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Extrapolation methods to solve non-autonomous retarded partial differential equations

by

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Abstract. Using extrapolation spaces introduced by Da Prato–Grisvard and Nagel we prove a non-autonomous perturbation theorem for Hille–Yosida operators. The abstract result is applied to non-autonomous retarded partial differential equations.

1. Introduction. The purpose of this note is to study non-autonomous perturbations of (not necessarily densely defined) Hille–Yosida operators. The abstract result of this paper will be illustrated by means of non-autonomous retarded partial differential equations.

Let X be a Banach space and A a linear operator on X with domain $D(A)$. We say that $(A, D(A))$ is a *Hille–Yosida* operator if there exists $\omega \in \mathbb{R}$ such that every $\lambda > \omega$ is in the resolvent set $\varrho(A)$ of A and

$$\sup\{\|(\lambda-\omega)^n(\lambda-A)^{-n}\| : \lambda \geq \omega,\ n \in \mathbb{N}\} < \infty.$$

If the constant ω can be chosen negative, then A is called of *negative type*.

It follows from the Hille–Yosida theorem that a Hille–Yosida operator generates a C_0-semigroup on the closure of its domain. More precisely, we have (cf. [Hi-Ph], Thm. 12.2.4).

PROPOSITION 1.1 *Let $(A, D(A))$ be a Hille–Yosida operator on the Banach space X. Then the part $(A_0, D(A_0))$ of A in $X_0 := (\overline{D(A)}, \|\cdot\|)$ given by*

$$D(A_0) := \{x \in D(A) : Ax \in X_0\}, \quad A_0x := Ax \quad \text{for } x \in D(A_0),$$

generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0. Moreover, $\varrho(A) \subset \varrho(A_0)$ and $(\lambda - A_0)^{-1}$ is the restriction of $(\lambda - A)^{-1}$ to X_0 for $\lambda \in \varrho(A)$.

For the rest of this paper we assume without loss of generality that $(A, D(A))$ is a Hille–Yosida operator of negative type on X.

Now we summarize some basic facts on extrapolation spaces and Favard classes, which will be used throughout this paper. For more details we refer to [Na1], [Wa] and [Na-Si] where also the missing proofs can be found.

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On the space X_0 we introduce a new norm by

$$\|x\|_{-1} := \|A_0^{-1}x\|, \quad x \in X_0.$$

The completion of $(X_0, \|\cdot\|)$ will be called the *extrapolation space* of X_0 associated with A_0 and it will be denoted by X_{-1}.

Since $A_0^{-1}T_0(t) = T_0(t)A_0^{-1}$ for all $t \geq 0$, one has

$$\|T_0(t)x\|_{-1} \leq \|T_0(t)\|_{\mathcal{L}(X)}\|x\|_{-1} \quad \text{for } x \in X_0 \text{ and } t \geq 0.$$

This shows that the operator $T_0(t)$ can be uniquely extended to a bounded operator on the Banach space X_{-1}. The result is a C_0-semigroup on X_{-1} denoted by $(T_{-1}(t))_{t\geq 0}$. The semigroup $(T_{-1}(t))_{t\geq 0}$ will be called the *extrapolated semigroup* of $(T_0(t))_{t\geq 0}$. If we denote by $A_{-1} : D(A_{-1}) \to X_{-1}$ the generator of $(T_{-1}(t))_{t\geq 0}$, then we have the following properties (see [Na-Si], Prop. 1.3 and Thm. 1.4).

(i) $\|T_{-1}(t)\|_{\mathcal{L}(X_{-1})} = \|T_0(t)\|_{\mathcal{L}(X_0)}$.

(ii) $D(A_{-1}) = X_0$.

(iii) $A_{-1} : X_0 \to X_{-1}$ is the unique extension of $A_0 : X_0 \supset D(A_0) \to X_{-1}$ to an isometry $X_0 \to X_{-1}$.

(iv) A_{-1} is invertible with $(A_{-1})^{-1} \in \mathcal{L}(X_{-1})$.

The original space X now fits into this scheme of spaces X_0 and X_{-1} (see [Na-Si], Thm. 1.7).

THEOREM 1.2. *Let $(A, D(A))$ be a Hille–Yosida operator of negative type on the Banach space X. Then $X_0 := \overline{D(A)}$ is dense in X with respect to the norm*

$$\|x\|_{-1} := \|A^{-1}x\|, \quad x \in X.$$

Hence, the extrapolation space X_{-1} is also the completion of $(X, \|\cdot\|_{-1})$ and therefore $X \hookrightarrow X_{-1}$. Moreover, the operator A_{-1} is an extension of A, hence $(A_{-1})^{-1}X = D(A)$.

Let us also recall the definition of the Favard class of a C_0-semigroup (cf. [Bu-Be], Chap. 3). The *Favard class* of the generator A_0 is the space

$$F(A_0) := \left\{ x \in X_0 : \limsup_{t\to 0} \frac{1}{t}\|T_0(t)x - x\| < \infty \right\}$$

equipped with the norm

$$\|x\|_{F(A_0)} := \limsup_{t\to 0} \frac{1}{t}\|T_0(t)x - x\| \quad \text{for } x \in F(A_0).$$

It is easy to see that $F(A_0)$ is invariant under $(T_0(t))$ and $D(A_0) \subset F(A_0)$. Similarly we have $X_0 \subset F(A_{-1})$. In [Na-Si], Prop. 3.2, it is shown that $F(A_{-1})$ is the extrapolation space of $F(A_0)$.

