# Extremal Graphs for Intersecting Cliques

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**Abstract**

For any two positive integers $n \geq r \geq 1$, the well-known Turán Theorem states that there exists a least positive integer $ex(n, K_r)$ such that every graph with $n$ vertices and $ex(n, K_r) + 1$ edges contains a subgraph isomorphic to $K_r$. We determine the minimum number of edges sufficient for the existence of $k$ cliques with $r$ vertices each intersecting in exactly one common vertex.

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## 1 Introduction

With integers $n \geq r \geq 1$, we let $T_{n,r}$ denote the *Turán graph*, i.e., the complete $r$-partite graph on $n$ vertices where each partite set has either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ vertices and the edge set consists of all pairs joining distinct parts. The number of edges in $T_{n,r}$ is denoted by $ex(n, K_{r+1})$, where $K_r$ represents the complete graph on $r$ vertices.

For a graph $G$ and a vertex $x \in V(G)$, the *neighborhood* of $x$ in $G$ is denoted by $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, or when clear, simply $N(x)$, and let $\overline{N_G(x)} = V(G) - N_G(x)$. The *degree* of $x$ in $G$, denoted by $d_G(x)$, or $d(x)$, is the

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size of $N_G(x)$. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degrees, respectively, in $G$. The order of $G$ is often denoted by $|G| = |V(G)|$. For a subset $X \subset V(G)$, let $G[X]$ denote the subgraph of $G$ induced by $X$. A *matching* in $G$ is a set of edges from $E(G)$, no two of which share a common vertex, and the *matching number* of $G$, denoted by $\nu(G)$, is the maximum number of edges in a matching in $G$.

Suppose that we are given some fixed graph $H$. What is the maximum number, $ex(n, H)$, of edges in a graph $G$ on $n$ vertices that does not contain a copy of $H$ as a subgraph (often said to *forbid* $H$)? A graph $G$ on $n$ vertices with $ex(n, H)$ edges and without a copy of $H$ is called an *extremal graph* for $H$. For $n \geq |V(H)|$, adding one more edge to any one of the extremal graphs will produce a copy of $H$.

A graph on $2k + 1$ vertices consisting of $k$ triangles which intersect in exactly one common vertex is called a $k$-fan and denoted by $F_k$. For each $k$, the chromatic number of $F_k$ is three, and so by the Erdős-Stone theorem [4], $ex(n, F_k) = (1 + o(1))n^2/4$. The following result is due to Erdős, Füredi, Gould, and Gunderson [3].

**Theorem 1** *For every $k \geq 1$, and for every $n \geq 50k^2$, if a graph G on $n$ vertices has more than*

$$\lfloor \frac{n^2}{4} \rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even} \end{cases},$$

*edges, then G contains a copy of a k-fan. Further, the number of edges is best possible.*

A graph on $(r - 1)k + 1$ vertices consisting of $k$ cliques each with $r$ vertices, which intersect in exactly one common vertex, is called a $K_r$-fan and denoted by $F_{k,r}$. The purpose of this article is to generalize Theorem 1, when $k$ and $r$ are fixed and $n$ is large, as follows.

**Theorem 2** *For every $k \geq 1$ and $r \geq 2$, and for every $n \geq 16k^3r^8$, if a graph $G$ on $n$ vertices has more than*

$$ex(n, K_r) + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even} \end{cases},$$

*edges, then $G$ contains a copy of an $F_{k,r}$-fan. Further, the number of edges is best possible.*

Note that the number $ex(n, K_r) = |E(T_{n,r-1})|$. To show the lower bound for $ex(n, F_{k,r})$ we present the following graph, $G_{n,k,r}$. For odd $k$ (where $n \geq$

$(2k-1)(r-1)+1)$ $G_{n,k,r}$ is constructed by taking a Turan graph $T_{n,r-1}$ and embedding two vertex disjoint copies of $K_k$ in one partite set. For even $k$ (where now $n \geq (2k-2)(r-1)+1$) $G_{n,k,r}$ is constructed by taking a Turán graph $T_{n,r-1}$ and embedding a graph with $2k-1$ vertices, $k^2-(3/2)k$ edges with maximum degree $k-1$ in one partite set.

## 2 Lemmas

In this section we give preparatory lemmas for the proof of the main theorem.

Define $f(\nu, \Delta) = \max\{|E(G)| : \nu(G) \leq \nu, \Delta(G) \leq \Delta\}$. Chvátal and Hanson [2] proved the following theorem.

