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Extracting quantum coherence via steering

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As the precious resource for quantum information processing, quantum coherence can be created remotely if the involved two sites are quantum correlated. It can be expected that the amount of coherence created should depend on the quantity of the shared quantum correlation, which is also a resource. Here, we establish an operational connection between coherence induced by steering and the quantum correlation. We find that the steering-induced coherence quantified by such as relative entropy of coherence and trace-norm of coherence is bounded from above by a known quantum correlation measure defined as the one-side measurement-induced disturbance. The condition that the upper bound saturated by the induced coherence varies for different measures of coherence. The tripartite scenario is also studied and similar conclusion can be obtained. Our results provide the operational connections between local and non-local resources in quantum information processing.

Quantum coherence, being at the heart of quantum mechanics, plays a key role in quantum information processing such as quantum algorithms¹ and quantum key distribution². Inspired by the recently proposed resource theory of quantum coherence^{3,4}, researches are focused on the quantification^{5,6} and evolution^{7,8} of quantum coherence, as well as its operational meaning^{5,9} and role in quantum information tasks^{10–12}. When multipartite systems are considered, coherence is closely related to the well-established quantum information resources, such as entanglement¹³ and discord-type quantum correlations¹⁴. It is shown that the coherence of an open system is frozen under the identical dynamical condition where discord-type quantum correlation is shown to freeze¹⁵. Further, discord-type quantum correlation can be interpreted as the minimum coherence of a multipartite system on tensor-product basis¹⁶. An operational connection between local coherence and non-local quantum resources (including entanglement¹⁷ and discord¹⁸) is presented. It is shown that entanglement or discord between a coherent system and an incoherent ancilla can be built by using incoherent operations, and the generated entanglement or discord is bounded from above by the initial coherence. The converse procedure is of equal importance: to extract coherence locally from a spatially separated but quantum correlated bipartite state. The extraction of coherence with the assistance of a remote party has been studied in the asymptotical limit¹⁹. In this paper, we ask how we extract coherence locally from a *single copy* of a bipartite state.

The quantum steering has long been noted as a distinct nonlocal quantum effect²⁰ and has attracted recent research interest both theoretically and experimentally^{21–31}. It demonstrates that Alice can remotely change Bob's state by her local selective measurement if they are correlated, and is hence a natural candidate to accomplish the task of remote coherence extraction.

In this paper, we present the study of coherence extraction induced by quantum steering and the involved quantum correlation. Precisely, we introduce the quantity of steering-induced coherence (SIC) for bipartite quantum states. Here Bob is initially in an incoherent state but quantum correlated to Alice. Alice's local projective measurement can thus steer Bob to a new state which might be coherent. The SIC \bar{C} is then defined as the maximal average coherent of Bob's steered states that can be created by Alice's selective projective measurement. When there is no obvious incoherent basis for Bob, (for example, Bob's system is a polarized photon), the definition can be generalized to arbitrary bipartite system where Bob's incoherent basis is chosen as the eigenbasis of his reduced state. In this case, the SIC can be considered as a basis-free measure of Bob's coherence. The main result of this paper is building an operational connection between the SIC and the shared quantum correlation between Alice and Bob. We prove that the SIC can not surpass the initially shared *B*-side quantum correlation, which is a known quantum correlation measure named as measurement-induced disturbance (MID) Q_B ³². States whose relative entropy SIC \bar{C} can reach its upper bound Q_B are identified as maximally correlated states. For two-qubit states,

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while the trace-norm SIC \overline{C}^t can always reach the corresponding Q_B^t , we find an example of two-qubit state whose \overline{C}^t is strictly less than Q_B^t . This indicates that the condition for \overline{C}^t to reach the upper bound strongly depends on the measure of coherence. We further generalize the results to a tripartite scenario, where Alice can induce entanglement between Bob and Charlie in a controlled way. Since coherence of a single party is generally robust than quantum correlations involving two parties, our work provides a way to “store” quantum correlation as coherence. Besides, the coherent state induced by steering can be widely used for quantum information processing. Our results establish the intrinsic connection between coherence and quantum correlation by steering.

