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Extremal Problems for Geometric Hypergraphs*

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Abstract. A *geometric hypergraph* H is a collection of i-dimensional simplices, called *hyperedges* or, simply, *edges*, induced by some $(i+1)$-tuples of a *vertex set* V in general position in d-space. The topological structure of geometric *graphs*, i.e., the case $d = 2$, $i = 1$, has been studied extensively, and it proved to be instrumental for the solution of a wide range of problems in combinatorial and computational geometry. They include the k-set problem, proximity questions, bounding the number of incidences between points and lines, designing various efficient graph drawing algorithms, etc. In this paper, we make an attempt to generalize some of these tools to higher dimensions. We will mainly consider extremal problems of the following type. What is the largest number of edges (i-simplices) that a geometric hypergraph of n vertices can have without containing certain *forbidden* configurations? In particular, we discuss the special cases when the forbidden configurations are k intersecting edges, k pairwise intersecting edges, k crossing edges, k pairwise crossing edges, k edges that can be stabbed by an i-flat, etc. Some of our estimates are tight.

1. Introduction

In recent years, the study of graph drawings has become a rich separate discipline within computational geometry. Much of the research has been motivated by applications,

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including software engineering, CAD, database design, cartography, circuit schematics, automatic animation, visual interfaces, etc. (See [27].) It is quite remarkable that classical graph theory proved to be rather powerless to tackle many of the arising problems. Instead, one often had to develop new topological tools to deal with families of curves, i.e., graphs drawn in the plane or in some other surfaces. Perhaps the best-known example is the Lipton–Tarjan Separator Theorem for planar graphs [14], which has many extensions, generalizations, and a broad spectrum of applications ranging from numerical analysis to complexity theory. In particular, it enables us to use the divide-and-conquer paradigm to construct various geometric representations of abstract graphs and networks. Another important example is the following result discovered independently by Ajtai–Chvátal–Newborn–Szemerédi and Leighton. It can be used to obtain, e.g., sharp bounds for the area requirement of graph layouts. Let $\kappa(G)$ denote the *crossing number* of a graph G, i.e., the minimum number of crossing pairs of edges over all planar drawings of G.

Theorem 1.1 [2], [13]. *Let G be a simple graph with n vertices and $e(G)$ edges. If $e(G) \geq 4n$, then $\kappa(G) \geq e(G)^3/100n^2$.*

As Székely [24] pointed out, this result almost immediately implies the Szemerédi–Trotter theorem [25], [26] on the number of incidences between points and lines. His argument is so nice and short that we cannot resist adapting it to establish the following generalization of the Szemerédi–Trotter theorem, which was found by Clarkson–Edelsbrunner–Guibas–Sharir–Welzl and has numerous algorithmic consequences. (For an improved version of Theorem 1.1 and for some other applications of Székely's idea, see [22] and [18], respectively.)

Theorem 1.2 [7]. *The total number of sides of n distinct cells determined by m lines in general position in the plane is at most $O(m^{2/3}n^{2/3} + m)$.*

Proof. Notice that it is sufficient to prove the assertion for a system of cells $\mathcal{C}$, no two of which share an edge. Pick a point p_i in each cell $c_i \in \mathcal{C}$. For any pair (s_i, s_j) of collinear edges belonging to $c_i \in \mathcal{C}$ and $c_j \in \mathcal{C}$, respectively, connect p_i to p_j by a polygonal chain of length three via the midpoints of the segments s_i and s_j, provided that this polygon is not adjacent to any other member of $\mathcal{C}$. The collection of these polygonal chains can be regarded as the edge set of a graph G' whose vertices are $p_1, \ldots, p_n$. If a line is adjacent to k cells in $\mathcal{C}$, then it contributes to exactly $k-1$ edges of G'. Hence, X, the total number of sides of all cells in $\mathcal{C}$, differs from the number of edges of G' by at most m. Removing the multiple edges from G', we obtain a simple graph whose number of edges satisfies $e(G) \geq e(G')/4 \geq (X - m)/4$. In view of the fact that any crossing between two edges of G occurs at a crossing of some pair of lines of the arrangement, Theorem 1.1 implies that either $e(G) < 4n$ or

$$\binom{m}{2} \geq \kappa(G) \geq \frac{e(G)^3}{100n^2}.$$

Since $X \leq 4e(G) + m$, Theorem 1.2 follows. □

Given a set of n points in general position in the plane, join two of them by a line segment if there are exactly k points on one side of the line connecting them. Let G denote the resulting graph. Lovász [15] proved that no straight line can cross more than $2k$ edges of G. Now Theorem 1.1 implies that the number of edges (the number of so-called *k-sets*) is at most $O(k^{1/2}n)$. Indeed, if the number of edges is e, then either we have $e < 4n$ or there exists an edge crossing at least $e^2/(50n^2)$ other edges. Thus, $e^2/(50n^2) \leq 2k$, as required. (See [20] for a slight improvement.) It was shown by Dey and Edelsbrunner [9] that a similar approach can be used to establish an $O(n^{8/3})$ upper bound on the number of *halving planes* in 3-space, which improved some earlier results of [6] and [4].