**Theorem 3** *For every $\nu \geq 1$ and $\Delta \geq 1$,*

$$f(\nu, \Delta) = \nu\Delta + \lfloor \frac{\Delta}{2} \rfloor \lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \rfloor \leq \nu\Delta + \nu.$$

We will frequently use the following special case proved by Abbott et al. [1].

$$f(k-1, k-1) = \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

The extremal graphs are exactly those we embedded into $T_{n,r-1}$ in the previous section to obtain the extremal $F_{k,r}$-free graph $G_{n,k,r}$.

Let $a$ be a positive integer and let $X$ and $Y$ be two disjoint vertex sets of $V(G)$. We say that $X$ *dominates* $Y$ *with $a$-deficiency* if $d_Y(x) \geq |Y| - a$ for each $x \in X$. Let $V_1, V_2, \ldots, V_m$ be disjoint subsets of $V(G)$. We say that $\{V_1, V_2, \ldots, V_m\}$ *is $a$-deficiency complete* if $V_i$ dominates $V_j$ with deficiency $a$ for every pair $i \neq j$ with $i, j = 1, 2, \ldots, m$.

The following lemma will be used very heavily in our proof of the main Theorem.

**Lemma 2.1** *Let $a$ be a positive integer. Let $G$ be a graph and let $\{X_1, X_2, \ldots, X_m\}$ be an $a$-deficiency complete partition of $V(G)$ with $|X_i| \geq ma + 2t$ for each $i$. Suppose that $C_1, C_2, \ldots, C_t$ are $t$ cliques of $G$ with the properties:*

*(1) $|C_i \cap X_j| \leq 2$ for each pair $i$ and $j$,*
*(2) $|C_i \cap X_j| = 2$ for at most one $j$ for each $i$.*

*Then, there exist $t$ cliques $D_1, D_2, \ldots, D_t$ satisfying:*

*(1)* $C_i \subseteq D_i$ *for each* $i$,
*(2)* $D_1 - C_1$, $D_2 - C_2$, $\ldots$, $D_t - C_t$ *are mutually disjoint,*
*(3) For each* $i$ *we have that* $|D_i \cap X_j| = 1$ *for all* $j$ *except possibly one at which* $|D_i \cap X_j| = |C_i \cap X_j| = 2$.

*Proof:* We need to show that, if $C_i \cap X_j = \emptyset$, there exists a vertex $v_j \in X_j - \bigcup_{\ell=1}^{t} C_\ell$ such that $v_j$ is adjacent to all vertices in $C_i$. Iteration of this argument will then provide the statement. Without loss of generality, we may assume that $i = j = 1$.

Since $d_{X_1}(v) \geq |X_1| - a$ for each $v \in C_1$,

$$|\bigcap_{v \in C_1} N_{X_1}(v)| \geq |X_1| - |C_1|a \geq ma + 2t - ma \geq 2t.$$

By our assumptions, we have that $|(\bigcup_{i=2}^{t} C_i) \cap X_1| \leq 2(t-1)$, thus $\bigcap_{v \in C_1} N_{X_1}(v) - \bigcup_{i=2}^{k} C_i \neq \emptyset$. Lemma 2.1 now follows. □

**Lemma 2.2** *Let* $G$ *be a graph and* $Y_1, Y_2, \ldots, Y_m$ *be* $m$ *vertex disjoint subsets of* $V(G)$ *and* $Y_0 \subseteq V(G) - \bigcup_{i=1}^{m} Y_i$ *such that* $|Y_i| \geq (i-1)a + k$ *for each* $i = 1, \ldots, m$. *If* $Y_i$ *dominates* $Y_j$ *with* $a$*-deficiency for every* $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, m$, *and* $i \neq j$, *then, there are* $k$ *vertex disjoint cliques* $C_1$, $C_2$, $\ldots$, $C_k$ *satisfying* $|C_i| = m$ *and* $|C_i \cap Y_j| = 1$ *for each* $i$ *and* $j \geq 1$. *Furthermore, if* $|Y_0| \geq ma + k$, *then there are* $k$ *vertex disjoint cliques* $D_1$, $D_2$, $\ldots$, $D_k$ *with the property that* $|D_i| = m + 1$ *and* $|D_i \cap Y_j| = 1$ *for each* $i = 1, \ldots, k$ *and* $j = 0, 1, \ldots, m$.