PROPOSITION 1.3. *For the Favard classes $F(A_0)$ and $F(A_{-1})$ the following holds.*

(i) $A_{-1}F(A_0) = F(A_{-1})$.

(ii) $\|A_{-1}x\|_{F(A_{-1})} = \|x\|_{F(A_0)}$ *for* $x \in F(A_0)$.

(iii) $D(A_0) \subseteq D(A) \hookrightarrow F(A_0) \hookrightarrow X_0 \subseteq X \hookrightarrow F(A_{-1}) \hookrightarrow X_{-1}$.

(iv) $T_{-1}(t)F(A_{-1}) \subseteq F(A_{-1})$.

Let now $C : X_0 \to F(A_{-1})$ be a bounded operator. Then it is proved in a recent paper [Ni-Rh] that the operator

$$D(B_0) := \{x \in X_0 : A_{-1}x + Cx \in X_0\}, \quad B_0x := A_{-1}x + Cx, \quad x \in D(B_0),$$

generates a C_0-semigroup $(S_0(t))_{t\geq 0}$ on X_0. This semigroup is given by the variation of constants formula

$$S_0(t)x = T_0(t)x + \int_0^t T_{-1}(t-s)CS_0(s)x\,ds, \quad x \in X_0,\ t \geq 0. \tag{1}$$

In this paper we shall generalize this result to the case of time depending perturbations. We prove that if $C(\cdot)$ is a strongly continuous function from $[0, T]$ into the space of bounded linear operators $\mathcal{L}(X_0, F(A_{-1}))$, then the variation of constants formula

$$U(t,s)x = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)C(\sigma)U(\sigma,s)\,d\sigma, \tag{2}$$
$$x \in X_0,\ 0 \leq s < t \leq T,$$

gives a strongly continuous evolution family on X_0.

As an application we prove in Section 3 that the variation of constants formula (2) solves the non-autonomous retarded partial differential equation

$$\text{(NRDE)} \qquad \begin{cases} x'(t) = Bx(t) + L(t)x_t \\ x(\tau) = f(\tau - s), \quad s - r \leq \tau \leq s,\ 0 \leq s < t \leq T, \end{cases}$$

on $C([-r, 0], E)$, where E is any Banach space, B generates a C_0-semigroup on E and $L(\cdot)$ is a strongly continuous operator-valued function from $[0, T]$ into $\mathcal{L}(C([-r, 0], E), E)$. We mention here that Clément *et al.* (see [Cl], [Cl1] and [Cl2]) proved a similar variation of constants formula by using duality methods in the sun-reflexive case. The non-autonomous retarded partial differential equation (NRDE) can also be solved by using the theory of multiplicative perturbations (see [De-Sch-Zh]).

Abstract extrapolation spaces have been introduced by Da Prato–Grisvard [DaP-Gr] and Nagel [Na1] and used for various purposes (cf. [Am], [vNe], [Na-Si] and [Ni-Rh]).

Concerning evolution families and their connection with non-autonomous Cauchy problems we refer to [Pa] and [Ta]. For terminology and basic results on semigroup theory we follow [Go], [Na] and [Pa] and for retarded differential equations we refer to [Ha] and [Di-vGi-Lu-Wa].

2. Non-autonomous perturbation of Hille–Yosida operators. The aim of this section is to prove that the variation of constants formula (2) defines an evolution family.

DEFINITION 2.1. Let $T > 0$ and $\Delta := \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}$. A family $(U(t,s))_{(t,s)\in\Delta}$ of bounded linear operators on a Banach space X is called an *evolution family* on X if the following conditions are satisfied.

(a) $U(t,t) = I$ and $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T$,
(b) $\Delta \ni (t,s) \mapsto U(t,s)x$ is continuous for each $x \in X$.

Throughout this section we consider a Hille–Yosida operator $(A, D(A))$ of negative type and as in the previous section we consider the scale of spaces $D(A_0) \subseteq D(A) \hookrightarrow F(A_0) \hookrightarrow X_0 \subseteq X \hookrightarrow F(A_{-1}) \hookrightarrow X_{-1}$ and the C_0-semigroups $(T_0(t))_{t\ge0}$ and $(T_{-1}(t))_{t\ge0}$. Moreover, let $C(\cdot) \in C([0,T], \mathcal{L}_s(X_0, F(A_{-1})))$ be the Banach space of all strongly continuous functions $F(\cdot) : [0,T] \to \mathcal{L}(X_0, F(A_{-1}))$.

In order to prove the main result of this section, we will need the following lemma due to Nagel and Sinestrari (see [Na-Si], Prop. 3.1).

