} + z - W^i(x_i)\right] U^i_{\ell\lambda}(x_i, y_i)$$

$$= W^i(x_i) \sum_{s=j,k} \int_{|\phi_i-\mu_{is}|}^{\min(\phi_i+\mu_{is},\, \pi-\phi_i-\mu_{is})} d\phi_s \sum_{\ell'\lambda'} K^{is}_{\ell\lambda\ell'\lambda'}(\phi_i, \phi_s)\, U^s_{\ell'\lambda'}(r\cos\phi_s, r\sin\phi_s)$$

$$\equiv W^i(x_i)\, S^i_{\ell\lambda}(x_i, y_i) \qquad (3)$$

$$r = (x_i^2 + y_i^2)^{\frac{1}{2}} \qquad \phi_i = \tan^{-1}\frac{y_i}{x_i} \qquad \cos\mu_{is} = \left[\frac{2m_i m_s}{(m_i + m_s)(m_s + m_s')}\right]^{\frac{1}{2}}$$

$$s = j,\ s' = k \ \text{ or } \ s = k,\ s' = j$$

$$z = E_{LAB} - P^2/2(m_1 + m_2 + m_3)$$

The geometrical factor coupling in the two other channels is, explicitly

$$K^{is}_{\ell\lambda\ell'\lambda'}(\phi_i, \phi_s) = \frac{8\pi^2}{(2J+1)^{\frac{1}{2}}\sin 2\mu_{is}} \sum_{MM'} Y^{M*}_{J\ell\lambda}(\xi, 0, \zeta_i, 0)\, Y^{M'}_{J\ell'\lambda'}(\xi + \zeta_{is}, 0, \xi + \zeta_{is} + \zeta_s, 0)$$

$$\cos\zeta_i = (\cos 2\mu_{is}\cos 2\phi_i - \cos 2\phi_s)/\sin 2\mu_{is}\sin 2\phi_i$$

$$\cos\zeta_{is} = (\sin\mu_{is}\sin\phi_i\cos\zeta_i - \cos\mu_{is}\cos\phi_i)/\cos\phi_s \qquad (4)$$

$$\cos\zeta_s = (\cos 2\mu_{is}\sin 2\phi_i\cos\zeta_i + \sin 2\mu_{is}\cos 2\phi_i)/\sin 2\phi_s$$

and is independent of the angle ξ which $\underline{x}_i$ makes with some arbitrary axis fixed in the plane of the triangle. For the states of zero total and relative angular momentum, K is just $1/\sin 2\mu_{is}$ and for three particles of the same mass, μ_{is} is $\pi/3$.

Since the two-particle forces W^i are assumed known, we can construct the Green's function for the left hand side of Eq. (3) in terms of product wave functions in x and y, $u_p^{\ell i}(x)\, f_\lambda(qy)$, which vanish at x = 0 or y = 0 and which we choose to normalize asymptotically to $u_p^{\ell i}(x) \rightarrow \sin(px + \delta_p^{\ell i} - \ell\pi/2)$, $f_\lambda(qy) \rightarrow \sin(qy - \lambda\pi/2)$ (i.e. $f_\lambda(qy) = qy\, j_\lambda(qy)$). They therefore have the orthonormality properties

$$\frac{2}{\pi} \sum\!\!\!\!\!\!\int dp\, u_p^{\ell i}(x)\, u_p^{\ell i}(x') = \delta(x - x'); \quad \frac{2}{\pi} \int_0^\infty dq\, f_\lambda(qy)\, f_\lambda(qy') = \delta(y - y') \tag{5}$$

where the symbol $\sum\!\!\!\!\!\!\int$ is written to remind us that if W^i is strong enough to support bound states at $p = i\gamma$, these discrete terms must also be included. Since, when this Green's function is applied to the source term on the right, we encounter $u_p^{\ell i}(x')W^i(x')$, we can eliminate explicit reference to the potential in favor of the half off-shell t matrix by the relation

$$\begin{aligned} u_p^{\ell i}(x) W^i(x) &= -(2p\, e^{-i\delta_p}/\pi) \int_0^\infty dk\, k f_\ell(kx)\, t_\ell^i(k, p; p^2) \\ t_\ell^i(p, p; p^2) &= e^{i\delta_p^{\ell i}} \sin \delta_p^{\ell i}/p \equiv \tau(p) \end{aligned} \tag{6}$$