*Proof:* Let $y_{1,1}, y_{1,2}, \ldots, y_{1,k}$ be $k$ arbitrary vertices in $Y_1$. Since $|N(y_{1,i}) \cap Y_2| \geq |Y_2| - a \geq k$, there are $k$ vertices $y_{2,1}, y_{2,2}, \ldots, y_{2,k}$ in $Y_2$ such that $y_{1,i}y_{2,i} \in E$ for all $i = 1, \ldots, k$. Since $|N(y_{1,i}) \cap N(y_{2,i}) \cap Y_3| \geq |Y_3| - 2a \geq k$, there are $k$ vertices $y_{3,1}$, $y_{3,2}$, $\ldots$, $y_{3,k}$ in $Y_3$ such that $y_{3,i} \in N(y_{1,i}) \cap N(y_{2,i})$ for all $i = 1$, $2, \ldots, k$. Continuing in the same fashion, we see that Lemma 2.2 follows. □

The case $k = 1$ of the main theorem is Turan's theorem, the case of $r = 2$ is trivial, and the case of $r = 3$ is Theorem 1. We assume that $k \geq 2$ and $r \geq 4$. The aim of this section it to prove the following lemma.

**Lemma 2.3** *Let* $G$ *be an extremal graph for* $F_{k,r}$ *on* $n$ *vertices with* $n \geq 4k^2r^4$, *and with minimum degree* $\delta \geq (\frac{r-2}{r-1})n - k$. *Then there exists a partition* $V(G) = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_{r-2}$, *so that* $V_i \neq \emptyset$ *for all* $i = 0, \ldots, r-2$ *and for every* $x \in V_i$, *the following hold:*

$$\sum_{j \neq i} \nu(G[V_j]) \leq k - 1 \quad \textit{and} \quad \Delta(G[V_i]) \leq k - 1; \tag{1}$$

$$d_{G[V_i]}(x) + \sum_{j \neq i} \nu(G[N(x) \cap V_j]) \leq k - 1. \qquad (2)$$

*Proof:* Since $G$ plus any edge contains a copy of $F_{k,r}$, $G$ contains $k$ edge disjoint cliques $D_1$, $D_2$, …, $D_k$ sharing one vertex $v_0$ with $|D_1| = r - 1$ and $|D_j| = r$ for all $j \geq 2$. Let $V(D_1) = \{v_0, v_1, \ldots, v_{r-2}\}$. Denote the graph induced by $\bigcup_{i=1}^{k} D_i$ by $D$. Clearly, $|D| = k(r-1)$. For each $i = 0, \ldots, r-2$, we define $X_i = \bigcap_{j \neq i} N(v_j) - V(D)$. Since $G$ does not contain $F_{k,r}$ as a subgraph,

$$X_i \cap X_j = \emptyset \text{ for } i \neq j.$$

Since the minimum degree $\delta(G) \geq \frac{r-2}{r-1}n - k$,

$$|X_i \cup V(D)| \geq \frac{n}{r-1} - (r-2)k.$$

Thus,

$$|X_i| \geq \frac{n}{r-1} - (r-2)k - k(r-1) = \frac{n}{r-1} - k(2r-3). \qquad (3)$$

For each $i \geq 1$, if there is an edge $uv \in E(G[X_i])$, replacing $v_i$ by the edge $uv$ in $D$ we obtain a copy of $F_{k,r}$, a contradiction. Thus,

$$E(G[X_i]) = \emptyset, \text{ for each } i = 1, 2, \ldots, r-2.$$

For every $x_i \in X_i$ and $i \neq 0$, since $d(x_i) \geq \frac{r-2}{r-1}n - k$, $d_{X_i}(x_i) = 0$, and $|X_i| \geq \frac{n}{r-1} - k(2r-3)$, then

$$\begin{aligned}
|\overline{N_{G-X_i}(x_i)}| &= (n - d(x_i)) - |X_i| \\
&\leq \left(\frac{n}{r-1} + k\right) - \left(\frac{n}{r-1} - k(2r-3)\right) \\
&= 2k(r-1).
\end{aligned}$$

Thus,

$$d_{G-X_i}(x_i) \geq |G - X_i| - 2k(r-1),$$

for each $x \in X_i$ where $i = 1, 2, \ldots, r-2$. In particular, we have that

$$d_{X_j}(x) \geq |X_j| - 2k(r-1) \qquad (4)$$

for each $x \in X_i$, i.e., $X_i$ dominates $X_j$ with $2k(r-1)$-deficiency, where $i = 1, 2, \ldots, r-2$, $j = 0, 1, \ldots, r-2$ and $j \neq i$.

