EXTREMAL METRICS ON GRAPHS AND MANIFOLDS

By

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Abstract

We review basic results in the field of finding extremal metrics for spectral invariants of the Laplacian on both graphs and manifolds. Special attention is given to the special case of the Klein bottle. The nececery theory is developed to produce the result of Jakobson et all [J-N-P] regarding λ_1 on the Klein bottle. Using similar techniques, a new result is established in proving that there is only one extremal metric of a certain kind for λ_2 on the Klein bottle.

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Chapter 1

General Theory for Graphs

We will first focus our attention on graphs. A graph can be thought of as the 1-skeleton of a simplicial complex on a manifold. In this sense, the problem of finding extremal metrics on a graph is the discretizeation of the problem of finding extremal metrics on a manifold; however, in practice we rarely attempt to find an extremal metric for a manifold by discretizing.

Instead, the extremal metric problem for graphs can be used to develop strategies for dealing with the more complicated case on manifolds. Many tools and results developed for the relatively simple structure of graphs, are quite similar to those on manifolds.

In addition, the problem of finding extremal metrics for graphs is often interesting in and of itself. There are many useful applications to fields such as information networking.

1.1 Basic Definitions

We begin with the basic definitions for weighted graphs, then construct two important geometric invariants.

A graph G is a set of vertices (denoted V(G)) together with a set of edges (denoted E(G)). We denote the edge between v_1 and v_2 by $\overline{(v_1, v_2)}$. Unless otherwise specified, we assume all graphs to have a finite number of edges and vertices. We will also restrict our attention to simple graphs: those in which each edge has two distinct endpoints and no two vertices have more than one common edge.

We wish to translate the concept of a metric to graphs. Geometrically, a metric prescribes the lengths of curves on a manifold, a metric for a graph should do the same. The edges of the graph correspond to distances between points, thus the most natural candidate for a metric on a graph would define lengths of edges.

Definition 1.1.1. A weight function for a graph G is a function $w : E(G) \to \mathbb{R}^+$ which assigns a positive real number $w(e_j)$ to every edge $e_j \in E(G)$
A weighted graph is a graph along with a weight function on that graph.

Once a weight function has been chosen for a particular graph, the usual geometric notions such as distance and volume can be constructed. These will always depend on the particular choice of weight function.

Definition 1.1.2. for two vertices $v_1, v_2 \in V(G)$, define the distance between them to be

$$dist(v_1, v_2) := \min_{p \in P} \sum_{e_k \in p} w(e_k)$$

where P is the set of paths from v_1 to v_2. In other words, the distance is the length of the shortest path connecting the two points.

Definition 1.1.3. The volume of a graph G is defined as the sum of the weights of the edges.

$$vol(G) := \sum_{e_j \in E(G)} w(e_j)$$

We will be particularly interested in the case that the weight function is constant.

Definition 1.1.4. If $w(e_j) = 1 \ \forall j$, then G is said to be a combinatorial graph and w the combinatorial weight for G.

1.1.1 Girth

The first geometric invariant we will work with is girth. Intuitively, girth of an object is the shortest distance around it. For graphs, this is interpreted as the length shortest (non-trivial) cycle.

Definition 1.1.5. A systole is a closed non-trivial path of minimum length.

By non-trivial, it is meant that the path must contain at least three distinct vertices, and there is no backtracking sub-cycle; that is, there is no sub-cycle of length two that traverses only one edge.

Proposition 1.1.1. *A systole has no self-intersections, i.e. every vertex is traversed precisely once.*

Proof. Let G be a weighted graph and suppose s is a systole for G that traverses some vertex more than once. Then for some vertices $v_{i_j}, v_{i_k} \in V(G)$, we have $v_{i_j} = v_{i_k}$.

W.l.o.g. $s = (v_{i_1}, v_{i_2}, ..., v_{i_{j-1}}, v_{i_j}, v_{i_{j+1}}, ..., v_{i_{k-1}}, v_{i_k}, v_{i_{k+1}}, ...v_{i_l})$. Consider the sub chain $s' = (v_{i_j}, v_{i_{j+1}}, ..., v_{i_{k-1}}, v_{i_k})$. There must be at least one edge in s not contained in s', since otherwise, s would contain a closed chain of length one or two. The first possibility is excluded since we require that the graph G be a simple graph. The second is impossible since the definition of systole precludes backtracking.

Since the collection of edges contained in s' is a proper subset of the collection of edges contained in s, and since each edge must have positive weight, the length of s' is strictly less than the length of s. This contradicts the fact that s is a systole.

$\square$

Definition 1.1.6. The girth of a graph G with weight w, denoted $\gamma(G,w)$ is the length of a systole. Given a predetermined graph G, $\gamma(w)$ denotes girth as a function of the weight w.

It is clear from the definition that $\gamma(G,w)$ is invariant under isometries of graphs.

1.1.2 Laplacian

The second geometric invariant we shall consider is the Laplacian. For a function $f : \mathbb{R}^n \to \mathbb{R}$, the Laplacian $\triangle$ is taken to be the divergence of the gradient, or

$$\triangle f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

If f is twice continuously differentiable, we have the following characterizations of the Laplacian:

Mean Value Property

$$\triangle f \equiv 0 \ in \ \Omega \subset \mathbb{R}^n \Leftrightarrow f(x_0) = (\frac{1}{area(\partial B_r(x_0))}) \int_{\partial B_r(x_0)} f(x) dA \qquad (1.1.1)$$

$$\triangle f < 0 \ in \ \Omega \subset \mathbb{R}^n \Leftrightarrow f(x_0) < (\frac{1}{area(\partial B_r(x_0))}) \int_{\partial B_r(x_0)} f(x) dA. \qquad (1.1.2)$$

$$for \ any \ B_r(x_0) \subset \Omega$$

A rigorous definition of the Laplacian for a graph or arbitrary manifold should take into account both its formal definition, and its geometric property of measuring difference between a function and its average over some region.

The Laplacian in $\mathbb{R}^n$ acts on functions of the points in $\mathbb{R}^n$. Thus it is reasonable to assume that the Laplacian for graphs should act on functions of the vertices of the graph, as these correspond to points. Also, since the Laplacian in $\mathbb{R}^n$ is a linear operator, the Laplacian for graphs should be a linear operator acting on $\mathbb{R}^{V(G)}$.

Given a function $f : \mathbb{R}^{V(G)} \to \mathbb{R}$, we would like to define $\triangle f : \mathbb{R}^{V(G)} \to \mathbb{R}$ so that the i^{th} coordinate of $\triangle f$ measures the difference between f at the vertex i, and the average value of f on the set of vertices connected to i.

In order to take the weight function into account, we must formulate a way to interpret the weight function at vertices.

Definition 1.1.7. The degree of vertex v_i on graph G with weight w is defined as the sum of weights of edges adjacent to v_i, $deg(v_i) := \sum_{e_k \sim v_i} w(e_k)$.

The Laplacian is defined so that i^{th} coordinate measures the difference between $f(v_i)$ multiplied by the degree of i, and the weighted average value of f on the vertices confected to i. The weights used in the average are the weights of the connecting edge. Finally we have an explicit representation of the Laplacian for a weighted graph G.

Definition 1.1.8. For a graph G with vertices $v_1, v_2, ..., v_k$ and weight w, the Laplacian is defined as:

$$\triangle_{G,w} = \begin{pmatrix} deg(v_1) & -w(\overline{v_1, v_2}) & ... & -w(\overline{v_1, v_k}) \\ -w(\overline{v_2, v_1}) & deg(v_2) & ... & -w(\overline{v_2, v_k}) \\ \vdots & \vdots & \ddots & \vdots \\ -w(\overline{v_k, v_1}) & -w(\overline{v_k, v_2}) & ... & deg(v_k) \end{pmatrix}$$

where we take the convention that $w(\overline{(v_i, v_j)})$ is equal to 0 if there is no edge connecting these two vertices.

When the graph and weight are understood from context, the Laplacian shall be referred to simply as $\triangle$.

From the definition, it is clear that $\triangle$ on graphs satisfies a version of the mean value property. What is less clear is that it can also be defined in a similar manner to the Laplacian in $\mathbb{R}^n$, in terms of the operator D which serves the role of the gradient.

The gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ is comprised of its partial derivatives. Partial derivatives in $\mathbb{R}^n$ are defined as the limiting case of a difference quotient.

$$\partial_j f(x) = \lim_{h \to 0} [f(x) - f(x + he_j)] \|he_j\|^{-1}$$

where e_j is standard basis vector. We can construct a similar definition for graphs. The only difference (for the sake of having $\triangle$ agree with the definition given above) is that we will take the weight to the power $1/2$ instead of -1. We must also prescribe an arbitrary orientation σ on the edges of G. The choice of orientation will not effect the Laplacian, and is a necessary part of the definition which corresponds to the natural orientation given to $\mathbb{R}^n$.

We take the discrete differential of f in the direction of edge e_j to be defined as $\partial_j f := [f(e_j^+) - f(e_j^-)]$, where e_j^+ is the vertex at the head of edge e_j, and e_j^- is the vertex at the tail.

Definition 1.1.9. Fixing an arbitrary orientation σ on the edges of G with weight w, we can define the D operator in imitation of the gradient in $\mathbb{R}^n$ as

$$D_\sigma := \begin{pmatrix} w(e_1)^{1/2} \partial_1 \\ w(e_2)^{1/2} \partial_2 \\ \vdots \\ w(e_n)^{1/2} \partial_n \end{pmatrix}$$

More explicitly

$$D_\sigma := \begin{pmatrix} D_{1,1} & D_{1,2} & \dots & D_{1,m} \\ D_{2,1} & D_{2,2} & \dots & D_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ D_{n,1} & D_{n,2} & \dots & D_{n,m} \end{pmatrix} \text{Where } D_{i,j} = \left\{ \begin{array}{ll} \sqrt{w(e_i)} & v_j = e_i^+ \\ -\sqrt{w(e_i)} & v_j = e_i^- \\ 0 & v_j \notin \partial e_i \end{array} \right\}$$

In most cases the choice of orientation is inconsequential and we refer simply to D without mention of σ. As is the case with the Laplacian in $\mathbb{R}^n$, we have a representation of $\triangle$ in terms of the operator D:

Proposition 1.1.2. $\triangle := D^T D$ *for any choice of orientation.*

Proof.

$$D^T D = \left(\begin{array}{cccc} D^T \begin{bmatrix} D_{1,1} \\ D_{2,1} \\ \vdots \\ D_{n,1} \end{bmatrix} & D^T \begin{bmatrix} D_{1,2} \\ D_{2,2} \\ \vdots \\ D_{n,2} \end{bmatrix} & \dots & D^T \begin{bmatrix} D_{1,m} \\ D_{2,m} \\ \vdots \\ D_{n,m} \end{bmatrix} \end{array} \right)$$

Where

$$D^T \begin{bmatrix} D_{1,i} \\ D_{2,i} \\ \vdots \\ D_{n,i} \end{bmatrix} = \begin{bmatrix} \langle (D_{1,1}, D_{2,1}, ..., D_{n,1}), (D_{1,i}, D_{2,i}, ..., D_{n,i}) \rangle \\ \vdots \\ \langle (D_{1,i}, D_{2,i}, ..., D_{n,i}), (D_{1,i}, D_{2,i}, ..., D_{n,i}) \rangle \\ \vdots \\ \langle (D_{1,m}, D_{2,m}, ..., D_{n,m}), (D_{1,i}, D_{2,i}, ..., D_{n,i}) \rangle \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n D_{k,1} D_{k,i} \\ \vdots \\ \sum_{k=1}^n D_{k,i}^2 \\ \vdots \\ \sum_{k=1}^n D_{k,m} D_{k,i} \end{bmatrix}$$

First consider the off-diagonal terms. In the first coordinate position, $D_{k,1} = 0$ unless vertex v_1 is in the boundary of e_k. Similarly $D_{k,i} = 0$ unless v_1 is in the boundary of e_k. Thus the only way a term in this sum could be nonzero is if $\partial e_k = \{v_1, v_i\}$. Since G is assumed to have no chains of length 2, there can be at most one edge between v_1 and v_i, hence the sum has at most one non-zero term. We conclude that $\sum_{k=1}^n D_{k,1} D_{k,i} = \left(\pm\sqrt{w(e_k)}\right)\left(\mp\sqrt{w(e_k)}\right)$ if $\partial e_k = \{v_1, v_i\}$, and 0 otherwise. In other words, this sum is equal to $-w(\overline{v_1, v_i})$. All other off-diagonal terms are similar.

The diagonal term $\sum_{k=1}^n D_{k,i}^2$ is clearly equal to $\sum_{e_k \sim v_i} (\pm\sqrt{w(e_k)})^2$. That is the sum of the weights $w(e_k)$ for each edge k attached to v_i. This is precisely the degree of v_i. $\square$

With this representation of the Laplacian, we can formulate an important characterization of the spectrum of $\triangle$.

Proposition 1.1.3. *All eigenvalues of $\triangle$ are nonnegative, and for connected graphs, the multiplicity of the eigenvalue 0 is equal to 1.*

Proof. Since $\triangle$ is a real symmetric n by n matrix, it has n real eigenvalues. To see that these eigenvalues are nonnegative, note that for any function
$f : V(G) \to \mathbb{R}$, we have that $\| f \|^2 \geq 0$. Thus if f is an eigenvector for eigenvalue λ, we have:

$$\lambda \| f \|^2 = \lambda\langle f, f\rangle = \langle \lambda f, f\rangle = \langle \triangle f, f\rangle = \langle D^T Df, f\rangle = \langle Df, Df\rangle = \| Df \|^2 \geq 0$$

So λ must be greater than or equal to 0 and we conclude that $\triangle$ is nonnegative definite.

The multiplicity of $\lambda_0 = 0$ is at least 1 since the rows of $\triangle$ each sum to 0, so $\triangle \overrightarrow{1} = 0$. Now suppose that f is an eigenvector of λ_0. Then from above, we have that $0 = \lambda_0 \| f \|^2 = \| Df \|^2$. Thus $Df = \overrightarrow{0}$. As is the case in $\mathbb{R}^n$, if D of a function is zero, it must be constant. To see this note that $Dg = \overrightarrow{0}$ implies that the j^{th} coordinate, $\partial_j f$, is equal to 0. So for each edge $0 = w(e_j)^{1/2}\partial_j := [g(e_j^+) - g(e_j^-)]w(e_j)^{1/2}$. In other words, on any edge with non-zero weight, the difference between the values of g on the endpoints must be 0. If the graph G is connected, this implies g is constant. If G is not connected, it is clear that g must be constant on connected components, thus the multiplicity of λ_0 is equal to the number of connected components of G.

$\square$

1.2 General Graph Theory Results

1.2.1 Spanning Trees and Kirckhoff's Theorem

Trees and spanning trees play an important role in graph theory and information networks. As we shall see, they are also closely related to the Laplace operator.

Definition 1.2.1. A tree is a connected graph with no cycles.

Definition 1.2.2. A spanning tree for a connected graph G is a subgraph of G that is a tree and contains every vertex of G.

Remark 1.2.1. Note that if G has n vertices, then any spanning tree t must necessarily have n-1 edges, since t is connected and t has no cycles. Similarly, any subtree of G with n-1 edges is a spanning tree.

In the study of un-weighted graphs, an important characteristic of a graph G is the number of spanning trees it contains. We would like to define an invariant for weighted graphs, related to the number of spanning trees, that takes into account the weights of the edges traversed by those trees.

Definition 1.2.3.

$$\tau(G, w) = \sum_{t \in T(G)} \prod_{e_j \in t} w(e_j)$$

where $T(G)$ is the set of spanning trees for G.

Note that if w is the combinatorial weight, the product becomes 1 for each spanning tree, and $\tau(G)$ is just the number of spanning trees.

We are mostly interested in $\tau(G)$ due to Kirckhoff's Theorem, a result that relates $\tau(G)$ to the Laplacian.

Theorem 1.2.1. ***Kirckhoff's Theorem***

$$\tau(G) = \mid \triangle_{i,i} \mid$$

That is, the minor of $\triangle$ obtained by removing the i^{th} row and column.

Note that since the multiplicity of 0 as an eigenvalue of $\triangle$ is one, we can show that $\mid \triangle_{i,i} \mid$ is just the product of the non-zero eigenvalues.

Proof. Let G be a weighted graph with n edges and m vertices and let w be the weight function on G. For any matrix A, let $A_{i,i}$ denote the matrix obtained by removing the i^{th} row and i^{th} column from A. Similarly, let $A_{H,i}$ denote the matrix obtained by removing the i^{th} column and rows indexed by the set $H \subset E(G)$, etc.

By proposition 1.1.2, $\triangle = D^T D$. Thus any minor $\mid \triangle_{i,i} \mid$ of the Laplacian matrix can be written as $\mid (D^T D)_{i,i} \mid$.

By the Cauchy-Binet theorem

$$\mid (D^T D)_{i,i} \mid = \sum_K \mid D^T_{i,H} \mid \mid D_{H,i} \mid$$

Where the sum is taken over all sets H of $n - m + 1$ edges. Note that $n \geq m - 1$ with equality in the case that G is a tree. The formula above is still valid however, if we allow H to be empty.

By lema 1.2.2 below,

$$\mid D^T_{i,H} \mid = \left\{ \begin{array}{ll} \pm\sqrt{\prod_{e_j \in t} w(e_j)} & if\ the\ columns\ of\ D^T_{i,H}\ form\ tree\ t \\ 0 & if\ the\ columns\ of\ D^T_{i,H}\ do\ not\ form\ a\ tree \end{array} \right\}$$

Combining these equations with the fact that $\mid D^T_{i,H} \mid = \mid D_{H,i} \mid$, we have:

$$\mid \triangle_{i,i} \mid = \sum_{t \in T(G)} \prod_{e_j \in t} w(e_j) + \sum_{G_{n-1} \setminus T(G)} 0$$

Where G_{n-1} is the set of subgraphs with $n-1$ edges, and $T(G)$ is the subset of trees in G_{n-1}. Thus

$$\mid \triangle_{i,i} \mid = \sum_{t \in T(G)} \prod_{e_j \in t} w(e_j) = \tau(G)$$

as was to be proved.

$\square$

Lemma 1.2.2.

$$\mid D^T_{i,H} \mid = \left\{ \begin{array}{ll} \pm\sqrt{\prod_{e_j \in t} w(e_j)} & if\ the\ columns\ of\ D^T_{i,H}\ form\ tree\ t \\ 0 & if\ the\ columns\ of\ D^T_{i,H}\ do\ not\ form\ a\ tree \end{array} \right\}$$

Proof. Again, let G be a weighted graph with weight w. Suppose G has n edges and m vertices. Let $M = D^T_{i,H}$ denote a matrix obtained by removing row i and $n-m+1$

columns indexed by the set H from D^T. This corresponds to excluding one vertex and all but m-1 edges.

The columns of M represent the edges of a subgraph. Let $e_{(\overline{v_j,v_k})}$ denote the column of M corresponding to the edge $(\overline{v_j,v_k})$ between v_j and v_k where $j<k$.

Case 1 If the columns of M do not correspond to a tree, then they must contain a subset of columns that correspond to a cycle of edges. In such a cycle, each vertex must be contained in precisely two edges.

If the cycle does not include vertex v_i then consider the linear combination

$$\sum_{edges\ in\ cycle} c\, e_{(\overline{v_j,v_k})}$$

where the constant $c = \pm\frac{1}{w(\overline{v_j,v_k})}$. The sign of c is taken so that that the two appearances of a vertex in the sum have opposite sign. Since each vertex appears in the sum precisely twice, the j^{th} coordinate of the sum will be 0 if no edge in the cycle touches vertex j, and $1-1=0$ otherwise. The coefficients c are clearly nonzero, thus the columns $e_{(v_j,v_k)}$ forming a cycle are linearly dependant and $\det M = 0$.

If v_i is included in the cycle, then the coefficients of the linear combination are adjusted so that for the edges that contain v_i, $c = \pm\frac{1}{\sqrt{w(\overline{v_j,v_i})}}$. As before $\det M = 0$.

Case 2 Now suppose the columns of M correspond to a tree. Since there are m-1 edges in M, the tree must be a spanning tree for G by remark 1.2.1. Thus vertex v_i is necessarily contained in said tree.

In order to simplify the process of taking the determinant of M, recall that adding a non-zero multiple of one column of M to another does not change the determinant. We may then take the determinant of a simpler matrix obtained from M following a process that essentially traces each vertex back to v_i.

Any column c of M must correspond to some edge $(\overline{v_j,v_k})$ (one of v_j and v_k could possibly be v_i). Since each column corresponds to an edge through two vertices, it must have at most two non-zero entries $\pm\sqrt{w(\overline{v_j, v_k})}$ and $\mp\sqrt{w(\overline{v_j, v_k})}$ in the j^{th} and k^{th} rows. If one of these vertices is v_i, the column has only one non-zero entry since the i^{th} row has been removed.

