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EXTREMAL FUNCTIONS FOR MOSER'S INEQUALITY

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ABSTRACT. Let Ω be a bounded smooth domain in R^n, and $u(x)$ a C^1 function with compact support in Ω. Moser's inequality states that there is a constant c_o, depending only on the dimension n, such that

$$\frac{1}{|\Omega|}\int_\Omega e^{n\omega_{n-1}^{\frac{1}{n-1}} u^{\frac{n}{n-1}}}\, dx \le c_o,$$

where $|\Omega|$ is the Lebesgue measure of Ω, and ω_{n-1} the surface area of the unit ball in R^n. We prove in this paper that there are extremal functions for this inequality. In other words, we show that the

$$\sup\{\frac{1}{|\Omega|}\int_\Omega e^{n\omega_{n-1}^{\frac{1}{n-1}} u^{\frac{n}{n-1}}}\, dx : u \in W_o^{1,n}, \|\nabla u\|_n \le 1\}$$

is attained. Earlier results include Carleson-Chang (1986, Ω is a ball in any dimension) and Flucher (1992, Ω is any domain in 2-dimensions).

1. INTRODUCTION

Let Ω be a bounded smooth domain in R^n, and $u(x)$ a C^1 function supported in Ω with $\|\nabla u\|_q < n$. Sobolev's Imbedding Theorem says that if $1 \le q < n$, then

$$\|u\|_p \le C(n,q), \tag{1}$$

where $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, and $C(n,q)$ is a constant independent of the function u, as well as the domain Ω. The imbedding is no longer valid when $q = n$. Indeed, there are unbounded functions whose gradients are in L^n. However, Trudinger [14] in 1967 proved that if $\|\nabla u\|_n \le 1$, then u is in an exponential class. More precisely, the integral

$$\int_\Omega e^{\beta_o u^{\frac{n}{n-1}}}\, dx,$$

is uniformly bounded, for some positive β_0 depending only on dimension. Moser [12] in 1971 then found the best exponent β_0. He showed if $\|\nabla u\|_n \le 1$, then

$$\frac{1}{|\Omega|}\int_\Omega e^{n\omega_{n-1}^{\frac{1}{n-1}} u^{\frac{n}{n-1}}}\, dx \le c_0, \tag{2}$$

where c_0 is a constant depending only on n. (ω_{n-1} is the surface area of the unit ball in R^n.)

The aim of this paper is to prove the following:

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Theorem 1. *There are extremal functions for Moser's inequality* (2)*. In other words, the*

$$\sup\{\frac{1}{|\Omega|}\int_\Omega e^{n\omega_{n-1}^{\frac{1}{n-1}}u^{\frac{n}{n-1}}}\,dx : u\in W_o^{1,n}, \|\nabla u\|_n\le 1\}$$

is attained.

The first result in this direction is due to Carleson-Chang [3], who proved in 1986 that there are extremals when Ω is a ball in any dimension. Their result came as a surprise, since it was known at that time that no extremals exist for Sobolev's inequality (1) when Ω is a ball. (See an account of this in the more expository article [10].) In 1992, M. Flucher [5] proved the same existence for any bounded smooth domain in 2-dimensions. Though our result is an improvement, the method of the proof relies on both heavily. The key ingredient is the use of n-Green's functions, the singular solutions to the n-Laplacian. As to the solvability of the corresponding Euler equation, see Adimurthi [1] and Struwe [6].

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2. Outline of Proof

By $W_o^{1,n}(\Omega)$ we mean the Sobolev space of functions vanishing on the boundary $\partial\Omega$ with $\|\nabla u\|_n<\infty$, and we denote by $F_\Omega(u)$ the Moser functional

$$\int_\Omega\left(e^{n\omega_{n-1}^{\frac{1}{n-1}}u^{\frac{n}{n-1}}}-1\right)dx.$$

(the term -1 in the integrand is introduced for convenience). We now describe the outline of the proof. Let $\{u_j\}$ be a maximizing sequence for (2), that is, $\{u_j\}\subset W_o^{1,n}(\Omega)$, $\|\nabla u_j\|_n\le 1$, and $F_\Omega(u_j)$ tends to the supremum. We get for free from funtional analysis that we can extract a subsequence, still denoted by $\{u_j\}$, which satisfies

$$\|\nabla u_j\|_n\le 1,\ u_j\rightharpoonup u \textit{ weakly, and } |\nabla u(x)|^n dx\rightharpoonup d\mu \textit{ weakly,} \tag{3}$$

where u is a function in $W_o^{1,n}(\Omega)$, and $d\mu$ a finite measure on Ω. Our goal is to prove that $F_\Omega(u_j)\to F_\Omega(u)$ (u will then be an extremal).