**Claim 4** *Let $x_1$, $x_2$, …, $x_{r-2}$ be $r-2$ vertices such that $x_i \in X_i$ for each $i = 1$, …, $r-2$. Then, for any $Y_0 \subseteq X_0$ with $|Y_0| \geq 2k(r-1)^2 \geq 2k(r-1)(r-2)+k$,* *we have the following inequality*

$$|\bigcap_{i=1}^{r-2} N(x_i) \cap Y_0| \geq k.$$

*Proof:* By (4), $d_{X_0}(x_i) \geq |X_0| - 2k(r-1)$, and so

$$|\bigcap_{i=1}^{r-2} N(x_i) \cap X_0| \geq |X_0| - 2k(r-1)(r-2).$$

Claim 4 follows. □

Let $X_0^*$ denote the set of all vertices of $X_0$ of degree at least $2k(r-1)^2$ in $X_0$.

**Claim 5** $|X_0^*| \leq 2k(r-1)(r-2)$.

*Proof:* Suppose, to the contrary, $|X_0^*| > 2k(r-1)(r-2)$. For each $i$, let

$$X_0^i = \{x \in X_0^* \mid d_{X_i}(x) \geq |X_i|/(2k(r-1)+1)\}.$$

By (4), $d_{X_0}(x_i) \geq |X_0| - 2k(r-1)$ for every $x_i \in X_i$ , thus $N(S) \supseteq X_i$ for every $S \subseteq X_0^*$ with $|S| = 2k(r-1)+1$, which implies that $|X_0^i| \geq |X_0^*| - 2k(r-1)$. Therefore,

$$|\bigcap_{i=1}^{r-2} X_0^i| \geq |X_0^*| - 2k(r-1)(r-2) > 1.$$

There is an $x_0 \in X_0^*$ such that $|N(x_0) \cap X_i| \geq |X_i|/(2k(r-1)+1)$ for each $i = 1, 2, \ldots, r-2$. Recall that by (3) we have $|X_i| \geq n/(r-1) - k(2r-3)$ for each $i = 1, \ldots, r-2$. Since $n \geq 4k^2r^4$, the following inequality holds.

$$|N_{X_i}(x_0)| \geq |X_i|/(2k(r-1)+1) \geq 2k(r-1)(r-2) + k.$$

Applying Lemma 2.2 with $Y_0 = N(x_0) \cap X_0$, $Y_1 = N(x_0) \cap X_1, \ldots, Y_{r-2} = N(x_0) \cap X_{r-2}$, and $a = 2k(r-1)$, we obtain $k$ vertex disjoint cliques $C_1, C_2, \ldots, C_k$ of sizes $r-1$ in $N(x_0)$. Then, a copy of $F_{k,r}$ is found, a contradiction. □

Let $Z_0 = X_0 - X_0^*$ and $Z_i = X_i$ for each $i = 1, 2, \ldots, r-2$. By Claim 5 and (3), we have that

$$|V - \bigcup_{i=0}^{r-2} X_i| \leq k(2r-3)(r-1).$$

Thus,

$$|V - \bigcup_{i=0}^{r-2} Z_i| \leq k(2r-3)(r-1) + 2k(r-1)(r-2) < 4k(r-1)^2.$$

Further, the following inequality holds.

$$|Z_0| \geq n/(r-1) - k(2r-3) - 2k(r-1)(r-2) = n/(r-1) - k(2r^2 - 4r + 1).$$

Since $\delta(G) \geq \frac{r-2}{r-1}n - k$, the following inequalities hold for every $z_0 \in Z_0$ (recall that $Z_0 = X_0 - X_0^*$ and thus by the definition of $X_0^*$ we have $\Delta(G[Z_0]) \leq 2k(r-1)^2$).

$$\begin{aligned}|\overline{N_{G-Z_0}(z_0)}| &\leq (n - d(z_0)) - (|Z_0| - \Delta(G[Z_0])) \\ &\leq \left(\frac{n}{r-1} + k\right) - \left(\frac{n}{r-1} - k(2r^2 - 4r + 1) - 2k(r-1)^2\right) \\ &\leq 4kr(r-1).\end{aligned}$$

In particular, for each $z_0 \in Z_0$, we have that for $i > 0$

$$d_{Z_i}(z_0) \geq |Z_i| - 4kr(r-1).$$

That is, $Z_0$ dominates $Z_i$ with $4kr(r-1)$-deficiency.