Results

Coherence and measurement-induced disturbance. A state is said to be incoherent on the reference basis $\Xi = \{|\xi_i\rangle\}$, if it can be written as³

$$\sigma_{\Xi} = \sum_i p_i |\xi_i\rangle\langle\xi_i|. \quad (1)$$

Let \mathcal{I}_{Ξ} be the set of incoherent state on basis Ξ . The incoherent completely positive trace-preserving (ICPTP) channel is defined as

$$\Lambda_{\text{ICPTP}(\cdot)} = \sum_n K_n(\cdot)K_n^{\dagger}, \quad (2)$$

where the Kraus operators K_n satisfy $K_n\mathcal{I}_{\Xi}K_n^{\dagger} \subset \mathcal{I}_{\Xi}$. According to ref. 3, a proper coherence measure $C(\rho, \Xi)$ of a quantum state ρ on a fixed reference basis Ξ should satisfy the following three conditions. (C1) $C(\rho, \Xi) = 0$ iff $\rho \in \mathcal{I}_{\Xi}$. (C2) Monotonicity under selective measurements on average: $C(\rho, \Xi) \geq \sum_n p_n C(\rho_n, \Xi)$, $\forall \{K_n\}$ satisfying $K_n\mathcal{I}_{\Xi}K_n^{\dagger} \subset \mathcal{I}_{\Xi}$ and $\sum_n K_n^{\dagger}K_n = I$, where $\rho_n = K_n\rho K_n^{\dagger}/p_n$, occurring with probability $p_n = \text{tr}[K_n\rho K_n^{\dagger}]$, is the state corresponding to outcome n . (C3) Convexity: $\sum_n p_n C(\rho_n, \Xi) \geq C(\sum_n p_n \rho_n, \Xi)$.

A candidate of coherence measure is the minimum distance between ρ and the set of incoherent states

$$C(\rho, \Xi) = \min_{\sigma \in \mathcal{I}_{\Xi}} D(\rho, \sigma), \quad (3)$$

where $D(\cdot, \cdot)$ is a distance measure on quantum states and satisfies the following five conditions. (D1) $D(\rho, \sigma) = 0$ iff $\rho = \sigma$. (D2) Monotonicity under selective measurements on average: $D(\rho, \sigma) \geq \sum_n p_n D(\rho_n, \sigma)$, $\forall \{K_n\}$. (D3) Convexity: $\sum_n p_n D(\rho_n, \sigma) \geq D(\sum_n p_n \rho_n, \sigma)$. (D4) $D(\rho, \Lambda^{\Xi}(U\rho U^{\dagger})) \geq D(\rho, \Lambda^{\Xi}(\rho))$, $\forall \Xi, U$, where U is a unitary operation, and $\Lambda^{\Xi}(\cdot) \equiv \sum_i |\xi_i\rangle\langle\xi_i|(\cdot)|\xi_i\rangle\langle\xi_i|$. (D5) $D(\rho, \sigma) = D(\rho \otimes \rho_a, \sigma \otimes \rho_a)$. Conditions (D1-D3) make sure that (C1-C3) is satisfied by the coherence measure defined in Eq. (3). When (D4) is satisfied, the coherence of ρ on the reference basis Ξ can be written as

$$C(\rho, \Xi) = D(\rho, \Lambda^{\Xi}(\rho)). \quad (4)$$

As proved in ref. 3, the relative entropy $D^r(\rho, \sigma) = S(\rho\|\sigma) \equiv \text{Tr}(\rho \log_2 \rho - \rho \log_2 \sigma)$ and the l_1 matrix norm $D^l(\rho, \sigma) = \|\rho - \sigma\|_{l_1} \equiv \sum_{ij} |\rho_{ij} - \sigma_{ij}|$ satisfies all the conditions (D1-D4), which makes the corresponding coherence measures $C^r(\rho, \Xi) = D^r(\rho, \Lambda^{\Xi}(\rho))$ and $C^l(\rho, \Xi) = D^l(\rho, \Lambda^{\Xi}(\rho))$ satisfy the conditions (C1-C3). As discovered recently³³, the trace-norm distance $D^t(\rho, \sigma) \equiv \text{tr}\sqrt{(\rho_B - \sigma_B)^{\dagger}(\rho_B - \sigma_B)}$ does not satisfy (D2).