The main difficulty in this type of problem is that the Moser funtional $F_\Omega(u)$ is not compact. In other words, there exists a sequence of functions $\{u_j\}$ which satisfies all the conditions in (3), but $F_\Omega(u_j)$ fails to converge to $F_\Omega(u)$. Here is an example. Take Ω to be the unit ball in R^n, and define u_a to be $c_a\log\frac{1}{|x|}$ for $a\le|x|\le 1$, and a constant d_a for $0\le|x|\le a$, where d_a is chosen so that the functions are continuous, and c_a chosen so that $\|\nabla u_a\|_n=1$. It is easy to see that as $a\to 0, u_a\rightharpoonup u=0$ weakly, $|\nabla u_a(x)|^n dx\rightharpoonup\delta_0=$ the Dirac measure at 0 weakly, and that $\limsup F_\Omega(u_a)>F_\Omega(0)$.

All is not lost, however. P. L. Lions [11] was able to show that this is the only thing that can go wrong.

Theorem 2 (P. L. Lions)**.** *Suppose* $\{u_j\}$ *is a sequence satisfying* (3)*. Then, either* (a) *the compactness holds, i.e.,* $F_\Omega(u_j)\to F_\Omega(u)$*; or* (b) $\{u_j\}$ *concentrates at some point* x_0*, i.e.,* $u_j\rightharpoonup u=0$ *weakly, and* $|\nabla u(x)|^n dx\rightharpoonup\delta_{x_0}$.

This is the so-called the *concentration-compactness principle* for the Moser functional. See Flucher [3] for another proof. So far, we haven't used the condition that u_j is *maximizing*. In the following, we'll show that maximizing sequences *never* concentrate. To do this, we first quantify the concentration phenonmenon. The following notion was introduced in Flucher [5].

Definition 1. Let x_0 be a point in $\bar{\Omega}$. The *concentration function* at x_0 is defined to be

$$C_\Omega(x_0) = \sup\{\limsup F_\Omega(u_j) : \|\nabla u\|_n \leq 1, \{u_j\} \textit{ concentrates at } x_0\}.$$

Obviously, we have $\sup_u F_\Omega(u) \geq \sup_x C_\Omega(x)$. And, in view of Lions's concentration-compactness principle, it now suffices to prove

$$\sup_u F_\Omega(u) > \sup_x C_\Omega(x).$$

In fact, this was how Carleson-Chang [3] proved their theorem on a ball:

Theorem 3 (Carleson-Chang)**.** *Let B be the unit ball in R^n. Then*
(a) $\sup_x C_B(x) = C_B(0) = e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}}|B|$,
(b) $\sup_u F_B(u) > e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}}|B|$.

Now we switch from balls to a general domain Ω. When we do so, both the sup of Moser functional and that of the concentration function will change. The key observation is that the *ratio* of functional over concentration will only *increase*.

Theorem 4.

$$\frac{\sup_u F_\Omega(u)}{\sup_x C_\Omega(x)} \geq \frac{\sup_u F_B(u)}{\sup_x C_B(x)}.$$

Thus our Theorem 1 is a consequence of Lions's Theorem 2, Carleson-Chang's Theorem 3 and Theorem 4. The appearance of Theorem 4 bears some resemblance to the classical isoperimetric inequality. Indeed, somewhere in the proof, we do use the classical isoperimetric inequality.

The proof of Theorem 4 consists of two parts: one is the comparison of the concentration function on Ω and on the ball B; the other is the comparison of the Moser functional on these two domains. More, precisely, we'll prove

Theorem 5. (a) *For every x in Ω, $C_\Omega(x) = r_\Omega^n(x) C_B(0)$,*
(b) $\sup_u F_\Omega(u) \geq (\sup_x r_\Omega^n(x)) \sup_u F_B(u)$.

The factor (without the n-th power), $r_\Omega(x)$, that appears in both formulas is what we call the *n-harmonic radius*, which will depend only on the point x and the domain Ω. It is obvious that Theorem 5 implies Theorem 4. We now digress to define the n-Green's functions and the n-harmonic radius.