We define a new matrix M' by replacing the columns of M as follows:

Let $p = (e_{(v_{k_1}, v_{k_2})}, e_{(v_{k_2}, v_{k_3})}, ...e_{(v_{k_{p-1}}, v_{k_p})})$ where $v_{k_1} = v_i$ and $v_{k_p} = v_j$ be a path of edges from $e_{(v_j, v_k)}$ to an edge that passes through v_i.

Replace the column c with the following linear combination of columns corresponding to the edges in p.

$$\sum_{e \in p} \frac{w(v_{k_\alpha}, v_{k_{\alpha+1}})}{w(v_{k_{\alpha-1}}, v_{k_\alpha})} e_{(v_{k_\alpha}, v_{k_{\alpha+1}})}$$

The weights in this sum are chosen so that the α^{th} row of one column is canceled by the α^{th} row of the next column. Since the final column has only one non-zero row, the result of the sum is to complectly cancel all but on of the original non-zero rows of c, which has the value $\pm\sqrt{w(e_{(v_j, v_k)})}$.

As a result, M' is an $(n-1)\times(n-1)$ matrix with precisely one non-zero element in each column. Permuting the columns only changes the sign of the determinant, and will result in the diagonal matrix M'' with the value $\pm\sqrt{w(e_{(v_j, v_k)})}$ on the j^{th} diagonal position. $\mid M \mid = \mid M' \mid = \pm \mid M'' \mid = \pm \prod \sqrt{w(e_{(v_j, v_k)})} = \pm\sqrt{\prod w(e_{(v_j, v_k)})}$ $\square$

1.2.2 Perturbations of Symmetric Matrices and Eigenvalues

In the following chapter we will be interested in the effect of perturbing the weight function on the eigenvalues of the Laplacian. The eigenvalues of the Laplacian are the roots of the characteristic polynomial, who'se coefficients depend analytically on the entries of the Laplacian, and thus also depend analytically on the weight. Since the roots of a polynomial depend analytically on the coefficients of that polynomial, it is reasonable to conjecture that a linear perturbation of the weight function will result in an analytic perturbation of the eigenvalues and eigenvectors. What is not as clear is that said eigenvectors can be taken to be orthonormal at all times.

Theorem 1.2.3. *Let $\mathbf{Q}(\varepsilon) = Q + \varepsilon B$ be a linear perturbation of Q in the space of symmetric matrixes. Then for each eigenvalue λ of Q with multiplicity k. There exists a set of k real valued $\mathcal{C}^\omega$ functions $\lambda_j(\varepsilon)$ and a set of k $\mathcal{C}^\omega$ vector valued functions $\varphi_j(\varepsilon)$ such that the following hold:*

- *for $0 < \varepsilon < \delta$, $\mathbf{Q}(\varepsilon)\varphi_{\mathbf{j}}(\varepsilon) = \lambda_j(\varepsilon)\varphi(\varepsilon)$*
- *for $0 < \varepsilon < \delta$ $\exists \alpha$ such that any eigenvalue of $\mathbf{Q}(t)$ in $(\lambda - \alpha, \lambda + \alpha)$ must be one of the $\lambda_j(\varepsilon)'s$*
- *for any fixed ε, the vectors $\varphi_j(\varepsilon)$ form an orthonormal set.*

Chapter 2

Extremal Weight Problems for Graphs

2.1 Basic Problem of Extremal Weights

In this chapter, we shall discuss some basic results in the general theory of finding extremal weights for invariants on graphs. A weight is extremal for a functional on a Graph if no other weight function respecting certain conditions yields a larger (or smaller) value for that functional. The invariants in question include the girth operator $\gamma(G, w)$ discussed in chapter 1 and two spectral invariants. The spectral invariants are so called because they are functionals of the eigenvalues of the Laplacian. This category includes $\lambda_1(G, w)$ - the first nonzero eigenvalue, and $\log \det \triangle^*(G, w)$ the logarithm of the product of the nonzero eigenvalues of the Laplacian.

We wish to consider the following basic problems a fixed geometric invariant $\alpha(G, w)$:

1. Which weight functions are extremal for $\alpha(w)$ on some fixed graph G?

2. What bounds or approximations are there for the extreme value of $\alpha(w)$?

3. For which Graphs G is the combinatorial weight extremal for $\alpha(w)$?

Before stating results for these particular invariants, we cover some general results.

Remark 2.1.1. Throughout this chapter, we shall make frequent use of the fact that the set of all weight functions on a fixed graph G is a subset of the vector space $\mathbb{R}^{E(G)}$, which is isomorphic to $\mathbb{R}^n$ where n is the number of edges. This fact allows us to introduce geometric intuition as the set of weights we wish to optimize over is a subset of a finite dimensional vector space. Also note that in general we will reserve n to be the number of edges in G and m to be the number of vertices for the remainder of the chapter.

2.1.1 Normalization Condition

The aforementioned problem of maximizing girth, λ_1, and $\log\det\triangle^*$ over the set of all possible weight functions is trivial unless other restrictions are in place. Note that multiplying the weight function by a nonzero scaler c has the effect of increasing the length of every path by a factor of c. Thus for any fixed graph G, $\gamma(cw) = c\gamma(w)$ and the girth can be made as large as desired. Similar arguments can be used to show that λ_1, and $\log\det\triangle^*$ can also be made as large as desired by simply scaling the weight function.

Because of this we must put restrictions on the set of weights to be considered to disallow scaling by an arbitrary constant. The simplest way to do this is to require the volume of graph G to be a fixed constant.

Normalization Condition

$$Vol(G) = n = \mid E(G) \mid \tag{2.1.1}$$

With the normalization condition, the set of weights under consideration is the interior of an n-simplex in $\mathbb{R}^{E(G)} \cong \mathbb{R}^n$.

Note that it is also the intersection of $(\mathbb{R}^+)^n$ with the 1-norm sphere of radius n. It may be possible to impose other normalization conditions, such as requiring that the weight function vectors lie on the p-norm sphere for various p.

2.1.2 Concavity and Perturbations of Metrics

We will show in the following sections that the invariants we wish to consider are all concave functionals on the set of weights.

Definition 2.1.1. A real-valued functional α on a vector space X is said to be concave if and only if $\forall \mathbf{u}, \mathbf{v} \in X,\ \forall c \in (0,1) \quad \alpha(c\mathbf{v} + (1-c)\mathbf{u}) \geq c\alpha(\mathbf{v}) + (1-c)\alpha(\mathbf{u})$

If a functional on the set of weights is concave, any locally maximal weight is globally maximal. Thus a weight w is extremal for the functional if and only if no sufficiently small perturbation of w respecting the normalization condition increases its value.

As mentioned above, the set of weights upon which the functionals $\gamma(w)$, $\lambda_1(w)$, and $\log\det \triangle(w)^*$ are taken to act is the interior $int(S)$ of some n-simplex S. The closure of this set is the simplex S, which is compact. This corresponds to the set of all possible non-negative (as opposed to strictly positive) weight functions obeying the normalization condition. It is possible that for certain graphs, the functionals under consideration achieve their maximal value on the boundary of S. In this case, the extremal value is obtained by allowing the weights of certain edges to become zero.

The following lemmas will be useful in proving the concavity of certain functionals.

Lemma 2.1.1. *The point-wise minimum of a set of concave operators is also a concave operator.*

Proof.

$$\textit{Let } \alpha(w) := \min_{j \in I} \alpha_j \quad \textit{with } \alpha_j \textit{ concave } \forall j$$

For any weight functions $w, u \in \mathbb{R}^{E(G)}$, we have that for any $0 \le c \le 1$

$\alpha(cw + (1-c)u) = \min \alpha_j(cw + (1-c)m)$.

So for some $i \in I$,

$\alpha(cw + (1-c)\,u) = \alpha_i(cw + (1-c)\,u)$

$\ge c\,\alpha_i(w) + (1-c)\,\alpha_i(u)$ (by concavity of α_i)

$\ge c\,\alpha(w) + (1-c)\,\alpha(u)$ (by minimality of α)

Thus α is a concave function.

□

2.2 Extremal Weights for Girth

The first geometric invariant we will work with is girth. We begin by showing that it is a concave functional on the set of weights respecting the normalization condition. We use this to formulate some necessary and sufficient conditions for maximal girth.

2.2.1 Concavity of Girth

Proposition 2.2.1. *$\gamma(w)$ is a concave function of w.*

Proof. It is clear from the definition that the girth of G can be expressed as

$$\gamma(w) = \min_{\alpha \in \mathcal{G}} \sum_{e_k \in \alpha} w(e_k)$$

where $\mathcal{G}$ is the set of closed geodesics on G.

For any fixed geodesic α, $\sum_{e_k \in \alpha} w(e_k)$ is a linear functional on $\mathbb{R}^n$, and hence is concave. Thus $\gamma(w)$ is the minimum of a set of concave operators. By lemma 2.1.1 it is concave. □

2.2.2 Conditions for Maximal Girth

We begin with a lemma that gives a necessary condition for maximality. This lemma is interesting in that the necessary condition implies a certain amount of symmetry in the weight function on G.

The proof of the lemma uses the fact that γ is concave, thus a weight is extremal for girth if and only if no perturbation of the weight increases girth.

Lemma 2.2.2. *For any fixed graph G, if the weight w maximizes the girth of G, then every nonzero edge of G is contained in some systole.*

We prove the contrapositive: Suppose $\exists e_j \in E(G)$ such that $w(e_j) \neq 0$ and for any systole s of G, $e_j \notin s$. Then it is possible to increase $\gamma(w)$ by changing w.

Proof. Fix a graph G with weight w, and let $w_j := w(e_j)$ for brevity. Suppose with weight w, G has k systoles $s_1, s_2, ...s_k$. Suppose as well that there is a nonzero edge e_0 not in any of these systoles.

Let d be the difference between the length of a systole and the length of the next shortest closed loop. Choose $\varepsilon > 0$ s.t. $2\varepsilon < d$ and $\varepsilon < w_0 = w(e_0)$.

Now define the weight function $\mathbf{w}(\varepsilon)$ - where again $\mathbf{w}(\varepsilon)(e_j)$ is abbreviated as $\mathbf{w_j}(\varepsilon)$ - as follows:

$$\mathbf{w_j}(\varepsilon) := \left\{ \begin{array}{ll} w_j + \frac{\varepsilon}{\sigma} & e_j \in s_i \\ w_j - \varepsilon & e_j = e_0 \\ w_j & otherwise \end{array} \right\}$$

where σ is the total number of edges belonging to at least one systole.

$\mathbf{w}(\varepsilon)$ is a permissible weight function since the sum of the weights of the edges is that same as that of w, and since all weights are still strictly positive. Also, this perturbation increases the length of each w-systole, since the only edge to lose weight is not in any such systole, and every edge in each systole gains weight.

Also, the systoles for weight $\mathbf{w}(\varepsilon)$ must be among the systoles of w, since each systole gains an overall weight of at most ε, and $\varepsilon < d/2$. Thus no w systole gains enough weight to be longer than any non-systole loop.

$\therefore \gamma(\mathbf{w}(\varepsilon)) > \gamma(w)$ $\square$

We will now develop a necessary and sufficient condition for a weight w to be extremal for girth on a given graph G.

Fix graph G with $|E(G)| = n$, and again let $w_j := w(e_j)$ for brevity. Suppose the weight w is extremal for girth. For some fixed $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^{E(G)}$, consider the linear perturbation given by $\mathbf{w_j}(\varepsilon) = w_j + \varepsilon b_j$. Note that $\mathbf{w_j}(0) = w_j$ and that $\mathbf{w_j}(\varepsilon)$ is linear with respect to ε.

The choice of b must respect the normalization condition.

$$\sum_{j=1}^{n} \mathbf{w_j}(\varepsilon) = n \qquad \forall \varepsilon$$

Thus we have

$$\sum_{j=1}^{n} b_j = 0$$

So $\langle b, (1, 1, ..., 1)\rangle = 0$ or $b \perp \overrightarrow{1}$.

Now suppose the systoles for weight w are given by $S_1, S_2, ..., S_k$, where $S_i = (e_{i_1}, e_{i_2}, ...e_{i_p})$. Then the length of S_i for weight $\mathbf{w}(\varepsilon)$ is given by

$$w_{i_1} + \varepsilon b_{i_1} + w_{i_2} + \varepsilon b_{i_2} + ... + w_{i_p} + \varepsilon b_{i_p}$$

$$= \sum_{j=1}^{p} (w_{i_j} + \varepsilon b_{i_j}) = \sum_{j=1}^{p} w_{i_j} + \varepsilon \sum_{j=1}^{p} b_{i_j} = \gamma(w) + \varepsilon \sum_{j=1}^{p} b_{i_j}$$

since the girth is the length of any systole.

If w is indeed maximal, then perturbing by sufficiently small ε will not increase the length of a systole. Thus we have that $\sum_{j=1}^{p} b_{i_j} \leq 0$.

Similarly, if w is not maximal, there is some perturbation $\mathbf{w_j}(\varepsilon) : \varepsilon \mapsto w_j + \varepsilon b_j$ such that $\sum_{j=1}^{p} b_{i_j} > 0$ for every systole. Thus $\langle b, \chi_{s_i} \rangle > 0 \ \forall i$, where χ_{s_i} is the characteristic function of the systole s_i

We have reformulated the problem of determining if a weight is extremal into a linear algebra problem:

the weight w is not extremal for girth if and only if there exists a nontrivial linear perturbation given by $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^{E(G)}$ such that.

$\langle b, \overrightarrow{1} \rangle = 0$ and $\langle b, \chi_{s_i} \rangle > 0 \quad \forall s_i$ a systole of w.

In other words, if and only if the system:

$$\left(\begin{array}{c} \left[\quad \overrightarrow{1} \quad \right] \\ \left[\quad \chi_{s_1} \quad \right] \\ \vdots \\ \left[\quad \chi_{s_k} \quad \right] \end{array} \right) \begin{bmatrix} \\ b \\ \\ \end{bmatrix} \begin{array}{c} = \\ > \\ \vdots \\ > \end{array} 0$$

has a solution.

The theorem below gives a simple condition for when this system had no solution.

Theorem 2.2.3. *A weight w for the graph G is extremal for girth if and only if $\overrightarrow{1}$ lies in the positive cone generated by the characteristic functions χ_{s_i}, where the s_i's are the systoles of w.*

Proof. The theorem and proof are essentially an extension and application of Farkas Lemma

let

$$\mathfrak{C} = \left\{ \sum_{i=1}^{k} \alpha_i \chi_{s_i} | \alpha_i \geq 0, \ \textit{not all } 0 \right\}$$

that is, the positive open cone of the characteristic functions χ_{s_i}.

proof of $\Leftarrow$

suppose $\overrightarrow{1} \in \mathfrak{C}$, then $\overrightarrow{1} = \Sigma_{i=1}^{k} \alpha_i \chi_{s_i}$ for some $\alpha_i \geq 0$ not all equal to 0. Let b be a solution to the system of equations $\langle b, \overrightarrow{1} \rangle = 0$ and $\langle b, \chi_{s_i} \rangle > 0 \quad \forall s_i$. Then $\langle b, \overrightarrow{1} \rangle$, so from above:

$0 = \langle b, \overrightarrow{1} \rangle = \langle b, \alpha_1 \chi_{s_1} + \alpha_2 \chi_{s_2} + ...\alpha_k \chi_{s_k} \rangle = \alpha_1 \langle b, \chi_{s_1} \rangle + \alpha_2 \langle b, \chi_{s_2} \rangle + ...\alpha_k \langle b, \chi_{s_k} \rangle > 0$ since $\langle b, \chi_{s_i} \rangle > 0 \ \forall s_i$ and $\alpha_i \geq 0$ not all equal to 0. Contradiction.

proof of $\Rightarrow$

Suppose $\overrightarrow{1} \notin \mathfrak{C}$ Then consider the orthogonal projection $prj\mathfrak{C}$ of $\mathfrak{C}$ onto $\overrightarrow{1}^{\perp}$. Note that $prj\mathfrak{C}$ is also a cone in the subspace $\overrightarrow{1}^{\perp}$, To verify this, note that 0 is not in $prj\mathfrak{C}$ since no multiple of $\overrightarrow{1}$ is $\mathfrak{C}$.

Thus we have that $prj\mathfrak{C}$ is contained entirely within a half-plane of $\overrightarrow{1}^{\perp}$. Let b be the positive normal to the boundary of this half plane. That is, b has positive inner product with every vector in the open half plane containing $prj\mathfrak{C}$.

Since $b \in \overrightarrow{1}^{\perp}$, we have $\langle b, \overrightarrow{1} \rangle = 0$. Also, since χ_{s_i} is the direct sum of $prj\chi_{s_i}$ (it's projection onto $\overrightarrow{1}^{\perp}$) and $prp\chi_{s_i}$(it's component in the direction of $\overrightarrow{1}$), we have $\langle b, \chi_{s_i} \rangle = \langle b, prj\chi_{s_i} + prp\chi_{s_i} \rangle = \langle b, prj\chi_{s_i} \rangle + \langle b, \alpha \overrightarrow{1} \rangle = \langle b, prj\chi_{s_i} \rangle + 0 > 0$ as desired. $\square$

The result gives us a powerful tool in determining weather a weight w is extremal for girth. It is a simple matter of finding the systoles of the graph with the given weight and performing a computation. Although the results do not give us a simple algorithm for finding an extremal weight for girth, they do indicate many properties of such a weight. For instance such a weight must have a relatively plentiful amount of systoles so that each edge is in at least one and the positive cone spanned by their characteristic functions contains the vector $\overrightarrow{1}$. As we shall see, extremal weights and metrics will often have similarly symmetric properties for other invariants.

2.2.3 Combinatorial Weight

The results above lead us to the following result:

Theorem 2.2.4. *If the group A of automorphisms of G acts transitively on the edges of G, then the combinatorial weight ($w(e_j) = 1\ \forall j$)is extremal.*

Recall that A acts transitively on the edges of G means that for any two edges e_i and e_j, $\exists \alpha \in A$ such that $\alpha : e_i \mapsto e_j$.

Proof. Suppose A acts transitively on the edges of G, and take w to be the combinatorial weight on $E(G)$.

First we note that the image of a systole under any automorphism α is another systole since the automorphism must be a bijection of edges and all edges have the same weight. By transitivity, we have an automorphism from any one edge to any other edge; and since this automorphism preserves systoles, we have that the number of systoles containing an edge must be the same for any edge of the graph.

Let k be the number of systoles containing an arbitrary edge of G. Consider the vector sum $\chi_{s_1} + \chi_{s_2} + ...\chi_{s_k}$ of the characteristic functions of all k systoles. Since each edge passes through k systoles, each component of this vector sum must be k. Thus $\chi_{s_1} + \chi_{s_2} + ... + \chi_{s_k} = [k, k, ...k] = k\overrightarrow{1}$, and $\overrightarrow{1} = \frac{1}{k}\chi_{s_1} + \frac{1}{k}\chi_{s_2} + ... + \frac{1}{k}\chi_{s_k}$ is in the positive cone generated by the characteristic functions χ_{s_i}. By Theorem 2.2.3 , w must be extremal.

□

2.3 Extremal Weights for λ_1

The second invariant we shall work with is the first nonzero eigenvalue of the Laplacian. For a fixed graph G we would like to find the weighs w that maximize λ_1, the first nonzero eigenvalue of $\triangle(w)$, while satisfying the normalization condition .

2.3.1 Concavity of λ_1 and the Rayleigh-Ritz Theorem

The first important result is the concavity of $\lambda_1(w)$ for any fixed graph G. This is established by a variant of the Rayleigh-Ritz theorem.

Theorem 2.3.1. ***Rayleigh-Ritz Theorem***

For any symmetric, positive definite matrix m by m A, we have the following estimate on the smallest and largest eigenvalues of A:

$$\lambda_0 = \min_{\|x\|=1} \langle Ax, x\rangle$$

$$\lambda_m = \max_{\|x\|=1} \langle Ax, x\rangle$$

Proof. Since A is symmetric and positive definite, $A = Q^T BQ$ where Q is orthogonal and B is diagonal with the eigenvalues of A along the diagonal.

$\langle Ax, x\rangle = x^T Ax = x^T Q^T BQx = (Qx)^T B(Qx)$

Consider the change of variables $y = Qx$. Since Q is orthogonal,$\| y \|=\| Qx \|=\| x \|$.

