

# EXTREMAL LENGTH AND CONFORMAL CAPACITY<sup>(1)</sup>

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1. **Introduction.** The theory of extremal length has been successfully applied to conformal mappings, analytic functions of a complex variable, and in recent years it has found application in the study of quasiconformal mappings in space. Its application in quasiconformal mapping theory begins essentially with a theorem proved by Gehring [G3], that the conformal capacity of a space ring  $R$  is directly related to the extremal length of the family of curves that join the boundary components of  $R$ . He has also shown that the conformal capacity is related to the extremal length of a family of surfaces that separate the boundary components of  $R$ . Gehring assumes that the separating surfaces are sufficiently smooth; Krivov [K] establishes a similar result under the assumption that the extremal metric is well-behaved. Under similar assumptions, other authors have dealt with the extremal length of separating surfaces, cf. [FU, Theorem 9], [SA], and [H].

The purpose of this paper is to eliminate the need for these assumptions. Thus, we consider the case of two disjoint compact sets  $C_0, C_1$  contained in the closure of a bounded, open, connected set  $G$ . It is proved that the conformal capacity  $C$  of  $C_0, C_1$ , relative to  $G$  is related to the  $n/n-1$ -dimensional module  $M$  of all closed sets that separate  $C_0$  from  $C_1$  in the closure of  $G$  by

$$C^{-1/(n-1)} = M.$$

This is accomplished by using a technique of Gehring's [G1, Lemma 3] which eliminates all assumptions concerning the behavior of the extremal metric. Then, a surface-theoretic approximation theorem, first developed in [FF, 8.23], permits the consideration of arbitrary closed separating sets.

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## 2. Preliminaries.

2.1.  $E^n$  will denote Euclidean  $n$ -space and  $L_n$  is  $n$ -dimensional *Lebesgue measure*. *Hausdorff  $k$ -dimensional measure* in  $E^n$  will be denoted by  $H^k$  (for its definition and properties see [F2]) and in this paper, only  $H^1$  and  $H^{n-1}$  will be used. If  $A \subset E^n$ , then *bdry  $A$*  will mean the boundary of  $A$  and for  $x \in E^n$ ,  $\delta(x, A)$  will be the *distance from  $x$  to  $A$* . More generally,  $\delta(A, B)$  will denote the *distance between*

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the sets  $A$  and  $B$ .  $S(x, r)$  will stand for the open  $n$ -ball of radius  $r$  and centered at  $x$ . If  $A$  is an  $L_n$ -measurable subset of  $E^n$ , then  $\mathcal{L}^p(A)$  will denote those functions  $f$  for which  $|f|^p$  is  $L_n$ -integrable over  $A$  and  $\|f\|_p$  will be its  $\mathcal{L}^p$  norm.

2.2. A real valued function  $u$  defined on an open set  $U \subset E^n$  is said to be *absolutely continuous in the sense of Tonelli* (ACT) on  $U$  if it is ACT on every interval  $I \subset U$ , cf. [SS, p. 169]. Thus, the gradient of  $u$ ,  $\nabla u$ , exists at  $L_n$ -almost every point in  $U$ . The following co-area formula, which was proved in [Z2], will be used frequently throughout this paper.

2.2.1. THEOREM. *If  $A$  is an  $L_n$ -measurable subset of  $U$  and  $u$  is ACT on  $U$ , then*

$$\int_A |\nabla u(x)| dL_n(x) = \int_{-\infty}^{\infty} H^{n-1}[u^{-1}(s) \cap A] dL_1(s).$$

Therefore,

$$\int_U f(x) |\nabla u(x)| dL_n(x) = \int_{-\infty}^{\infty} \int_{u^{-1}(s)} f(x) dH^{n-1}(x) dL_1(s)$$

whenever  $f \in \mathcal{L}^1(U)$ .

2.3. *Conformal capacity.* The following will be standard notation throughout.  $G$  is an open, bounded, connected set in  $E^n$  and  $C_0, C_1$ , are disjoint compact sets in the closure of  $G$ . We will let  $R = G - (C_0 \cup C_1)$  and  $R^* = R \cup C_0 \cup C_1$ . The *conformal capacity of  $C_0, C_1$  relative to the closure of  $G$*  is defined as

$$(1) \quad C[G, C_0, C_1] = \inf \int_R |\nabla u|^n dL_n$$

where the infimum is taken over all functions  $u$  which are continuous on  $R^*$ , ACT on  $R$ , and assume boundary values 1 on  $C_1$  and 0 on  $C_0$ . Such functions are called *admissible* for  $C[G, C_0, C_1]$ . Sometimes we will write  $C$  for  $C[G, C_0, C_1]$ .

If  $C_0 \cup C_1 \subset G$  and if  $C_0 \cup C_1$  consists of only a finite number of nondegenerate components, then the arguments of [G2, §§3-7] can be applied with only slight modifications to prove that the infimum in (1) is attained by a unique admissible function  $u$  which is ACT in  $G$ . (By using the methods of Chapter III in [FU], one can prove the existence of an extremal for more general situations.) This extremal function  $u$  satisfies the variational condition

$$(2) \quad \int_R |\nabla u|^{n-2} \nabla u \cdot \nabla w dL_n = 0$$

for any function  $w$  which is ACT on  $G$ , assumes boundary value 0 on  $C_0 \cup C_1$ , and for which  $|\nabla w| \in \mathcal{L}^n(R)$ .