Introduced in ref. 32, MID characterizes the quantumness of correlations. MID of a bipartite system ρ is defined as the minimum disturbance caused by local projective measurements that do not change the reduced states $\rho_A \equiv \text{Tr}_B(\rho)$ and $\rho_B \equiv \text{Tr}_A(\rho)$

$$\mathcal{Q}(\rho) = \inf_{\mathbb{E}_A, \mathbb{E}_B} D(\rho, \Lambda_A^{\mathbb{E}_A} \otimes \Lambda_B^{\mathbb{E}_B}(\rho)), \quad (5)$$

where the infimum is taken over projective measurements which satisfy $\Lambda_A^{\mathbb{E}_A}(\rho_A) = \rho_A$ and $\Lambda_B^{\mathbb{E}_B}(\rho_B) = \rho_B$, and $D(\cdot, \cdot)$ is a distance on quantum states, which satisfies conditions (D1-D5) and further (D6) $D(U\rho U^{\dagger}, U\sigma U^{\dagger}) = D(\rho, \sigma)$. It can be checked that (D6) can be satisfied by relative entropy but not satisfied by l_1 -norm. Comparing Eq. (5) with Eq. (4), we find MID is just the coherence of the bipartite state ρ on the local eigenbasis $\mathbb{E}_A \otimes \mathbb{E}_B$.

For later convenience, we introduce B -side MID as

$$\mathcal{Q}_B(\rho) = \inf_{\mathbb{E}_B: \Lambda_B^{\mathbb{E}_B}(\rho_B) = \rho_B} D(\rho, I_A \otimes \Lambda_B^{\mathbb{E}_B}(\rho)). \quad (6)$$

\mathcal{Q}_B goes to zero for B -side classical states, which can be written as $\rho_{B-cla} = \sum_i \rho_i^A \otimes |e_i^B\rangle\langle e_i^B|$, while \mathcal{Q} is strictly positive for ρ_{B-cla} if $\exists i, [\rho_A, \rho_i^A] \neq 0$. Notice that for \mathcal{Q}_B one do not have a coherence interpretation.

Definition of steering-induced coherence. As shown in Fig. 1, Alice and Bob initially share a quantum correlated state ρ , and Bob's reduced state ρ_B is incoherent on his own basis. Now Alice implements a local projective measurement on basis Ξ_A . When she obtains the result i (which happens with probability $p_i^{\xi_i}$), Bob is “steered” to a coherent state $\rho_B^{\xi_i}$. We introduce the concept of SIC for characterizing Alice's ability to create Bob's coherence on average using her local selective measurement.

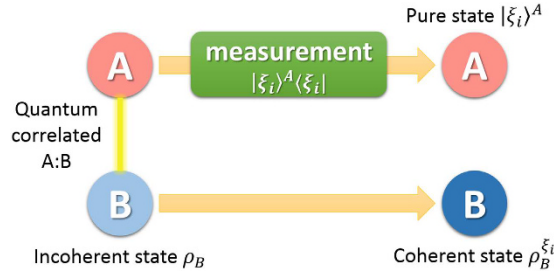


Figure 1. Scheme for creating Bob's coherence by Alice's local measurement and classical communication. When Alice implements local projective measurement on basis $\Xi_A = \{|\xi_i^A\rangle\}$, she gets result i with probability p^{ξ_i} and meanwhile steer Bob's state to $\rho_B^{\xi_i}$ which can be coherent on Bob's initial eigenstate \mathbb{E}_B . SIC is defined as the maximal average coherence of states $\rho_B^{\xi_i}$ that can be created by Alice's local selective measurement.

Definition (Steering-induced coherence, SIC). For a bipartite quantum state ρ , Alice implements projective measurement on basis $\Xi_A = \{|\xi_i^A\rangle\} (i = 0, \dots, d_A - 1)$. With probability $p^{\xi_i} = \text{tr}[\rho(\xi_i^A \otimes I)]$, she obtains the result $\xi_i^A = |\xi_i^A\rangle\langle\xi_i^A|$, which steers Bob's state to $\rho_B^{\xi_i} = (\xi_i^A \otimes \rho) / p^{\xi_i}$. Let $\mathbb{E}_B = \{|e_j^B\rangle\} (j = 0, \dots, d_B - 1)$ be the eigenbasis of reduced states ρ_B . The steering-induced coherence is defined as the maximum average coherence of Bob's steered states on the reference basis \mathbb{E}_B

$$\bar{\mathcal{C}}(\rho) = \inf_{\mathbb{E}_B} [\max_{\Xi_A} \sum_i p^{\xi_i} C(\rho_B^{\xi_i}, \mathbb{E}_B)]. \tag{7}$$

where the maximization is taken over all of Alice's projective measurement basis Ξ_A , and the infimum over \mathbb{E}_B is taken when ρ_B is degenerate and hence \mathbb{E}_B is not unique.