If the right hand side of Eq. (3) were bounded in both x and y, the Green's function solution

$$U_{\ell\lambda}^i(x, y) = X_{\ell\lambda}^i(x, y) + \frac{4}{\pi^2} \sum\!\!\!\!\!\!\int dp \int_0^\infty dq \int_0^\infty dx' \int_0^\infty dy'\, \frac{u_p^{\ell i}(x)\, f_\lambda(qy) u_p^{\ell i}(x')\, f_\lambda(qy')}{z + i\epsilon - p^2 - q^2}\, W^i(x') S_{\ell\lambda}^i(x'y') \tag{7}$$

would have the exterior representation

$$U^{i\,ext}_{\ell x}(x,y) = X^{i}_{\ell\lambda}(x,y) + \sum\!\!\!\!\!\!\int_0^\infty dq\, e_\ell(\sqrt{z-q^2}\,x)\, f_\lambda(qy)\, F^{i}_{\ell\lambda}(q)$$

$$e_\ell(\sqrt{z-q^2}\,x) = i\sqrt{z-q^2}\,x\, h^{(1)}_\ell(\sqrt{z-q^2}\,x) \rightarrow e^{i(\sqrt{z-q^2}\,x - \ell\pi/2)} \tag{8}$$

with

$$F^{i}_{\ell\lambda}(q) = \int_0^\infty dk\, k\, t(k, \sqrt{z-q^2}; z-q^2) \iint_{\text{interior}} dx\, dy\, f_\ell(kx)\, f_\lambda(qy)\, S^{i}_{\ell\lambda}(x,y) \tag{9}$$

Clearly, the $F^{i}_{\ell\lambda}(q)$ are simply proportional to the Faddeev T^{i} in the $J\ell\lambda$ representation, and include elastic scattering and rearrangement collisions via the discrete terms in the sum. Unfortunately, the source term, although bounded in x if $W^{i}(x)$ has a finite range, falls off only like $1/y$ for reasons discussed in the first paragraph.

The key to a separation of Eq. (3) into exterior and interior parts is to note that the limits of integration on ϕ_s plus the assumption that W^{i} vanishes for $x > R$ for all i is sufficient to limit the region for $x_s < R$ in which $U^{s}_{\ell\lambda}(x_s, y_s)$ need be known to compute $S(x,y)$ to the <u>finite</u> domain

$$0 < x_s < R; \quad 0 < y_s < (R + x_s \cos\mu_{is})/\sin\mu_{is} \tag{10}$$

while in the contribution coming from $x_s > R$ (which lies in the strip bounded by $y_s = (R \pm x_s \cos\mu_{is})/\sin\mu_{is}$), can be computed from the one-variable representation given in Eq. (8). Hence, if we assume the wave function known in this interior region, for example, in terms of some complete set $A_n(x,y)$ over this finite domain, i.e.

$$U^i_{\ell\lambda}(x,y) = X^i_{\ell\lambda}(x,y) + \theta(R-x)\sum_n a_n A_n(x,y) \\ + \theta(x-R) \sum\!\!\!\!\!\!\int dq\, e_\ell(\sqrt{z-q^2}\,x)\, f_\lambda(qy)\, F^i_{\ell\lambda}(q) \tag{11}$$

The $F^i_{\ell\lambda}(q)$ can be determined by solving the one-variable integral equation

$$F^i_{\ell\lambda}(q) = \tilde{X}^i_{\ell\lambda}(q) + \tilde{X}^{iB}_{\ell\lambda}(q) + \sum_n a_n X^{in}_{\ell\lambda}(q) + \sum\!\!\!\!\!\!\int dp \sum_{s\ell'\lambda'} Q^{is}_{\ell\lambda\ell\lambda'}(q,p)\, F^s_{\ell'\lambda'}(p) \tag{12}$$