**Claim 6** *For every $v \in V - \bigcup_{i=0}^{r-2} Z_i$, there exists a $j = j(v)$ such that $d_{Z_j}(v) < 2k(r-1)^2 + k < 2kr(r-1)$. Further, such a $j(v)$ is unique.*

*Proof:* Suppose, to the contrary, there is a $v \in V - \bigcup_{i=0}^{r-2} Z_i$ such that $d_{Z_j}(v) \geq 2k(r-1)^2 + k$ for every $j = 0, 1, \ldots, r-2$. Set $a = 2k(r-1)$ and $m = r-1$, then for all $0 \leq j \leq r-2$

$$\begin{aligned}|N_{Z_j}(v)| &= d_{Z_j}(v) \geq ma + k, \text{ and} \\ d_{Z_j}(z_i) &\geq |Z_j| - a \text{ for } z_i \in Z_i,\ i > 0,\ i \neq j.\end{aligned}$$

Applying Lemma 2.2, we see that there are $k$ vertex disjoint cliques of order $r-1$ whose vertex sets are in $N(v)$, a contradiction.

To show the uniqueness of $j(v)$, suppose there are two distinct $j_1$ and $j_2$ such that $d_{Z_{j_i}}(v) < 2k(r-1)^2 + k$ for both $i = 1$ and 2. Since $n \geq 4k^2r^4 \geq 4kr^2(r-1)^2$, we have that

$$\begin{aligned}d(v) &\leq n - |Z_{j_1} \cup Z_{j_2}| + 4k(r-1)^2 + 2k \\ &\leq n - \left[\left(\frac{n}{r-1} - 2k(r-1)^2\right) + \left(\frac{n}{r-1} - k(2r-3)\right)\right] + 4k(r-1)^2 + 2k \\ &= \frac{r-2}{r-1}n - \frac{n}{r-1} + 2k(r-1)^2 + k(2r-3) + 4k(r-1)^2 + 2k \\ &< \frac{r-2}{r-1}n - k,\end{aligned}$$

a contradiction. □

Adding each $v \in V - \bigcup_{i=0}^{r-2} Z_i$ to $Z_{j(v)}$, we obtain a partition of $V = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_{r-2}$.

Clearly, for each $i = 0, \ldots, r-2$,

$$|V_i| \geq |Z_i| \geq \frac{n}{r-1} - 2k(r-1)^2. \tag{5}$$

For each $i$ and each $v_i \in V_i$, since

$$\Delta(G[V_i]) \leq \Delta(G[Z_i]) + |V - \bigcup_{i=0}^{r-2} Z_i| \leq 2k(r-1)^2 + 4k(r-1)^2,$$

we have that:

$$\begin{aligned}
|\overline{N_{G-V_i}(v_i)}| &\leq (n - d(v_i)) - (|V_i| - \Delta(G[V_i])) \\
&\leq \left(\frac{n}{r-1} + k\right) - \left(\frac{n}{r-1} - 2k(r-1)^2 - 6k(r-1)^2\right) \\
&= k + 2k(r-1)^2 + 6k(r-1)^2 \\
&< 8kr^2
\end{aligned}$$

In particular, we have that:

$$d_{V_j}(v_i) \geq |V_j| - 8kr^2. \tag{6}$$

We will show that $V_0, V_1, \ldots, V_{r-2}$ satisfy (1) and (2). Let $a = 8kr^2$. Since $n \geq 4k^2r^4 \geq 8kr^4$, for any $j$, we have that

$$|V_j| \geq \frac{n}{r-1} - 2k(r-1)^2 \geq (r-1)a + 2k.$$

*Proof of (1).* Suppose for some $y \in V_i$, $|N(y) \cap V_i| \geq k$, say the neighbors are $y_1, y_2, \ldots, y_k$ in $V_i$. By Lemma 2.1, there are $k$ cliques $D_1, D_2, \ldots, D_k$ such that $y, y_j \in D_j$ and $|D_j| = r$ for each $j$. Further, $D_j \cap D_\ell = \{y\}$ for all $j \neq \ell$. Thus, a copy of $F_{k,r}$ is found, a contradiction.