LEMMA 2.2. *For $f \in L^1(\mathbb{R}_+, F(A_{-1}))$ and $t \ge 0$ we define*

$$(T_{-1} * f)(t) := \int_0^t T_{-1}(t-s)f(s)\,ds.$$

For this convolution integral the following properties hold with a constant $M_1 < \infty$ independent of f and t:

(i) $(T_{-1} * f)(t) \in X_0$.
(ii) $\|(T_{-1} * f)(t)\| \le M_1 \|f\|_{L^1((0,t),F(A_{-1}))}$.
(iii) $\lim_{t\searrow 0} \|(T_{-1} * f)(t)\| = 0$.

From Lemma 2.2 follows by successive approximation the existence and uniqueness of the evolution family $(U(t,s))_{(t,s)\in\Delta}$ satisfying (2).

THEOREM 2.3. *Let $(A, D(A))$ be a Hille–Yosida operator of negative type on the Banach space X and consider $X_0 := \overline{D(A)}$. Moreover, let $C(\cdot) \in C([0,T], \mathcal{L}_s(X_0, F(A_{-1})))$. The expansion*

$$U(t,s) = \sum_{n=0}^{\infty} U_n(t,s), \quad \textit{where} \tag{3}$$

$$U_0(t,s) := T_0(t-s) \quad \textit{and} \quad U_{n+1}(t,s) := \int_s^t T_{-1}(t-\sigma)C(\sigma)U_n(\sigma,s)\,d\sigma, \tag{4}$$

converges in the uniform operator topology of $\mathcal{L}(X_0)$ uniformly on Δ and defines an evolution family $(U(t,s))_{(t,s)\in\Delta}$ on X_0, satisfying

$$\|U(t,s)\| \le Me^{Mv(t,s)(t-s)} \quad \textit{for } (t,s) \in \Delta, \tag{5}$$

where M is a constant and $v(t,s) := \sup_{s\le\sigma\le t} \|C(\sigma)\|_{\mathcal{L}(X_0,F(A_{-1}))}$. In addition, the variation of constants formula (2) holds.

Proof. One can see by Lemma 2.2(i), (iii), that for each $n \in \mathbb{N}$, $U_n(t,s)X_0 \subset X_0$ for all $(t,s) \in \Delta$ and $\Delta \ni (t,s) \mapsto U_n(t,s)$ is strongly continuous on X_0, where $(U_n(t,s))_{(t,s)\in\Delta}$ is the family of operators given by (4). From Lemma 2.2(ii) it follows that

$$\|U_1(t,s)x\| = \Big\| \int_0^{t-s} T_{-1}(t-s-\sigma)C(\sigma+s)T_0(\sigma)x\,d\sigma \Big\|$$
$$\le M \int_0^{t-s} \|C(\sigma+s)T_0(\sigma)x\|_{F(A_{-1})}\,d\sigma \le M^2 v(t,s)(t-s)\|x\|,$$

where $v(t,s) := \sup_{s\le\sigma\le t} \|C(\sigma)\|_{\mathcal{L}(X_0,F(A_{-1}))}$, $x \in X_0$ and $(t,s) \in \Delta$. By induction we obtain

$$\|U_n(t,s)\| \le M^{n+1} v(t,s)^n \frac{(t-s)^n}{n!}, \quad (t,s) \in \Delta,\ n \in \mathbb{N}.$$

This implies that the expansion

$$U(t,s) = \sum_{n=0}^{\infty} U_n(t,s)$$

converges in the uniform operator topology of $\mathcal{L}(X_0)$ uniformly on Δ and defines a strongly continuous family $(U(t,s))_{(t,s)\in\Delta}$ on X_0, since $\Delta \ni (t,s) \mapsto U_n(t,s)$ is strongly continuous on X_0 for each $n \in \mathbb{N}$. Moreover, the estimate (5) holds. From (3) and (4) it is easy to see that the family $(U(t,s))_{(t,s)\in\Delta}$ satisfies the variation of constants formula (2) and therefore $U(t,t) = I$ for all $t \in [0,T]$.

It now remains to show that

$$U(t,r)U(r,s) = U(t,s) \quad \text{for } 0 \le s \le r \le t \le T.$$

Since

$$U(t,r)U(r,s) = \sum_{n=0}^{\infty} U_n(t,r) \sum_{m=0}^{\infty} U_m(r,s) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} U_{n-j}(t,r)U_j(r,s),$$

we only have to prove that

$$\sum_{j=0}^{n} U_{n-j}(t,r)U_j(r,s) = U_n(t,s) \quad \text{for } 0 \le s \le r \le t \le T.$$

The assertion is true for $n=0$ and by induction we have

$$\begin{aligned}\sum_{j=0}^{n+1} U_{n+1-j}(t,r)U_j(r,s) &= \sum_{j=0}^{n} U_{n+1-j}(t,r)U_j(r,s) + T_0(t-r)U_{n+1}(r,s)\\ &= \sum_{j=0}^{n}\int_r^t T_{-1}(t-\sigma)C(\sigma)U_{n-j}(\sigma,r)U_j(r,s)\,d\sigma\\ &\quad + \int_s^r T_{-1}(t-\sigma)C(\sigma)U_n(\sigma,s)\,d\sigma\\ &= \Big(\int_r^t + \int_s^r\Big)T_{-1}(t-\sigma)C(\sigma)U_n(\sigma,s)\,d\sigma\\ &= \int_s^t T_{-1}(t-\sigma)C(\sigma)U_n(\sigma,s)\,d\sigma\\ &= U_{n+1}(t,s) \quad \text{for } 0\le s\le r\le t\le T. \ \blacksquare\end{aligned}$$