Since Bob's initial state ρ_B is incoherent on its own basis \mathbb{E}_B , the SIC $\bar{\mathcal{C}}(\rho)$ describes the maximum ability of Alice's local selective measurement to create Bob's coherence on average. We verify the following properties for $\bar{\mathcal{C}}(\rho)$.

- (E1) $\bar{\mathcal{C}}(\rho) \geq 0$, and $\bar{\mathcal{C}}(\rho) = 0$ iff ρ is a B -side classical state.
- (E2) Non-increasing under Alice's local completely-positive trace-preserving channel: $\bar{\mathcal{C}}(\Lambda_A \otimes I(\rho)) \leq \bar{\mathcal{C}}(\rho)$.
- (E3) Monotonicity under Bob's local selective measurements on average: $\bar{\mathcal{C}}(\rho) \geq \sum_n p_n \bar{\mathcal{C}}(\rho_n)$, $\forall \{K_n^B\}$ satisfying $K_n^B \mathcal{I}_{\mathbb{E}_B} K_n^{B\dagger} \subset \mathcal{I}_{\mathbb{E}_B}$, where $\rho_n = I_A \otimes K_n^B \rho (I_A \otimes K_n^{B\dagger}) / p_n$ and $p_n = \text{tr}[I_A \otimes K_n^B \rho (I_A \otimes K_n^{B\dagger})]$.
- (E4) Convexity: $\sum_n p_n \bar{\mathcal{C}}(\rho_n) \geq \bar{\mathcal{C}}(\sum_n p_n \rho_n)$.

Proof. Condition (E1) can be proved using the method in ref. 31, where it is proved that $\mathcal{C}_i(\rho) \equiv \max_{\xi_i^A} C(\rho_B^{\xi_i}, \mathbb{E}_B)$ vanishes iff ρ is a B -side classical state. (E2) is verified by noticing that the local channel Λ_A can not increase the set of Bob's steered states, and hence the optimal steered states $\{\rho_B^{\xi_i}\}$ may not be steered to after the action of channel Λ_A . The conditions (E3) and (E4) are directly derived from conditions (C2) and (C3) for coherence. \square

Relation between SIC and MID. Intuitively, Alice's ability to extract coherence on Bob's side should depend on the quantum correlation between them. The following theorem gives a quantitative relation between the SIC $\bar{\mathcal{C}}(\rho)$ and quantum correlation measured by B -side MID $\mathcal{Q}_B(\rho)$.

Theorem 1. When the distance measure in the definition of MID and coherence satisfies conditions (D1-D6), the SIC is bounded from above by the B -side MID, i.e.,

$$\bar{\mathcal{C}}(\rho) \leq \mathcal{Q}_B(\rho). \tag{8}$$

Proof. We start with the situation that ρ_B is non-degenerate and hence one do not need to take the infimum in Eqs (5) and (7). By definition, we have

$$\mathcal{Q}_B(\rho) = D(\rho, \rho^{\mathbb{E}_B}), \tag{9}$$

where $\rho^{\mathbb{E}_B} = I \otimes \Lambda^{\mathbb{E}_B}(\rho)$.

After Alice implements a selective measurement on basis Ξ_A , the average coherence of Bob's state becomes

$$\begin{aligned} \bar{\mathcal{C}}_{\Xi_A}(\rho) &= \sum_i p^{\xi_i} D(\rho_B^{\xi_i}, \Lambda^{\mathbb{E}_B}(\rho_B^{\xi_i})) \\ &= \sum_i p^{\xi_i} D\left(\frac{\xi_i^A \rho \xi_i^{A\dagger}}{p^{\xi_i}}, \frac{\xi_i^A \rho^{\mathbb{E}_B} \xi_i^{A\dagger}}{p^{\xi_i}}\right). \end{aligned} \tag{10}$$

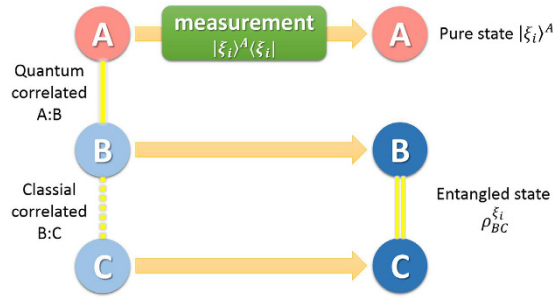


Figure 2. Scheme for creating entanglement between Bob and Charlie by Alice’s local selective measurement. When Alice implements local projective measurement on basis $\Xi_A = \{|\xi_i^A\rangle\}$, she gets result i with probability p^{ξ_i} and meanwhile steer the state shared between Bob and Charlie to $\rho_{BC}^{\xi_i}$ which can be entangled.