2.4. *Integral currents.* The following notion of exterior normal, which was first introduced in [F1, p. 48], will be used later.

An  $L_n$ -measurable set  $E \subset E^n$  has the unit vector  $n(x)$  as exterior normal to  $E$  at  $x$  if, letting

$$\begin{aligned} S(x, r) &= \{y : |y-x| < r\}, \\ S_+(x, r) &= S(x, r) \cap \{y : (y-x) \cdot n(x) \geq 0\} \\ S_-(x, r) &= S(x, r) \cap \{y : (y-x) \cdot n(x) \leq 0\} \\ \alpha &= L_n[S(x, 1)] \end{aligned}$$

we have

$$2 \lim_{r \rightarrow 0} L_n[S_-(x, r) \cap E] / \alpha r^n = 1, \quad 2 \lim_{r \rightarrow 0} L_n[S_+(x, r) \cap E] / \alpha r^n = 0.$$

The set of points  $x$  for which  $n(x)$  exists is called the *reduced boundary of  $E$*  and will be denoted by  $\beta(E)$ . Obviously,  $\beta(E) \subset \text{bdry } E$ . The importance of the exterior normal is seen in the following general version of the Gauss-Green Theorem [F3], [DG].

2.4.1. THEOREM. *If  $E$  is a bounded  $L_n$ -measurable set with  $H^{n-1}[\beta(E)] < \infty$ , then*

$$\int_E \text{div } f(x) \, dL_n(x) = \int_{\beta(E)} f(x) \cdot n(x) \, dH^{n-1}(x)$$

whenever  $f: E^n \rightarrow E^n$  is continuously differentiable.

This theorem enables us to regard a bounded, open set  $U \subset E^n$  with

$$H^{n-1}(\text{bdry } U) < \infty$$

as an *integral current mod 2* (or integral class), i.e., an integral current  $T$  with coefficients in the group of integers mod 2, [Z1, §3.6] or [FL]. Thus, if  $\phi$  is a differential  $n$ -form of class  $C^\infty$ , then

$$T(\phi) = \int_U \phi \, dL_n.$$

The *boundary of  $T$* ,  $\partial T$ , is defined as  $\partial T(\omega) = T(d\omega)$  whenever  $\omega$  is an  $n-1$  form and  $d\omega$  is its exterior derivative. Now  $\beta(U)$  is a Hausdorff  $(n-1)$ -rectifiable set and therefore, 2.4.1 allows us to identify  $\beta(U)$  with  $\partial T$ , [Z1, §3.6]. Thus, the *support of  $\partial T$*  is  $\text{cl } (\beta(U)) \subset \text{bdry } U$ , the *mass of  $T$*  is  $M(T) = L_n(U)$ , and  $M(\partial T) = H^{n-1}[\beta(U)]$  [Z1, §3.6].

In view of this identification the following theorem is an immediate consequence of [Z1, 6.2] although the original version was given in [FF, 8.23]. An open set is called a *convex cell* if it can be expressed as the finite intersection of open half-spaces and an  *$n$ -dimensional polyhedral set* is the union of a finite number of convex cells.

2.4.2. THEOREM. *If  $U \subset E^n$  is a bounded, open set with  $H^{n-1}(\text{bdry } U) < \infty$ , then there is a sequence of  $n$ -dimensional polyhedral sets  $P_i$  such that*

- (i)  $P_i \subset \{x : \delta(x, U) < i^{-1}\}$ ,
- (ii)  $L_n(P_i) \rightarrow L_n(U)$  as  $i \rightarrow \infty$ ,
- (iii)  $H^{n-1}(\text{bdry } P_i) \rightarrow H^{n-1}[\beta(U)]$  as  $i \rightarrow \infty$ .

Moreover, by employing an argument similar to that of [FL, (5.6) and (5.7)], it can be arranged that

$$(iv) \text{ bdry } P_i \subset \{x : \delta[x, \beta(U)] < i^{-1}\}.$$

Of course, one could apply [FL, 5.6 and p. 170] directly after verifying that the boundary of  $U$  in Fleming's sense is  $\beta(U)$ . Finally, defining the measure  $\mu_i$  as the restriction of  $H^{n-1}$  to bdry  $P_i$  and  $\mu$  as the restriction of  $H^{n-1}$  to  $\beta(U)$ , (iii) above and [Z1, 5.7] or [FL, 4.2] imply that

$$(v) \mu_i \rightarrow \mu \text{ weakly as } i \rightarrow \infty,$$

that is,

$$\int f d\mu_i \rightarrow \int f d\mu \text{ for every continuous } f.$$

2.5. *The module of a system of measures.* Instead of dealing with extremal length, we prefer to work with the module as developed in [FU].

Let  $\mathcal{M}$  be the class of all Borel measures on  $E^n$ . With an arbitrary system  $E \subset \mathcal{M}$  of measures is associated a class of nonnegative Borel functions  $f$  subject to the condition

$$(3) \int_{E^n} f d\mu \geq 1 \quad \text{for every } \mu \in E.$$

We will write  $f \wedge \mu$  if (3) is satisfied and  $f \wedge E$  if (3) is satisfied for every  $\mu \in E$ . For  $0 < p < \infty$ , the *module of  $E$* ,  $M_p(E)$ , is defined as

$$(4) M_p(E) = \inf \left\{ \int_{E^n} f^p dL_n : f \wedge E \right\}.$$

A statement concerning measures  $\mu \in \mathcal{M}$  is said to hold for  $M_p$ -a.e.  $\mu$  if the statement fails to hold for only a set of measures  $E_0$  where  $M_p(E_0) = 0$ . The proofs of the following statements can be found in [FU, Chapter 1].