Remark 2.4. If in Theorem 2.3, $(A, D(A))$ is an arbitrary Hille–Yosida operator on X, then it is easy to see, using the rescaling procedure (cf. [Na], A-I, 3.1), that all assertions of Theorem 2.3 are true except the estimate (5) which is replaced by

$$\|U(t,s)\| \le Me^{(\omega+Mv(t,s))(t-s)} \quad \text{for } (t,s)\in\Delta, \tag{6}$$

where M and ω are such that $\|T_0(t)\|\le Me^{\omega t}$ for $t\ge 0$.

Let $A(t)$ be the part of $A_{-1}+C(t)$ in X_0, i.e.

$$\begin{aligned} D(A(t)) &:= \{x\in X_0 : A_{-1}x + C(t)x \in X_0\}\\ A(t)x &:= A_{-1}x + C(t)x, \quad x\in D(A(t)),\ t\in[0,T].\end{aligned}$$

As in [Cl2], Lemma 2.4, one can see that if $x\in D(A(t))$ for all $t\in[0,T]$, then

$$\lim_{\substack{s,r\to t\\ r<s}} \frac{1}{s-r}(U(s,r)x - x) = A(t)x. \tag{7}$$

In fact, let $t\in[0,T]$ and $x\in D(A(t))$. Then from Lemma 2.2 we have

$$\begin{aligned}&\frac{1}{s-r}\Big\|\int_0^{s-r} T_{-1}(s-r-\sigma)[C(\sigma+r)U(\sigma+r,r)x - C(t)x]\,d\sigma\Big\|\\ &\qquad \le \frac{M}{s-r}\int_0^{s-r}\|C(\sigma+r)U(\sigma+r,r)x - C(t)x\|_{F(A_{-1})}\,d\sigma.\end{aligned}$$

Since $C(\cdot)\in C([0,T],\mathcal{L}_s(X_0,F(A_{-1})))$, we obtain

$$\lim_{\substack{s,r\to t\\ r<s}} \frac{1}{s-r}\Big\|\int_0^{s-r} T_{-1}(s-r-\sigma)[C(\sigma+r)U(\sigma+r,r)x - C(t)x]\,d\sigma\Big\| = 0.$$

So by (2) follows

$$\begin{aligned} 0 &= \lim_{\substack{s,r\to t\\ r<s}} \frac{1}{s-r}\Big[(U(s,r)x - x) - (T_0(s-r)x - x) - \int_r^s T_{-1}(s-\sigma)C(t)x\,d\sigma\Big]\\ &= \lim_{\substack{s,r\to t\\ r<s}} \frac{1}{s-r}\Big[(U(s,r)x-x) - \int_0^{s-r} T_{-1}(s-r-\sigma)(A_{-1}x + C(t)x)\,d\sigma\Big].\end{aligned}$$

From $A_{-1}x + C(t)x\in X_0$ the assertion follows.

Since (7) holds and with the same proof as in Theorem 2.5 of [Cl2] we obtain the following relation between $A(\cdot)$ and the evolution family $(U(t,s))_{(t,s)\in\Delta}$ given by Theorem 2.3.

COROLLARY 2.5. *Let the assumptions be as in Theorem 2.3. Let $s\in[0,T]$ and $x\in D(A(s))$. Then the evolution family $(U(t,s))_{(t,s)\in\Delta}$ satisfies*

(a) $\displaystyle \frac{\partial^+}{\partial t}U(t,s)x\Big|_{t=s} = A(s)x,$

(b) $\displaystyle \frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x,$

where the right derivative in (a) *and the derivative in* (b) *are in the norm topology of X_0.*

3. Non-autonomous retarded partial differential equations. In this section we consider a C_0-semigroup $(S(t))_{t\ge0}$ on a Banach space E with generator $(B, D(B))$. Denote by $C_E := C([-r,0],E)$ the Banach space of all continuous functions from $[-r,0]$ into E, where r is a positive constant. Let $L(\cdot)\in C([0,T],\mathcal{L}_s(C_E,E))$.