The second equality holds because $D(\rho_B^{\xi_i}, \Lambda^{\mathbb{E}_B}(\rho_B^{\xi_i})) = D(\xi_i^A \otimes \rho_B^{\xi_i}, I_A \otimes \Lambda^{\mathbb{E}_B}(\xi_i^A \otimes \rho_B^{\xi_i}))$ (condition (D5)) and $\xi_i^A \otimes \rho_B^{\xi_i} = \frac{\xi_i^A \rho_B^{\xi_i \dagger}}{p^{\xi_i}}$. Since selective measurement does not increase the state distance (condition (D2)), we have $\bar{C}_{\Xi_A}(\rho) \leq Q_B(\rho), \forall \Xi_A$, and hence Eq. (8) holds.

The generalization to degenerate state is straightforward. We choose \mathbb{E}_B^o to reach the infimum of Q_B , which may not be the optimal eigen-basis for \bar{C} . Hence we have $Q_B(\rho) = D(\rho, I_A \otimes \Lambda_{\mathbb{E}_B^o}(\rho)) \geq \max_{\Xi_A} \sum_i p^{\xi_i} C(\rho_B^{\xi_i}, \mathbb{E}_B^o) \geq \bar{C}(\rho)$. According to ref. 17, the coherence of a quantum system B can in turn be transferred to the entanglement between the system and an ancilla C by incoherent operations. The established entanglement, measured by the minimum distance between the state ρ^{BC} and a separable state $\sigma^{BC} := \sum_k p_k \rho_k^B \otimes \rho_k^C$ as $E(\rho_{BC}) = \min_{\sigma^{BC} \in \mathcal{S}} D(\rho^{BC}, \sigma^{BC})$, is bounded from above by the initial coherence of B . Here \mathcal{S} is the set of separable states and the state distance D is required not to increase under trace-preserving channels $D(\Lambda(\rho), \Lambda(\sigma)) \leq D(\rho, \sigma)$, which is automatically satisfied when we combine conditions (D2) and (D3).

This leads to the three-party protocol as shown in Fig. 2, where Alice’s local selective measurement can create entanglement between Bob and Charlie. In this protocol, Bob and Charlie try to build entanglement between them from a product state $\rho^{AB} \otimes |e_0^C\rangle\langle e_0^C|$, but are limited to use incoherent operations. Since ρ_B is incoherent on his eigenbasis \mathbb{E}_B , Bob and Charlie can build only classically correlated state without Alice’s help. Now Alice implement projective measurement $\{|\xi_i^A\rangle\langle \xi_i^A|\}$ and on the outcome i , the state shared between Bob and Charlie is steered to $\rho_{BC}^{\xi_i}$ which can be entangled. The following corollary of theorem 1 gives the upper bound of the steering-induced entanglement.

Corollary 1 Alice, Bob and Charlie share a tripartite state ρ , which is prepared from the product state $\rho^{AB} \otimes |e_0^C\rangle\langle e_0^C|$ using an ICPTP channel on BC : $\rho = I_A \otimes \Lambda_{ICPTP}^{BC}(\rho^{AB} \otimes |e_0^C\rangle\langle e_0^C|)$. Here $\mathbb{E}_B \otimes \mathbb{E}_C = \{|e_i^B\rangle \otimes |e_j^C\rangle\}$ is the reference basis of coherence. Alice’s local selective measurement $\{|\xi_i^A\rangle\langle \xi_i^A|\}$ can establish entanglement between Bob and Charlie, and the established entanglement on average is bounded from above by the initial B -side MID between Alice and Bob

$$\sum_i p^{\xi_i} E(\rho_{BC}^{\xi_i}) \leq Q_B(\rho^{AB}). \tag{11}$$

Proof. Before Alice implement the measurement, the state shared between Bob and Charlie is incoherent on basis $\mathbb{E}_B \otimes \mathbb{E}_C$ and hence can be written as $\rho^{BC} = \sum_{ij} q_{ij} |e_i^B\rangle\langle e_i^B| \otimes |e_j^C\rangle\langle e_j^C|$. Apparently, $Q(\rho^{BC}) = 0$, so Bob and Charlie is classically correlated.