Definition 2. Let x_0 be a point in Ω. The *n-Green's function* $G = G_{x_0} = G_{\Omega,x_0}$ on Ω with pole at x_0 is the singular solution to the *n-Laplacian*:

$$\Delta_n G = Div(|\nabla G|^{n-2}\nabla G) = \delta_{x_0} \textit{ in } \Omega, \quad G = 0 \textit{ on } \partial\Omega. \tag{4}$$

In terms of distributions, equation (4) means

$$\int_\Omega |\nabla G|^{n-2}\nabla G \cdot \nabla\phi \, dx = \phi(x_0), \tag{5}$$

for every compactly supported smooth function $\phi(x)$ on Ω. Of course, when $n = 2$, the 2-Laplacian is the usual Laplacian, and 2-Green's function is the usual Green's function. In higher dimensions, the existence and uniqueness of this G is also well-known, see [6] and [7], for example. The n-Green's function on the ball B with pole at the origin is $G_0 = -\omega_{n-1}^{-\frac{1}{n-1}} \log |x|$, and for general domain we have the following asymptotic expansion:

$$G_{x_0}(x) = -\omega_{n-1}^{-\frac{1}{n-1}} \log |x - x_0| - H_{x_0}(x), \tag{6}$$

where $H_{x_0}(x)$ is a continuous function on Ω and is $C^{1,\alpha}$ in $\Omega \setminus \{x_0\}$.

Definition 3. The *n-harmonic radius* at x_0 is defined to be

$$r_\Omega(x_0) = e^{-\omega_{n-1}^{\frac{1}{n-1}} H_{x_0}(x_0)}.$$

Remark 1. When $n = 2$, and Ω simply-connected, one can use the invariance of Green's functions under conformal mappings to see that the n-harmonic radius is nothing but $|f'(0)|$, where $f(z)$ is a conformal mapping from the unit disc to Ω with $f(0) = x_0$. See [2].

Remark 2. In higher dimensions, n-Green's functions are invariant under Möbius transformations. (The usual Green's functions are not.) As a consequence, one can compute the conformal radius at $x_0 \in B$ as $1 - |x_0|^2$.

We return to the discussion of the proof of our Theorem 1, which now reduces to that of Theorem 5. To prove Theorem 5, we'll need to transplant functions, either from a general domain Ω to a ball, or from a ball to Ω. In doing so, we have to keep the functions in the same class, i.e., $\|\nabla u\|_n \leq 1$, and, at the same time, to obtain a relation between the functional on the two domains. For the direction from Ω to Ω^*(the symmetrized domain of Ω, which is a ball), the *classical rearrangement* u^* is the main tool. (See [7].) Recall that $|\{u^* > t\}| = |\{u > t\}|$, and

Theorem 6. *For* $u \in W_0^{1,n}(\Omega)$, *we have*
(a) $\|\nabla u^*\|_{L^n(\Omega^*)} \leq \|\nabla u\|_{L^n(\Omega)}$,
(b) $F_{\Omega^*}(u^*) = F_\Omega(u)$.

For the other direction, from B to the unit ball Ω, we use the *n-harmonic transplantation*, which is defined via the level sets of n-Green's functions.

Definition 4. Let x_0 be a point in Ω, and v_0 a decreasing, radial function on B. The *n-harmonic transplantation* of v_0 on Ω at x_0 is defined to be $v_{x_0} = v_{\Omega,x_0} = v_0 \circ G_{B,0}^{-1} \circ G_{\Omega,x_0}$.

So, v_{x_0} has the same level sets as G_{x_0} does. Furthermore, v_{x_0} and v_0 agree on the corresponding level sets of G_{x_0} and G_0. The analogous result to Theorem 6, when we move functions from B to Ω, is:

Theorem 7. *For a radial, decreasing function* v_0 *in* $W_0^{1,n}(B)$, *we have*
(a) $\|\nabla v_{x_0}\|_{L^n(\Omega)} = \|\nabla v_0\|_{L^n(B)}$,
(b) $F_\Omega(v_{x_0}) \geq r_\Omega^n(x_0) F_B(v_0)$.

We will prove some properties about n-Green's functions in the next section. The proofs of Theorems 7 and 5 are presented in the last section.

3. n-Green's Functions

We develop in this section some important properties about the n-Green's function.