Next suppose that $\sum_{j \neq i} \nu(V_j) \geq k$. Let $y_1z_1, y_2z_2, \ldots, y_kz_k$ be a $k$-matching with the property that $y_j$ and $z_j$ are in the same $V_\ell$ for some $\ell \neq i$. Now, since $n \geq 4k^2r^4 \geq 16k^2r^3$,

$$|\bigcap_{j=1}^{k} (N_{V_i}(y_j) \cap N_{V_i}(z_j))| > |V_i| - 2k(8kr^2) \geq \left(\frac{n}{r-1} - 2k(r-1)^2\right) - 16k^2r^2 \geq 1.$$

Therefore, there exists a vertex $y \in V_i$, such that $\bigcup_{j=1}^{k} \{y_j, z_j\} \subseteq N(y)$. By Lemma 2.1, there are $k$ cliques $D_1, D_2, \ldots, D_k$ such that $y, y_j, z_j \in D_j$ and $|D_j| = r$ for each $j$. Further, $D_j \cap D_\ell = \{y\}$ for all $j \neq \ell$. Thus, a copy of $F_{k,r}$ is found, a contradiction. □

*Proof of (2).* Let $v \in V_i$ have neighbors $x_1, x_2, \ldots, x_s$ in $V_i$ and neighbors $y_1$, $z_1, y_2, z_2, \ldots, y_t$, and $z_t$ in $V - V_i$ where, for each $j = 1, \ldots, t$, $y_j$ and $z_j$ in the same $V_\ell$ for some $\ell \neq i$ and $y_j z_j \in E(G)$. By (1), both $s$ and $t$ are less than $k$. Suppose for the moment that $s + t \geq k$. Consider $k$ of the cliques $\{v, x_1\}$, $\ldots$, $\{v, x_s\}$, $\{v, y_1, z_1\}$, $\ldots$, $\{v, y_t, z_t\}$. Applying Lemma 2.1 again, we obtain $k$ cliques $D_1$, $D_2$, $\ldots$, $D_k$ which induce a copy of $F_{k,r}$, a contradiction, which completes the proof of Lemma 2.3. □

## 3 Proof of the Main Lemma

The following lemma was obtained in [3].

**Lemma 3.1** *Let $H$ be a graph and $b$ a nonnegative integer such that $b \leq \Delta(H) - 2$, and let $\nu = \nu(H)$, $\Delta = \Delta(H)$. Then*

$$\sum_{x \in V(H)} \min\{d_H(x), b\} \leq \nu(b + \Delta). \tag{7}$$

Let $G$ be a graph with a partition of the vertices into $r - 1$ non-empty parts

$$V(G) = V_0 \dot{\cup} V_1 \dot{\cup} \ldots \dot{\cup} V_{r-2}.$$

Let $G_i = G[V_i]$ for each $i = 0$, $1$, $\ldots$, $r - 2$, and define

$$G_{cr} = (V(G), \{v_i v_j \ : \ v_i \in V_i, v_j \in V_j, i \neq j\}),$$

where "cr" denotes "crossing". For each $i \in \{0, 1, \ldots, r-2, cr\}$ let $d_i(x) = d_{G_i}(x)$ and $\nu_i = \nu(G_i)$. We generalized Lemma 6.2. in [3] to the following lemma.

**Lemma 3.2** *Suppose $G$ is partitioned as above so that (1) and (2) are satisfied. If $G$ is $F_{k,r}$-free, then*

$$\sum_{i=0}^{r-2} |E(G_i)| - \left( \sum_{0 \leq i < j \leq r-2} |V_i||V_j| - |E(G_{cr})| \right) \leq f(k-1, k-1). \tag{8}$$

*Proof:* Observe that $G_{cr}$ is an $(r-1)$-partite graph, and $\sum_{0 \leq i < j \leq r-2} |V_i||V_j| - |E(G_{cr})|$ is the number of edges missing from the complete $(r-1)$-partite graph. By (1) and the definition of $f$, we see that $|E(G_i)| \leq f(k-1, k-1)$, so the left hand side of (8) is bounded above by $(r-1)f(k-1, k-1)$. Delete vertices of $G$ so that the left hand side of (8) is maximal, let $G$ be minimal in this case.