This gives us the following formula:

$$\min_{\|x\|=1} \langle Ax, x\rangle = \min_{\|y\|=1} y^T By$$

Since B is a diagonal matrix with the eigenvalues λ_i along the diagonal, computing the last term yields:

$$\min_{\|y\|=1} \sum_{i=1}^{m} y_i^2 \lambda_i$$

But since $\| y \|^2= 1$, the sum of the terms y_i^2 must equal 1, so this minimum is achieved by having the only nonzero coefficient in the sum be the y_i^2 which is multiplied by the smallest eigenvector, λ_1. So this minimum is equal to $1\lambda_1 + 0\lambda_2 + 0\lambda_3 + ... + 0\lambda_m = \lambda_1$.

Similarly the maximum is achieved by having the only non-zero coefficient correspond to λ_m. $\square$

The theorem as stated does not directly give an estimate on λ_1 of $\triangle$ since $\triangle$ is only nonnegative definite, and λ_1 is actually the second eigenvalue after 0. To rectify this, we restrict our attention to the quotient space of the eigenspace of λ_0 in $\mathbb{R}^{V(G)} \cong \mathbb{R}^m$. We expect the first eigenvalue of $\triangle \restriction_{E_{\lambda_0}^{\perp}}$ to be the first nonzero eigenvalue of $\triangle$. From before, the eigenspace E_{λ_0} is spanned by the constant function $\overrightarrow{1}$. This leads us to the following proposition:

Proposition 2.3.2. ***Rayleigh-Ritz Formula (version 2)***

$$\lambda_1 = \min_{\substack{\| x \| = 1 \\ x \perp \overrightarrow{1}}} \langle \triangle x, x \rangle$$

Proof. We follow the same proof as above. Since Q is orthogonal, $\langle x, \overrightarrow{1} \rangle = \langle y, Q\overrightarrow{1} \rangle$. Since the rows of Q are given by eigenvectors of $\triangle$, and since such eigenvectors are orthogonal, $Q\overrightarrow{1} = [c, 0, 0, .., 0]$. Thus as before we have

$$\min_{\substack{\| x \| = 1 \\ x \perp \overrightarrow{1}}} \langle \triangle x, x \rangle = \min_{\substack{\| y \| = 1 \\ y \perp [c,0,0,...,0]}} \sum_{i=0}^{n-1} y_{i+1}^2 \lambda_i$$

Since we are forced to take y such that the first coefficient is 0, this sum is now minimized by taking $y = [0, 1, 0, ..0]$, hence the sum becomes λ_1 $\square$

The Rayleigh-Ritz Theorem can be used to show that λ_1 is a concave functional of weight.

Theorem 2.3.3. *$\lambda_1(w)$ is a concave function of w.*

Proof. The entries in $\triangle$ are all linear combinations of weights, thus each is a linear functional of weight. For any fixed vector x in $\mathbb{R}^{V(G)}$, $\langle \triangle x, x \rangle$, is a linear combination of the entries in $\triangle$, and so again a linear functional of weight. The Rayleigh-Ritz Theorem then, states that λ_1 is the minimum of a set of linear (hence concave) functionals of weight. By lemma 2.1.1, it is itself a concave functional of weight. $\square$

From the proof of theorem 2.3.2, it seems logical to attempt to characterize λ_k as the largest eigenvalue of $\triangle$ restricted to the subspace obtained by quotienting out all higher eigenspaces. In this way, the Rayleigh-Ritz Theorem can be generalized to give a representation of higher eigenvalues.

Theorem 2.3.4. ***Min-Max Theorem***

$$\lambda_k = \min_{\dim(V)=k} \{ \max_{\substack{\| x \|= 1 \\ x \in V}} \langle \triangle x, x \rangle \}$$

Proof. Let $\{v_1, v_2, ..., v_k, ..., v_m\}$ be an orthonormal set of eigenvectors. As before, $\langle x, v_j \rangle = \langle y, Qv_j \rangle$ since Q is orthogonal. Also, $Qv_j = e_j$ since the rows of Q are taken to be orthogonal unit eigenvectors. Thus we have:

$$\max_{\substack{\| x \|= 1 \\ x \perp span\{v_{k+1}, ..., v_m\}}} \langle \triangle x, x \rangle = \max_{\substack{\| y \|= 1 \\ y \perp span\{e_{k+1}, ..., e_m\}}} \sum_{i=1}^{m} y_i^2 \lambda_i$$

Since the $k+1^{st}$ to m^{th} coordinates of y must be zero, the maximum is achieved when $y = e_k$ thus the sum is equal to λ_k. Since the dimension of $\mathbb{R}^{V(G)}/span\{v_{k+1}, ..., v_m\}$ is clearly k, we have:

$$\lambda_k \geq \min_{\dim(V)=k} \{ \max_{\substack{\| x \|= 1 \\ x \in V}} \langle \triangle x, x \rangle \}$$

Now let V be any subspace of dimension k and let $\{w_{k+1}, w_{k+2}, ..., w_m\}$ be a basis for $V^{\perp}$. We define

$$x_0 = \sum_{j=k}^{m} c_j v_j$$

taking the coefficients c_j so that x_0 is orthogonal to $w_{k+1}, w_{k+2}, ...,$and w_m and $\| x_0 \|= 1$. This can always be done since the system of equations $\langle w_{k+1}, x_0 \rangle, = 0$ $..., \langle w_m, x_0 \rangle = 0$ can be thought of as a system with $m-k$ equations for the $m-k+1$ unknowns c_j, and the resulting solution can easily be normalized without effecting orthogonality.

Thus we have the following:

$$\max_{\substack{\| x \|= 1 \\ x \perp span\{w_{k+1},\ldots,w_m\}}} \langle \triangle x, x\rangle \geq \langle \triangle x_0, x_0\rangle$$

Expanding by the definition we have:

$$= \left\langle \triangle \sum_{j=k}^{m} c_j v_j, \sum_{i=k}^{m} c_i v_i \right\rangle = \sum_{j=k}^{m}\sum_{i=k}^{m} c_j c_i \langle \triangle v_j, v_i\rangle = \sum_{j=k}^{m}\sum_{i=k}^{m} \lambda_j c_j c_i \langle v_j, v_i\rangle$$

By the orthonormality of eigenvectors, we have that this is equal to the following:

$$= \sum_{j=k}^{m} \lambda_j c_j^2 \geq \lambda_k \sum_{j=k}^{m} c_j^2 = \lambda_k \parallel x_0 \parallel^2 = \lambda_k$$

Since the choice of V was arbitrary, we have

$$\lambda_k \leq \min_{\dim(V)=k} \{ \max_{\substack{\| x \|= 1 \\ x \in V}} \langle \triangle x, x\rangle\}$$

□

Unfortunately we can not use this result to show λ_k is a concave functional as we did for λ_1. This is due to the fact that - although the eigenspace for λ_1 is independent on the choice of weight function, the eigenspaces of higher eigenvalues does depend on the weight function. Thus we can not characterize arbitrary λ_k as the minimum over some fixed subspace (independent of w) of the linear functional $\langle \triangle x, x\rangle\}$ using the min-max theorem. It is possible to show that λ_k in general is concave. In general, λ_k need not be concave.

Lastly, it is clear from the proof that an alternate representation of λ_k exists. Namely, λ_k is the maximum over all subspaces of dimension $k-1$ of $min\langle \triangle x, x\rangle$.

2.3.2 Conditions for Extremal λ_1

We now deduce a necessary and sufficient condition for extremal λ_1.

Theorem 2.3.5. *The weight w is extremal for λ_1 if and only if $\exists\{\varphi_1, \varphi_2, ..., \varphi_p\}$ an orthogonal basis of eigenvectors of E_{λ_1} (the eigenspace of λ_1) such that there exists a set of positive scalers $\{c_1, c_2, ..., c_p\}$ with*

$$\sum_{k=1}^{p} c_k(\partial_j \varphi_k)^2 = 1 \tag{2.3.1}$$

for any edge e_j.

Recall that the partial derivative of φ in the direction of edge e_j to be defined as $\partial_j \varphi := [\varphi(e_j^+) - \varphi(e_j^-)]$, so the sum above can be rewritten as

$$\sum_{k=1}^{p} c_k[\varphi_k(e_j^+) - \varphi_k(e_j^-)]^2 = 1$$

Remark 2.3.1. The above condition can be thought of as follows. There exists some basis of eigenvectors $\{\varphi_k\}$ so that the function $\Phi : E(G) \to \mathbb{R}^p$ whose k^{th} component function is given by $e_j \mapsto \sqrt{c_k}\,[\varphi_k(e_j^+) - \varphi_k(e_j^-)]$, maps $E(G)$ to the unit sphere in $\mathbb{R}^p$.

Proof. By theorem 1.2.3, all eigenvalues are smooth functions of weight, thus sice λ_1 is concave, a weight is extremal for λ_1 if and only if it is a critical point. We prove that condition 2.3.1 above is necessary and sufficient for the weight w to be a critical point for λ_1. The theorem follows immediately.

Fix graph G and weight w and let $\triangle$ denote the Laplacian on G with weight w. Let $n =\mid E(G) \mid$, $m =\mid V(G) \mid$, and p be the multiplicity of λ_1.

As was done in section 2.2.2, we consider a linear perturbation of weight given by $\mathbf{w} : \varepsilon \mapsto w + \varepsilon b$, for some fixed $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$. As before, this perturbation must respect the normalization condition, thus $b \perp \overrightarrow{1}$.

Note that if we add ε to the weight of any edge $e_i = \overline{(v_j, v_k)}$, the resulting change in the Laplacian is given by adding ε to the j^{th} and k^{th} diagonal terms and subtracting ε from the jk^{th} and kj^{th} off-diagonal terms. In other words, we add the matrix $\varepsilon B_{(e_i)}$ where $B_{(e_i)}$ is the Laplacian on graph G with weight function χ_{e_i}, the characteristic function of e_i which assigns weight 1 to that edge and 0 to all others.

Thus the perturbation $\mathbf{w}(\varepsilon)$ results in the perturbation $\mathbf{L} : \varepsilon \mapsto \triangle + \varepsilon B$ on the space of symmetric matrices. Where

$$B = \sum_{i=1}^{n} b_i B_{(e_i)}$$

Now theorem 1.2.3tells us that $\mathbf{L}(\varepsilon)$ induces the analytic perturbations:

$\lambda_j(\varepsilon) = \lambda_j + \varepsilon\mu_{j1} + \varepsilon^2\mu_{j2} + ...$

$\phi_j(\varepsilon) = \varphi_j + \varepsilon\, f_{j1} + \varepsilon^2 f_{j2} + ...$

where for any sufficiently small ε, $\mathbf{L}(\varepsilon)\phi_j(\varepsilon) = \lambda_j(\varepsilon)\phi_j(\varepsilon)$ and $\phi_j(\varepsilon)$ is a unit vector. Since the $\lambda_j(\varepsilon)$ are analytic they must be equal to their Taylor expansions. Thus $\mu_{j1} = \lambda_j'(0)$, $\mu_{j2} = \frac{1}{2}\lambda_j''(0)$,... and so on.

Since $\mathbf{L}(\varepsilon)\phi(\varepsilon) = \lambda_j(\varepsilon)\phi(\varepsilon)$ we have that

$(\triangle + \varepsilon B)(\varphi_j + \varepsilon\, f_{j1} + \varepsilon^2 f_{j2} + ...) = (\lambda_j + \varepsilon\mu_{j1} + \varepsilon^2\mu_{j2} + ...)(\varphi_j + \varepsilon\, f_{j1} + \varepsilon^2 f_{j2} + ...)$.

Multiplying through by linearity and collecting like powers of ε we have the following system of equalities:

for ε^0, $\triangle\varphi_j = \lambda_j\varphi_j$ as expected

for ε^1, $B\varphi_j + \triangle f_{j1} = \mu_{j1}\varphi_j + \lambda_j f_{j1}$

$\vdots$

We take the product with φ_j.

$\langle B\varphi_j + \triangle f_{j1}, \varphi_j\rangle = \langle \mu_{j1}\varphi_j + \lambda_j f_{j1}, \varphi_j\rangle$

$\langle B\varphi_j, \varphi_j\rangle + \langle \triangle f_{j1}, \varphi_j\rangle = \mu_{j1}\langle \varphi_j, \varphi_j\rangle + \lambda_j\langle f_{j1}, \varphi_j\rangle$

Since $\triangle$ is self-adjoint, $\langle \triangle f_{j1}, \varphi_j\rangle = \langle f_{j1}, \triangle\varphi_j\rangle = \lambda_j\langle f_{j1}, \varphi_j\rangle$. Combined with the condition that $\| \varphi_j \|^2 = 1$ this gives us:

$\langle B\varphi_j, \varphi_j\rangle + \lambda_j\langle f_{j1}, \varphi_j\rangle = \mu_{j1} + \lambda_j\langle f_{j1}, \varphi_j\rangle$

or simply $\lambda_j'(0) =: \mu_{j1} = \langle B\varphi_j, \varphi_j\rangle$

Recalling the definition of B, we have $\left\langle \left(\sum_{i=1}^n b_i B_{(e_i)}\varphi_j\right), \varphi_j\right\rangle$, which expanded linearly gives us $\sum_{i=1}^n b_i\langle B_{(e_i)}\varphi_j, \varphi_j\rangle$. Since $B_{(e_i)}$ is the Laplacian of G with weight function given χ_{e_i}, by proposition 1.1.2 we can decompose $B_{(e_i)}$ into $D^T D$ where D is the gradient matrix for χ_{e_i}. Thus we have

$\langle B_{(e_i)}\varphi_j, \varphi_j\rangle = \langle D\varphi_j, D\varphi_j\rangle = \| D\varphi_j \|^2 = (\partial_i\varphi_j)^2$

We conclude $\lambda_j'(0) = \sum_{i=1}^n b_i\langle B_{(e_i)}\varphi_j, \varphi_j\rangle = \sum_{i=1}^n b_i(\partial_i\varphi_j)^2$

The weight w is critical for λ_j if and only if given any $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$ respecting the normalization condition $b \perp \overrightarrow{1}$, we have for the resulting perturbations $\mathbf{w}(\varepsilon)$ and $\lambda_{\mathbf{j}}(\varepsilon)$, that $\lambda_j'(0) = 0$.

$0 = \lambda_j'(0) = \sum_{i=1}^n b_i\langle B_{(e_i)}\varphi_j, \varphi_j\rangle = \sum_{i=1}^n b_i(\partial_i\varphi_j)^2$

So $b \perp \overrightarrow{1} \Rightarrow b \perp [(\partial_1\varphi_j)^2, ..., (\partial_n\varphi_j)^2]$

This implies that $[(\partial_1\varphi_j)^2, ..., (\partial_n\varphi_j)^2] \in span\,\overrightarrow{1}$, that is, $(\partial_i\varphi_j)^2 = (\partial_k\varphi_j)^2$ for any two edges e_i and e_k.

Now take any set of scalars $\{c_1, c_2, ..., c_p\}$, with

$$\sum_{k=1}^p c_k(\partial_1\varphi_k)^2 = 1$$

Without loss of generality we can take each c_j to be positive, since if some c_j is negative, we can correct this by multiplying the corresponding φ_j by -1, resulting in a new orthonormal basis of eigenvectors for which the coefficients are all positive.

From above, this set of coefficients will have the same sum

$$\sum_{k=1}^p c_k(\partial_1\varphi_k)^2$$

for any edge e_j. $\square$

2.4 Extremal Weights for $\log\det\triangle^*$

The final invariant functional to be considered is $\log\det\triangle^*$, the logarithm of the product of nonzero eigenvalues of $\triangle$. The problem is simplified by again noting that this is a smooth function of the weight. From this we can construct a somewhat involved but straightforward brute force proof of concavity. We then provide a simple necessary and sufficient condition for extremal $\log\det\triangle^*$ in terms of spanning trees based on the representation of the determinant of a minor given by Kirckhoff's Theorem (1.2.1).

2.4.1 Concavity of $\log\det\triangle^*$

Since it is an $m-1$ by $m-1$ minor determinant of $\triangle$, $\log\det\triangle^*$ is a smooth function of w. Thus proving concavity is a simple matter of restricting the function to a line in the subspace of weight function satisfying the normalization condition, and showing that the second directional derivative is negative. Since $\log\det\triangle^*$ is smooth, we may compute the second derivative by taking the second term of the Taylor expansion.

Theorem 2.4.1. $\log\det\triangle^*$ *is a concave function of* w.

Proof. First we show that $\log\det$ is a concave function on the space of symmetric positive definite matrices, by showing that it is concave when restricted to any line of $m-1$ by $m-1$ symmetric positive definite matrices. For any positive definite symmetric matrix Q, let $\mathbf{Q}(\varepsilon)$ be the line defined by $\mathbf{Q}(\varepsilon) = Q + \varepsilon B$. $\log\det$ is concave if and only if the second derivative of $\log\det\mathbf{Q}(\varepsilon)$ at $\varepsilon = 0$ is negative for any such line.

By lemma 2.4.2 below, we have that $\frac{d^2}{d\varepsilon^2}\log\det(\mathbf{Q}(\varepsilon))\mid_{\varepsilon=0} = -trace(BQ^{-1}BQ^{-1})$. Since Q^{-1} is positive definite and symmetric, we can diagonalize Q^{-1} as OPO^T where the matrix O is orthogonal, and P is strictly positive diagonal. This gives us $trace(BQ^{-1}BQ^{-1}) = trace(BOPO^TBOPO^T)$, or since trace is independent of the order of multiplication, $trace(O^TBOPO^TBOP)$. Let $C = O^TBO$. This is still clearly a symmetric matrix. We have $trace(BQ^{-1}BQ^{-1}) = trace(CPCP)$.

$$CPCP = (CP)^2$$
$$= \left(C \begin{bmatrix} p_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad C \begin{bmatrix} 0 \\ p_{22} \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad C \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p_{m-1\ m-1} \end{bmatrix} \right)^2$$
$$= \left(p_{11} \begin{bmatrix} C_{i1} \end{bmatrix} \quad p_{22} \begin{bmatrix} C_{i2} \end{bmatrix} \quad \dots \quad p_{m-1\ m-1} \begin{bmatrix} C_{i\,m-1} \end{bmatrix} \right)^2$$

The trace of the product of this matrix with itself can be computed directly. The j^{th} diagonal entry is given by
$$\sum_{i=1}^{m-1} p_{ii}p_{jj}(c_{ij}^2)$$

Since the entries in P are all positive, we have that the trace must be greater than zero. Thus $trace(BQ^{-1}BQ^{-1}) > 0$ so $\frac{d^2}{d\varepsilon^2} \log\det(\mathbf{Q}(\varepsilon)) < 0$ and $\log\det$ is a concave function on the entries of $\mathbf{Q}(\varepsilon)$. Since the entries of $\triangle^*$ depend linearly on w, we have that the composition of functions, $\log\det \triangle^*$, is a concave functional of w. □

Now we compute the second derivative used in the proof above.