If the resolvent for $Q(q,p)$ exists, application to the inhomogeneous term in Eq. (12) immediately gives T^i in terms of known functions with coefficients a_n. These known functions are completely determined by the two-body half off-shell t matrices and the complete set $A_n(x,y)$, so the a_n are the analog of phase shifts for the three-particle system. Further, if this exterior representation is used in Eq. (11) and reinserted in Eq. (3), we obtain an equation for $U^i_{\ell\lambda}(x,y)$ over the finite domain (Eq. (10)); hence, using the orthonormality of the $A_n(x,y)$, this gives immediately a matrix equation for the a_n. (If there are three-body forces, these must be explicitly introduced into Eq. (3) at this point.) It remains only to show that $Q(q,p)$ falls off more rapidly than const/$(qp)^{\frac{1}{2}}$ for large q and p. Explicitly

$$Q^{is}_{\ell\lambda\ell'\lambda'}(q,p) = \int_0^\infty dk\, k\, t^i_\ell(k, \sqrt{z-q^2};\ z-q^2)\, G^{is}_{\ell\lambda\ell'\lambda'}(k,q,p) \tag{13}$$

where G is a purely geometrical factor given by

$$G^{is}_{\ell\lambda\,\ell'\lambda'}(k,q,p) = \int_0^\infty r\,dr \int_0^{\pi/2} d\phi_i \int_{|\mu_{is}-\phi_i|}^{\min(\mu_{is}+\phi_{is},\,\pi-\mu_{is}-\phi_i)} d\phi_s\, K^{is}_{\ell\lambda\ell'\lambda'}(\phi_i,\phi_s)$$

$$\theta(r\sec\phi_s - R)\, f_\ell(kr\cos\phi_i)\, f_\lambda(qr\sin\phi_i)\, e_{\ell'}(\sqrt{z-p^2}\, r\cos\phi_s) \qquad (14)$$

$$f_{\lambda'}(pr\sin\phi_s)$$

The factor K is a product of spherical harmonics of angles in the physical range, so can only improve the convergence of the integrals and can be safely ignored. If we perform the k integration over the off-shell extension of t first, the fact that this comes from the difference between the wave function and its asymptotic form[5] ensures that it will be bounded by a factor proportional to $1/(k^2+\beta^2)$ with $\beta \sim 1/R$, and the integral of this times $kf_\ell(kx)$ will be bounded by something proportional to $e^{-\beta x}$, which is no surprise if we look at the left hand side of Eq. (6). Hence (ignoring bounded factors), the q dependence will be determined by $t((z-q^2)^{\frac{1}{2}}, (z-q^2)^{\frac{1}{2}}; z-q^2)$, which falls off at least a rapidly as const/iq for large q; (it is also easy to see from other ways of writing Q(q, p) that there is no difficulty at $q^2 = z$). Similarly, for large p, the exponential term becomes $\exp(-(p^2-z)^{\frac{1}{2}}\, r\cos\phi_s)$, and an asymptotic behavior at least as rapidly decreasing as const/p is also guaranteed once we note that $r\cos\phi_s$ is bounded from below by R. Hence $Q^{is}_{\ell\lambda\ell'\lambda'}(p,q) \leq C/qp$, which guarantees the existence of a resolvent kernel for Eq. (12).

It is also important to note that if we make the decomposition[5] $t(k,\sqrt{z-q^2};\, z-q^2) = \tau(\sqrt{z-q^2})\, f_{\sqrt{z-q^2}}(k)$ in Eqs. (13) and (14), we can investigate in which kinematic regions the geometrical factor will make the result sensitive to the off-shell extension f, and where the resolvent kernels will depend primarity on the on-shell dependence $\tau(\sqrt{z-q^2})$. This will determine where the optimum regions lie for determining the on-shell factor for unstable systems (e.g. π-π phase shifts in $\pi N \rightarrow \pi\pi N$ final states), and where to investigate off-shell behavior for systems in which the on-shell behavior is known (e.g. the three-nucleon system). As developed above, the analysis is non-relativistic, but since the final equation depends only on the half off-shell t matrices, the external analysis can be used in relativistic systems to the extent that one has confidence in covariant definitions of the off-shell extension of two-particle t matrices (e.g. via the Blankenbecler-Sugar equation). Hence, it can be immediately applied to problems of overlapping resonances in the Dalitz plot and the determination of elementary particle resonance parameters; note that there is no double counting, and all relative phases are explicitly given. This application, and the inclusion of spin, do not affect the compactness proof given above, but are obviously too complicated to be developed in a short article.