We now claim that for each $i = 0$, $\ldots$, $r - 2$ and every $x \in V_i$,

$$d_i(x) - (|V - V_i| - d_{cr}(x)) > 0. \tag{9}$$

In fact, if for some $x \in V_i$, $d_i(x) - (|V - V_i| - d_{cr}(x)) \leq 0$ holds, then

$$\begin{aligned}
&|E(G_i - x)| + \sum_{j \neq i} |E(G_j)| - \left( \sum_{j \neq i} |V_i - x||V_j| + \sum_{i \neq j < \ell \neq i} |V_j||V_\ell| - |E(G_{cr} - x)| \right) \\
&= \sum_{j=0}^{r-2} |E(G_j)| - \left( \sum_{0 \leq j < \ell \leq r-2} |V_j||V_\ell| - |E(G_{cr})| \right) - (d_i(x) - |V - V_i| + d_{cr}(x)) \\
&\qquad \geq \sum_{j=0}^{r-2} |E(G_j)| - \left( \sum_{0 \leq j < \ell \leq r-2} |V_j||V_\ell| - |E(G_{cr})| \right),
\end{aligned}$$

contradicting the minimality of $G$. Hence (9) holds.

We also claim that for each $i = 0, \ldots, r-2$,

$$d_i(x) - (|V - V_i| - d_{cr}(x)) \leq k - 1 - \sum_{j \neq i} \nu_j. \tag{10}$$

To see (10), we need only observe that,

$$\begin{aligned}
&d_i(x) - (|V - V_i| - d_{cr}(x)) \\
&\qquad \leq k - 1 - \sum_{j \neq i} [\nu(G_j[N(x) \cap V_j]) + |V_j| - d_j(x)] \quad \text{by (2)} \\
&\qquad \qquad \leq k - 1 - \sum_{j \neq i} \nu_j,
\end{aligned}$$

where the last inequality holds since any matching in $G_j$ has at most $|V_j| - d_j(x)$ edges with one or both endpoints outside $N(x) \cap V_j$. This proves (10).

We can also assume that for each $i = 0, 1, \ldots, r-2$

$$1 \leq \sum_{j \neq i} \nu_j \leq k - 2, \tag{11}$$

by the following arguments. If $\sum_{j \neq i} \nu_j = 0$, then $G_j$ is empty for every $j \neq i$, and in this case by (1),

$$|E(G_i)| - \left( \sum_{j < \ell} |V_j| \, |V_\ell| - E(G_{cr})| \right) \leq |E(G_i)| \leq f(k-1, k-1);$$

thus (8) holds trivially, verifying the lemma. If $\sum_{j \neq i} \nu_j = k - 1$, then by (9) and (10), we would have

$$0 < d_i(x) - (|V - V_i| - d_{cr}(x)) \leq 0,$$

a contradiction.

We may further suppose that

$$2 \le \nu_i \text{ for each } i = 0, \ldots, r-2. \tag{12}$$

To the contrary, without loss of generality, assume that $\nu_0 \le 1$, then (11) implies that $\sum_{i=0}^{r-2} \nu_i \le k-1$. As

$$\sum_{i=0}^{r-2} f(\nu_i, \Delta) \le \ f\left(\sum_{i=0}^{r-2} \nu_i \ , \ \Delta\right)$$

always holds, we get that $\sum_{i=0}^{r-2} |E(G_i)| \le f(k-1, k-1)$ and (8) follows.

Now apply Lemma 3.1 for the graph $G_i$ ($i = 0, \ldots, r-1$) with $\Delta = k-1$ and $b = k - 1 - \sum_{j \ne i} \nu_j \le \Delta - 2$ (by (12)). Using (10) and (7) we get

$$\begin{aligned}
\sum_{x \in V_i} \left[ d_i(x) - \left( \sum_{j \ne i} |V_j| - d_{cr}(x) \right) \right] \\
\le \sum_{x \in V_i} \min \left\{ d_i(x) \ , \ k - 1 - \sum_{j \ne i} \nu_j \right\} \\
\le \nu_i \left( 2(k-1) - \sum_{j \ne i} \nu_j \right).
\end{aligned} \tag{13}$$