- (i)  $M_p(E) \leq M_p(E')$  if  $E \subset E'$ ,
- (ii)  $M_p(E) \leq \sum_{i=1}^{\infty} M_p(E_i)$  if  $E = \bigcup_{i=1}^{\infty} E_i$ .
- (iii) If  $\bar{\mu}$  is the completion of  $\mu$  and  $L_n(A) = 0$ , then  $\bar{\mu}(A) = 0$  for  $M_p$ -a.e.  $\mu \in \mathcal{M}$ .
- (iv) If  $f \in \mathcal{L}^p(E^n)$ , then  $f$  is  $\bar{\mu}$  integrable for  $M_p$ -a.e.  $\mu \in \mathcal{M}$ .
- (v) If  $\|f_i - f\|_p \rightarrow 0$ , then there is a subsequence  $f_{i_j}$  such that

$$\int_{E^n} |f_{i_j} - f| d\bar{\mu} \rightarrow 0 \quad \text{for } M_p\text{-a.e. } \mu \in \mathcal{M}.$$

(vi) If  $E \subset \mathcal{M}$ , then  $M_p(E) = 0$  if and only if there is a nonnegative Borel function  $f \in \mathcal{L}^p$  such that

$$\int_{E^n} f d\mu = \infty \quad \text{for every } \mu \in E.$$

(vii) If  $p > 1$  and  $E \subset \mathcal{M}$ , then there is a nonnegative Borel function  $f$  such that

$$\int_{E^n} f^p dL_n = M_p(E)$$

and  $\int f d\mu \geq 1$  for  $M_p$ -a.e.  $\mu \in E$ .

For the applications in this paper, the measures  $\mu$  are complete (in fact, they are the restrictions of Hausdorff measure to compact sets) and for such measures, we have the following

2.5.1 THEOREM. *If  $p \geq 2$ ,  $E_1 \subset E_2 \subset \dots$  are sets of complete measures, and  $E = \bigcup E_i$ , then*

$$M_p(E) = \lim_{i \rightarrow \infty} M_p(E_i).$$

**Proof.** In view of (i) above, observe that the limit exists and is dominated by  $M_p(E)$ . Therefore, only the case where the limit is finite needs to be considered.

For each  $i$ , let  $f_i$  be the Borel function associated with  $E_i$  as in (vii) above. Clarkson's inequality [C] states, for any  $i$  and  $j$ ,

$$(5) \quad \int_{E^n} \left| \frac{f_i + f_j}{2} \right|^p dL_n + \int_{E^n} \left| \frac{f_i - f_j}{2} \right|^p dL_n \leq 2^{-1} \int_{E^n} |f_i|^p dL_n + 2^{-1} \int_{E^n} |f_j|^p dL_n.$$

If  $i > j$ , then  $2^{-1} \cdot (f_i + f_j) \wedge \mu$  for  $M_p$ -a.e.  $\mu \in E_j$ . Therefore, because of (5),

$$(6) \quad \int_{E^n} \left| \frac{f_i - f_j}{2} \right|^p dL_n \leq 2^{-1} M_p(E_i) - 2^{-1} M_p(E_j).$$

The right side of (6) tends to zero as  $i, j \rightarrow \infty$  with  $i > j$  and therefore, there is a nonnegative function  $f$  such that  $\|f_i - f\|_p \rightarrow 0$ . Thus, from the above properties of module (especially (v)), we have that  $f \wedge \mu$  for  $M_p$ -a.e.  $\mu \in E$ . This implies

$$M^p(E) \leq \int_{E^n} f^p dL_n = \lim_{i \rightarrow \infty} M_p(E_i)$$

which is all that is required to prove.

**3. Module of separating sets and conformal capacity.** In this section we will establish the relationship between conformal capacity and arbitrary closed separating sets.  $G, R^*, R, C_1$ , and  $C_0$  will be as defined in §2.3.

We will say that a set  $\sigma \subset E^n$  separates  $C_0$  from  $C_1$  in  $R^*$  if  $\sigma \cap R$  is closed in  $R$  and if there are disjoint sets  $A$  and  $B$  which are open in  $R^* - \sigma$  such that  $R^* - \sigma = A \cup B$ ,  $C_0 \subset A$ , and  $C_1 \subset B$ . Let  $\Sigma$  denote the class of all sets that separate  $C_0$  from  $C_1$  in  $R^*$ . With every  $\sigma \in \Sigma$ , associate a complete measure  $\mu$  in the following way: for every  $H^{n-1}$ -measurable set  $A \subset E^n$ , define

$$\mu(A) = H^{n-1}(A \cap \sigma \cap R).$$

From the properties of Hausdorff measure, it is clear that the Borel sets of  $E^n$  are  $\mu$ -measurable and therefore the module of  $\Sigma$  can be as defined as in 2.5. Thus, for  $n' = n/n - 1$ ,

$$M_{n'}(\Sigma) = \inf \left\{ \int_{E^n} f^{n'} dL_n : f \wedge \Sigma \right\}$$

where  $f \wedge \Sigma$  means that  $f$  is a nonnegative Borel function on  $E^n$  such that

$$\int_{\sigma \cap R} f dH^{n-1} \geq 1 \quad \text{for every } \sigma \in \Sigma.$$

3.1. As in [V1, 2.3] it can be shown that if  $\Sigma'$  denotes those  $\sigma \in \Sigma$  for which  $H^{n-1}(\sigma \cap R) = \infty$ , then  $M_{n'}(\Sigma') = 0$ .