Lemma 1. *Let $G = G_{x_0}(x)$.*

(a)

$$\int_{\{G<t\}} |\nabla G|^n \, dx = t \ \ for\ every\ t,$$

(b)

$$\int_{\partial\{G>t\}} |\nabla G|^{n-1} \, dx = 1 \ \ for\ every\ t.$$

(c) *The sets $\{G > t\}$ form a sequence of approximately small balls of radii $\rho_t = r_\Omega(x_0)e^{-\omega_{n-1}^{\frac{1}{n-1}} t}$. In other words, $B(x_0, \rho_t - r_t) \subset \{G > t\} \subset B(x_0, \rho_t + r_t)$, with $r_t/\rho_t \to 0$ as $t \to \infty$. In particular,*

$$\lim_{t\to\infty} \frac{|\{G > t\}|}{\alpha_n e^{-n\omega_{n-1}^{\frac{1}{n-1}} t}} = r_\Omega^n(x_0),$$

where α_n is the volume of the unit ball in R^n. (d) *On the set $\{G = t\}$, we have*

$$|\nabla G(x)| = \omega_{n-1}^{\frac{-1}{n-1}} \frac{1}{\rho_t} + O(1)\ uniformly,\ as\ t \to \infty.$$

Proof. (a) Choose a smooth approximation of the function $\phi(x) = \inf\{G(y), t\}$ in equation (5).

(b) follows from equation (4) via an integration by parts.

(c) Solving for $|x - x_0|$ in (6), we get

$$|x - x_0| = e^{-\omega_{n-1}^{\frac{1}{n-1}} t} e^{-\omega_{n-1}^{\frac{1}{n-1}} H_{x_0}(x)} = e^{-\omega_{n-1}^{\frac{1}{n-1}} t} e^{-\omega_{n-1}^{\frac{1}{n-1}} H_{x_0}(x_0)}$$
$$+ (e^{-\omega_{n-1}^{\frac{1}{n-1}} H_{x_0}(x)} - e^{-\omega_{n-1}^{\frac{1}{n-1}} H_{x_0}(x_0)}) e^{-\omega_{n-1}^{\frac{1}{n-1}} t} = \rho_t + r_t.$$

It is easy to see that $r_t/\rho_t \to 0$ as $t \to \infty$, by the continuity of $H_{x_0}(x)$ at x_0.

(d) On $\{G = t\}$, we have

$$|\nabla G(x)| = |-\omega_{n-1}^{\frac{-1}{n-1}} \frac{x - x_0}{|x - x_0|^2} - \nabla H_{x_0}(x)| = \omega_{n-1}^{\frac{-1}{n-1}} \frac{1}{\rho_t} + O(1),$$

by the $C^{1,\alpha}$ property of $H_{x_0}(x)$ in $\Omega \setminus \{x_0\}$ and (c). □

Lemma 2. *For domains Ω in R^n,*

$$\sup_x r_\Omega(x) \le \sup_x r_{\Omega^*}(x) = r_{\Omega^*}(0).$$

Proof. We have from (c) of Lemma 1:

$$r_\Omega^n(x) = \lim_{t\to\infty} \frac{|\{G_{\Omega,x} > t\}|}{\alpha_n e^{-n\omega_{n-1}^{\frac{1}{n-1}} t}},$$

and

$$r_{\Omega^*}^n(0) = \lim_{t\to\infty} \frac{|\{G_{\Omega^*,0} > t\}|}{\alpha_n e^{-n\omega_{n-1}^{\frac{1}{n-1}} t}}.$$

($G_{\Omega^*,0}$ is the n-Green's function on Ω^* with pole at 0.)

Now we compare the two sets, $\{G_{\Omega,x} > t\}$ and $\{G_{\Omega^*,0} > t\}$. Part (a) of Lemma 1 and Theorem 6 implies

$$t = \int_{\{G_{\Omega,x}<t\}} |\nabla G_{\Omega,x}|^n \, dx \geq \int_{\{G^*_{\Omega,x}<t\}} |\nabla G^*_{\Omega,x}|^n \, dx \geq \int_{\{v_t<t\}} |\nabla v_t|^n \, dx,$$

where v_t is the n-harmonic function sharing the same boundary values as $G^*_{\Omega,x}$ on $\{G^*_{\Omega,x} < t\}$, and the last inequality is Dirichlet's principle. This v_t must be a constant multiple of $G_{\Omega^*,0}$, say, $v_t = \lambda_t G_{\Omega^*,0}$. So, we have

$$t \geq \int_{\{G_{\Omega^*,0}<t/\lambda_t\}} \lambda_t^n |\nabla v_t|^n \, dx = t\lambda_t^{n-1}.$$