On the measurement outcome i , the entanglement between Bob and Charlie becomes $E(\rho_{BC}^{\xi_i})$ which satisfies $E(\rho_{BC}^{\xi_i}) \leq Q(\rho_{BC}^{\xi_i}) \leq C(\rho_{BC}^{\xi_i}, \mathbb{E}_B \otimes \mathbb{E}_C)$. Notice that $\rho_{BC}^{\xi_i} = \Lambda_{ICPTP}^{BC}(\rho_B^{\xi_i} \otimes |e_0^C\rangle\langle e_0^C|)$ and hence $C(\rho_{BC}^{\xi_i}, \mathbb{E}_B \otimes \mathbb{E}_C) \leq C(\rho_B^{\xi_i} \otimes |e_0^C\rangle\langle e_0^C|, \mathbb{E}_B \otimes \mathbb{E}_C) = C(\rho_B^{\xi_i}, \mathbb{E}_B)$. Eq. (11) is arrived by noticing that $\sum_i p^{\xi_i} C(\rho_B^{\xi_i}, \mathbb{E}_B) \leq Q_B(\rho^{AB})$ from theorem 1.

Now we consider a general tripartite state ρ . If the reduced state $\rho^{BC} = \text{tr}_A \rho$ is non-degenerate, one can follow the same steps and prove that

$$\sum_i p^{\xi_i} E(\rho_{BC}^{\xi_i}) \leq Q_{\{BC\}}(\rho), \tag{12}$$

whenever ρ^{BC} is incoherent on basis $\mathbb{E}_B \otimes \mathbb{E}_C$. Here $Q_{\{BC\}}$ is the $\{BC\}$ -side MID between Alice and the combination of Bob and Charlie. However, when ρ^{BC} is degenerate, the condition that the tripartite state ρ is prepared from $\rho^{AB} \otimes |e_0^C\rangle\langle e_0^C|$ by an ICPTP channel on BC is stringent. For example, the state $\rho^X = \frac{1}{2}|0\rangle^A \langle 0| \otimes |\Psi^+\rangle^{BC} \langle \Psi^+| + \frac{1}{2}|1\rangle^A \langle 1| \otimes |\Psi^-\rangle^{BC} \langle \Psi^-|$ where $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, with ρ^{BC} incoherent on basis $\{|ij\rangle^{BC}\}, (i, j = 0, 1)$, violates Eq. (12), since $Q_{\{BC\}}(\rho) = 0$ but the left-hand-side reaches unity for Alice’s

measurement basis $\Xi_A = \{|0\rangle, |1\rangle\}$. It indicates that the state ρ^X can not be prepared from a product state in the form $\rho^{AB} \otimes |e_0^C\rangle\langle e_0^C|$ using only incoherent operations.

States to reach the upper bound. According to theorem 1, Bob’s maximal coherence that can be extracted by Alice’s local selective measurement is bounded from above by the initial quantum correlation between them. Since the relative entropy is the only distance measure found to date which satisfies all the conditions (D1–D6), we employ relative entropy as the distance in the definition of coherence and MID, and discuss the states which can reach the upper bound of theorem 1.

Theorem 2. *The SIC can reach B-side MID*

$$\bar{C}^r(\rho) = \mathcal{Q}_B^r(\rho) = S(\rho_B) - S(\rho). \tag{13}$$

for maximally correlated states $\rho^{mc} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|$.