The left side in (13) equals

$$2|E(G_i)| + \sum_{j \ne i} |E(V_i, V_j)| - \sum_{j \ne i} |V_i||V_j|,$$

so adding these $r-1$ sums (for $i = 0, \ldots, r-2$) gives

$$\begin{aligned}
2|E(G)| &= 2\sum_{i=0}^{r-2} |E(G_i)| + 2|E(G_{cr})| \\
&= \sum_{i=0}^{r-2} \left( 2|E(G_i)| + \sum_{i \ne j} |E(V_i, V_j)| - \sum_{j \ne i} |V_i||V_j| \right) + 2\sum_{i<j} |V_i||V_j| \\
&\le \sum_{i=0}^{r-2} \nu_i \left( 2(k-1) - \sum_{j \ne i} \nu_j \right) + 2\sum_{i<j} |V_i||V_j| \\
&= 2\left[ k^2 - 2k + 1 - (k - 1 - \nu_0)\left( k - 1 - \sum_{j>0} \nu_j \right) - \sum_{0 \ne j \ne \ell \ne 0} \nu_j \nu_\ell \right] \\
&\quad + 2\sum_{i<j} |V_i||V_j|.
\end{aligned}$$

This yields $|E(G)| \le k^2 - 2k + \sum_{i<j} |V_i||V_j|$ (by (11), $k - 1 - \nu_0 \ge 1$ and $k - 1 - \sum_{i \ne 0} \nu_i \ge 1$), and since $f(k-1, k-1) > k^2 - 2k$, this implies (8), finishing the proof of Lemma 3.2. $\square$

## 4 Proof of The Theorem

We can summarize Lemma 3.2 and Lemma 2.3 as follows.

**Lemma 4.1** *Suppose that $G$ is an $F_{k,r}$-free graph on $n$ vertices with $n \geq 4k^2r^4$, and with minimum degree $\delta \geq \frac{r-2}{r-1}n - k$, then $|E(G)| \leq ex(n, K_r) + f(k-1, k-1)$.*

*Proof:* We can assume that $G$ has the maximum number of edges under the conditions of Lemma 4.1 and apply Lemma 2.3 to get a decomposition of $G$ into $G_0$, $G_1$, …, $G_{r-2}$, $G_{cr}$. The graph $G_{cr}$ consists of the edges between $V_i$ and $V_j$ for all distinct pairs $i$ and $j$. Lemma 3.2 implies that

$$|E(G)| = \sum_{i=0}^{r-2} |E(G_i)| + |E(G_{cr})|$$
$$\leq \sum_{i<j} |V_i||V_j| + f(k-1, k-1)$$
$$\leq ex(n, K_r) + f(k-1, k-1),$$

and we are done. $\square$

Since $ex(n, K_r) - ex(n-1, K_r) = \left\lfloor \frac{r-2}{r-1} n \right\rfloor$, we see that the following lemma holds.

**Lemma 4.2** *Let $G$ be a graph of order $n$, let $k$ be an integer and $c$ some constant independent from $n$. If $|E(G)| \geq ex(n, K_r) + c$ and $d(x) \leq \frac{r-2}{r-1}n - k$, then $|E(G - x)| \geq ex(n-1, K_r) + c + k$.*

*Proof of Theorem 2.* Suppose that $n \geq 16k^3r^8$, and that $G$ is an $F_{k,r}$-free graph on $n$ vertices. We need to show that $G$ has at most $ex(n, K_r) + f(k-1, k-1)$ edges. Suppose, to the contrary, that $|E(G)| > ex(n, K_r) + f(k-1, k-1)$. By Lemma 4.1, there exists a vertex $x = x_n$ with degree $d_G(x_n) < \frac{r-2}{r-1}n - k$.

Denote $G$ by $G^n$, and let $G^{n-1} = G^n - x_n$. By Lemma 4.2,

$$|E(G^{n-1})| \geq ex(n-1, K_r) + f(k-1, k-1) + k.$$

If there exists a vertex $x_{n-1} \in V(G^{n-1})$ with degree $d_{G^{n-1}}(x_{n-1}) < \frac{r-2}{r-1}(n-1) - k$, then delete it to obtain $G^{n-2} = G^{n-1} - x_{n-1}$. Continue this process as long as $\delta(G^i) < \frac{r-2}{r-1}i - k$, and after $n - \ell$ steps we get a subgraph $G^\ell$ with $\delta(G^\ell) \geq \frac{r-2}{r-1}\ell - k$. Note that

$$\ell(\ell-1)/2 \geq |E(G_\ell)| \geq ex(\ell, K_r) + k(n-\ell) + f(k-1, k-1) \geq k(n-\ell).$$

We have that $\ell > \sqrt{kn} \geq 4k^2r^4$, a contradiction to Lemma 4.1. $\square$

## 5 Remark

To avoid tedious calculations, we did not attempt to lower the bound $n \geq 16k^3r^8$ in the proof, although we strongly believe the bound can be lowered substantially.

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