Lemma 2.4.2. $\frac{d^2}{d\varepsilon^2} \log\det(\boldsymbol{Q}(\varepsilon)) = -trace(Q^{-1}BQ^{-1}B)$

Proof. We use the fact that $\log\det(\mathbf{Q}(\varepsilon))$ is an analytic function of ε to compute the second derivative using the Taylor expansion.

$$\log\det(\mathbf{Q}(\varepsilon))$$
$$= \log\det(Q + \varepsilon B)$$
$$= \log\det(Q(I + \varepsilon Q^{-1}B))$$
$$= \log\left(\det(Q) \times \det(I + \varepsilon Q^{-1}B)\right)$$
$$= \log\det(Q) + \log\det(I + \varepsilon Q^{-1}B)$$

Since $\log\det(Q)$ is independent of ε, we have

$$\frac{d^2}{d\varepsilon^2}\log\det(\mathbf{Q}(\varepsilon)) = \frac{d^2}{d\varepsilon^2}\log\det\begin{pmatrix} 1+\varepsilon\, a_{11} & \dots & \varepsilon\, a_{1m-1} \\ \vdots & \ddots & \vdots \\ \varepsilon\, a_{m-1 1} & \dots & 1+\varepsilon\, a_{m-1\ m-1} \end{pmatrix}$$

Where $A = [a_{ij}] = Q^{-1}B$. If we take the determinant of $I + \varepsilon A$ we get

$$1 + \varepsilon \sum_{i=1}^{m-1} a_{ii} + \varepsilon^2 \sum_{i<j}(a_{ii}a_{jj} - a_{ij}a_{ji}) + O(t^3)$$

So, by Taylor's theorem, at $\varepsilon = 0$ we have $\det(\mathbf{Q}(\varepsilon)) = 1$, $\det(\mathbf{Q}(\varepsilon))' = \sum_{i=1}^{m-1} a_{ii}$, and $\det(\mathbf{Q}(\varepsilon))'' = 2\sum_{i<j}(a_{ii}a_{jj} - a_{ij}a_{ji}) = \sum_{i\neq j}(a_{ii}a_{jj} - a_{ij}a_{ji})$. For any positive, twice differentiable function f, the second derivative of $\log(f)$ is given by $\frac{f\,f'' - (f')^2}{f^2}$. In this case we have $\frac{d^2}{d\varepsilon^2}\log\det(\mathbf{Q}(\varepsilon)) =$

$$1 \times \sum_{i\neq j}(a_{ii}a_{jj} - a_{ij}a_{ji}) - \left(\sum_{i=1}^{m-1} a_{ii}\right)\left(\sum_{j=1}^{m-1} a_{jj}\right)$$

$$= \sum_{i\neq j}(a_{ii}a_{jj}) - \sum_{i\neq j}(a_{ij}a_{ji}) - \sum_{i=1}^{m-1} a_{ii}^2 - \sum_{i\neq j}(a_{ii}a_{jj})$$

$$= -\sum_{i\neq j}(a_{ij}a_{ji}) - \sum_{i=1}^{m-1} a_{ii}^2$$

$$= -\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} a_{ij}$$

$$= -trace(A^2)$$

□

2.4.2 Conditions for Extremal $\log\det\triangle^*$ and Combinatorial Weight

As mentioned earlier, Kirckhoff's Theorem (1.2.1), gives a representation for any minor determinant in terms of $\tau(G,w)$, the weighted spanning tree number. Since log is an increasing function, any weight function w will be locally maximal for $\log\det\triangle^*$ if and only if it is locally maximal for $\det\triangle^*$. Thus, the problem becomes equivalent to finding critical points for

$$\tau(w) = \sum_{t\in T(G)} \prod_{e_j\in t} w(e_j)$$

The problem of finding extremal points for $\tau(w)$ is a simple matter of calculous. The set of spanning trees for a fixed graph is independent of the weight functions, so $\tau(w)$ is clearly a smooth function of w in $\mathbb{R}^{E(G)}$, being a polynomial of the weights of edges. The problem then becomes one of finding extremal points for a differentiable function on the subspace $V = \{w \in \mathbb{R}^{E(G)} \mid \langle w, \overrightarrow{1}\rangle = n\}$. This subspace is a level set of the linear function given by $\Lambda(w) = \langle w, \overrightarrow{1}\rangle$, so we can use the method of Lagrange multipliers.

We have that w is an extremal point on V if and only if $grad(\tau(w)) = c\, grad(\Lambda) = c\overrightarrow{1}$ Thus we have that the partial derivatives of $\tau(w)$ are all equal to some constant.

Computing $\frac{\partial\tau}{\partial w(e_j)}$, we note that holding all weights but $w(e_j)$ constant, we have $\tau(w(e_j))$ is linear. A simple computation reveals that:

$$\frac{\partial\tau}{\partial w(e_j)} = \sum_{t\in T(G)\ s.t.\ e_j\in t} \ \prod_{e_i\in t,\ e_i\neq e_j} w(e_i)$$

On combinatorial graphs, this is equal to the number of spanning trees containing e_j. Thus we have the simple condition that the combinatorial weight is extremal for $\log\det\triangle^*$ if and only if each edge contains the same number of spanning trees. Again, we see that extremal graphs often have strong symmetry properties.

In the case that the weight is not combinatorial, we still have the condition that the partial derivatives $\frac{\partial\tau}{\partial w(e_j)}$ are all equal to some constant. This constant divided by $\tau(w)$ is sometimes called the effective resistance of edge e_j. Thus w is extremal for $\log\det\triangle^*$ is and only if the effective resistance on all edges is the same.

Chapter 3

Basic Results for The Laplacian on Manifolds

We now turn our attention to the analogous problem of finding extremal metrics for geometric invariants on manifolds. We will concentrate entirely on the functional λ_1, although similar results can be developed for girth and $\log\det\triangle^*$. The development of several results for λ_1 on manifolds parallels much of the work done in chapter 2. Specifically, Rayleigh's theorem, min-max, and the condition of extremal λ_1 inducing an immersion into a sphere all reappear with similar proofs for the case of manifolds. As mentioned in the introduction to chapter 1, this is primarily due to the fact that a graph can be thought of as the discretization of a manifold.

We begin with a brief overview of the construction of differentiable manifolds, and quickly move on to the construction of the Laplacian before proving some results related to extremal metrics for λ_1

3.1 Basic Constructions for Riemanian Manifolds

Some familiarity with smooth manifolds, charts, smooth maps, and tangent spaces is assumed. For a full development, refer to [B]. The basic constructions of tangent bundles and metrics are reviewed below.

Definition 3.1.1. Tangent space at a point p of M is the space of linear functionals on germs of $\mathcal{C}^\infty$ functions based at p obeying the Leibnitz rule.

This can be thought of as the space of first order partial differential operators acting on functions at p. A basis for this vector space given a chart φ is given by the set $\{\partial_j\}$ where
$\partial_j f := \varphi^{-1}(\frac{\partial}{\partial x_j})f := \frac{\partial}{\partial x_j}(f \circ \varphi^{-1}) \mid_{x=\varphi(p)}$.

The tangent bundle TM is the disjoint union of tangent spaces given the obvious manifold structure induced by M.

We are primarily concerned with C^∞ manifolds endowed with a Riemanian metric.

Definition 3.1.2. A Riemanian metric associates to every point $p \in M$ a map $G : T_pM \times T_pM \to \mathbb{R}$ Which is a symmetric, bilinear, and non-degenerate.

Thus the metric allows us to take an inner product of two vectors in the tangent space based at the same point $p \in M$. If $u, v \in T_pM$, we write $G : u, v \mapsto \langle u, v, \rangle_G$ and $g_{i,j} = \langle \partial_i, \partial_j \rangle_G$. Moreover, if G depends smoothly on p, then it is said to be a smooth Riemanian metric.

At a point $p \in M$, the metric is given by the matrix G denoted:

$$G_{nxn} = \begin{pmatrix} g_{1,1} & g_{1,n} \\ & \\ g_{n,1} & g_{n,n} \end{pmatrix}$$

Also, we have that the inverse matrix of G is given by:

$$G^{-1} = \begin{pmatrix} g^{1,1} & g^{1,n} \\ & \\ g^{n,1} & g^{n,n} \end{pmatrix}$$

Definition 3.1.3. A vector field on a manifold M is a map $\alpha : M \to TM$ such that $\alpha(p) \in T_pM$.

In other words, a vector field associates to every point p on M a vector in the tangent space at p. This is also referred to as a section on the fiber bundle TM with base space M.

In local coordinates a vector field associates to a point $(x_1, x_2, ...x_n)$ in M the vector $\sum a_j(x)\partial_j$ where $a_j(x)$ is a function of x. The vector field is smooth if and only if the functions $a_j(x)$ are smooth for an arbitrary coordinate chart. We will mostly be interested in smooth vector fields.

3.2 The Laplacian on a Manifold

The Laplacian on manifolds is defined again as the divergence of the gradient. In turn, both of these concepts are defined to mimic the behavior of their simpler counterparts in Euclidian space.

3.2.1 Gradient

Given a smooth function $f : M \to \mathbb{R}$ and a vector $\xi = \sum \xi_j \partial_j$ in $T_pM \cong \mathbb{R}^n$, we can define a C^∞ function ξf that takes any point p to the directional derivative of f in the direction ξ evaluated at p. In other words, $\xi f(p)$ will be the number

$$\sum \xi_{x_j}(\partial_j f \mid_p) := \sum \xi_{x_j}(\frac{\partial}{\partial x_j}(f \circ \varphi^{-1}) \mid_{\varphi(p)})$$

It is clear that the functions $\xi_{x_j}(\frac{\partial}{\partial x_j} f \mid_x)$ are all smooth as long as f is. So for instance, if ξ is the standard basis vector e_j, the map $\xi f = e_j f$ takes a point p on m to the value of the j^{th} partial of f (in terms of local coordinates) evaluated at p.

Functions of the form ξf play the role of directional derivatives for manifolds, thus they are intimately tied with the concept of gradient.

Consider the the function ξf from the previous example in the manifold $\mathbb{R}^n$. This example is exactly the directional derivative function of f which takes any point p to the directional derivative of f evaluated at p. In $\mathbb{R}^n$, such a function is given by the map $\xi f : x \mapsto \langle \nabla f(x), \xi \rangle$

This motivates us to define the gradient of a smooth function on a manifold to satisfy a similar condition.

Definition 3.2.1. For any smooth function f on the manifold M, define gradient of f such that for any vector ξ, $\langle \nabla f(x), \xi \rangle := \xi f$

Some of the most important properties of the gradient in $\mathbb{R}^n$ remain true under this definition:

Theorem 3.2.1. *$grad(f + h) = grad(f) + grad(h)$*

$grad(fh) = h\, grad(f) + f\, grad(h)$

Proof. for any ξ we have that $\langle grad(f + h), \xi \rangle$

$= \xi(f + h)$

$= \xi f + \xi h$

$= \langle grad(f), \xi \rangle + \langle grad(h), \xi \rangle$

$= \langle grad(f) + grad(h), \xi \rangle$

also we have that $\langle grad(fh), \xi \rangle = \xi(fh)$. In local coordinates, this gives us $\sum \xi_{x_j} (\frac{\partial}{\partial x_j} fh \circ \varphi^{-1} \mid_{\varphi(x)})$, which is equal to $\sum \xi_{x_j} [(\frac{\partial}{\partial x_j} f \circ \varphi^{-1} \mid_{\varphi(x)})h + (\frac{\partial}{\partial x_j} h \circ \varphi^{-1} \mid_{\varphi(x)})f]$ by the Leibniz rule since we are just in $\mathbb{R}^n$. Expanding everything out linearly, we see that $\xi(fh) = h(\xi f) + f(\xi h)$

$$= h\langle grad(f), \xi\rangle + f\langle grad(h), \xi\rangle$$
$$= \langle h\, grad(f) + f\, grad(h), \xi\rangle$$

$\square$

Note: in the above proof we use the fact that two vector fields ξ and η are equal if and only if $\langle \xi, \varphi\rangle = \langle \eta, \varphi\rangle$ for any test vector field φ.

We will now derive an expression for the gradient in local coordinates. Recall that if $g_{i,j} = \langle \partial i, \partial j\rangle_G$ is the element in the i^{th} row and j^{th} column of the metric $G = [g_{i,j}]$, then $g^{i,j}$ is the element in the i^{th} row and j^{th} column of G^{-1} Thus since $[g_{i,j}][g^{i,j}] =$ I, we have that $g_{j,k}g^{k,l} = 1$ if $j = l$ and 0 otherwise. We may use this to rewrite:

$$\xi f = \sum_{j=1}^{n} \xi^j \partial_j f = \sum_{j=1}^{n} \sum_{k,l=1}^{n} \xi^j g_{j,k} g^{k,l} \partial_l f$$
$$= \left\langle \sum_{j=1}^{n} \xi^j \partial_j, \sum_{k,l=1}^{n} (g^{kl}\partial_l f)\partial_k \right\rangle_G$$
$$= \left\langle \xi, \sum_{k,l=1}^{n} (g^{kl}\partial_l f)\partial_k \right\rangle_G$$

From the definition of gradient as ξ, $\langle \nabla f(x), \xi\rangle := \xi f$, it follows that in local coordinates we have the following representation of grad f with respect to the metric g_{ij},

$$grad(f) = \sum_{k,l} (g^{kl}\partial_l f)\partial_k$$

This is akin to the usual gradient in $\mathbb{R}^n$, taken through the inverse matrix for the metric. In $\mathbb{R}^n$, $[g^{kl}] = $ I

$\therefore grad(f) = \sum_k (\partial_k f)\partial_k = \begin{pmatrix} \partial_1 f, & \partial_2 f, & \ldots, & \partial_n f \end{pmatrix}$ as we would expect.

3.2.2 Connections and Divergence

In order to define the concept of divergence for a vector field on a manifold, we will need to extend the definition of directional derivative of vector field. In particular, we will define an object called a covariant derivative (or connection) which will play the role of the Jacobian of a vector field on $\mathbb{R}^n$.

Definition 3.2.2. Given a Riemanian Manifold M, a connection, (or covariant derivative) given $p \in M$, $\xi \in T_pM$, and X a smooth vector field, is a map $(\xi, X) \mapsto \nabla_\xi X \in T_pM$, such that:

•$\nabla_\xi(X + Y) = \nabla_\xi X + \nabla_\xi Y$

•$\nabla_{\xi+\eta} X = \nabla_\xi X + \nabla_\eta X$

•$\nabla_\xi(fX) = (\xi f)X(p) + f(p)\nabla_\xi X)$

In other words, given a vector field X, the connection takes a vector based at p (ξ) to another vector ($\nabla_\xi X$) based at p that depends linearly on both ξ and X and follows the Leibnitz rule. Such a thing can be thought of as the directional derivative of X in the direction of ξ. Sure enough, in $\mathbb{R}^n$ The directional derivative is indeed a connection.

In particular, given a connection and two C^∞ vector fields X and Y, We can define another C^∞ vector field on M by $\nabla_X Y$.

Definition 3.2.3. Given a Riemanian Manifold M, the Levi-Civita connection is the unique connection on M that satisfies the properties:

•$\nabla_X(Y) - \nabla_Y(X) = [X, Y] \quad (where\, [X, Y]f := X(Yf) - Y(Xf)\, for\, any\, function\, f)$

•$\forall \xi \in T_pM,\ \xi\langle X, Y\rangle_G = \langle \nabla_\xi(X), Y\rangle_G + \langle X, \nabla_\xi(Y)\rangle_G$

The Levi-Civita connection gives us an unambiguous notion of differentiation for a vector field on M. As mentioned before, $\nabla_\xi X$ can be thought of as the directional derivative of the vector field X in the direction of ξ. Thus in $\mathbb{R}^n$, we have

$$\nabla_\xi X = (\xi X_1, \xi X_2, \ldots, \xi X_n) = \begin{pmatrix} \frac{\partial X1}{\partial x_1} & \cdots & \frac{\partial X1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial Xn}{\partial x_1} & \cdots & \frac{\partial Xn}{\partial x_n} \end{pmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$$

The Levi-Civita connection operator is analogous to the Jacobian of a vector field in the $\mathbb{R}^n$ case. Thus, since divergence in $\mathbb{R}^n$ is taken to be the trace of the Jacobian matrix, the Levi-Civita connection motivates the following generalization of divergence.

Definition 3.2.4. Given A smooth vector field X on a manifold M, $[div(X)](p) := trace(\xi \mapsto \nabla_\xi X)$

3.2.3 The Laplacian

We now have all the technical machinery necessary to define the Laplace operator for manifolds. We will define this operator formally, give a representation in local coordinates, then give an intuitive description of how the Laplacian behaves. As before we will formally define the Laplacian as the Divergence of the Gradient.

Definition 3.2.5. $\triangle f := div(grad(f))$

If we are to work with the Laplacian in local coordinates, we must develop a representation for the divergence of a vector field. In order to simplify notation, we will use Christofel symbols to represent the simplest case and build the general case using linearity.

Definition 3.2.6. The Christofel symbol $\Gamma_{i,j}^k$ is defined so that:

$\nabla_{\partial_i} \partial_j = \sum \Gamma_{i,j}^k$

In other words, $\Gamma^k_{i,j}$ is the k^{th} component of $\nabla_{\partial_i}\partial_j$.

Since the covariant derivative is linear with respect to both the input vector and vector field, we can use these to build $\nabla_\xi X$ for any vector field X and for any vector ξ. Thus if:

$$\xi = \sum_{i=1}^{n} \xi^i \partial_i, \ and \ X = \sum_{j=1}^{n} \eta^j(x)\partial_j$$

Then by linearity:

$$\begin{aligned}
\nabla_\xi X &= \nabla_{\sum \xi^i \partial_i} X \\
&= \sum_{i=1}^{n} \xi^i \nabla_{\partial_i}(X) \\
&= \sum_{i=1}^{n} \xi^i \nabla_{\partial_i}(\sum_{j=1}^{n} \eta^j(x)\partial_j) \\
&= \sum_{i=1}^{n} \xi^i (\sum_{j=1}^{n} \nabla_{\partial_i} \eta^j(x)\partial_j)
\end{aligned}$$

Now by the Leibniz rule we have

$$= \sum_{i=1}^{n} \xi^i \sum_{j=1}^{n} [\partial_i \eta^j(x)\partial_j + \eta^j(x)\nabla_{\partial_i}\partial_j]$$

$$= \sum_{i,k=1}^{n} \xi^i [\partial_i \eta^j(x) + \sum_{j=1}^{n} \eta^j \Gamma^k_{i,j}]$$

This gives us a representation of the covariant derivative in local coordinates in terms of Christofel symbols. We can use this to derive a formula for the divergence in local coordinates, since the divergence is the trace of $\xi \mapsto \nabla_{xi} X$.

Take $\xi = \partial_l$ (ie, $\xi^l = 1$ and $\xi^j = 0$ for $j \neq l$). $\nabla \partial_l X = \sum_k (\partial_l \eta^l \sum_j \eta^j \Gamma^k_{l,j})\partial_k$. Taking the sum along the diagonal we obtain:

$$\sum_{l=1}^{n}(\partial_l \eta^l + \sum_{j=1}^{n} \eta^j \Gamma^l_{l,j})$$

The first summation in this expression is easily recognized as the usual divergence in $\mathbb{R}^n$. This leads us to believe (as we shall verify next) that $\Gamma^k_{i,j} \equiv 0$ in $\mathbb{R}^n$.

In order to fully express things in local coordinates, we must find an expression for Christofel symbols in local coordinates.

Proposition 3.2.2.

$$\Gamma^k_{i,j} = \frac{1}{2}\sum_l g^{k,l}[\partial_i g_{j,l} + \partial_j g_{i,l} - \partial_l g_{i,j}]$$

Proof.

$$\frac{1}{2}[\partial_i g_{j,l} + \partial_j g_{i,l} - \partial_l g_{i,j}]$$

$$= \frac{1}{2}\sum_m(\Gamma^m_{i,l} g_{m,j} + \Gamma^m_{i,j} g_{m,l}) + \frac{1}{2}\sum_m(\Gamma^m_{j,i} g_{m,l} + \Gamma^m_{j,l} g_{m,i}) - \frac{1}{2}\sum_m(\Gamma^m_{l,j} g_{m,i} + \Gamma^m_{l,i} g_{m,j})$$

$$= \frac{1}{2}\sum_m(\Gamma^m_{i,j} g_{m,l} + \Gamma^m_{i,j} g_{m,l})$$

since Christofel symbols are symmetric in the lower indices.

$$\therefore \frac{1}{2}\sum_l g^{k,l}[\partial_i g_{j,l} + \partial_j g_{i,l} - \partial_l g_{i,j}] = \sum_l g^{k,l}\sum_m(\Gamma^m_{i,j} g_{m,l})$$

$$= \sum_{l,m}\Gamma^m_{i,j} g^{k,l} g_{m,l}$$

Since $[g^{k,l}]$ and $[g_{m,l}]$ are inverse matrices, their product will be the identity, and $g^{k,l}g_{m,l}$ will be 1 if $k = m$ and 0 otherwise, thus the sum becomes $\sum_{k=m}\Gamma^m_{i,j} \times 1 + \sum_{k\neq m}\Gamma^m_{i,j} \times 0$ Which gives us precisely $\Gamma^k_{i,j}$

$\square$

This formula allows us to conclude that in $\mathbb{R}^n$, $\Gamma^k_{i,j} \equiv 0$ as expected, since $g_{i,l}$ will be a constant function, so it's derivatives will all be identically zero.

In general, we have that in local coordinates

$$div(X) = \sum_{l=1}^{n}(\partial_l \eta^l + \sum_{j=1}^{n} \eta^j \Gamma^l_{l,j})$$

$$= \sum_{l=1}^{n}(\partial_l \eta^l + \frac{1}{2}\sum_{j,k=1}^{n} \eta^j g^{l,k}[\partial_l g_{j,k} + \partial_j g_{l,k} - \partial_k g_{l,j}])$$

After some computation, this simplifies to:

$$= (\frac{1}{\sqrt{\det[G]}})\sum_{l} \partial_l(\eta^l \sqrt{\det[G]})$$

Now we compute a representation of the Laplacian in local coordinates. Using the expressions we have derived for divergence and gradient in local coordinates, we have that:

$$\triangle f = \frac{1}{\sqrt{\det[G]}} \sum_{k,j=1}^{n} \partial_k g^{kj} \sqrt{\det[G]} \partial_j f \tag{3.2.1}$$

3.3 Extremal Metrics on Manifolds

3.3.1 Normalization Condition & Conformal Classes

As before, the problem of finding extremal metrics for λ_1 on a fixed manifold M is trivial if further restrictions are not imposed. We must formulate a normalization condition.