3.2. LEMMA. *Let  $u$  be an admissible function for  $C[G, C_0, C_1]$  (see 2.3) and let  $S \subset [0, 1]$  be an  $L_1$ -measurable set. If  $\Sigma(S) = \{u^{-1}(s) : s \in S\}$  and  $M_{n'}[\Sigma(S)] = 0$ , then  $L_1(S) = 0$ .*

**Proof.** Since  $M_{n'}[\Sigma(S)] = 0$ , (vi) of 2.5 provides a nonnegative Borel function  $f \in \mathcal{L}^{n'}$  such that

$$\int_{u^{-1}(s) \cap R} f dH^{n-1} = \infty \quad \text{for every } s \in S.$$

However, Hölder's inequality and the co-area formula (2.2.1) imply

$$\infty > \int_R f |\nabla u| dL_n \geq \int_0^1 \int_{u^{-1}(s) \cap R} f(x) dH^{n-1}(x) dL_1(s)$$

and therefore  $L_1(S) = 0$ .

3.3. THEOREM.  $M_{n'}(\Sigma) \geq C[G, C_0, C_1]^{-1/n-1}$ .

**Proof.** Choose  $\varepsilon > 0$  and let  $f$  be any function for which  $f \wedge \Sigma$ . Let  $u$  be an admissible function for  $C = C[G, C_0, C_1]$  such that

$$\int_R |\nabla u|^n dL_n < C + \varepsilon.$$

$R^*$  is connected since  $G$  is and therefore it is clear that  $u^{-1}(s) \in \Sigma$  for all  $0 < s < 1$ . Hence, by Hölder's inequality and 2.2.1, we have

$$\begin{aligned} \left( \int_{E^n} f^{n'} dL_n \right)^{1/n'} (C + \varepsilon)^{1/n} &\geq \left( \int_{E^n} f^{n'} dL_n \right)^{1/n'} \left( \int_R |\nabla u|^n dL_n \right)^{1/n} \\ &\geq \int_R f |\nabla u| dL_n \\ &\geq \int_0^1 \int_{u^{-1}(s)} f dH^{n-1}(x) dL_1(s) \geq 1. \end{aligned}$$

Since  $\varepsilon$  is arbitrary,

$$\int_R f^{n'} dL_n \geq C^{-1/n-1}$$

which is also true for the infimum over all  $f \wedge \Sigma$ , and thus the result follows.

3.4. The opposite inequality will be established by means of a sequence of approximations and we will begin by first assuming that  $C_0 \cup C_1$  consists only of a finite number of nondegenerate continua and that  $C_0 \cup C_1 \subset G$ . We will also assume initially that  $H^{n-1}(\text{bdry } G) < \infty$ .

Let  $V$  be an open connected set such that  $G \supset \text{cl } V \supset V \supset C_0 \cup C_1$  and let  $v$  be the extremal function for  $C[V, C_0, C_1]$  (see 2.3). Since  $v$  satisfies the variational condition (2), the proof of the following lemma is very similar to that of Lemma 3 in [G3] and will not be given here. It is possible to obtain a stronger result, but the following is sufficient for our purposes. Recall from 2.4 the definition of an  $n$ -dimensional polyhedral set.

3.5. LEMMA. *Let  $\pi$  be the boundary of an  $n$ -dimensional polyhedral set  $P$  ( $P$  not necessarily contained in  $V$ ) such that  $C_0 \subset P$  and  $C_1 \subset E^n - \text{cl } P$ . Then  $\pi$  separates  $C_0$  from  $C_1$  in  $V$  and*

$$\int_{\pi(b)} |\nabla v|^{n-1} dL_n \geq 2bC[V, C_0, C_1]$$

whenever  $0 < b < \delta(\pi, C_0 \cup C_1)$  and where  $\pi(b) = \{x : \delta(x, \pi) < b\}$ .

3.6. REMARK. In Lemmas 3.7 and 3.8, the integral average,  $f_r$ , of  $|\nabla v|^{n-1}$  will be used. Thus, defining  $\nabla v = 0$  on the complement of  $V$ , for each  $r > 0$  we have

$$f_r(x) = \alpha(r)^{-1} \int_{S(x,r)} |\nabla v(y)|^{n-1} dL_n(y)$$

where  $\alpha(r) = L_n[S(x, r)]$ . It is well known that  $f_r$  is continuous and that

$$f_r \rightarrow |\nabla v|^{n-1} \text{ } L_n\text{-a.e. as } r \rightarrow 0.$$

Also, by a result of K. T. Smith [S] and Lebesgue's dominated convergence theorem,  $\|f_r\|_n \rightarrow \|\nabla v|^{n-1}\|_n$  as  $r \rightarrow 0$  and consequently,  $\|f_r - |\nabla v|^{n-1}\|_n \rightarrow 0$ .