Proof. Any maximally correlated state can be written in a pure state decomposition form $\rho = \sum_i q_i |\Psi_i\rangle\langle\Psi_i|$ with $|\Psi_i\rangle = \sum_j \lambda_{ij} |jj\rangle$ and $\langle\Psi_i|\Psi_{i'}\rangle = \delta_{ii'}$. Here $\rho_B = \sum_j (\sum_i q_i |\lambda_{ij}|^2) |j\rangle\langle j|$ has eigenbasis $\mathbb{E}_B = \{|i\rangle\}$. In order to calculate the B-side MID, we consider Bob’s projective measurement $\Lambda_B^{\mathbb{E}_B}$, which takes the bipartite state to $\rho^{\mathbb{E}_B} = \sum_j (\sum_i q_i |\lambda_{ij}|^2) |jj\rangle\langle jj|$. Apparently, $S(\rho^{\mathbb{E}_B}) = S(\rho_B)$. By definition, we have

$$\mathcal{Q}_B^r(\rho) = S(\rho \| \rho^{\mathbb{E}_B}) = S(\rho^{\mathbb{E}_B}) - S(\rho) = S(\rho_B) - S(\rho). \tag{14}$$

In order to extract the maximum average coherence on Bob’s side, Alice measures her quantum system on basis Ξ_A , where $|\xi_k^A\rangle = \frac{1}{\sqrt{d_A}} \sum_{j=0}^{d_A-1} e^{-i\frac{2\pi kj}{d_A}} |j\rangle, k = 0, \dots, d_A - 1$ and d_A is the dimension of A . On the measurement result k , Bob’s state is steered to $\rho_B^{\xi_k} = \sum_i q_i |\phi_i^{\xi_k}\rangle\langle\phi_i^{\xi_k}|$ where $|\phi_i^{\xi_k}\rangle = \sum_j e^{i\frac{2\pi kj}{d_A}} \lambda_{ij} |j\rangle$, which happens with probability $p^{\xi_k} = \frac{1}{d}$. Apparently, $\langle\phi_i^{\xi_k}|\phi_{i'}^{\xi_k}\rangle = \delta_{ii'}$ and hence $S(\rho_B^{\xi_k}) = S(\rho) = -\sum_i q_i \log_2 q_i$. Meanwhile, we have $\Lambda_B^{\mathbb{E}_B}(\rho_B^{\xi_k}) = \sum_j (\sum_i q_i |\lambda_{ij}|^2) |j\rangle\langle j| = \rho_B$. The coherence of steered state $\rho_B^{\xi_k}$ is then

$$C^r(\rho_B^{\xi_k}, \mathbb{E}_B) = S(\Lambda_B^{\mathbb{E}_B}(\rho_B^{\xi_k})) - S(\rho_B^{\xi_k}) = S(\rho_B) - S(\rho), \tag{15}$$

for any outcome k . Therefore we arrive at Eq. (13).

Any pure bipartite state can be written in a Schmidt decomposition form $|\Psi\rangle = \sum_j \lambda_j |jj\rangle$, and hence belongs to the set of maximally correlated states. As introduced in ref. 17, a maximally correlated state ρ^{mc} is prepared from an product state $\rho_B \otimes |0\rangle_C\langle 0|$ using an incoherent unitary operator, and its entanglement $E(\rho^{mc})$ can reach the initial coherence of ρ_B . Further, for maximally correlated states, one can check the equality, $E(\rho^{mc}) = \mathcal{Q}_B(\rho^{mc})$. Therefore, ρ^{mc} can be used in a scenario where coherence is precious and entanglement is not as robust as single-party coherence. Precisely, consider the situation where Alice and Bob share a maximally correlated state ρ_{AB}^{mc} but they are not use it in a hurry. To store the resource for latter use, she can transfer the entanglement between them into Bob’s coherence using her local selective measurement. Bob stores his coherent state as well as Alice’s measurement results. When required, Bob can perfectly retrieve the entanglement by preparing a maximally correlated state using only incoherent operations.

Two-qubit case, relation between l_1 -norm of SIC and trace-norm distance of B-side MID. One cannot define MID based on the l_1 -norm distance, since it does not satisfy (D6) in general. However, it can be checked that for single-qubit states ρ_B and $\sigma_B, \|\rho_B - \sigma_B\|_{l_1} = D^t(\rho_B, \sigma_B) = |\mathbf{r}^\rho - \mathbf{r}^\sigma|^4$, where \mathbf{r}^ρ and \mathbf{r}^σ are Bloch vectors of ρ_B and σ_B respectively. Hence the l_1 -norm of coherence for a single-qubit state ρ_B can be written as

$$C^{l_1}(\rho_B, \Xi) = D^t(\rho_B, \bar{\Lambda}^\Xi(\rho_B)). \tag{16}$$

Besides, D^t , which satisfies condition (D6), is proper to be used as a distance measure for MID. Therefore, when the Bob’s particle is a qubit, it is meaningful to study the relation between l_1 -norm of SIC and trace-norm distance of B-side MID. Now we consider a two-qubit state ρ , and employ C^{l_1} in the definition of $\bar{C}(\rho)$ as in Eq. (7) and prove the following theorem.