From the representation of the Laplacian given by equation 3.2.1 we see that for an arbitrary manifold M, scaling the metric by a constant c will have the effect of scaling the Laplacian by $\frac{1}{c^2}$. To avoid this we must again adopt a normalization condition.

Definition 3.3.1. The volume form of a manifold M with metric $[G]$ is taken to be:
$dV := \sqrt{det[G]}dx_1dx_2\dots dx_n$

From now on, integration will typically be taken to be against the volume form unless otherwise indicated.

Definition 3.3.2. The volume of a manifold M with metric $[G]$ is:
$Vol(M) = \int_M dV$

Requiring that the volume remain constant will preclude the possibility of simply scaling the metric by a constant. Thus we take the following normalization condition:

we wish to maximize λ_1 on a fixed manifold M such that $Vol(M) = c$ for some fixed constant c.

Definition 3.3.3. On some fixed manifold M, the two metrics g_1 and g_2 are called conformally equivalent if for $g_1 = f\, g_2$ for some positive smooth function f on M called the conformal factor. The additional restriction that M have the same volume under both metrics is also imposed, thus $\int f dV \int = 0$.

It is immediate from the definition that this defines an equivalence relation on the space of smooth metrics on M. An equivalence class of conformal metrics on M defines a conformal class on M.

It is often useful to restrict the problem of finding extremal metrics for λ_1 to that of finding metrics that are extremal for λ_1 within their conformal class. Such metrics will be referred to as c-extremal.

3.3.2 Spaces of Functions and Green's Formulas

Several important theorems in chapter 2 relied on the fact that we could decompose the graph Laplacian into $D^T D$. Thus the quadratic form $\langle \triangle v, v\rangle$ became $\langle Dv, Dv\rangle$. An analogous decomposition for Manifolds comes in the form of Green's formulas; corollaries to the divergence theorem.

Theorem 3.3.1. ***The Divergence theorem (for manifolds with no boundary)***

If M is a manifold and if X is a C^1 vector field on M with compact support in M, then:

$$\int_M (div(X))dV = 0$$

Corollary 3.3.2. ***Green's formulas (for manifolds with no boundary)***

For the manifold M, suppose $h \in C^1(M), f \in C^2(M)$, and $h\, grad(f)$ has compact support.

$$\int_M h\triangle(f)dV = -\int_M \langle grad(f), grad(h)\rangle_g dV$$

and when $f, h \in C_c^2(M)$

$$\int_M h\triangle(f) - f\triangle(h)dV = 0$$

Note that this last statement can be rewritten $\langle h, \triangle(f)\rangle_{\mathcal{L}^2} - \langle f, \triangle(h)\rangle_{\mathcal{L}^2} = 0$. This is a statement of the fact that $\triangle$ is a symmetric operator on the subspace $C_c^2(M)$ of the Hilbert space $\mathcal{L}^2(M)$, or for a compact manifold, just $C^2(M)$.

Proof. Note that $div(\varphi X) = \varphi\, div(X) + \langle grad(\varphi), X\rangle_g$

Letting $\varphi = h$ and $X = grad(f)$ we have:

$div(h\, grad(f)) = h\, div(grad(f)) + \langle grad(h), grad(f)\rangle_g$

We obtain the first formula by integrating both sides over M. Since $h\, grad(f)$ has compact support, the integral of the left side becomes zero by the divergence theorem. We are left with $0 = \int_M h\triangle(f) + \langle gradf, gradh\rangle_g dV$. the linearity of integration gives us the first Green's formula.

The second formula is obtained from the first by subtracting the two integrals and noting that $\langle grad(f), grad(h)\rangle_g - \langle grad(h), grad(f)\rangle_g = 0$ by the symmetry of g_{ij}. □

Similar results hold for compact manifolds with boundary. The form dA is the measure induced on ∂M by $g_{i,j}$.

Theorem 3.3.3. ***The Divergence theorem (for manifolds with boundary)***

If M is an orientable manifold with boundary and if $X \in C_0^1(\overline{M})$ is vector field on M-closure, then:

$$\int_M (div(X))dV = \int_{\partial M} \langle X, \nu\rangle dA$$

Where ν is the outward pointing normal.

Corollary 3.3.4. ***Green's formulas (for manifolds with boundary)***

When $h \in C^1(\overline{M}), f \in C^2(\overline{M})$, and $hgrad(f)$ has compact support on $\overline{M}$.

$$\int_M h\triangle(f) + \int_M \langle gradf, gradh\rangle_g dV = \int_{\partial M} h(\nu f)dA$$

and when $f, h \in C_0^2(\overline{M})$

$$\int_M h\triangle(f) - f\triangle(h)dV = \int_{\partial M} h(\nu f) - f(\nu h)dA$$

Proof. The proof is the same as before with the observation that $\langle grad(f), \nu\rangle := \nu f$ by the definition of gradient. □

From the above formulae, it is clear that if we wish to use Green's Formulas on Manifolds with boundary, we must do something about the boundary terms that arise. To that end, when dealing with a manifold with boundary, we usually restrict the space of functions that $\triangle$ is taken to act on:

Closed Eigenfunction

For a compact manifold M, We take the Laplacian to act on the space of functions $C^2(M)$ with the usual $\mathcal{L}^2$ inner product.

If on the other hand M is a compact manifold with boundary where $M \cup \partial M$ is compact, then we have several choices as to which space of functions to work with.

Dirichlet

We take the space of functions on which the Laplacian acts to be $C^2_D(M) = \{f \in C^2(M) \cap C^0(\overline{M}) \mid f \equiv 0 \ on \ \partial M\}$.

Neuman

We take the space of functions on which the Laplacian acts to be $C^2_N(M) = \{f \in C^2(M) \cap C^0(\overline{M}) \mid \nu f \equiv 0 \ on \ \partial M\}$.

It is clear from the definition in both of these cases that the boundary terms in Green's formulas for manifolds with boundary are equal to zero. Thus the Laplacian is a symmetric operator in all three cases above

It is also possible to study the Robin boundary condition, where f and νf are taken to be proportionate, so again the term $h(\nu f) - f(\nu h)$ is identically zero. Lastly one can study a mixed problem where different boundary conditions hold for different regions of the boundary.

One unfortunate problem with the above spaces is that they are not generally complete under the $\mathcal{L}^2$ inner product.

Definition 3.3.4. For compact manifold (possibly with boundary) M, let $\mathcal{H}(M)$ be the completion of the space of $\mathcal{C}^\infty$ functions with compact support in M with respect to the inner product

$D[f, h] := \int_M \langle grad(f), grad(h) \rangle_g dV.$

As the suggestive notation would indicate, this is indeed a Hilbert space, (see [C]). The Laplacian is not defined on the whole space.

3.3.3 Spectral Decomposition of the Laplacian

From Green's formulas $\triangle$ is a symmetric operator on some subspace of $\mathcal{H}(M)$,for the closed, Dirichlet, and Neuman cases.

By the spectral theorem for the Laplace operator we have the following theorem:

Theorem 3.3.5. *The spectrum of The Laplacian on a compact manifold consists entirely of eigenvalues with finite multiplicity.*

Moreover, eigenfunctions corresponding to distinct eigenvalues of $\triangle$ are orthogonal. All eigenvalues are non-negative real numbers, and if φ is an eigenfunction for 0, then φ is constant.

There exists a set of eigenfunctions for the Laplacian that form an orthonormal basis of $\mathcal{L}^2(M)$.

Proof. The proof that the spectrum consists of isolated simple eigenvalues is omitted.

To see that eigenfunctions are orthogonal, suppose $\lambda_i \neq \lambda_j$. Let u_i be an eigenfunction for λ_i and u_j be an eigenfunction for λ_j. Then by Green's formula

$0 = \int_M u_i \triangle(u_j) dV - \int_M u_j \triangle(u_i) dV$
$= \lambda_j \int_M u_i u_j dV - \lambda_i \int_M u_j u_i dV$
$= (\lambda_j - \lambda_i) \int_M u_j u_i dV$
$\therefore 0 = (\lambda_j - \lambda_i) < u_j, u_i >_{\mathcal{L}^2}$

Since $\lambda_i \neq \lambda_j$,we have $< u_j, u_i >_{\mathcal{L}^2} = 0$ and u_j, and u_i are orthogonal.

To see that eigenvalues are real nonnegative numbers and that the eigenspace of 0 is spanned by the constant function note that:

$\lambda \int_M u^2 dV$

$= - \int_M u(\triangle u) dV$

$= \int_M < grad(u), grad(u) >_g dV$

$\therefore \lambda ||u||_{\mathcal{L}^2} = \int_M ||grad(u)||_g dV$

So $\lambda \geqslant 0$ and $\lambda = 0 \Leftrightarrow ||grad(u)||_g = 0 \Leftrightarrow$ u is constant on M.

It follows that an orthonormal basis of $\mathcal{C}^2(M)$ can be constructed from eigenfunctions. The last claim follows from the fact that this space is dense in $\mathcal{L}^2(M)$

□

Thus using the spectral theorem, we can use the above results to decompose $\mathcal{L}_2(M)$ into an orthonormal basis of eigenfunctions.

Definition 3.3.5. Given a function f in $\mathcal{L}_2(M)$, define the j^{th} Fourier coefficient of f to be $\alpha_j := \langle f, \varphi_j \rangle_{\mathcal{L}^2}$, where, as in theorem 3.3.5, the set $\{\varphi_j\}$ forms an orthonormal basis of eigenfunctions of $\triangle$, and specifically, φ_j corresponds to λ_j.

3.4 Estimates on λ_1

Many of the theorems that gave us representations for the graph Laplacian have analogs for the Laplacian on manifolds. Specifically, the Rayleigh-Ritz theorem and the min-max theorem can both be formulated for manifolds. The proofs of these theorems are similar to those in the case of graphs and rely mostly on the spectral decomposition of $\triangle$ on Hilbert spaces.

3.4.1 Rayleigh's Theorem

Rayleigh's theorem is essentially the same for Manifolds as it is for graphs. Of course, in the case of manifolds, the theorem provides only a representation of λ_1 since there is no largest eigenvalue for the Laplacian on a manifold.

Theorem 3.4.1. ***Rayleigh's Theorem***

$$\lambda_1 = \min_{f \in \mathcal{H}(M)} \frac{D[f,f]}{\| f \|^2}$$

or alternatively

$$\lambda_1 = \min_{\substack{\| f \| = 1 \\ f \in \mathcal{H}(M)}} D[f,f]$$

Note the similarity to the Rayleigh-Ritz Theorem for graphs. The similarity extends to the proof. The proof of theorem 2.3.2 relied on the decomposition of the Laplacian into $D^T D$ to change the quadratic form for $\triangle$ into a basis of eigenvectors in which the result became immediate. In this case we use Green's Formulas to change to a basis of eigenfunctions to accomplish the same result. The only non-trivial difference comes from the necessity of approximating the infinite sum for the crucial step.

Proof. Given $f \in \mathcal{H}(M)$ and let α_k be the first non-zero Fourier coefficient of f as defined above. Fix any $r \in \mathbb{N}$ with $k \leq r$. Then we have

$$0 \leq D[f - \sum_{j=k}^{r} \alpha_j \varphi_j, f - \sum_{j=k}^{r} \alpha_j \varphi_j]$$

by linearity,

$$= D[f,f] - 2\sum_{j=k}^{r} \alpha_j D[f, \varphi_j] + \sum_{j=k}^{r} \alpha_j^2 D[\varphi_j, \varphi_j]$$

By Green's theorem, we have $\int_M (\triangle\varphi_j) f dV = -\int \langle grad(\varphi_j), grad(f)\rangle_G dV \quad \forall f \in \mathcal{H}(M)$ i.e. $\langle \triangle\varphi_j, f\rangle_{\mathcal{L}^2} = -D[\varphi_j, f]$

So the above expression gives us

$$= D[f,f] - 2\sum_{j=k}^{r} \alpha_j \langle \triangle\varphi_j, f\rangle_{\mathcal{L}^2} + \sum_{j=k}^{r} \alpha_j^2 \langle \triangle\varphi_j, \varphi_j\rangle_{\mathcal{L}^2}$$

and since φ_j is an eigenfunction this becomes

$$= D[f,f] - 2\sum_{j=k}^{r} \alpha_j \lambda_j \langle \varphi_j, f\rangle_{\mathcal{L}^2} + \sum_{j=k}^{r} \alpha_j^2 \lambda_j \langle \varphi_j, \varphi_j\rangle_{\mathcal{L}^2}$$

By the orthonormality of the set of eigenfunctions, $\langle \varphi_j, \varphi_j\rangle_{\mathcal{L}^2} = 1$. And by definition,$\langle \varphi_j, f\rangle_{\mathcal{L}^2} = \alpha_j$. So we are left with

$$= D[f,f] - 2\sum_{j=k}^{r} \alpha_j^2 \lambda_j + \sum_{j=k}^{r} \alpha_j^2 \lambda_j$$

$$\therefore 0 \leq D[f,f] - \sum_{j=k}^{r} \alpha_j^2 \lambda_j$$

And $\forall r \geq k, \lambda_k(\sum_{j=k}^{r} \alpha_j^2) \leq \sum_{j=k}^{r} \alpha_j^2 \lambda_j \leq D[f,f]$

If we let $r \to \infty$, then we have

$\lambda_k(\sum_{j=k}^{\infty} \alpha_j^2) \leq D[f,f]$

and from spectral decomposition, we have $\sum_{j=k}^{\infty} \alpha_j^2 = \| f \|_{\mathcal{L}^2}^2$

So $\lambda_k \| f \|_{\mathcal{L}^2}^2 \leq D[f,f]$. The coefficient k of the first nonzero Fourier coefficient α_k will vary from function to function, but since $\lambda_0 = 0$, it will always be at least 1. Thus $\forall f \in \mathcal{H}(M), \quad \lambda_1 \leq \frac{D[f,f]}{\|f\|_{\mathcal{L}^2}^2}$.

Finally we note that we have equality when $f = \varphi_1$, thus

$$\lambda_1 = \min_{f \in \mathcal{H}(M)} \frac{D[f,f]}{\| f \|_{\mathcal{L}^2}^2}$$

$\square$

3.4.2 The Min-Max Theorem

Remark 3.4.1. In the Min-Max theorem for manifolds, we have λ_k characterized as the maximum of a minimum rather than the other way around, again because we do not have finitely many eigenvalues of the Laplacian as we did for graphs.

Theorem 3.4.2.

$$\lambda_k = \max_{\substack{V \subset \mathcal{H}(M) \\ dim(V) = k-1}} \left\{ \min_{f \in V^{\perp}} \frac{D[f,f]}{\| f \|_{\mathcal{L}^2}^2} \right\}$$

Proof. Again we let $\{\varphi_1, \varphi_2, ...\}$ be an orthonormal set of eigenfunctions corresponding to the eigenvalues $\{\lambda_1, \lambda_2, ...\}$ listed with multiplicity of the Laplacian.

Let $W = span\{\varphi_1, \varphi_2, ...\varphi_{k-1}\}$ be the eigenspace of the first $k-1$ eigenvalues. Then if $f \in W^{\perp}$ then clearly the first $k-1$ Fourier coefficients of f must be zero. Thus from the proof of theorem 3.4.1 above, we have $\lambda_k \| f \|_{\mathcal{L}^2}^2 \leq D[f,f]$ for all f in $\mathcal{H}(M) \cap W^{\perp}$. This gives us

$$\lambda_k \leq \min_{f \in W^{\perp}} \frac{D[f,f]}{\| f \|_{\mathcal{L}^2}^2} \quad thus \quad \lambda_k \leq \max_{\substack{V \subset \mathcal{H}(M) \\ dim(V) = k-1}} \left\{ \min_{f \in V^{\perp}} \frac{D[f,f]}{\| f \|_{\mathcal{L}^2}^2} \right\}$$

Now, for fixed $V \subset \mathcal{H}(M)$ of dimension k-1, let $\{w_1, w_2, ..., w_{k-1}\}$ be a basis for V, and define the function

$$f_0 = \sum_{j=1}^{k} c_j \varphi_j$$

We choose the scalers c_j so that f_0 is orthogonal to $w_1, w_2, ...,$and w_{k-1}. This is always possible since the $(k-1)$ orthogonality conditions define a system of $(k-1)$ linear equations in k unknowns $c_1, ..., c_k$

Once this is done, we normalize f_0 noting that it is still orthogonal to vectors spanning V.

$$\min_{f \in V^{\perp}} D[f,f] \leq D[f_0, f_0]$$

Expanding by the definition we have:

$$= D\left[\sum_{j=1}^{k} c_j\varphi_j, \sum_{i=1}^{k} c_i\varphi_i\right] = \sum_{j=1}^{k}\sum_{i=1}^{k} c_jc_i \left\langle grad(\varphi_j), grad(\varphi_i)\right\rangle_{\mathcal{L}^2}$$

applying Green's formula, we have:

$$= \sum_{j=1}^{k}\sum_{i=k}^{m} \lambda_j c_j c_i \langle \triangle\varphi_j, \varphi_i\rangle_{\mathcal{L}^2} = \sum_{j=1}^{k}\sum_{i=k}^{m} \lambda_j c_j c_i \langle \varphi_j, \varphi_i\rangle_{\mathcal{L}^2}$$

By the orthonormality of eigenvectors, we have that this is equal to the following:

$$= \sum_{j=1}^{k} \lambda_j c_j^2 \leq \lambda_k \sum_{j=1}^{k} c_j^2 = \lambda_k \parallel f_0 \parallel^2 = \lambda_k$$

Since the choice of V was arbitrary, we have

$$\lambda_k \geq \max_{\substack{V \subset \mathcal{H}(M) \\ dim(V) = k-1}} \left\{ \min_{f \in V^{\perp}} \frac{D[f,f]}{\parallel f \parallel_{\mathcal{L}^2}^2} \right\}$$

$\square$

3.4.3 Weyl's Law

Definition 3.4.1. Let $N(\lambda)$ denote the number of eigenvalues λ_j between 0 and λ inclusive.

Theorem 3.4.3. ***Weyl's Law***

$$N(\lambda) \sim \frac{vol(D^n)\, vol(M)\lambda^{n/2}}{(2\pi)^n} \quad Where\ vol(D^n)\ is\ the\ volume\ of\ the\ unit\ disk\ in\ \mathbb{R}^n$$

By this we mean that $N(\lambda)$ asymptotically approaches the value on the right side of the equation as λ grows without bound.

For a proof, refer to [C]

As a direct corollary, We also have, for the case n=2, the following estimate on λ_k

$$\lambda_k \sim \frac{(2\pi)^2 N(\lambda_k)}{vol(D^2)\, vol(M)} = \frac{(2\pi)^2 k}{vol(D^2)\, vol(M)}$$

This tells us that λ_k grows approximately linearly on two dimensional manifolds.

Note that this approximation improves with larger values of λ_k, and as such is not necessarily the best tool for estimating λ_1.

3.4.4 Cheeger's Inequality

Cheeger's Inequality is another formula relating eigenvalues of the Laplacian to purely geometric quantities. In this case a bound on λ_1 related to the isoperimetric dimension of the Manifold is given in the form of Cheeger's constant.

Cheeger's constant is defined as

Definition 3.4.2.

$$\mathfrak{h}(M) := \inf_D \frac{A(\partial D)}{V(D)}$$

where the infimum is taken over all domains D.

Theorem 3.4.4. $\lambda_1(\Omega) \geq \frac{\mathfrak{h}^2}{4}$ *for any domain* Ω *in* M.

Proof. The proof begins with The Rayleigh-Ritz theorem. Specifically the case of equality for eigenfunctions φ.

$\lambda_1(\Omega) = \frac{D[\varphi,\varphi]}{\|\varphi\|^2_{\mathcal{L}^2(\Omega)}}$

We would like to estimate this term from below by $\frac{1}{4}\left(\frac{A(\partial D)}{V(D)}\right)^2$. This can be done using the co-area formulas (lemma 3.4.5 below) which for a smooth function f, relate the integral of $\| grad(f) \|$ to the area of a level curve of f and the volume of a region enclosed by it.