3.7. LEMMA. *With  $\pi$  as in 3.5,*

$$\int_{\pi} f_r dH^{n-1} \geq C[V, C_0, C_1]$$

whenever  $r < \delta(\pi, C_0 \cup C_1)$ .

**Proof.** Choose  $b > 0, r > 0$  so that  $b + r < \delta(\pi, C_0 \cup C_1)$ . If  $\pi_y$  denotes the translation of  $\pi$  through the vector  $y$ , then Fubini's Theorem and 3.5 imply

$$\begin{aligned} \int_{\pi(b)} f_r(x) dL_n(x) &= \alpha(r)^{-1} \int_{S(0,r)} \int_{\pi(b)} |\nabla v(x+y)|^{n-1} dL_n(x) dL_n(y) \\ (7) \qquad \qquad \qquad &= \alpha(r)^{-1} \int_{S(0,r)} \int_{\pi_y(b)} |\nabla v(x)|^{n-1} dL_n(x) dL_n(y) \\ &\geq 2bC[V, C_0, C_1] \end{aligned}$$

since  $\pi_y$  satisfies the conditions of 3.5. In addition to this, if  $d(x) = \delta(x, \pi)$ , then  $|\nabla d| = 1$   $L_n$ -a.e. on  $E^n - \pi$  [F4, (3) of 4.8] and therefore 2.2.1 gives

$$(8) \qquad \int_{\pi(b)} f_r(x) dL_n(x) = \int_0^b \int_{d^{-1}(s)} f_r(x) dH^{n-1}(x) dL_1(s).$$

Let  $F(s)$  denote the inner integral on the right. Since  $f_r$  is continuous on  $E^n$  and  $\pi$  is the boundary of a polyhedral set, it is clear that

$$\lim_{s \rightarrow 0} F(s) = 2 \int_{\pi} f_r dH^{n-1}.$$

Hence, from (7) and (8) we have

$$C[V, C_0, C_1] \leq \lim_{b \rightarrow 0} (2b)^{-1} \int_0^b F(s) dL_1(s) = \int_{\pi} f_r dH^{n-1}.$$

3.8. LEMMA. *If  $\Sigma$  is the class of sets that separate  $C_0$  from  $C_1$  in  $G$ , then*

$$\int_{\sigma \cap R} |\nabla v|^{n-1} dH^{n-1} \geq C[V, C_0, C_1]$$

for  $M_n$ -a.e.  $\sigma \in \Sigma$ .

**Proof.** Select  $\sigma \in \Sigma$  and let  $U$  be that part of a partition of  $G - \sigma$  that contains  $C_0$ . Since  $H^{n-1}(\text{bdry } G) < \infty$  by assumption, by appealing to 3.1 we can take  $H^{n-1}(\text{bdry } U) < \infty$ . Recall that  $G \supset \text{cl } V$  and that  $\nabla v = 0$  on the complement of  $V$ . Hence, we can choose  $r_0$  so small that the support of  $f_{r_0}$  is contained in  $G$  (and therefore for all  $r \leq r_0$ ) and  $r_0 < \delta(\text{bdry } U, C_0 \cup C_1)$ .

By applying 2.4.2 to the set  $U$ , we obtain a sequence of  $n$ -dimensional polyhedral sets  $P_i$ . Let  $\pi_i = \text{bdry } P_i$ . From properties (i), (ii), and (iv) of 2.4.2 it is clear that eventually  $C_0 \subset P_i$  and  $C_1 \subset E^n - \text{cl } P_i$ . Thus Lemma 3.7 applies to  $\pi_i$  for all large  $i$ . Now,  $\beta(U) \subset \text{bdry } U \subset (\text{bdry } G) \cup \sigma$  and since the support of  $f_r$  is contained in  $G$  for all  $r \leq r_0$ , it is clear that

$$(9) \quad \int_{\sigma} f_r dH^{n-1} \geq \int_{\beta(U)} f_r dH^{n-1} \quad \text{for all } r \leq r_0.$$

Since  $f_r$  is continuous, (v) of 2.4.2 and 3.7 imply

$$\int_{\beta(U)} f_r dH^{n-1} = \lim_{i \rightarrow \infty} \int_{\pi_i} f_r dH^{n-1} \geq C[V, C_0, C_1] \quad \text{for } r \leq r_0.$$

Thus, from (9), we have

$$(10) \quad \int_{\sigma} f_r dH^{n-1} \geq C[V, C_0, C_1] \quad \text{for } r \leq r_0.$$

In light of the fact that  $\|f_r - |\nabla v|^{n-1}\|_n \rightarrow 0$  as  $r \rightarrow 0$  (3.6), the result follows from (iv) and (v) of 2.5 and (10).

3.9. LEMMA. *Let  $u$  be the extremal function for  $C[G, C_0, C_1]$  (see 3.4 and 2.3). Then for  $M_n$ -a.e.  $\sigma \in \Sigma$ ,*

$$\int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \geq C[G, C_0, C_1] = C.$$



**Proof.** Let  $V_i$  be a sequence of open connected sets such that

$$G \supset \text{cl } V_{i+1} \supset V_{i+1} \supset \text{cl } V_i \supset V_i \supset C_0 \cup C_1$$

and  $G = \bigcup V_i$ . Let  $v_i$  be the extremal function for  $C[V_i, C_0, C_1] = C_i$ .