Using extrapolation methods and especially the variation of constants formula (2), we solve the following non-autonomous retarded partial differential equation:

$$\text{(NRDE)}\qquad \begin{cases} x'(t) = Bx(t) + L(t)x_t\\ x(\tau) = f(\tau - s), \quad s-r\le\tau\le s,\ 0\le s<t\le T,\end{cases}$$

on C_E, where f is a given function in C_E and $x_t\in C_E$ denotes the function

$$x_t(\tau) := x(t+\tau) \quad \text{for } \tau\in[-r,0].$$

We consider the Banach space $X := E \times C_E$ equipped with the norm

$$\left\|\begin{pmatrix}\eta\\ g\end{pmatrix}\right\| := \|\eta\|_E + \|g\|_{C_E} = \|\eta\|_E + \sup_{-r\le\tau\le0}\|g(\tau)\|_E \quad \text{for } \begin{pmatrix}\eta\\ g\end{pmatrix}\in X.$$

Let $M \ge 1$, $\omega \ge 0$ be two constants such that

$$\|S(t)\| \le Me^{\omega t} \quad \text{for } t\ge 0.$$

We denote by δ_0 the Dirac measure concentrated at zero and $\delta_0' f := f'(0)$ for $f \in C_E^1 := \{f\in C_E : f \text{ is differentiable and } f'\in C_E\}$.

The following lemma will be used to prove well-posedness of the Cauchy problem (NRDE).

LEMMA 3.1. *With the notations introduced above, the matrix operator*

$$\mathcal{A} := \begin{pmatrix} 0 & B\delta_0-\delta_0' \\ 0 & d/d\tau\end{pmatrix} \quad \textit{with } D(\mathcal{A}) := \{0\}\times\{f\in C_E^1 : f(0)\in D(B)\}$$

on X satisfies

$$(\omega,\infty)\subset\varrho(\mathcal{A}) \quad \textit{and} \quad \|(\lambda-\mathcal{A})^{-n}\| \le \frac{M}{(\lambda-\omega)^n}$$

for all $\lambda>\omega$ and $n\in\mathbb{N}$. This means that $\mathcal{A}$ is a Hille–Yosida operator on X.

Proof. For $\lambda>\omega$, $\binom{\eta}{g}\in X$ and $\binom{0}{f}\in D(\mathcal{A})$ we have

$$\begin{pmatrix}\eta\\ g\end{pmatrix} = (\lambda-\mathcal{A})\begin{pmatrix}0\\ f\end{pmatrix} = \begin{pmatrix} f'(0)-Bf(0)\\ (\lambda-d/d\tau)f\end{pmatrix}.$$

Hence,

$$\tag{8} \begin{cases} f(\tau) = e^{\lambda\tau}f(0) + \int_\tau^0 e^{\lambda(\tau-\sigma)}g(\sigma)\,d\sigma,\\ f'(0)-Bf(0)=\eta,\end{cases}$$

for $\tau\in[-r,0]$. Since $(\omega,\infty)\subset\varrho(B)$, it follows from (8) that

$$\tag{9} f(\tau) = e^{\lambda\tau}(\lambda-B)^{-1}(g(0)+\eta) + \int_\tau^0 e^{\lambda(\tau-\sigma)}g(\sigma)\,d\sigma \quad \text{for } \tau\in[-r,0].$$

This implies $(\omega,\infty)\subset\varrho(\mathcal{A})$ and

$$(\lambda-\mathcal{A})^{-1}\begin{pmatrix}\eta\\ g\end{pmatrix} = \begin{pmatrix}0\\ f\end{pmatrix},$$

where f is given by (9). One can show by induction that for $n\in\mathbb{N}$,

$$(\lambda-\mathcal{A})^{-n}\begin{pmatrix}\eta\\ g\end{pmatrix} = \begin{pmatrix}0\\ f_n\end{pmatrix},$$

where

$$f_n(\tau) = \sum_{p=0}^{n-1} e^{\lambda\tau}\frac{(-\tau)^p}{p!}(\lambda-B)^{-(n-p)}(g(0)+\eta) + \frac{1}{(n-1)!}\int_\tau^0 (\sigma-\tau)^{n-1}e^{\lambda(\tau-\sigma)}g(\sigma)\,d\sigma$$

for $\tau\in[-r,0]$. Since B generates a C_0-semigroup on E, it follows from the Hille–Yosida theorem that

$$\begin{aligned}
&\|f_n(\tau)\|_E\\
&\le \frac{M}{(\lambda-\omega)^n}e^{\lambda\tau}\|\eta\|_E\sum_{p=0}^{n-1}\frac{[(\omega-\lambda)\tau]^p}{p!}\\
&\quad + \|g\|_{C_E}\left[\frac{M}{(\lambda-\omega)^n}e^{\lambda\tau}\sum_{p=0}^{n-1}\frac{[(\omega-\lambda)\tau]^p}{p!} + \frac{1}{(n-1)!}\int_\tau^0(\sigma-\tau)^{n-1}e^{\lambda(\tau-\sigma)}\,d\sigma\right]\\
&\le \frac{M}{(\lambda-\omega)^n}e^{\lambda\tau}e^{(\omega-\lambda)\tau}\|\eta\|_E\\
&\quad + \|g\|_{C_E}\left[\frac{M}{(\lambda-\omega)^n}e^{\lambda\tau}\sum_{p=0}^{n-1}\frac{[(\omega-\lambda)\tau]^p}{p!} + \frac{M}{(n-1)!}\int_\tau^0(-\sigma)^{n-1}e^{(\lambda-\omega)\sigma}\,d\sigma\right]\\
&\le \frac{M}{(\lambda-\omega)^n}\|\eta\|_E + \|g\|_{C_E}[\ldots]
\end{aligned}$$