$\frac{D[\varphi,\varphi]}{\|\varphi\|^2_{\mathcal{L}^2(\Omega)}} = \int_\Omega \left(\frac{\|grad(\varphi)\|}{|\varphi|}\right)^2$

We can not apply the co-area formulas while the integrand is being squared, thus we manipulate the equation as follows: By the chain rule, $grad(\varphi^2) = 2\varphi grad(\varphi)$, so the above is equal to:

$$\int_\Omega \left(\frac{| 2\varphi | \| grad(\varphi) \|}{| 2\varphi | | \varphi |}\right)^2 dV = \frac{1}{4}\int_\Omega \left(\frac{\| grad(\varphi^2) \|}{\varphi^2}\right)^2 dV$$

By Cauchy-Schwartz we have

$$\geq \frac{1}{4}\left(\frac{\int_\Omega \parallel grad(\varphi^2) \parallel dV}{\int_\Omega \varphi^2 dV}\right)^2$$

Now applying co-area formula 3 with the smooth function $f = \varphi^2$

$$\int_\Omega \parallel grad(\varphi^2) \parallel dV = \int_0^\infty A(t)dt$$

Also

$$\int_\Omega \varphi^2 dV = \int_\Omega \left(\varphi^2 \parallel grad(\varphi^2) \parallel^{-1}\right) \parallel grad(\varphi^2) \parallel dV$$

By co-area formula 2 we have

$$= \int_0^\infty \int_{\Gamma(t)} \left(\varphi^2 \parallel grad(\varphi^2) \parallel^{-1}\right) dA_t dt$$

On a level set $\Gamma(t)$, by definition φ^2 is identically the constant t. So we have

$$= \int_0^\infty t \int_{\Gamma(t)} \parallel grad(\varphi^2) \parallel^{-1} dA_t dt$$

and from co-area formula 1

$$= -\int_0^\infty tV'(t)dt$$

integrating by parts gives us

$$= \int_0^\infty V(t)dt$$

putting both halves together we have

$$\lambda_1(\Omega) \geq \frac{1}{4}\left(\frac{\int_\Omega \parallel grad(\varphi^2) \parallel dV}{\int_\Omega \varphi^2 dV}\right)^2 \geq \frac{1}{4}\left(\frac{\int_0^\infty A(t)dt}{\int_0^\infty V(t)dt}\right)^2$$

Since $\mathfrak{h}(\Omega)$ is the infemum of $\frac{A(\partial D)}{V(D)}$, we have that $1 = \mathfrak{h}(\Omega)\frac{1}{\mathfrak{h}(\Omega)} \geq \mathfrak{h}(\Omega)\frac{V(t)}{A(t)}$. Thus since $\mathfrak{h}(\Omega)$ does not depend on t, we have

$$\frac{1}{4}\left(\frac{\int_0^\infty A(t)dt}{\int_0^\infty V(t)dt}\right)^2 \geq \frac{1}{4}\left(\frac{\mathfrak{h}(\Omega)\int_0^\infty V(t)dt}{\int_0^\infty V(t)dt}\right)^2 = \frac{1}{4}(\mathfrak{h}(\Omega))^2$$

$\square$

Lemma 3.4.5. ***co-area formulas***

Let $f : \Omega \to \mathbb{R}$ *be in* $c^\infty(\Omega) \cup c^0(\overline{\Omega})$

$$(1) \qquad V'(t) = -\int_{\Gamma(t)} \| \, grad(f) \, \|^{-1} \, dA_t$$

$$(2) \quad \int_\Omega h \, \| \, grad(f) \, \| \, dV = \int_0^\infty \int_{\Gamma(t)} h \, dA_t dt$$

$$(3) \qquad \int_\Omega \| \, grad(f) \, \| \, dV = \int_0^\infty A(t)dt$$

Proof. By the implicit function theorem, for any non-critical value $t \in \mathbb{R}$ of f, $f^{-1}(t)$ is an $n-1$ manifold. Let $(\alpha, \beta) \subset \mathbb{R}$ be an interval of non-critical values for f with $t \in (\alpha, \beta)$.

One suspects that we can find a change of coordinates from $f^{-1}(\alpha, \beta)$ to the product of the manifolds $f^{-1}(\mu)$ and (α, β). If this is true we can define the volume of $f^{-1}(\alpha, \beta)$ in terms of iterated integration along a level set and (α, β).

We consider the mapping of the n-cylinder $f^{-1}(\mu) \times (\alpha, \beta)$ into $f^{-1}(\alpha, \beta)$ given by.

$\Psi : f^{-1}(\mu) \times (\alpha, \beta) \to f^{-1}(\alpha, \beta)$

where $\Psi(q, t)$ is the flow (local one-dimensional group action) at point q on the level set $f^{-1}(\mu)$. By construction, this is a diffeomorphism for which $f \circ \Psi(q, t) = t$, that is, $\Psi(q, t)$ is in the t^{th} level set. Also, it is clear from construction that $\mid \frac{\partial \Psi}{\partial t} \mid = \frac{1}{\| grad(f) \|}$

□

Chapter 4

Tools for Computing Extremal λ_1 on manifolds

In the following, we develop the tools necessary to derive extremal metrics for λ_k on the Klein bottle. As before, we will consider a metric extremal if and only if any analytic metric perturbation results in a smaller value for λ_k. In order to make use of this, we show that an analytic metric perturbation induces an analytic perturbation of an orthonormal basis of eigenfunctions for λ_k. We then develop a necessary set of requirements on such a basis for extremality. These requirements come from two major theorems. The first is an analog of the condition for extremal λ_k for graphs, involving a minimal immersion via the eigenfunctions into a sphere. The second condition is on the zeroes of the eigenfunctions,and is given by Courant's nodal domain theorem.

4.1 Perturbations of symmetric operators

The following result tells us that an analytic perturbation of the Laplacian will result in an analytic perturbation of its eigenvalues and an orthonormal basis of eigenfunctions.

Theorem 4.1.1. *Let A be a bounded self adjoint operator on a Hilbert space $\mathcal{H}$, and let $\mathbf{A}(t)$ be equal to the convergent power series:*

$$\mathbf{A}(t) = \sum_{j=0}^{\infty} t^j A_j$$

where $A_0 = A$.

Suppose that λ is an isolated eigenvalue of A with finite multiplicity equal to κ and orthonormal eigenvectors $\varphi_1, ...\varphi_\kappa$.

Then for $i = 1, ...\kappa$, there exist power series

$$\lambda_i(t) = \sum_{j=0}^{\infty} t^j \lambda_{i_j} \quad \text{and} \quad \phi_i(t) = \sum_{j=0}^{\infty} t^j \varphi_{i_j}$$

with $\lambda_{i_0} = \lambda$ and $\varphi_{i_0} = \varphi_i$,

such that for any sufficiently small $t > 0$,

- $\mathbf{A}(t)\phi_{\mathbf{i}}(t) = \lambda_{\mathbf{i}}(t)\phi_{\mathbf{i}}(t)$
- *Any element of spect$\mathbf{A}(t) \cap (\lambda - \varepsilon, \lambda + \varepsilon)$ must be $\lambda_{\mathbf{i}}(\mathbf{t})$ for some i.*
- $\{\phi_{\mathbf{i}}(t)\}$ *is an orthonormal set.*

We prove the more simple case with multiplicity of λ equal to one below. As such the subscript i is omitted for clarity. For the full proof of the theorem refer to [R].

Proof. Since $\mathcal{H}$ is a Hilbert space, we may decompose $\mathcal{H} = \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda^\perp$. Here $\mathcal{H}_\lambda$ is the eigenspace of λ, *span*$\{\varphi\}$. Let $P = \langle \varphi, \cdot \rangle \varphi$ be the orthogonal projection onto $\mathcal{H}_\lambda$.

By definition, $\ker(A - \lambda) = \mathit{span}\{\varphi\} = \mathcal{H}_\lambda$, so $A - \lambda$ is injective when restricted to $\mathcal{H}_\lambda^\perp$. We define the pseudo-inverse R as:

$R := 0 \oplus (A \restriction_{\mathcal{H}_\lambda^\perp} - \lambda)^{-1}$. In other words, R sends $\mathcal{H}_\lambda$ to zero, and is the inverse of $A - \lambda$ for components orthogonal to $\mathcal{H}_\lambda$, thus $RP = 0$ and $R(A - \lambda) = (A - \lambda)R = I - P$.

Now let$\widetilde{A}(t) := \mathbf{A}(t) - A = \sum_{j=1}^{\infty} t^j A_j$

Define the following function of variables t, and μ:

$$f(t,\mu) = \left\langle \varphi_0, \sum_{k=0}^{\infty} (\mu - \widetilde{A}(t))[R(\mu - \widetilde{A}(t))]^k \varphi_0 \right\rangle$$

We have that f is analytic in both t and μ in some neighborhood of $(0,0)$. Also, a computation shows that $f(0,\mu) = \mu$, so we have $\frac{\partial f}{\partial \mu}(0,0) = 1$ and $f(0,0) = 0$. By the implicit function theorem, in a neighborhood of $t = 0$ there exists an analytic function $\widetilde{\lambda}(t)$ such that $f(t, \widetilde{\lambda}(t)) = 0$.

At this point we would like to show that $\widetilde{\lambda}(t)$ is in fact equal to $\lambda(t) - \lambda$ with $\lambda(t)$ as in the statement of the theorem. We do this as follows.

Define $S(t) := R(\widetilde{\lambda}(t) - \widetilde{A}(t))$

We have that $(I - S(t))$ is invertible for sufficiently small t. We use the inverse to recover $\phi(t)$ from its component in $\mathcal{H}_\lambda$, namely φ_0.

Let $\phi(t) := (I - S(t))^{-1}\varphi_0$. Then

$\phi(t) = P\phi(t) + Prp(t)\phi(t) = \varphi_0 \oplus R(\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t)$

Multiplying both sides by $(A_0 - \lambda)$, recalling that φ_0 is an eigenvector for λ:

$(A_0 - \lambda)\phi(t) = 0 + (A_0 - \lambda)R(\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t)$

Since R is the pseudo-inverse of $A_0 - \lambda$, we have:

$(A_0 - \lambda)R = I - P$ thus:

$(A_0 - \lambda)\phi(t) = (\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t) - P((\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t))$.

By definition this is equal to

$(\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t) - \left\langle \varphi_0, (\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t) \right\rangle \varphi_0$

Next we have that $\left\langle \varphi_0, (\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t) \right\rangle \varphi_0 = 0$ since this is the projection of $(\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t)$ onto $\mathcal{H}_\lambda$ and $(\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t) \in \mathcal{H}_\lambda^\perp$.

Thus

$(A_0 - \lambda)\phi(t) = (\widetilde{\lambda}(t) - \widetilde{A}(t))\phi(t)$

$(A_0 + \widetilde{A}(t))\phi(t) = (\lambda + \widetilde{\lambda}(t))\phi(t)$

Letting $\lambda(t) = \lambda + \widetilde{\lambda}(t)$ we obtain the first part of the theorem. Part 2 follows from the spectral theorem for self adjoint operators, and part 3 results from the fact that if $\lambda(t)$ is analytic then so is $\| \lambda(t) \|$, thus we may normalize and still retain analyticity. $\square$

4.2 Minimal Immersions in Spheres

As was the case with graphs, The eigenfunctions of λ_k induce an immersion of M into a sphere. We have the following result:

Theorem 4.2.1. *A metric g for a given a Manifold M is c-extremal for λ_k, if and only if there exists an orthogonal basis of eigenfunctions $\varphi_1, \varphi_2, ..., \varphi_l$ such that:*

$$\sum (\varphi_i)^2 \equiv 1 \tag{4.2.1}$$

$$\sum (d\varphi_i)^2 \equiv \frac{1}{2}\lambda_k \tag{4.2.2}$$

This is essentially the statement that the eigenfunctions induce a minimal immersion of M into the sphere of dimension equal to one less than the dimension of the eigenspace.

Proof. Given a smooth perturbation of the metric within its conformal class resulting in $\triangle(t) = (1 + t\,f)\triangle_0$ with f the conformal factor. From theorem 4.1.1, this induces the perturbations given by:

$\lambda_k(t) = \lambda_k + t\mu_{k_1} + t^2\mu_{k_2} + ...$

$\varphi_k(t) = \varphi_k + t\phi_{k_1} + t^2\phi_{k_2} + ...$

Where φ_k is any eigenvector of the eigenvalue λ_k.

This gives us

$$\lambda_k(t)\varphi_k(t) = \lambda_k\varphi_k + t(\lambda_k\phi_{k_1} + \mu_{k_1}\varphi_k) + t^2(\lambda_k\phi_{k_2} + \mu_{k_1}\phi_{k_1} + \mu_{k_2}\varphi_k)... \tag{4.2.3}$$

Theorem 4.1.1 also states that

$\lambda_k(t)\varphi_k(t) = \triangle_t\varphi_k(t)$

and

$\triangle_t\varphi_k(t) = (1 + t\,f)\triangle_0(\varphi_k + t\phi_{k_1} + t^2\phi_{k_2} + ...)$

By linearity and the fact that φ_k is an eigenfunction we have:

$= (1 + t\,f)(\lambda_k\varphi_k + t\triangle_0\phi_{k_1} + t^2\triangle_0\phi_{k_2} + ...)$

Distributing and collecting powers of t gives

$$= \lambda_k\varphi_k + t(f\lambda_k\varphi_k + \triangle_0\phi_{k_1}) + t^2(f\triangle_0\phi_{k_1} + \triangle_0\phi_{k_2}) + ... \tag{4.2.4}$$

Equating the coefficients of like powers of t from equations 4.2.3 and 4.2.4 gives us:

$f\lambda_k\varphi_k + \triangle_0\phi_{k_1} = \lambda_k\phi_{k_1} + \mu_{k_1}\varphi_k$

$f\triangle_0\phi_{k_1} + \triangle_0\phi_{k_2} = \lambda_k\phi_{k_2} + \mu_{k_1}\phi_{k_1} + \mu_{k_2}\varphi_k$

Taking the inner product with φ_k on both sides of the first equality results in

$\lambda_k\langle f\varphi_k, \varphi_k\rangle + \langle\triangle_0\phi_{k_1}, \varphi_k\rangle = \lambda_k\langle\phi_{k_1}, \varphi_k\rangle + \mu_{k_1}\langle\varphi_k, \varphi_k\rangle$

Since $\triangle_0$ is self-adjoint

$\langle\triangle_0\phi_{k_1}, \varphi_k\rangle = \langle\phi_{k_1}, \triangle_0\varphi_k\rangle = \lambda_k\langle\phi_{k_1}, \varphi_k\rangle$

and we are left with

$\lambda_k\langle f\varphi_k, \varphi_k\rangle = \mu_{k_1}$

And we conclude

$\lambda'_k = \lambda_k \int f\varphi_k^2$

This is true for any eigenfunction φ_j for λ_k in the basis given by theorem 4.1.1.

$$\sum_j \lambda'_{k_j} = \lambda_k \int f \sum_j \varphi_{k_j}^2$$

Now suppose that $\sum_j \varphi_{k_j}^2 \equiv 1$

Then $\sum_j \lambda'_{k_j} = \lambda_k \int f\, dV = 0$ due to the normalization condition that the volume of M be constant within a conformal class. Thus $\lambda'_{k_j} \mid_0$ are not all strictly positive, and the perturbation results in a smaller λ_k. Thus g is c-extremal.

Next suppose that $\sum_j \varphi^2_{k_j} \not\equiv 1$

The the constant function 1 is not in the positive cone spanned by the basis of eigenfunctions. $\exists f \in \mathcal{C}^\infty$ a conformal factor such that $\langle f, \varphi^2_{k_j} \rangle > 0 \;\; \forall j$ and $\langle f, 1 \rangle = 0$. This gives us a metric perturbation for which $\lambda'_{k_j} \mid_{t=0} = \lambda_k \langle f, \varphi^2_{k_j} \rangle > 0$ and g is not c-extremal.

□

4.3 Courant's Nodal Domain Theorem

Definition 4.3.1. For any smooth function $\varphi : M \to \mathbb{R}$ The nodal set of φ is defined to be the set $\{x \mid \varphi(x) = 0\}$. The connected components of the complement of the nodal set in $\overline{M}$ are referred to as nodal domains.

Theorem 4.3.1. *Given eigenvalues $\lambda_1, \lambda_2, ...$ of $\triangle$ and corresponding eigenfunctions $\varphi_1, \varphi_2....$ The number of nodal domains of φ_k is less than or equal to k.*

Proof. By the implicit function theorem (bibliography reference), the nodal set of φ_k is a piecewise smooth $n-1$ manifold in a neighborhood around any point where $grad(\varphi_k) \neq 0$, so the the nodal domains are in fact domains.

Suppose for the sake of contradiction that φ_k has more than k nodal domains $N_1, N_2, ..., N_k, N_{k+1}, ...$ each of which is a domain. Define the function

$$f = \sum_{i=1}^{k} c_j \varphi_k \chi_{N_i}$$

By theorem 3.3.5, The set $\{\varphi_1, \varphi_2 ...\}$ forms an orthonormal basis for $\mathcal{L}^2(M)$, so we have that $\langle f, \varphi_j \rangle_{\mathcal{L}^2} = 0$ for all j with $0 \leq j < k$.

$$\frac{D[f,f]}{\| f \|^2_{\mathcal{L}^2}}$$

$$= \frac{\int_M \| grad(f) \|^2 \, dV}{\int_M f^2 dV}$$

$$= \frac{\sum_{i=1}^k c_j^2 \int_M \| grad(\varphi_k \chi_{N_i}) \|^2 \, dV}{\sum_{i=1}^k c_j^2 \int_M \varphi_k \chi_{N_i}^2 dV}$$

$$\leq \sum_{i=1}^k \frac{c_j^2 \int_M \| grad(\varphi_k \chi_{N_i}) \|^2 \, dV}{c_j^2 \int_M (\varphi_k \chi_{N_i})^2 dV}$$

$$= \sum_{i=1}^k \frac{\int_{N_i} \| grad(\varphi_k) \|^2 \, dV}{\int_{N_i} \varphi_k^2 dV}$$

By Green's Formula, we have

$$\frac{\lambda_k}{\lambda_k} \int_{N_i} \varphi_k^2 dV = \frac{1}{\lambda_k} \int_{N_i} (\triangle \varphi_k) \varphi_k dV = \frac{1}{\lambda_k} \int_{N_i} \| grad(\varphi_k) \|^2 \, dV - \frac{1}{\lambda_k} \int_{\partial N_i} 0$$

since the boundary of N_i is contained in the nodal set for φ_k so $\varphi_k \equiv 0$ on ∂N_i.

Thus from above we have

$$\frac{D[f,f]}{\| f \|^2_{\mathcal{L}^2}} \leq \sum_{i=1}^k \frac{\int_{N_i} \| grad(\varphi_k) \|^2 \, dV}{\int_{N_i} \varphi_k^2 dV} = \sum_{i=1}^k \frac{\int_{N_i} \| grad(\varphi_k) \|^2 \, dV}{\frac{1}{\lambda_k} \int_{N_i} \| grad(\varphi_k) \|^2 \, dV} = \lambda_k \times 1$$

By the Min-Max theorem we also have that $\lambda_k \leq \frac{D[f,f]}{\|f\|^2_{\mathcal{L}^2}}$ and the resulting equality implies that f is an eigenfunction for λ_k. But f is identically zero on the nodal set N_{k+1} which then implies that f is identically zero on M by the maximum principal. Contradiction

$\square$

Chapter 5

λ_k on the Klein Bottle

We now use the tools developed in Chapter 4 to find extremal metrics for λ_k on the Klein Bottle. The general method is to use constraints on the eigenfunctions in order to set up a system of differential equations. The system will involve a parameter dependant on the value of λ_k. We then use further restrictions on eigenfunctions for extremal λ_k to determine which parameter gives the proper solution set to the system of differential equations.

The Klein bottle can be realized as a quotient space of $\mathbb{R}^2$ in the following way:

A rectangular lattice Γ in $\mathbb{R}^2$ is defined as a discrete subgroup generated by 2 independent vectors, u, v. The Torus can be realized as $\mathbb{R}^2/\Gamma$ where without loss of generality, Γ is generated by $(2\pi, 0)$ and $(0, a)$. Given any $p \in M$, there is a neighborhood U of p for which we have a natural map φ from U into the rectangle given by $(2\pi, 0)$, $(0, a)$, $(2\pi, a)$ and the origin in $\mathbb{R}^2$.

Furthermore, the action of $\mathbb{Z}^2$ on the torus generated by $\sigma : \varphi^{-1}(x, y) \mapsto \varphi^{-1}(x + \pi, a - y)$ is properly discontinuous and the resulting quotient space is a Klein bottle. This gives us a double cover of the Klein bottle by a taurus. The corresponding subgroup of $\mathbb{R}^2$ with $\mathbb{K}$ as the quotient space is generated by $(x, y) \mapsto (x + \pi, -y)$, and $(x, y) \mapsto (x, y + a)$. The result is a natural map from the manifold $(\mathbb{K}, g)$ to a rectangle in $\mathbb{R}^2$, with induced local coordinate chart $(x(p), y(p))$.