We will first show that  $C_i \rightarrow C$  as  $i \rightarrow \infty$ . Recall that  $C < \infty$ . If  $i > j$ , then the restriction of  $v_i$  to  $V_j$  is an admissible function for  $V_j$  and thus, so is  $2^{-1} \cdot (v_i + v_j)$ . As in the proof of 2.5.1, an application of Clarkson's inequality [C] gives

$$(11) \quad \int_R \left| \frac{\nabla v_i - \nabla v_j}{2} \right|^n dL_n \leq \frac{1}{2} C_i - \frac{1}{2} C_j \quad \text{for } i > j.$$

Since  $C_i$  is a monotonically increasing sequence bounded above by  $C$ , (11) implies the existence of a vector-valued function  $f$  such that

$$(12) \quad \int_R |\nabla v_i - f|^n dL_n \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

In fact, since  $C_0 \cup C_1$  consists only of a finite number of nondegenerate continua, an argument similar to that of [G2, p. 362] shows that there is an admissible function  $u'$  such that  $\nabla u' = f$   $L_n$ -a.e. on  $R$ . Thus, (12) shows that

$$(13) \quad \lim_{i \rightarrow \infty} C_i = C.$$

This also implies that  $u'$  is the extremal function for  $C$ , i.e.,  $u' = u$ .

Since  $\| |\nabla v_i| - |\nabla u| \|_n \rightarrow 0$  as  $i \rightarrow \infty$ , there is a subsequence of  $|\nabla v_i|$  (which will still be denoted by  $|\nabla v_i|$ ) such that  $|\nabla v_i| \rightarrow |\nabla u|$   $L_n$ -a.e. and therefore that  $|\nabla v_i|^{n-1} \rightarrow |\nabla u|^{n-1}$   $L_n$ -a.e. on  $R$ . This fact, along with

$$\| |\nabla v_i|^{n-1} \|_{n'} \rightarrow \| |\nabla u|^{n-1} \|_{n'}$$

leads to

$$\| |\nabla v_i|^{n-1} - |\nabla u|^{n-1} \|_{n'} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Thus, with (iv) and (v) of 2.5, we have (for another subsequence)

$$(14) \quad \lim_{i \rightarrow \infty} \int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} = \int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

Lemma 3.8 states that for each  $i$ ,

$$\int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} \geq C_i \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma,$$

and therefore, the result follows from (13), (14), and (ii) of 2.5.

**3.10. THEOREM.** *If  $G$  is a bounded, open, connected set, if  $C_0 \cup C_1$  consists of only a finite number of nondegenerate continua, and if  $C_0 \cup C_1 \subset G$ , then*

$$M_n(\Sigma) = C[G, C_0, C_1]^{-1/n-1}.$$

**Proof.** If it were the case that  $H^{n-1}(\text{bdry } G) < \infty$ , then the result would follow immediately from 3.3 and 3.9. For, if we let  $f = |\nabla u|^{n-1} \cdot C^{-1}$ , where  $u$  is the extremal for  $C$ , then by 3.9 there is  $\Sigma_0 \subset \Sigma$  such that  $f \wedge \Sigma_0$  and  $M_n(\Sigma_0) = M_n(\Sigma)$ . Hence, by 3.3,

$$C^{-1/n-1} \leq M_n(\Sigma) \leq \int_R f^{n'} dL_n = C^{-1/n-1}.$$

In order to eliminate the assumption  $H^{n-1}(\text{bdry } G) < \infty$ , select a sequence of open, connected sets  $V_1 \subset V_2 \subset \dots$  such that  $C_0 \cup C_1 \subset V_1$ ,  $H^{n-1}(\text{bdry } V_i) < \infty$ , and  $G = \bigcup V_i$ . As in the proof of 3.9, let  $v_i$  be the extremal for  $C[V_i, C_0, C_1] = C_i$  and again we have

$$(15) \quad C_i \rightarrow C, \quad \| |\nabla v_i|^{n-1} - |\nabla u|^{n-1} \|_{n'} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Thus, for a subsequence

$$(16) \quad \lim_{i \rightarrow \infty} \int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} = \int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

For each  $i$ , every  $\sigma \in \Sigma$  separates  $C_0$  from  $C_1$  in  $V_i$ , and thus applying 3.9 with  $V_i$  replacing  $G$ , we have

$$\int_{\sigma \cap R} |\nabla v_i|^{n-1} dH^{n-1} \geq \int_{\sigma \cap V_i} |\nabla v_i|^{n-1} dH^{n-1} \geq C_i \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

(Observe that 3.9, with  $V_i$  replacing  $G$ , applies only to  $\Sigma_i$ , which are those sets that separate  $C_0$  from  $C_1$  in  $V_i$ . However, a class in  $\Sigma$  which is  $M_n$ -zero relative to  $\Sigma_i$  is  $M_n$ -zero relative to  $\Sigma$ .) Hence, in view of (15) and (16),

$$(17) \quad \int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \geq C \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma$$

which, as we have seen from above, is sufficient to establish the theorem.