Hence $\lambda_t \leq 1$. Therefore,

$$\begin{aligned} r^n_\Omega(x) &= \lim_{t\to\infty} \frac{|\{G_{\Omega,x} > t\}|}{\alpha_n e^{-n\omega_{n-1}^{\frac{1}{n-1}} t}}, = \lim_{t\to\infty} \frac{|\{G^*_{\Omega,x} > t\}|}{\alpha_n e^{-n\omega_{n-1}^{\frac{1}{n-1}} t}} \\ &= \lim_{t\to\infty} \frac{|\{G_{\Omega^*,0} > t/\lambda_t\}|}{\alpha_n e^{-n\omega_{n-1}^{\frac{1}{n-1}} t}} \leq \lim_{t\to\infty} \frac{|\{G_{\Omega^*,0} > t/\lambda_t\}|}{\alpha_n e^{-n\omega_{n-1}^{\frac{1}{n-1}} t/\lambda_t}} = r^n_{\Omega^*}(0). \ \Box \end{aligned}$$

Lemma 3. *For every $0 < r \leq 1$, we have*

$$\frac{1}{(\omega_{n-1}^{\frac{1}{n-1}} r)^n} \int_{\partial\{G > -\omega_{n-1}^{-\frac{1}{n-1}} \log r\}} \frac{1}{|\nabla G|} \, ds \geq r^n_\Omega(x_0),$$

and the inequality tends to be an equality, as $r \to 0$.

Proof. The isoperimetric inequality for domains A in R^n says that

$$|A| \leq \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \left(\int_{\partial A} ds \right)^{\frac{n}{n-1}}.$$

If we take A to be $\{G > -\omega_{n-1}^{\frac{1}{n-1}} \log r\}$, then we have

$$\begin{aligned} |A| &\leq \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \left(\int_{\partial A} |\nabla G|^{\frac{n-1}{n}} \frac{1}{|\nabla G|^{\frac{n-1}{n}}} \, ds \right)^{\frac{n}{n-1}} \\ &\leq \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \left\{ \left(\int_{\partial A} |\nabla G|^{n-1} \, ds \right)^{\frac{1}{n}} \left(\int_{\partial A} \frac{1}{|\nabla G|} \, ds \right)^{\frac{n-1}{n}} \right\}^{\frac{n}{n-1}} \\ &= \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \int_{\partial A} \frac{1}{|\nabla G|} \, ds. \end{aligned}$$

On the other hand, we can estimate $|A|$ from below in terms of $r_\Omega(x_0)$. Since $G_{A,x_0}(x) = G_{\Omega,x_0}(x) + \omega_{n-1}^{\frac{1}{n-1}} \log r$, we have $H_{A,x_0}(x) = H_{\Omega,x_0}(x) - \omega_{n-1}^{\frac{1}{n-1}} \log r$. Thus $r_A(x_0) = r \cdot r_\Omega(x_0)$. And Lemma 2 gives

$$|A| \geq \alpha_n r^n_A(x_0) = \alpha_n r^n \cdot r^n_\Omega(x_0).$$

Combining these two inequalities gives the one in the lemma. Furthermore, we have from Lemma 1,

$$\frac{1}{|\nabla G|} \sim \omega_{n-1}^{\frac{1}{n-1}} r \cdot r_\Omega(x_0), \text{ and}$$

$$|\{G = -\omega_{n-1}^{\frac{1}{n-1}} \log r\}| \sim \omega_{n-1} r^{n-1} r^{n-1}_\Omega(x_0),$$

as $r \to 0$. The asymptotic equality then follows. $\Box$

4. Proofs of Theorems

Proof of Theorem 7. (a). By the co-area formula (see [2]), the definition of v_{x_0} (which yields $\nabla v_{x_0} = \frac{|\nabla v_0|}{|\nabla G_{B,0}|}\nabla G_{x_0}$), and part (b) of Lemma 1, we have

$$\begin{aligned}\|\nabla v_{x_0}\|^n_{L^n(\Omega)} &= \int_\Omega |\nabla v_{x_0}|^n\,dx = \int_0^\infty \int_{\partial\{v_{x_0}>t\}} |\nabla v_{x_0}|^{n-1}\,ds\,dt \\ &= \int_0^\infty \frac{|\nabla v_0|^{n-1}}{|\nabla G_{B,0}|^{n-1}} \int_{\partial\{v_{x_0}>t\}} |\nabla G_{x_0}|^{n-1}\,ds\,dt = \int_0^\infty \frac{|\nabla v_0|^{n-1}}{|\nabla G_{B,0}|^{n-1}}\,dt.\end{aligned}$$

The last integral is independent of domains, so it is equal to $\|\nabla v_0\|^n_{L^n(B)}$.