for $\tau\in[-r,0]$. If we compute the term $[\ldots]$ we obtain

$$\frac{M}{(\lambda-\omega)^n}e^{\lambda\tau}\sum_{p=0}^{n-1}\frac{[(\omega-\lambda)\tau]^p}{p!} + \frac{M}{(n-1)!}\int_\tau^0(-\sigma)^{n-1}e^{(\lambda-\omega)\sigma}\,d\sigma = \frac{M}{(\lambda-\omega)^n}$$

for $\tau\in[-r,0]$. Therefore,

$$\|(\lambda-\mathcal{A})^{-n}\| \le \frac{M}{(\lambda-\omega)^n} \quad \text{for all } \lambda>\omega \text{ and } n\in\mathbb{N}. \blacksquare$$

Since $D(B)$ is dense in E, we have $X_0 := \overline{D(\mathcal{A})} = \{0\}\times C_E$, which we identify with C_E. From Proposition 1.1 it follows that the part $\mathcal{A}_0$ of $\mathcal{A}$ in X_0 generates a C_0-semigroup on X_0, which we can identify with the following C_0-semigroup on C_E:

$$(T_0(t)f)(\tau) = \begin{cases} f(t+\tau) & \text{if } -r\le t+\tau\le 0,\\ S(t+\tau)f(0) & \text{if } t+\tau\ge 0,\end{cases}$$

for $\tau\in[-r,0]$ (see [Na], B-IV, p. 220). Let $(T_{-1}(t))_{t\ge0}$ denote the extrapolated semigroup of $(T_0(t))_{t\ge0}$ on X. Since $L(\cdot)\in C([0,T],\mathcal{L}_s(C_E,E))$ and

$X \hookrightarrow F(\mathcal{A}_{-1})$, we have

$$C(\cdot) := \begin{pmatrix} 0 & L(\cdot) \\ 0 & 0 \end{pmatrix} \in C([0,T], \mathcal{L}_s(X_0, F(\mathcal{A}_{-1}))).$$

Consequently, the assumptions of Theorem 2.3 are satisfied and hence there exists an evolution family $(U(t,s))_{(t,s)\in\Delta}$ on C_E satisfying

$$(10)\qquad \begin{pmatrix} 0 \\ U(t,s)f \end{pmatrix} = \begin{pmatrix} 0 \\ T_0(t-s)f \end{pmatrix} + \int_s^t T_{-1}(t-\sigma) \begin{pmatrix} L(\sigma)U(\sigma,s)f \\ 0 \end{pmatrix} d\sigma,$$

for $f \in C_E$ and $(t,s) \in \Delta$.

We can now state the main result of this section.

THEOREM 3.2. *Let $(B, D(B))$ be a generator of a C_0-semigroup $(S(t))_{t\ge 0}$ on a Banach space E. Let $L(\cdot) \in C([0,T], \mathcal{L}_s(C_E, E))$. For every $f \in C_E$, the non-autonomous retarded partial differential equation* (NRDE) *has a unique mild solution x on $[0,T]$. This means that $x \in C([0,T], E)$ and satisfies*

$$(11)\qquad \begin{cases} x(t) = S(t-s)f(0) + \int_s^t S(t-\sigma)L(\sigma)x_\sigma\, d\sigma, \\ x(\tau) = f(\tau - s), \quad s - r \le \tau \le s,\ 0 \le s < t \le T. \end{cases}$$

Moreover, the function x is given by

$$(12)\qquad x(t) = (U(t,s)f)(0), \quad 0 \le s \le t \le T,$$

where $(U(t,s))_{(t,s)\in\Delta}$ is the evolution family given by (10).

Proof. First, we remark that for all $\eta \in E$ we have

$$R_1 \begin{pmatrix} \eta \\ 0 \end{pmatrix} := \begin{pmatrix} 0 \\ e(\cdot)(I-B)^{-1}\eta \end{pmatrix} \in D(\mathcal{A}),$$

where $e(\tau) := e^\tau$, $\tau \in [-r, 0]$. It follows from (9) that

$$(13)\qquad (I - \mathcal{A})R_1 \begin{pmatrix} \eta \\ 0 \end{pmatrix} = \begin{pmatrix} \eta \\ 0 \end{pmatrix}.$$