The covering projection of $\mathbb{K}$ by $\mathbb{T}^2$ also induces a one to one correspondence between $\mathcal{L}^2(\mathbb{K})$ and the $\mathcal{L}^2$ functions f on $\mathbb{T}^2$ satisfying the condition $f \circ p(x, y) = f \circ p(x + \pi, -y)$ where p is the quotient projection. This is useful since we can expand $\mathcal{L}^2$ functions on $\mathbb{T}^2$ coordinate-wise into Fourier series in $\mathbb{T}^2$. Functions in $\mathcal{L}^2(\mathbb{K})$ can then be expressed as a series of functions of the form $\{\varphi_n(y)\sin(n\,x), \varphi_n(y)\cos(n\,x)\}$, with $\varphi_n(y + a) = \varphi_n(y)$ and $\varphi_n(-y) = (-1)^n \varphi_n(y)$.

It follows from [N] that an extremal metric for λ_1 on the Klein bottle must be a metric of revolution, that is a metric invariant under a $\mathbb{S}^1$ action. Without loss of generality, we can take this to be given by $(x, y) \to (x + t, y)$, so the metric on $\mathbb{K}$ must be given by $f(y)(dx^2 + dy^2)$ in local coordinates. Here, the function $f > 0$ is the conformal factor, and $f(y) = f(y + a) = f(-y)$ as above.

5.1 A basis of eigenfunctions for E_{λ_k}

We have that there must exist a basis of eigenfunctions all of which are of the form $\varphi_n(y)\sin(n\,x)$ or $\varphi_n(y)\cos(n\,x)$, where φ_n is an arbitrary smooth function with the same parity as $n \in \mathbb{N}$. For $\varphi_n(y(p)) = (\varphi_n \circ y)(p)$ to be a well defined function on $\mathbb{K}$, φ_n must necessarily be periodic.

Courant's nodal domain theorem places further restrictions on the eigenfunctions for extremal λ_k Specifically, they must each split $\mathbb{K}$ into $k + 1$ or fewer distinct nodal domains. The number of nodal domains must also be greater than one, since otherwise the eigenfunctions would not be orthogonal to the constant function, which is an eigenfunction for λ_0. Thus eigenfunctions for λ_1 need have precisely 2 nodal domains.

Since φ_n is independent of x and sin or cos is independent of y, the image of the nodal set of an eigenfunction under the quotient identification will consist of lines parallel to the x and y axes. These divide the image of $\mathbb{K}$ into rectangular regions which are the images of nodal domains. Let m_k denote the number of zeroes of φ_n on its period. We know that $\sin(n\,x)$ and $\cos(n\,x)$ achieve precisely n zeroes on the period $[0, \pi)$. Thus the number of nodal domains is determined by m_n and n.

The additional restriction of φ_n to being an odd or even function based on the parity of n allows us to list the various possibilities. For instance, it is not possible that $m_n = 0$ for odd values of n since a continuous odd function must pass through the origin. Also note that m must be even. This is because φ_n is periodic, and at any zero, φ_n must change sign since otherwise $\varphi_n = 0 = \varphi_n'$. This is not possible since $\triangle\varphi_n(y)\sin(n\,x) = \lambda_k\varphi_n(y)\sin(n\,x)$, thus $\varphi_n(y)$ is a nontrivial solution of the second order differential equation given by $\varphi_n'' = (2n - \lambda_k f)\varphi_n$ (see section 5.2.1 below).

5.2 λ_1

We shall first examine the case of λ_1.

As depicted in figure 5.1, in order to have two nodal domains the eigenfunctions of λ_1 must be of the form $\varphi_n(y)\sin(n\,x)$ or $\varphi_n(y)\cos(n\,x)$ with $n = 0, 1, 2$ and $m = 2, 2, 0$ respectively.

In other words elements of the basis of eigenfunctions described in theorem 4.2.1 must be in the following families of functions:

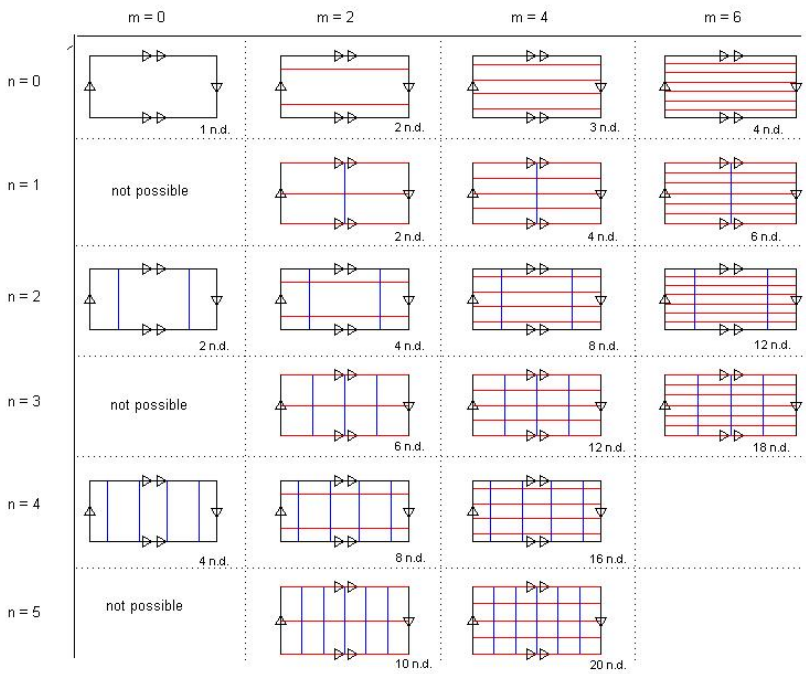

Figure 5.1: Number of Nodal Domains (n.d.)for $\varphi_m(y)cos(n\,x)$

$$\varphi_0(y)$$
$$\varphi_1(y)\sin(x)$$
$$\varphi_1(y)\cos(x)$$
$$\varphi_2(y)\sin(2x)$$
$$\varphi_2(y)\cos(2x)$$

Proposition 5.2.1. *The orthonormal basis of eigenfunction for extremal λ_1 that induces a minimal immersion into the sphere must be of the following form:*

$$\{\varphi_0(y), \varphi_1(y)\sin(x), \varphi_1(y)\cos(x), \varphi_2(y)\sin(2x), \varphi_2(y)\sin(2x)\}$$

The multiplicity of λ_1 is equal to 5.

Proof. No two elements of it can be of the same family of functions. thus the dimension of the eigenspace is at most 5. From [B] we have that the multiplicity of λ_1 is greater than 3. Suppose for the sake of contradiction that the multiplicity is 4. Then the only possibility is that $n = 1, 2$ and we have $\varphi_1(y)\sin(x)$, $\varphi_1(y)\cos(x)$, $\varphi_2(y)\sin(2x)$, and $\varphi_2(y)\cos(2x)$ as the basis.

Then by theorem 4.2.1, this induces an immersion into the sphere giving us

$\varphi_1^2(y)(\sin^2(x) + \cos^2(x)) + \varphi_2^2(y)(\sin^2(2x) + \cos^2(2x)) \equiv 1$

or just $\varphi_1^2 + \varphi_2^2 \equiv 1$

differentiating this as a function of y gives us

$2\varphi_1\varphi_1' + 2\varphi_2\varphi_2' \equiv 0$

Since both φ_1 and φ_2 functions are periodic, they must achieve a minimum. Thus their derivatives must vanish at some point.

From the above equation $\varphi_1' = 0 \Rightarrow \varphi_2\varphi_2' = 0$, and since φ_2 is never zero, $\Rightarrow \varphi_2' = 0$

Thus, φ_1' and φ_2' both vanish at the same point.

Again by theorem 4.2.1:

$$\sum_{i=0}^{4} d\psi_i \otimes d\psi_i \equiv \frac{1}{2}\lambda_k$$

Computing just the $dy \otimes dy$ portion results in

$[(\varphi_1')^2 + (\varphi_2')^2]dy \otimes dy = \frac{1}{2}\lambda_k f dy \otimes dy$

Thus, φ_1' and φ_2' can not both vanish since both f and λ_k are strictly positive.

This is a contradiction.

□

5.2.1 A System of Differential Equations for λ_k

We now use the second condition in theorem 4.2.1 to derive a system of equations for φ_n. The goal is to arrive at a system of equations that involves a parameter dependant on λ_k. Solutions to this system can then be checked against other conditions on extremality to find values of the parameter that give extremal λ_k.

From theorem 4.2.1:

$$\sum_{i=0}^{4} d\psi_i \otimes d\psi_i \equiv \frac{1}{2}\lambda_k$$

Computing the differentials gives us...

$$\begin{array}{ll}
\psi_0 = \varphi_0(y) & d\psi_0 = \varphi_0'(y)dy \\
\psi_1 = \varphi_1(y)\sin(x) & d\psi_1 = \varphi_1'(y)\sin(x)dy \,+\, \varphi_1(y)\cos(x)dx \\
\psi_2 = \varphi_1(y)\cos(x) & d\psi_2 = \varphi_1'(y)\cos(x)dy \,-\, \varphi_1(y)\sin(x)dx \\
\psi_3 = \varphi_2(y)\sin(2x) & d\psi_3 = \varphi_2'(y)\sin(2x)dy \,+\, 2\varphi_2(y)\cos(2x)dx \\
\psi_4 = \varphi_2(y)\sin(2x) & d\psi_4 = \varphi_2'(y)\cos(2x)dy \,-\, 2\varphi_2(y)\sin(2x)dx
\end{array}$$

and so

$$\begin{array}{l}
d\psi_0 \otimes d\psi_0 = (\varphi_0')^2 dy \otimes dy \\
d\psi_1 \otimes d\psi_1 = (\varphi_1')^2 \sin^2 dy \otimes dy \,+ 2\varphi_1\varphi_1' \sin\cos dx \otimes dy + \,(\varphi_1)^2\cos^2 dx \otimes dx \\
d\psi_2 \otimes d\psi_2 = (\varphi_1')^2 \cos^2 dy \otimes dy \,- 2\varphi_1\varphi_1' \cos\sin dx \otimes dy + \,(\varphi_1)^2\sin^2 dx \otimes dx \\
d\psi_3 \otimes d\psi_3 = (\varphi_2')^2 \sin^2 dy \otimes dy \,+ 4\varphi_2\varphi_2' \sin\cos dx \otimes dy + \,4(\varphi_2)^2\cos^2 dx \otimes dx \\
d\psi_4 \otimes d\psi_4 = (\varphi_2')^2 \cos^2 dy \otimes dy \,- 4\varphi_2\varphi_2' \cos\sin dx \otimes dy + \,4(\varphi_2)^2\sin^2 dx \otimes dx
\end{array}$$

Summing these we have

$[(\varphi_1)^2 + 4(\varphi_2)^2][\cos^2 + \sin^2]dx \otimes dx$

$+[(2-2)\varphi_1\varphi_1' \cos\sin + (4-4)\varphi_2\varphi_2' \cos\sin]dx \otimes dy$

$+[(\varphi_0')^2 + (\varphi_1')^2 + (\varphi_2')^2][\cos^2 + \sin^2]dy \otimes dy$

After some cancelations, the sum of the tensor products is equal to

$[(\varphi_1)^2 + 4(\varphi_2)^2]dx \otimes dx \,+ [(\varphi_0')^2 + (\varphi_1')^2 + (\varphi_2')^2]dy \otimes dy$

From the second condition in theorem 4.2.1, this must be equal to

$\frac{1}{2}\lambda_k f dx \otimes dx + \frac{1}{2}\lambda_k f dy \otimes dy$

We conclude that

$$(\varphi_1)^2 + 4(\varphi_2)^2 = (\varphi_0')^2 + (\varphi_1')^2 + (\varphi_2')^2 = \frac{1}{2} f \lambda_k \tag{5.2.1}$$

This system of equations is far from ideal. First, we have three functions to solve for and only two restraints. Moreover, the equations involve not only the unknown λ_k, but also the conformal parameter f. We must introduce further restraints in order to correct this. We will use the fact that $\varphi_n(y)\sin(n\,x)$ and $\varphi_n(y)\cos(n\,x)$ are eigenfunctions to derive further restrictions on φ_n.

From the expression of the Laplacian in local coordinates, we have $\triangle = -\frac{1}{f}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
Since $\varphi_n(y)\sin(n\,x)$ and $\varphi_n(y)\cos(n\,x)$ are eigenfunctions, we have that
$\lambda_1\varphi_0 = \triangle\varphi_0 = -\frac{1}{f}\varphi_0''$
so $\varphi_0'' = -f\lambda_1\varphi_0$

And for $n > 0$:

$\lambda_1\varphi_n\sin(n\,x) = \triangle\varphi_n\sin(n\,x) = -\frac{1}{f}[\varphi_n''\sin(n\,x) - 2n\varphi_n\sin(n\,x)]$
or $-\lambda_1 f\varphi_n\sin(n\,x) = \varphi_n''\sin(n\,x) - 2n\varphi_n\sin(n\,x)$.
dividing through by $\sin(n\,x)$ gives us $\varphi_n'' = 2n\varphi_n - \lambda_1 f\varphi_n$

To summarize,

$$\left\{\begin{array}{c} \varphi_0'' = -f\lambda_1\varphi_0 \\ \varphi_1'' = \varphi_1 - \lambda_1 f\varphi_1 \\ \varphi_2'' = 4\varphi_2 - \lambda_1 f\varphi_2 \end{array}\right\} \tag{5.2.2}$$

If we substitute the value for $\lambda_1 f$ given in equation 5.2.1 into the last two equations of 5.2.2we get the following system

$$\left\{ \begin{array}{l} \varphi_1'' = (1 - 2(\varphi_1^2 + 4\varphi_2^2))\varphi_1 \\ \varphi_2'' = (4 - 2(\varphi_1^2 + 4\varphi_2^2))\varphi_2 \end{array} \right\}$$

A system of equations involving φ_0 and φ_1 is obtained by substitute the value for $\lambda_1(f)$ given in equation 5.2.1 into the first two equations of 5.2.2.

$$\left\{ \varphi_0'' = -2(\varphi_1^2 + 4\varphi_2^2)\varphi_0\varphi_1'' = (1 - 2(\varphi_1^2 + 4\varphi_2^2))\varphi_1 \right\}$$

From the first condition in theorem 4.2.1

$\varphi_0^2(y) + \varphi_1^2(y)(\sin^2(x)\cos^2(x)) + \varphi_2^2(y)(\sin^2(2x)\cos^2(2x)) \equiv 1,$

thus $\varphi_2^2 \equiv 1 - \varphi_0^2 - \varphi_1^2$

substituting this into the above results in

$$\left\{ \varphi_0'' = -2(\varphi_1^2 + 4 - 4\varphi_0^2 - 4\varphi_1^2)\varphi_0\varphi_1'' = (1 - 2\varphi_1^2 - 8 + 8\varphi_0^2 + 8\varphi_1^2)\varphi_1 \right\}$$

Simplifying this, the system of equations for φ_0 and φ_1 becomes.

$$\left\{ \begin{array}{l} \varphi_0'' = (8\varphi_0^2 + 6\varphi_1^2 - 8)\varphi_0 \\ \varphi_1'' = (8\varphi_0^2 + 6\varphi_1^2 - 7)\varphi_1 \end{array} \right\}$$

5.2.2 Initial Conditions

Some other restrictions on the functions φ_n will result in initial conditions for this system of differential equations:

First we have that:

$$\varphi_1(0) = \varphi_0'(0) = \varphi_2'(0) = 0 \tag{5.2.3}$$

since these are all odd functions.

Substituting this into equation 5.2.1 at $t = 0$ we obtain:

$0 + 4(\varphi_2)^2(0) = 0 + (\varphi_1')^2(0) + 0 = \frac{1}{2}\lambda_k$, or $2\varphi_2(0) = \varphi_1'(0) = \sqrt{\frac{1}{2}\lambda_k}$

For convenience we make the substitution $p = p(\lambda_1) = \sqrt{\frac{1}{8}\lambda_1}$ giving us initial conditions $\varphi_2(0) = p$ and $\varphi_1'(0) = 2p$.

Lastly we may use the first condition from theorem 4.2.1: $\varphi_0^2(0)+\varphi_1^2(0)(\sin^2+\cos^2)+\varphi_2^2(0)(\sin^2+\cos^2)=1$

hence $0 \le p \le 1$, $\varphi_0^2(0)+0+p^2=1$ and $\varphi_0(0)=\sqrt{1-p^2}$

This gives us the following set of initial conditions for λ_1:

$$\begin{array}{lll} \varphi_0(0)=\sqrt{1-p^2}, & \varphi_1(0)=0, & \varphi_1(0)=p \\ \varphi_0'(0)=0, & \varphi_1'(0)=2p, & \varphi_2'(0)=0 \end{array} \tag{5.2.4}$$

5.2.3 Restrictions on the Parameter

The parameter p is taken to be nonnegative. Further restrictions on possible values of p can be obtained using the first integrals of the system of equations. Then solutions can be computed numerically for the remaining possible values of p.

It follows from K. [U] that the following are first integrals for the system:

$E_0=\varphi_0^2+(\varphi_0\varphi_1'-\varphi_1\varphi_0')^2+\frac{1}{4}(\varphi_0\varphi_2'-\varphi_2\varphi_0')^2$

$E_1=\varphi_1^2+\frac{1}{3}(\varphi_1\varphi_2'-\varphi_2\varphi_1')^2-(\varphi_1\varphi_0'-\varphi_0\varphi_1')^2$

$E_2=\varphi_2^2-\frac{1}{4}(\varphi_2\varphi_0'-\varphi_0\varphi_2')^2-\frac{1}{3}(\varphi_2\varphi_1'-\varphi_1\varphi_2')^2$

The first integrals are related in the following manner:

Lemma 5.2.2. $E_0+E_1+E_2=1$ $E_0+\frac{3}{4}E_1=1$ $E_2=-\frac{1}{4}E_1$

Proof. For the first equation

$E_0+E_1+E_2=\varphi_0^2+\varphi_1^2+\varphi_2^2+0(\varphi_0\varphi_1'-\varphi_1\varphi_0')^2+0(\varphi_0\varphi_2'-\varphi_2\varphi_0')^2+0(\varphi_1\varphi_2'-\varphi_2\varphi_1')^2$
$=\varphi_0^2+\varphi_1^2+\varphi_2^2\equiv 1$

For the second equation

$$E_0 + \tfrac{3}{4}E_1 = \varphi_0^2 + (\varphi_0\varphi_1' - \varphi_1\varphi_0')^2 + \tfrac{1}{4}(\varphi_0\varphi_2' - \varphi_2\varphi_0')^2$$
$$+\tfrac{3}{4}\varphi_1^2 + \tfrac{1}{4}(\varphi_1\varphi_2' - \varphi_2\varphi_1')^2 - \tfrac{3}{4}(\varphi_1\varphi_0' - \varphi_0\varphi_1')^2$$

$= \varphi_0^2 + \frac{3}{4}\varphi_1^2 + \frac{1}{4}\sum_{i<j\in\{0,1,2\}}(\varphi_i\varphi_j' - \varphi_j\varphi_i')^2$

Since $\varphi_0^2 + \varphi_1^2 + \varphi_2^2 = 1$, this expression becomes

$= 1 - \frac{1}{4}\varphi_1^2 - \varphi_2^2 + \frac{1}{4}\sum_{i<j\in\{0,1,2\}}(\varphi_i\varphi_j' - \varphi_j\varphi_i')^2$

$= 1 - \frac{1}{4}(\varphi_1^2 - 4\varphi_2^2) + \frac{1}{4}\sum_{i<j\in\{0,1,2\}}(\varphi_i\varphi_j' - \varphi_j\varphi_i')^2$

From equation 5.2.1, $\varphi_1^2 + 4\varphi_2^2 = \sum_{i=1}^3(\varphi_i')^2$

thus it suffices to show that $\sum_{i<j\in\{0,1,2\}}(\varphi_i\varphi_j' - \varphi_j\varphi_i')^2 = \sum_{i=1}^3(\varphi_i')^2$

A computation shows that the left side of this equation is equal to

$(\varphi_0')^2(\varphi_1^2 + \varphi_2^2) + (\varphi_1')^2(\varphi_0^2 + \varphi_2^2) + (\varphi_2')^2(\varphi_0^2 + \varphi_1^2) - 2\varphi_0\varphi_0'(\varphi_1\varphi_1' + \varphi_2\varphi_2')$

$-2(\varphi_1\varphi_1'\varphi_2\varphi_2')$

adding and subtracting $\sum_{i=1}^3 \varphi_i^2(\varphi_i')^2$ to this gives

$(\varphi_0')^2 + (\varphi_1')^2 + (\varphi_2')^2 + [\varphi_0\varphi_0' + \varphi_1\varphi_1' + \varphi_2\varphi_2']^2$

$= (\varphi_0')^2 + (\varphi_1')^2 + (\varphi_2')^2 + [\frac{1}{2}(\varphi_0^2 + \varphi_1^2 + \varphi_2^2)']^2$

$= (\varphi_0')^2 + (\varphi_1')^2 + (\varphi_2')^2$ as desired.