(b) We let $f(t) = e^{n\omega_{n-1}^{\frac{1}{n-1}} t^{\frac{n}{n-1}}} - 1$. Again, by the co-area formula,

$$\begin{aligned}F_\Omega(v_{x_0}) &= \int_0^\infty \int_{\partial\{v_{x_0}>t\}} \frac{f(t)}{|\nabla v_{x_0}|}ds\,dt \\ &= \int_0^\infty f(t) \int_{\partial\{v_{x_0}>t\}} \frac{|\nabla G_{B,0}(v(t))|}{|\nabla v_0(z(t))|}\frac{1}{|\nabla G_{x_0}|}\,ds\,dt \\ &= \int_0^\infty f(t)\frac{|\nabla G_{B,0}(z(t))|}{|\nabla v_0(z(t))|} \int_{\partial\{G>G_{B,0}(z(t))\}} \frac{1}{|\nabla G|}ds\,dt, \\ &= \int_0^\infty f(t)\left(\frac{1}{\omega_{n-1}^{\frac{1}{n-1}}|z(t)|}\right)\left(\frac{1}{\omega_{n-1}|z(t)|^{n-1}}\right) \\ &\quad\times\left(\int_{\partial\{v_0>t\}} \frac{1}{|\nabla v_0|}\,ds\right)\left(\int_{\partial\{G>G_{B,0}(z(t))\}} \frac{1}{|\nabla G|}\,ds\right)dt \\ &= \int_0^\infty f(t)\left(\int_{\partial\{v_0>t\}} \frac{1}{|\nabla v_0|}\,ds\right)\left(\frac{1}{\omega_{n-1}^{\frac{n}{n-1}}|z(t)|^n} \int_{\partial\{G>G_{B,0}(z(t))\}} \frac{1}{|\nabla G|}ds\right)dt \\ &\geq r^n_\Omega(x_0)\int_0^\infty f(t)\int_{\partial\{v_0>t\}} \frac{1}{|\nabla v_0|}ds\,dt = r^n_\Omega(x_0)\int_B f(v_0)dx = r^n_\Omega(x_0)F_B(v_0).\end{aligned}$$

In the above formulas, $z(t)$ is a point in B such that $v_0(z(t)) = t$. □

Proof of Theorem 5. We prove part (b) first. Let $v_0(x)$ be an extremal function which realizes $\sup_u F_B(u)$, as assured by Carleson-Chang's Theorem 3. We may assume this $v_0(x)$ is radial and decreasing on B, by Theorem 6. Now, Theorem 7 says every conformal rearrangement v_{x_0} satisfies $F_\Omega(v_{x_0}) \geq r^n_\Omega(x_0)F_B(v_0)$. Taking the supremum over x_0 in B gives us (b).

To prove (a), we first take a concentrating sequence $\{v_j\}$ on B which realizes $C_B(0)$. Theorem 7 gives us a sequence $\{v_{j,x_0}\}$ on Ω. The same argument for proving (a) of Theorem 7 yields

$$\int_{\{G_{\Omega,x_0}<t\}} |\nabla v_{j,x_0}|^n\,dx = \int_{\{G_{B,0}<t\}} |\nabla v_j|^n\,dx,$$

which tends to 0, as $j \to \infty$, for every t. So $\{v_{j,x_0}\}$ concentrates at x_0. Futhermore, as in the proof of (b) of Theorem 7, we have

$$F_\Omega(v_{j,x_0}) = \int_0^\infty f(t) \left(\int_{\partial\{v_j > t\}} \frac{1}{|\nabla v_j|} ds \right) \times \left(\frac{1}{\omega_{n-1}^{\frac{n}{n-1}} |z_j(t)|^n} \int_{\partial\{G_{x_0} > G(z_j(t))\}} \frac{1}{|\nabla G_{x_0}|} ds \right) dt.$$

The second inner integral converges to $r^n_\Omega(x_0)$ uniformly in t, as $j \to \infty$, by Lemma 3. And the rest of the integral is nothing but $F_B(v_j)$. So, $F_\Omega(v_{j,x_0}) \to r^n_\Omega(x_0) C_B(0)$. This gives $C_\Omega(x_0) \geq r^n_\Omega(x_0) C_B(0)$.