From (10) and (13) we obtain

$$(14)\qquad \begin{pmatrix} 0 \\ U(t,s)f \end{pmatrix} = \begin{pmatrix} 0 \\ T_0(t-s)f \end{pmatrix} + \int_s^t T_{-1}(t-\sigma)(I-\mathcal{A})R_1 \begin{pmatrix} L(\sigma)U(\sigma,s)f \\ 0 \end{pmatrix} d\sigma$$

$$= \begin{pmatrix} 0 \\ T_0(t-s)f \end{pmatrix} + (I - \mathcal{A}_{-1}) \int_s^t T_0(t-\sigma) \begin{pmatrix} 0 \\ e(\cdot)(I-B)^{-1}L(\sigma)U(\sigma,s)f \end{pmatrix} d\sigma$$

$$= \begin{pmatrix} 0 \\ T_0(t-s)f \end{pmatrix} + (I - \mathcal{A}_{-1}) \begin{pmatrix} 0 \\ \int_s^t T_0(t-\sigma)e(\cdot)(I-B)^{-1}L(\sigma)U(\sigma,s)f\, d\sigma \end{pmatrix}$$

for $(t,s) \in \Delta$. If we put $\eta(\sigma) := L(\sigma)U(\sigma,s)f$ for $\sigma \in [s,t]$ and $0 \le s < t \le T$, then we have

$$h(\tau) := \int_s^t (T_0(t-\sigma)e(\cdot)(I-B)^{-1}\eta(\sigma))(\tau)\, d\sigma$$

$$= \begin{cases} \int_s^{t+\tau} S(t+\tau-\sigma)(I-B)^{-1}\eta(\sigma)\, d\sigma \\ \quad + \int_{t+\tau}^t e^{t+\tau-\sigma}(I-B)^{-1}\eta(\sigma)\, d\sigma & \text{if } t+\tau-s \ge 0, \\ \int_s^t e^{t+\tau-\sigma}(I-B)^{-1}\eta(\sigma)\, d\sigma & \text{if } -r \le t+\tau-s \le 0, \end{cases}$$

for $\tau \in [-r, 0]$. Hence, $h \in C_E^1$, $h(0) \in D(B)$ and

$$h'(\tau) = \begin{cases} \int_s^{t+\tau} S(t+\tau-\sigma)B(I-B)^{-1}\eta(\sigma)\, d\sigma \\ \quad + \int_{t+\tau}^t e^{t+\tau-\sigma}(I-B)^{-1}\eta(\sigma)\, d\sigma & \text{if } t+\tau-s \ge 0, \\ \int_s^t e^{t+\tau-\sigma}(I-B)^{-1}\eta(\sigma)\, d\sigma & \text{if } -r \le t+\tau-s \le 0, \end{cases}$$

for $\tau \in [-r, 0]$ and $0 \le s < t \le T$. Consequently, $\binom{0}{h} \in D(\mathcal{A})$ and

$$(I - \mathcal{A}_{-1}) \begin{pmatrix} 0 \\ h \end{pmatrix}(\tau) = (I - \mathcal{A}) \begin{pmatrix} 0 \\ h \end{pmatrix}(\tau)$$

$$= \begin{pmatrix} h'(0) - Bh(0) \\ h - h' \end{pmatrix}(\tau) = \begin{pmatrix} 0 \\ h - h' \end{pmatrix}(\tau)$$

$$= \begin{pmatrix} 0 \\ \begin{cases} \int_s^{t+\tau} S(t+\tau-\sigma)\eta(\sigma)\, d\sigma & \text{if } t+\tau-s \ge 0 \\ 0 & \text{if } -r \le t+\tau-s \le 0 \end{cases} \end{pmatrix}$$

for $\tau \in [-r, 0]$ and $0 \le s < t \le T$. So by (14) we obtain

$$(15)\qquad (U(t,s)f)(\tau) = \begin{cases} S(t+\tau-s)f(0) \\ \quad + \int_s^{t+\tau} S(t+\tau-\sigma)L(\sigma)U(\sigma,s)f\, d\sigma & \text{if } t+\tau-s \ge 0, \\ f(t+\tau-s) & \text{if } -r \le t+\tau-s \le 0, \end{cases}$$

for $\tau \in [-r, 0]$ and $0 \le s < t \le T$. If, for $0 \le s < t \le T$, we put $x(t) := (U(t,s)f)(0)$, then it follows from (15) that

$$(16)\qquad x(t) = S(t-s)f(0) + \int_s^t S(t-\sigma)L(\sigma)U(\sigma,s)f\, d\sigma$$

and

$$(17)\qquad (U(t,s)f)(\tau) = \begin{cases} f(t+\tau-s) & \text{if } -r \le t+\tau-s \le 0, \\ x(t+\tau) & \text{if } t+\tau-s \ge 0. \end{cases}$$

If we extend x to the interval $[s-r,s]$ by

$$x(\tau)=f(\tau-s),\qquad \tau\in[s-r,s],$$

then from (17) we obtain

$$U(t,s)f=x_t\quad \text{for } (t,s)\in\Delta. \tag{18}$$

Therefore, it follows from (16) that (11) holds and the theorem is proved. ∎

Let I be an interval and F be a Banach space. We denote by $C^\theta(I,F)$ the space of all functions f Hölder continuous with exponent θ, $0<\theta<1$, on I, i.e., there is a constant C such that

$$\|f(t)-f(s)\|_F\le C|t-s|^\theta\quad \text{for } s,t\in I.$$

The Hölder space of order θ, $0<\theta<1$, associated with a C_0-semigroup $(S(t))_{t\ge0}$ with generator $(B,D(B))$ on a Banach space E is

$$D_B(\theta,\infty):=\Big\{x\in E:\ \sup_{t\in(0,T]}\frac{1}{t^\theta}\|S(t)x-x\|_E<\infty\Big\}.$$

Then $D_B(\theta,\infty)$ is the Lions interpolation space between $D(B)$ and E (see [Li]).