Lastly, for the third equation,

$E_0 + E_1 + E_2 = 1 = E_0 + \frac{3}{4}E_1$

so $E_1 + E_2 = 1 = \frac{3}{4}E_1$

and $E_2 = -\frac{1}{4}E_1$

$\square$

The first integrals can be used to rule out the possibility of extremal λ_1 for parameter p outside the interval $[0, \frac{\sqrt{3}}{2})$. The proof requires the following lemma.

Lemma 5.2.3. *For values of p in the interval $[0, \frac{\sqrt{3}}{2})$, $E_1(0) < 0$, while $E_1(0) > 0$ for $p > \frac{\sqrt{3}}{2}$.*

Proof. $E_1 = \varphi_1^2 + \frac{1}{3}(\varphi_1\varphi_2' - \varphi_2\varphi_1')^2 - (\varphi_1\varphi_0' - \varphi_0\varphi_1')^2$

Thus $E_1(0) = 0 + \frac{1}{3}(0 - p\,2p)^2 - (0 - \sqrt{1-p^2}\,2p)^2$

$= \frac{4}{3}p^4 - 4p^2(1-p^2) = 4p^2(\frac{4}{3}p^2 - 1)$

Setting this equal to 0, we have $\frac{4}{3}p^2 = 1$ so $p = \frac{\sqrt{3}}{2}$ (since p must be non-negative).

We have that $E_1(0) < 0$ for $0 \leq p < \frac{\sqrt{3}}{2}$, while $E_1(0) > 0$ for $p > \frac{\sqrt{3}}{2}$.

□

Proposition 5.2.4. *There are no extremal metrics for* λ_1 *with the value of* p *not in the interval* $[0, \frac{\sqrt{3}}{2}]$.

Proof. The proof is by contradiction. By lemma 5.2.3 $\frac{\sqrt{3}}{2}$ is where E_1 changes sign. We show that if E_1 is positive, then the curve $(\varphi_1(t), \varphi_2(t))$ rotates about the origin such that $\varphi_1(t)$ and $\varphi_1(t)$ have the same number of zeroes. To do this, a change of variables to polar coordinates is made.

Consider the spherical change of variables

$$\left\{\begin{array}{ll} \varphi_0 = \cos(\psi) & \varphi_0' = -\sin(\psi)\psi' \\ \varphi_1 = \sin(\psi)\sin(\theta) & \varphi_1' = \cos(\psi)\psi'\sin(\theta) + \sin(\psi)\cos(\theta)\theta' \\ \varphi_2 = \sin(\psi)\cos(\theta) & \varphi_2' = \cos(\psi)\psi'\cos(\theta) - \sin(\psi)\sin(\theta)\theta' \end{array}\right\}$$

This gives us

$(\varphi_1\varphi_0' - \varphi_0\varphi_1') = [-\sin^2(\psi)\sin(\theta)\psi'] - [\cos^2(\psi)\psi'\sin(\theta) + \cos(\psi)\sin(\psi)\cos(\theta)\theta']$

$= -[\sin^2(\psi) + \cos^2(\psi)]\sin(\theta)\psi' - \cos(\psi)\sin(\psi)\cos(\theta)\theta'$

$= -\sin(\theta)\psi' - \frac{1}{2}\sin(2\psi)\cos(\theta)\theta'$

$(\varphi_2\varphi_0' - \varphi_0\varphi_2') = [-\sin^2(\psi)\cos(\theta)\psi'] - [\cos^2(\psi)\psi'\cos(\theta) + \cos(\psi)\sin(\psi)\sin(\theta)\theta']$

$= -[\sin^2(\psi) + \cos^2(\psi)]\cos(\theta)\psi' - \cos(\psi)\sin(\psi)\sin(\theta)\theta'$

$= -\cos(\theta)\psi' - \frac{1}{2}\sin(2\psi)\sin(\theta)\theta'$

$(\varphi_1\varphi_2' - \varphi_2\varphi_1') = [\sin(\psi)\sin(\theta)\cos(\psi)\psi'\cos(\theta) - \sin^2(\psi)\sin^2(\theta)\theta'] -$

$[\sin(\psi)\cos(\theta)\cos(\psi)\psi'\sin(\theta) + \sin^2(\psi)\cos^2(\theta)\theta']$

$= 0 - sin^2(\psi)[\sin^2(\theta) + \cos^2(\theta)]\theta'$

$= -sin^2(\psi)\theta'$

and finally

$E_1 = \sin^2(\theta)(\sin^2(\psi) - (\psi')^2) - \frac{1}{2}\sin(2\theta)\sin(2\psi)\psi'\theta'$

$E_2 = -\frac{1}{3}(\theta')^2\sin^4(\psi) + \cos^2(\theta)(\sin^2(\psi) - \frac{1}{4}(\psi')^2) - \frac{1}{16}(\sin(2\psi)\sin(\theta)\theta')^2$
$+ \frac{1}{8}\sin(2\psi)\sin(2\theta)\theta'\psi'$

Suppose for the sake of contradiction that $p > \frac{\sqrt{3}}{2}$. Then from lemma 5.2.3 $E_1 > 0$. From lemma 5.2.2 $E_2 = -\frac{1}{4}E_1$, so $E_2 < 0$.

Since $\varphi_1(t)$, and $\varphi_2(t)$ are both $\mathcal{C}^\infty$, we have that $\theta(t)$ is also.

If $\theta' = 0$ then $E_1 = \sin^2(\theta)[\sin^2(\psi) - (\psi')^2]$ and $E_2 = \cos^2(\theta)[\sin^2(\psi) - \frac{1}{4}(\psi')^2]$

This implies that $\sin^2(\psi) - (\psi')^2 > 0$ and $\sin^2(\psi) - \frac{1}{4}(\psi')^2 < 0$ which leaves us with $\frac{1}{4}(\psi')^2 > (\psi')^2$. This is a contradiction.

If on the other hand θ' is never zero, then the curve $(\varphi_1(t), \varphi_2(t))$ must be a closed loop which rotates around the origin with monotone increasing or decreasing angle of rotation. This is also a contradiction since φ_1 has two zeros on its period while φ_2 has none.

The only possibility is that $p < \frac{\sqrt{3}}{2}$ as desired.

□

The following theorem gives valuable information on the nature of the solutions φ_0 and φ_1. As such, it is useful for determining which value of p results in extremal λ_1.

Theorem 5.2.5. *If λ_1 is extremal, the curve $(\varphi_0(t), \varphi_1(t))$, rotates around the origin in the $(\varphi_0(0), \varphi_1(0))$ plane.*

Proof. The angle of rotation can be parameterized by $\theta(t) = arc\tan(\frac{\varphi_1(t)}{\varphi_0(t)})$. In this case, $\theta' = \frac{1}{1+(\frac{\varphi_1}{\varphi_0})^2}\frac{\varphi_1\varphi_0' - \varphi_0\varphi_1'}{\varphi_0^2}$. Thus $\theta' = 0 \Leftrightarrow \varphi_1\varphi_0' - \varphi_0\varphi_1' = 0$.

It suffices to show that the Wronskian $\varphi_1\varphi_0' - \varphi_0\varphi_1' \neq 0$. However, we have that $0 > E_1 = \varphi_1^2 + \frac{1}{3}(\varphi_1\varphi_2' - \varphi_2\varphi_1')^2 - (\varphi_1\varphi_0' - \varphi_0\varphi_1')^2$

Since the first two terms of this sum are always strictly positive, the only possible way for E_1 to be strictly negative is for $\varphi_1\varphi_0' - \varphi_0\varphi_1'$ to be strictly negative.

□

Since φ_1 and φ_0 each have two zeroes on their period, we can conclude from the above theorems that the curve $\gamma(t) := (\varphi_0(t), \varphi_1(t))$ rotates precisely once around the origin in the $\varphi_0 - \varphi_1$ plane. Since the functions $\varphi_0(t)$ and $\varphi_1(t)$ are periodic, $\gamma(t)$ should form a closed loop on the same period. All of these theorems have been proven under the assumption that the metric g is extremal for λ_1. For non-extremal λ_1, we may still be able to solve the system of equations in **??**, however, we do not know anything about the corresponding curve given by the solutions.

This gives us a method to test numerically for the correct value of λ_1. If the parameter $p = \sqrt{\frac{1}{8}\lambda_1}$ in the initial conditions does not result in a closed curve $\gamma(t)$ rotating precisely once around the origin, we may conclude that λ_1 is not extremal. Our task then becomes to find out for which values of λ_1 this happens and for which it does not.

5.2.4 Solutions via Numerical methods

The system of equations with parameter has not been solved explicitly. Instead numerical integration methods have been used; implemented by a MATLAB program based on an explicit Runge-Kutta (4,5) formula.

Numerical experimentation indicates that the only value of the parameter p for

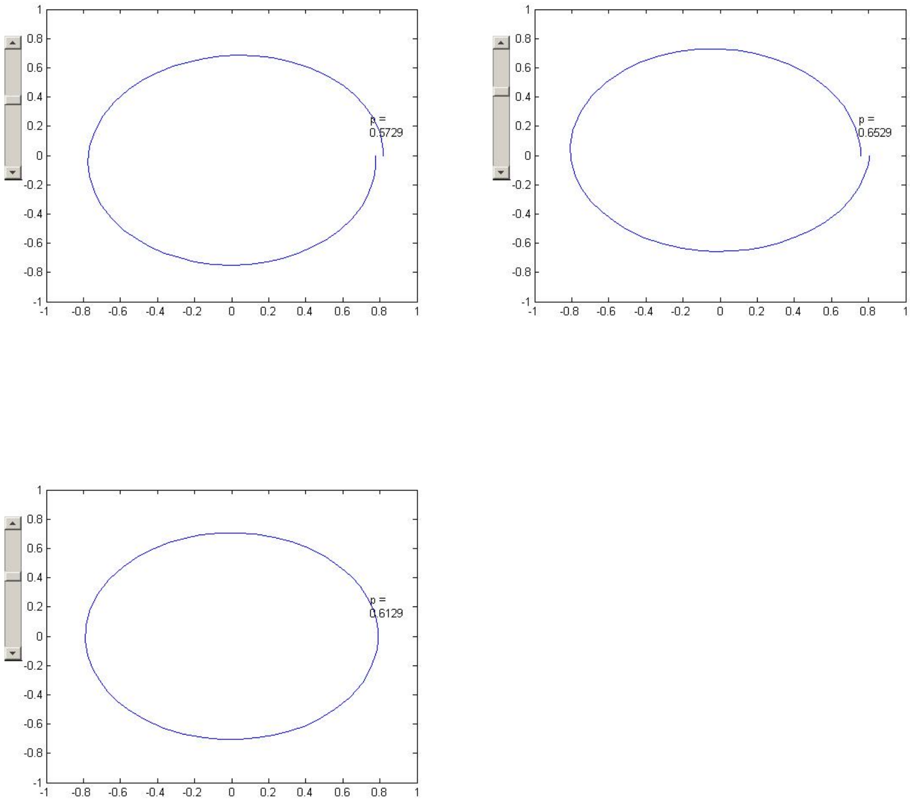

Figure 5.2: Graph of $\gamma(t) := (\varphi_0(t), \varphi_1(t))$ for Different Values of p

which $\gamma(t) := (\varphi_0(t), \varphi_1(t))$ forms a closed loop is $p = \frac{\sqrt{3}}{4} \approx 0.6124$. This is the only value of p for which φ_0 and φ_1 appear to be periodic on the same period with two zeroes each.

The difference between the initial position of γ and the position after a rotation by 2π for values of p on the interval $[0, \frac{\sqrt{3}}{2})$ is plotted below.

The smooth dependance on initial conditions of the system of equations provides

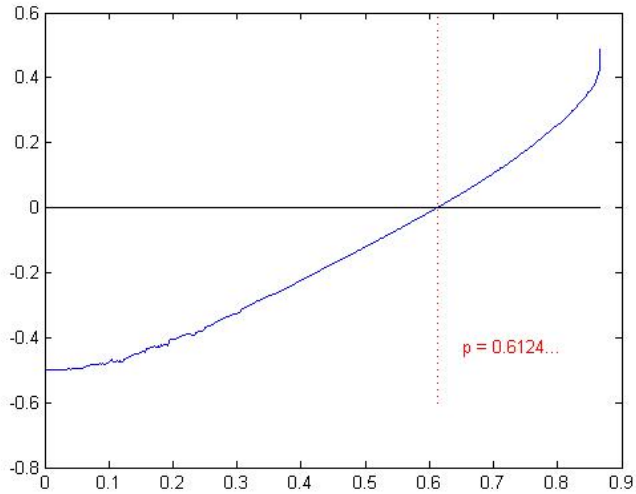

Figure 5.3: Gap in γ for Different Values of p

strong evidence that $p = \frac{\sqrt{3}}{4}$ is the only value of p which could yield valid eigenfunctions.

5.3 λ_2

5.3.1 ruling out other cases for λ_2

By Courant's nodal domain theorem, the eigenfunctions for λ_2 must have two or three nodal domains. This means that the basis of eigenfunctions derived for λ_1 is still a possibility. From figure 5.1, we see that there is only one possible case we have not checked; an eigenfunction of the form $\varphi_3(y)$ which is independent of x and achieves four zeroes on its period. Since the multiplicity of λ_2 is greater than 3, we have two new possibilities for the basis of eigenfunctions.

Theorem 5.3.1. *The only metric that is extremal for* λ_2 *is the extremal metric for* λ_1 *described in the previous section.*

Proof. Most of the conditions for extremal λ_2 are the same as for λ_1

This first possibility is a basis of the form
$\{\varphi_3(y), \varphi_1(y)\sin(x), \varphi_1(y)\cos(x), \varphi_2(y)\sin(2x)\, \varphi_2(y)\cos(2x)\}$. This is similar to the basis of eigenfunctions for extremal λ_1 except that in this case φ_0 has four zeroes on its period as opposed to φ_0 which has two.

This does not effect any of the derivations in section 5.2.3, and so again the curve $(\varphi_3(t), \varphi_1(t))$ is closed, rotates around the origin, and can be parameterized to have a strictly increasing angle of rotation. This is impossible since φ_3 has four zeroes on its period while $\varphi_2(t)$ has only two.

The second possibility is that λ_2 has multiplicity equal to 4 or 6, and that the basis of eigenfunctions is of the form

$\{\varphi_0(y), \varphi_3(y), \varphi_1(y)\sin(x), \varphi_1(y)\cos(x)\}$,

$\{\varphi_0(y), \varphi_3(y), \varphi_2(y)\sin(2x)\, \varphi_2(y)\cos(2x)\}$,

or

$\{\varphi_0(y), \varphi_3(y), \varphi_1(y)\sin(x), \varphi_1(y)\cos(x), \varphi_2(y)\sin(2x)\, \varphi_2(y)\cos(2x)\}$

Now we have that φ_0 and φ_3 both satisfy the same second order differential equation.

$$\left\{ \begin{array}{c} \varphi_0'' = -\lambda f \varphi_0 \\ \varphi_3'' = -\lambda f \varphi_3 \end{array} \right\}$$

We also have that the Wronskian $= (\varphi_0\varphi_3' - \varphi_0'\varphi_3)$ is constant, since

$(\varphi_0\varphi_3' - \varphi_0'\varphi_3)' = \varphi_0'\varphi_3' + \varphi_0\varphi_3'' - \varphi_0''\varphi_3 - \varphi_0'\varphi_3'$

$= \varphi_0\varphi_3'' - \varphi_0''\varphi_3 = -\lambda f(\varphi_0\varphi_3 - \varphi_0\varphi_3) = 0$

If the Wronskian is equal to zero, then we have that $\varphi_0\varphi_3' = \varphi_0'\varphi_3$. Since $\varphi_n(y)$ and $\varphi_n'(y)$ can not both be equal to zero, we have that $\varphi_3 = 0 \Rightarrow \varphi_0 = 0$ which is imposable since φ_0 has two zeroes on the period while φ_3 has four.

If the Wronskian is equal to some nonzero constant, then the curve $(\varphi_0(t), \varphi_3(t))$ rotates around the origin with a monotone increasing or decreasing angle of rotation θ. Since $(\varphi_0(t), \varphi_3(t))$ must form a closed loop it follows that (φ_0 and φ_3 have the same number of zeros on their period,which is again a contradiction.

$\square$

Conclusion

Numerical experimentation indicates that there is only one extremal metric for λ_1 on the Klein bottle. Further analysis shows that if this is the case, this same metric is the only extremal metric for λ_2 on the Klein bottle when restricted to metrics of revolution. It may be possible to use similar methods to those used in chapter five to deduce extremal metrics for λ_3 and so on, however, Courant's nodal domain theorem gives weaker restrictions on the eigenspace for higher eigenvalues of the Laplacian.

Appendix A

MATLAB Program

The following programs were implemented using MATLAB 7.0.

MATLAB has a pre-packaged set of functions for numerical analysis of ordinary differential equations. In order to use these, the system of equations must be coded as a separate function, which is passed to the o.d.e.-solvers.

– –

FILE: kleineigenpolar

The second order system of ordinary differential equations for φ_0 and φ_1 has been converted to polar coordinates so that the solutions may be parameterized by the angle of rotation θ. This allows for the plotting of one revolution around the origin to determine if the curve $(\varphi_1(t), \varphi_2(t))$ formes a closed loop.

– –

```
function dydt = kleineigenpolar(t,y)

% the system of differential equations for phi_1 and phi_2
% in polar coordinates
% y = [z, psi, z', psi']

dydt = [y(3) ; y(4) ; y(3)*(2*cot(y(2))*y(4)- sin(2*t)*(y(3)^2)/2);...
   2*cot(y(2))*(y(4)^2) + sin(2*y(2))/2 +...
   (y(3)^2)*((sin(2*y(2))*((sin(t))^2 - 4))/2 - sin(2*t)*y(4)/2)];
```

— —

FILE: plot_klein_eig4

The forth and final version of the program.

— —

```
%  This program creates a movie plotting the functions phi_0
% and phi_1 that arise in the system of ode's for eigenfunctions
% of the Laplacian on the Klein Bottle.  The functions have been
% re-parameterized in polar coordinates.  The parameter q in the
% system (determining the initial conditions) is dependant on
% lambda_1. It is varied between 0 and sqrt(3/4) by increments of
% one hundredth of 1/1024.  For extremal lambda_{1}, the solutions
% to the system should rotate around the origin once and form a
% closed curve.


parameter = [sqrt(3)/1536:sqrt(3)/1536:sqrt(3)/2];
%creates time scale
uppbound = numel(parameter);


% The following creates a slidebar that can be used to
% manually control the parameter at any time after the movie
% has finished


slideposition = uicontrol('style','slider','position',...
     [25 190 20 168],'Min',0,'Max',1,'Value',1, 'Callback',...
     'j = plot_klein_eigslide(q, slideposition);');

loop_gap = ones(1, 768);

% For reference, the system is given in terms of
% y = [z, psi, z', psi'], t = theta

% The following loop solves the system for various values of
% the parameter and stores the resulting graphs in M.

for j = 1:uppbound
```

```
   q = parameter(j);
   [t,y] = ode45(@kleineigenpolar,[0 (2*pi)],[0; acos(q);...
       sqrt(1 - q ^2)/(2*q); 0]);
   plot(cos(t).*sin(y(:,2)), sin(t).*sin(y(:,2)))
   axis([-1 1 -1 1])
   text(0.75,0.2,strvcat ('p =', num2str(q)))   %displays p
   set(slideposition,'Value', parameter(j));
   M(j) = getframe;
   loop_gap(j) = cos(t(numel(t)))*sin(y(numel(t),2)) - ...
   cos(t(1)).*sin(y(1,2));

% the loop_gap variable records the distance between the
% beginning and end of the curve after a rotation of 2pi.
% it can be plotted against the variable parameter to
% determine which values of the parameter appear to give
% a closed loop.

end
```

— —

FILE: plot_klein_eigslide

An auxiliary program used to scroll through values of p

— —

```
function new_j = plot_klein_eigslide(old_j, slideposition )

q = get(slideposition, 'Value');

[t,y] = ode45(@kleineigenpolar,[0 (2*pi)],[0; acos(q);...
    sqrt(1 - q ^2)/(2*q); 0]);
plot(cos(t).*sin(y(:,2)), sin(t).*sin(y(:,2)) )
axis([-1 1 -1 1])
text(0.75,0.2,strvcat ('p =', num2str(q)))

new_j = 1024*q;
```

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