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FIGURE CAPTION

Figure 1 - Scattering in the three channel (relative coordinate x_3) produces an outgoing wave which can scatter from the particles in the one channel (relative coordinate x_1) so long as they are within the range of forces, R. The effect falls off as $1/y_1$ regardless of the range of forces, and hence is non-local.

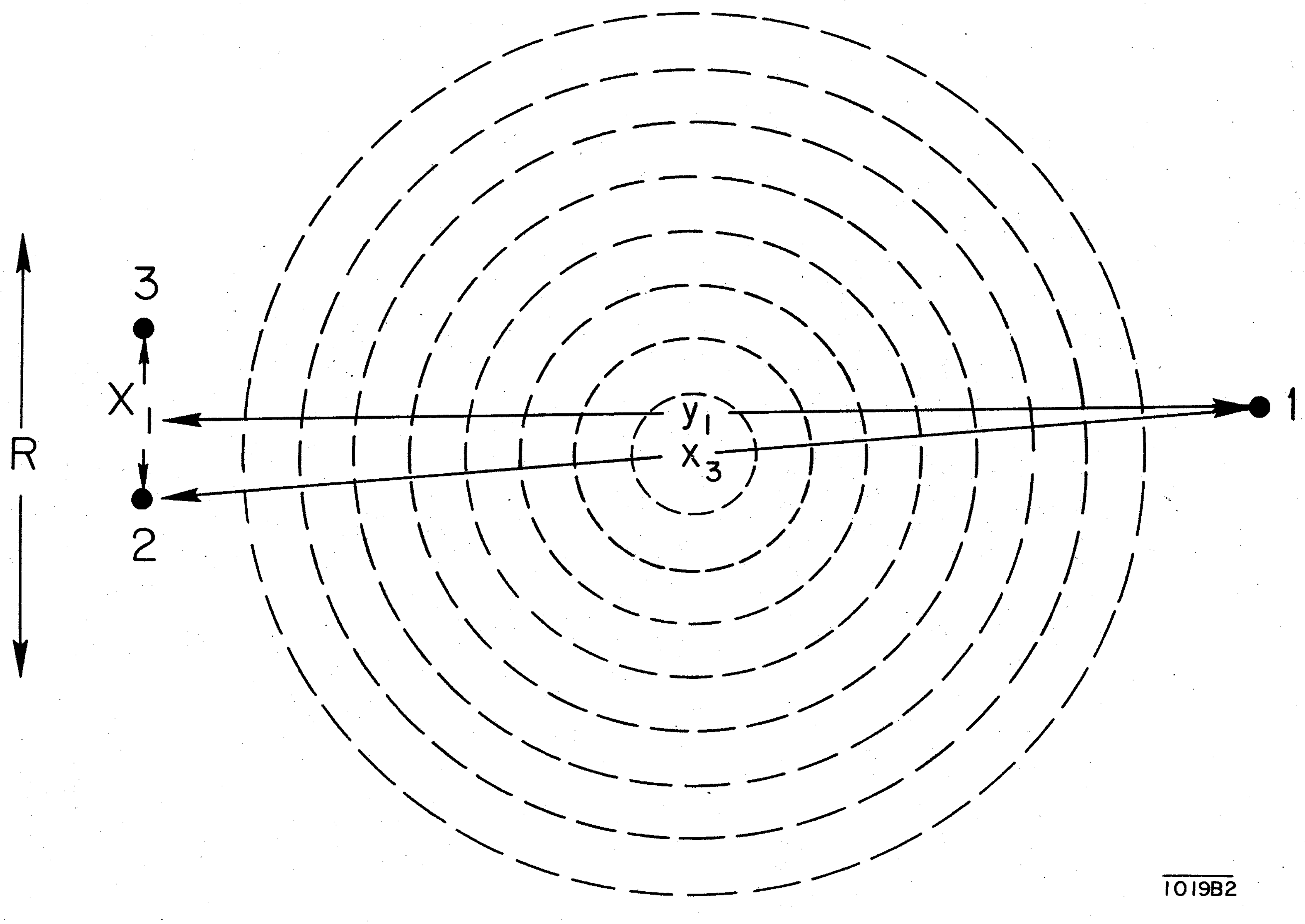

Fig. 1