Theorem 3. *For a two-qubit state ρ , we have*

$$\bar{C}^{l_1}(\rho) = \mathcal{Q}_B^t(\rho). \tag{17}$$

Proof. The state of a two-qubit state can be written as $\rho = \frac{1}{4} \sum_{i,j=0}^3 \Theta_{ij} \sigma_i^A \otimes \sigma_j^B$, where the coefficient matrix $\Theta_{ij} = \text{tr}(\rho \sigma_i^A \otimes \sigma_j^B)$ can be written in the block form $\Theta = \begin{pmatrix} \mathbf{1} & \mathbf{b}^T \\ \mathbf{a} & T \end{pmatrix}$.

For non-degenerate case $b \neq 0$, we choose the eigenbasis of ρ_B for the basis of density matrix and hence $\mathbf{b} = (0, 0, b_3)$. Further, a proper basis of qubit A is chosen such that the matrix T is in a triangle form with $T_{11} = T_{12} = T_{21} = 0$. We calculated the explicit form of $\mathcal{Q}_B^t(\rho)$ and $\bar{C}^{l_1}(\rho)$ and obtain

$$\mathcal{Q}_B^t(\rho) = \bar{\mathcal{C}}^t(\rho) = \left[\frac{T_{22}^2 + T_{31}^2 + T_{32}^2}{2} + \frac{\sqrt{(T_{32}^2 + T_{22}^2)^2 + 2T_{31}^2(T_{32}^2 - T_{22}^2) + T_{31}^4}}{2} \right]^{\frac{1}{2}}. \quad (18)$$

For degenerate case with $b=0$, we can always choose proper local basis such that T is diagonal. Here we impose $T_{11} \geq T_{22} \geq T_{33}$ without loss of generality. Direct calculations lead to

$$\mathcal{Q}_B^t(\rho) = \bar{\mathcal{C}}^t(\rho) = T_{22}. \quad (19)$$

We check that, for the state $\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|01\rangle\langle 01|$, we have $\bar{\mathcal{C}}^t(\rho) < \mathcal{Q}_B^t(\rho)$, but according to theorem 3, $\bar{\mathcal{C}}^t(\rho) = \mathcal{Q}_B^t(\rho)$. It means that relative entropy of coherence and l_1 -norm of coherence are truly different measures of coherence.

Discussion

In this paper, we have introduced the notion of SIC which characterizes the power of Alice's selective measurement to remotely create quantum coherence on Bob's site. Quantitative connection has been built between SIC and the initially shared quantum correlation measured by B -side MID. We show that SIC is always less than or equal to B -side MID. Our results are also generalized to a tripartite scenario where Alice can build the entanglement between Bob and Charlie in a controlled way.

Next, we discuss a potential application of SIC in secret sharing. Suppose Alice and Bob share a two-qubit state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, whose SIC reaches unity. When Alice measures her state on different basis, Bob's state is steered to, e.g., $\mathbb{E}_B^z = \{|0\rangle, |1\rangle\}$ or $\mathbb{E}_B^x = \{|+\rangle, |-\rangle\}$ with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The coherence of states in \mathbb{E}_B^z reach unity on basis \mathbb{E}_B^x and vice versa. Consequently, when we measure the states in the set \mathbb{E}_B^z on basis \mathbb{E}_B^x , the outcome is completely random. It is essential to quantum secret sharing using $|\Phi\rangle$. In this sense, the SIC is potentially related to the ability for Alice to share secret with Bob.

Coherence and various quantum correlations, such as entanglement and discord-like correlations, are generally considered as resources in the framework of resource theories^{9,35}. By coining the concept of SIC, we present an operational interpretation between measures of those two resources, SIC and MID, and open the avenue to study their (ir)reversibility. The applications of various coherence quantities like SIC in many-body systems, as in the case of entanglement^{36–38}, can be expected.

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Author Contributions

X.H. and H.F. contributed the idea. X.H. performed the calculations. X.H. and H.F. wrote the paper. All authors reviewed the manuscript and agreed with the submission.

Additional Information

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