For the other direction of (a), take a sequence $\{u_j\}$ on Ω realizing $C_\Omega(x_0)$. We first argue that that u_j must behave like $\lambda_j G_{x_0}$ off $\{x_0\}$, where $\lambda_j \to 0$. To see this, note the sets $\{u_j > 1\}$ are contained in balls $B(x_0, r_j)$, with $r_j \to 0$. We then replace u_j on $A_j = \{u_j \leq 1\}$ by an n-harmonic function which agrees with u_j on ∂A_j. (We still call the new sequence $\{u_j\}$.) This will not increase the norm of the gradient, by Dirichlet's principle. Futhermore, if we fix a point $y \neq x_0$, and set $\lambda_j = u_j(y)/G_{x_0}(y)$, then $\lambda_j \to 0$, and $u_j/\lambda_j \to G_{x_0}$ locally uniformly off x_0. To see the last statement, take a compact set K, containing y, but not x_0. Harnack's inequality (see [4]) says that the sequence $\{u_j/\lambda_j\}$ is uniformly bounded on K, so it is equicontinuous on K. Hence it converges uniformly on K. The limit must be n-harmonic, and equal to G_{x_0}.

Next, we obtain from Theorem 6 the sequence of symmetrized functions u_j^* on Ω^*, which satisfies $\|\nabla u_j^*\|_{L^n(\Omega^*)} \leq 1$, and $F_{\Omega^*}(u_j^*) = F_\Omega(u_j)$. It is easy to see that $\{u_j^*\}$ concentrates at 0. To get the conformal factor $r^n_\Omega(x_0)$, we would like to dilate $u_j(x)$ to $u_j^*(\frac{x}{r_\Omega(x_0)})$. This will not change the norm of the gradient, and the functional will have the desired conformal factor. However, the new function $u_j^*(\frac{x}{r_\Omega(x_0)})$ is supported on the set $1/r_\Omega(x_0) \cdot \Omega^*$, which is larger than the unit ball B. To remedy the situation, we take the part of u_j^*, where $u_j^* > 1$, over to the unit ball B, and dilate it so that it matches with $\lambda_j G_{B,0}$. (The latter is defined on the rest of B.) In other words, we are defining a function v_j on B, so that $v_j(z) = \lambda_j G_{B,0}(z)$ for values ≤ 1; and $v_j(z) = u_j^*(\eta_j z)$ for values > 1, where η_j is chosen so that the two pieces fit together. Notice, by part (c) of Lemma 1, that the radii of of the sets $\{u_j^* > 1\}$ and $\{\lambda_j G_{B,0} > 1\}$ are asymptotically equal to $r_\Omega(x_0) \exp(-\omega_{n-1}^{\frac{1}{n-1}} \frac{1}{\lambda_j})$ and $\exp(-\omega_{n-1}^{\frac{1}{n-1}} \frac{1}{\lambda_j})$, respectively. So, $\eta_j \to r_\Omega(x_0)$, as $j \to \infty$.

The sequence $\{v_j\}$ concentrates at 0, and $\|\nabla v_j\|_{L^n(B)} \leq \|\nabla u_j\|_{L^n(\Omega)} \leq 1$. Moreover, we have

$$\lim_{j\to\infty} F_\Omega(u_j) = \lim_{j\to\infty} \int_{\{u_j > 1\}} f(u_j)\, dx = \lim_{j\to\infty} \int_{\{u_j^* > 1\}} f(u_j^*)\, dx$$

$$\lim_{j\to\infty} \int_{\{u_j^*(x) > 1\}} f(v_j(x/\eta_j))\, dx = \lim_{j\to\infty} \eta_j^n \int_{\{v_j^*(x) > 1\}} f(v_j(x))\, dx$$

$$= r^n_\Omega(x_0) \lim_{j\to\infty} F_B(v_j) \leq r^n_\Omega(x_0) C_B(0).$$

This proves the other half of (a). The proof of Theorem 5 is now complete. ∎

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