Let $s\in[0,T]$. We say that (NRDE) has a classical solution x on $[0,T]$ if x is continuous on $[0,T]$, continuously differentiable on $(s,T]$, $x(t)\in D(B)$ for $t\in(s,T]$ and (NRDE) is satisfied on $[0,T]$.

We establish some sufficient conditions on the semigroup $(S(t))_{t\ge0}$ and $L(\cdot)$ to assure that the mild solution given by (11) is a classical solution of (NRDE).

PROPOSITION 3.3. *Let $(B,D(B))$ be a generator of an analytic semigroup $(S(t))_{t\ge0}$ on a Banach space E. Let $L(\cdot)\in C^\theta([0,T],\mathcal{L}(C_E,E))$. For every $f\in C_E$ such that $f(0)\in D_B(\theta,\infty)$ the mild solution x on $[0,T]$ given by* (11) *is a classical solution of the non-autonomous retarded differential equation* (NRDE).

Proof. Let $s\in[0,T]$. From (11) we have

$$x_s(t)=S(t)f(0)+\int_0^t S(t-\sigma)L_s(\sigma)(x_s)_\sigma\,d\sigma,\qquad t\in[0,T], \tag{19}$$

where $L_s(\sigma)=L(\sigma+s)$. It follows from (18) and Theorem 2.3 that the function $g:=L_s(\cdot)(x_s)_\cdot:[0,T]\to E$ is continuous. In particular $g\in L^{1/(1-\theta)}(0,T;E)$. So by Theorem 4.3.1 of [Pa], we have

$$v\in C^\theta([0,T],E),$$

where $v(t):=\int_0^t S(t-\sigma)L_s(\sigma)(x_s)_\sigma\,d\sigma$ $(t\in[0,T])$.

Since $f(0)\in D_B(\theta,\infty)$, the function $[0,T]\ni t\mapsto S(t)f(0)$ belongs to $C^\theta([0,T],E)$ and therefore the function x_s given by (19) is Hölder continuous with exponent θ from $[0,T]$ into E. From $L(\cdot)\in C^\theta([0,T],\mathcal{L}(C_E,E))$ we obtain $g\in C^\theta([0,T],E)$. Hence it follows from Corollary 4.3.3 of [Pa] that x_s is continuously differentiable on $(0,T]$, $x_s(t)\in D(B)$ for $t\in(0,T]$ and

$$x'(s+t)=Bx(s+t)+L(s+t)x_{t+s},\quad t\in(0,T].$$

This implies that x is a classical solution of (NRDE) on $[0,T]$. ∎

The following example gives a concrete application of Proposition 3.3.

EXAMPLE 3.4. Let Ω be a bounded open set of $\mathbb{R}^n$ with sufficiently smooth boundary Γ. On the Banach space $E:=C_0(\overline{\Omega})$ of all continuous functions vanishing at the boundary Γ we consider the Laplace operator Δ_x with maximal domain $D(\Delta_x)$. Then it is well known that Δ_x generates an analytic semigroup $(S(t))_{t\ge0}$.

Let $L\in\mathcal{L}(E)$ and $k:[0,T]\times\overline{\Omega}\ni(t,x)\mapsto k(x,t)\in\mathbb{R}$ be uniformly Hölder continuous with exponent $\theta\in(0,1)$ with respect to the variable t. For $t\in[0,T]$, we consider the bounded linear operator $L(t):C_E\to E$ defined by

$$(L(t)f)(x)=k(t,x)(Lf(-r))(x)\quad \text{for } x\in\overline{\Omega}.$$

Then

$$L(\cdot)\in C^\theta([0,T],\mathcal{L}(C_E,E)).$$

On the other hand, for $\theta\in(0,1)$ one has (see [Lu], Thm. 2.10, and [Lu1], Thm. 3.1.29)

$$D_{\Delta_x}(\theta,\infty)=\begin{cases}C_0^{2\theta}(\overline{\Omega}), & \theta\ne1/2,\\ C_0^{2}(\overline{\Omega}), & \theta=1/2,\end{cases}$$

where

$$C_0^{p+\alpha}(\overline{\Omega}):=\{u\in C^p(\overline{\Omega}),u^{(p)}\in C^\alpha(\overline{\Omega})\text{ and } u=0\text{ on }\Gamma\}\qquad (p\in\mathbb{N},\ \alpha\in(0,1)).$$

So by Proposition 3.3, we see that for $f(0)\in C_0^{2\theta}(\overline{\Omega})$, $\theta\in(0,1)$, $\theta\ne1/2$ (or $f(0)\in C_0^2(\overline{\Omega})$, if $\theta=1/2$), the retarded partial differential equation

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x)=\Delta_x u(t,x)+k(t,x)(Lu(t-r))(x), & x\in\overline{\Omega},\ 0\le s<t\le T,\\ u(\tau,\cdot)=f(\tau-s), & s-r\le\tau\le s,\end{cases}$$

has a unique classical solution.

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