**3.11. COROLLARY.** *With the hypotheses of 3.10, and if  $u$  is the extremal for  $C[G, C_0, C_1]$ , then*

- (i)  $0 \leq u(x) \leq 1$  for all  $x \in G$ ,
- (ii)  $\int_{\sigma \cap R} |\nabla u|^{n-1} dH^{n-1} \geq C[G, C_0, C_1]$  for  $M_n$ -a.e.  $\sigma \in \Sigma$ ,
- (iii)  $\int_{u^{-1}(s)} |\nabla u|^{n-1} dH^{n-1} = C[G, C_0, C_1]$ , for  $L_1$ -a.e.  $s \in [0, 1]$ .

**Proof.** By truncating  $u$  at levels 1 and 0 if necessary, a new admissible function would be formed whose gradient would be bounded above by the gradient of  $u$ . However, the extremal is unique and thus (i) follows. (ii) is just a restatement of (17).

In order to prove (iii), let

$$F(s) = \int_{u^{-1}(s)} |\nabla u|^{n-1} dH^{n-1} \quad \text{for } L_1\text{-a.e. } s,$$

and observe that since  $G$  is connected,  $u^{-1}(s) \in \Sigma$  whenever  $0 < s < 1$ . Thus, (ii) above and 3.2 imply that  $F(s) \geq C$  for  $L_1$ -a.e.  $s \in [0, 1]$ . However, (i) and an application of 2.2.1 give

$$C = \int_R |\nabla u|^n dL_n = \int_0^1 F(s) dL_1(s)$$

which implies that  $F(s) = C$  for  $L_1$ -a.e.  $s \in [0, 1]$ .

3.12. **REMARK.** The following observation has some interest in view of Theorem 2 of [G2].

*In addition to the hypotheses of 3.11, assume that  $H^{n-1}(C_0) = 0$ . Then there is a point  $x_0 \in C_0$  such that*

$$\limsup_{x \rightarrow x_0} |\nabla u(x)| = \infty.$$

For if this were not the case, then, since  $C_0$  is compact, there would be a constant  $K > 0$  and an open set  $U$  such that  $G - C_1 \supset \text{cl } U \supset U \supset C_0$  and  $|\nabla u|^{n-1} < K$   $L_n$ -a.e. on  $U$ . Choose  $\varepsilon > 0$ . Since  $H^{n-1}(C_0) = 0$ ,  $C_0$  can be covered by a countable number of open  $n$ -balls  $B_i$  such that

$$(18) \quad \bigcup B_i \subset U \quad \text{and} \quad \Sigma H^{n-1}(\text{bdry } B_i) < \varepsilon K^{-1}.$$

Since  $C_0$  is compact, a finite number of the  $B_i$  will cover  $C_0$ , say  $B_1, B_2, \dots, B_k$ . According to (vi) of 2.5, there is a nonnegative Borel function  $f \in \mathcal{L}^n(R)$  such that (ii) of 3.11 fails to hold for only those  $\sigma \in \Sigma$  for which

$$\int_{\sigma \cap R} f dH^{n-1} = \infty.$$

By employing 2.2.1, we can replace each  $n$ -ball  $B_i$ ,  $i = 1, 2, \dots, k$ , by a slightly larger one  $B'_i$  such that

$$\int_{S'_i} f dH^{n-1} < \infty \quad \text{where } S'_i = \text{bdry } B'_i,$$

$|\nabla u|^{n-1} < K$   $H^{n-1}$ -a.e. on  $S'_i$ , and (18) still holds. Now let  $\sigma = \text{bdry}(\bigcup B'_i)$ . Then  $\sigma \in \Sigma$  and (ii) of 3.11 implies that  $C[G, C_0, C_1] < \varepsilon$ , which means that it is zero since  $\varepsilon$  is arbitrary. This would mean that  $\nabla u = 0$   $L_n$ -a.e. on  $G$ . That is, since  $G$  is connected and  $u$  is ACT on  $G$ ,  $u$  would be constant, a contradiction.

In the following theorem, we will consider the general case of two disjoint compact sets  $C_0$  and  $C_1$  which are contained in the closure of an open, bounded, connected set  $G$ .

3.13. **THEOREM.**  $M_n(\Sigma) = C[G, C_0, C_1]^{-1/n-1}$ .

**Proof.** In view of 3.3, we may assume that  $C = C[G, C_0, C_1] \neq 0$ . For each positive integer  $i$  let

$$K_0(i) = \text{cl} \{x : \delta(x, C_0) < (2i)^{-1}\} \text{ and}$$

$$H_0(i) = \{x : \delta(x, C_0) < i^{-1}\}.$$

Define  $K_1(i)$  and  $H_1(i)$  similarly and let  $G_i = G \cup H_0(i) \cup H_1(i)$ . Since  $G$  is connected, it is clear that  $G_i$  is open and connected and notice that  $K_0(i) \cup K_1(i)$  consists only of a finite number of nondegenerate continua. We will consider only those  $i$  for which  $K_0(i)$  and  $K_1(i)$  are disjoint. Thus, let  $v_i$  be the extremal function for  $C_i = C[G_i, K_0(i), K_1(i)]$  and observe that if  $i > j$ , then the restriction of  $v_j$  to  $G_i$  is an admissible function for  $C_i$ . Finally, let  $\Sigma_i$  be those sets  $\sigma$  that separate  $K_0(i)$  from  $K_1(i)$  in  $G_i$  and subject to the condition that  $\sigma \cap [H_0(i) \cup H_1(i)] = \emptyset$ . The purpose for this requirement is that now an  $M_n$ -null class in  $\Sigma_i$  is also  $M_n$ -null in  $\Sigma$ . It is clear that  $\Sigma_1 \subset \Sigma_2 \subset \dots, \Sigma_i \subset \Sigma$  for all  $i$ , and

$$(19) \quad \bigcup_{i=1}^{\infty} \Sigma_i = \Sigma.$$

Since  $C_i$  is a monotonically decreasing sequence bounded below by  $C$ , we can employ again part of the argument of 3.9 to find a vector-valued function  $f$  such that

$$\lim_{i \rightarrow \infty} \int_{E^n} |\nabla v_i - f|^n dL_n = 0$$

and therefore, for a subsequence,

$$(20) \quad \lim_{i \rightarrow \infty} \|\nabla v_i|^{n-1} - |f|^{n-1}\|_{n'} = 0.$$

Hence, if  $L = \lim_{i \rightarrow \infty} C_i$ ,  $g_i = |\nabla v_i|^{n-1} \cdot C_i^{-1}$ , and  $g = |f|^{n-1} \cdot L^{-1}$  then (v) of 2.5 provides another subsequence such that

$$(21) \quad \lim_{i \rightarrow \infty} \int_{\sigma \cap R} g_i dH^{n-1} = \int_{\sigma \cap R} g dH^{n-1} \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

Now by employing 3.11 with  $G, C_0, C_1$  replaced by  $G_i, K_0(i), K_1(i)$  respectively, we have for each  $i$ ,

$$(22) \quad \int_{\sigma \cap R} g_i dH^{n-1} \geq 1 \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma_i.$$

Therefore, (ii) of 2.5, (19), and (21) imply

$$(23) \quad \int_{\sigma \cap R} g dH^{n-1} \geq 1 \quad \text{for } M_n\text{-a.e. } \sigma \in \Sigma.$$

Since  $v_i$  is the extremal for  $C_i$ , (ii) of 3.11 and 3.3 show that, for each  $i$ ,

$$\int_{E^n} (g_i)^{n'} dL_n = C_i^{-1/n-1}.$$

Thus, with (20), (22), and (23), we have

$$C^{-1/n-1} \geq \lim_{i \rightarrow \infty} C_i^{-1/n-1} = \lim_{i \rightarrow \infty} \int_{E^n} (g_i)^{n'} dL_n = \int_{E^n} g^{n'} dL_n \geq M(\Sigma).$$

Theorem 3.3 gives the opposite inequality, and thus the proof is complete.

We will conclude with a result concerning null sets for conformal capacity.

3.14. THEOREM. *Suppose  $C_0$  and  $C_1$  are disjoint compact sets in the closure of  $G$ . If  $E \subset G - (C_0 \cup C_1)$  is a compact set with  $H^{n-1}(E) = 0$ , then*

$$C[G, C_0, C_1] = C[G - E, C_0, C_1].$$

**Proof.** The topological dimension of  $E$  is no more than  $n - 2$  since  $H^{n-1}(E) = 0$  and therefore,  $G - E$  is connected. Thus, the right hand side of the equality has meaning. Clearly,

$$(24) \quad C[G, C_0, C_1] \geq C[G - E, C_0, C_1].$$

The opposite inequality will be established by considering separating sets. Let  $\Sigma^*$  be those sets that separate  $C_0$  from  $C_1$  in

$$[(G - E) - (C_0 \cup C_1)] \cup [C_0 \cup C_1]$$

and let  $\Sigma$  be those that separate  $C_0$  from  $C_1$  in  $R^*$ . In light of (24) and 3.13 it is sufficient to prove

$$(25) \quad M_n(\Sigma) \geq M_n(\Sigma^*).$$

To this end, let  $f$  be a function for which  $f \wedge \Sigma$ . In order to establish (25), we need only show  $f \wedge \Sigma^*$ . Choose  $\sigma^* \in \Sigma^*$  and notice that  $\sigma^* \cup E \in \Sigma$ . Thus,

$$\int_{\sigma^* \cup E} f dH^{n-1} \geq 1$$

and since  $H^{n-1}(E) = 0$ ,

$$\int_{\sigma^*} f dH^{n-1} \geq 1.$$

This shows that  $f \wedge \Sigma^*$  and consequently, proves the theorem.

If  $G$  is compactified  $E^n$ , Bagby has shown that  $C[G, C_0, C_1] = M_n(\Gamma)$  where  $\Gamma$  is the family of all arcs that meet both  $C_0$  and  $C_1$  (Ph.D. thesis, Harvard Univ., Cambridge, Mass., 1966). By using 3.13, [FU], and [W] one can show that this result is valid when  $G$  is an open, bounded, connected set and  $C_0 \cup C_1 \subset G$ . (Moreover, if  $C_0 \cup C_1 \subset \text{cl } G$ , the result is also valid if certain conditions are imposed on the tangential behavior of  $(\text{bdry } G) \cap (C_0 \cup C_1)$ .) Thus, if  $\Gamma^*$  is the family of arcs that join  $C_0$  to  $C_1$  in  $G - E$ , then 3.14 implies

$$M_n(\Gamma^*) = M_n(\Gamma).$$

This result was obtained by Väisälä [V2] in the case where  $C_0$  and  $C_1$  are non-degenerate continua.

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