A graph drawn in the plane by possibly crossing straight-line segments is called a *geometric graph*. More precisely, a geometric graph G consists of a set of points V in general position in the plane and a set of segments E whose endpoints belong to V. As was demonstrated above, for a number of applications it was necessary to solve some extremal problems for geometric graphs. The systematic study of these problems was initiated by P. Erdős, Y. Kupitz [12], and M. Perles. (For a recent survey, see [17].)

It seems plausible that to extend the incidence results to higher dimensions, to improve the upper bound for the number of times the unit distance can occur among n points in 3-space, or to make further progress concerning the k-set problem, one has to find the right generalizations of Theorem 1.1 to systems of surfaces or surface patches in d-space. For simplicity, we will only discuss the case when these surface patches are flat (simplices).

Definition 1.1. A *d-dimensional geometric r-hypergraph H_r^d* is a pair (V, E), where V is a set of points in general position in $\Re^d$, and E is a set of closed $(r-1)$-dimensional simplices induced by some r-tuples of V. The sets V and E are called the *vertex set* and *edge set* of H_r^d, respectively.

Akiyama and Alon [3] proved the following theorem. Let $V = V_1 \cup \cdots \cup V_d$ ($|V_1| = \cdots = |V_d| = n$) be a dn-element set in general position in $\Re^d$, and let E consist of all $(d-1)$-dimensional simplices having exactly one vertex in each V_i. Then E contains n disjoint simplices. Combining this with a result of Erdős [10], we obtain a nontrivial upper bound for the number of edges of a d-dimensional geometric d-hypergraph of n vertices that contains no k pairwise disjoint edges.

If we want to exclude *crossings* rather than disjoint edges, or want to generalize Theorem 1.1 to geometric hypergraphs, we face the following problem. Even if we restrict our attention to systems of triangles induced by three-dimensional point sets in general position, it is not completely clear how a "crossing" should be defined, let alone the notion of "crossing number." If two segments cross, they do not share an endpoint. Should this remain true for triangles? We have to clarify the terminology.

Definition 1.2. Two simplices are said to have a *nontrivial intersection,* if their relative interiors have a point in common. If, in addition, the two simplices are vertex disjoint, then they are said to *cross*.

More generally, k simplices are said to have a *nontrivial intersection,* if their relative interiors have a point in common. If, in addition, all simplices are vertex disjoint, then they are said to *cross*.

Consider k simplices. It is important to note that the fact that *every pair* of them has a nontrivial intersection does not imply that *all* of them do. To emphasize that this stronger condition is satisfied, we often say that the simplices have a *nontrivial intersection in the strong sense*, or simply that they *strongly intersect*. Similarly, a set of *pairwise* crossing simplices is not necessarily *crossing*. If want to emphasize that they *all* cross, we will say that they *cross in the strong sense*, or shortly that they *strongly cross*.

As we pick more and more distinct $(r-1)$-dimensional simplices induced by a set of n points in $\Re^d$, the number of crossings between them will usually increase. The aim of this paper is to generalize the planar results to obtain some information about the growth rate of this process. In the inverse formulation, one can ask for the maximum number of edges that a d-dimensional geometric r-hypergraph H_r^d of n vertices can have without containing some fixed crossing pattern. Throughout this paper, let $f_r^d(\mathcal{F}, n)$ denote this maximum, where $\mathcal{F}$ is the family of *forbidden* configurations, i.e., forbidden geometric subhypergraphs. Most of our bounds will be asymptotic: d and r are thought to be fixed, while n tends to infinity.

In the next two sections, we estimate $f_r^d(\mathcal{F}, n)$ for various families $\mathcal{F}$. In Section 4, we generalize Theorem 1.1. Finally, we discuss some related questions and give a few applications of our results.

2. Full-Dimensional Simplices

Let $\mathcal{I}_k^r$ (resp. $\mathcal{SI}_k^r$) denote the class of all geometric hypergraphs consisting of k $(r-1)$-dimensional simplices, any two of which have a nontrivial intersection (resp. all of which are strongly intersecting). Similarly, let $\mathcal{C}_k^r$ (resp. $\mathcal{SC}_k^r$) denote the class of all geometric hypergraphs consisting of k pairwise crossing (resp. strongly crossing) $(r-1)$-simplices.

Theorem 2.1. *For any fixed $k > 1$, one can select at most $O(n^{\lceil d/2 \rceil})$ d-dimensional simplices induced by n points in d-space with the property that no k of them share a common interior point. This bound cannot be improved. That is,*

$$f_{d+1}^d(\mathcal{I}_k^{d+1}, n) = \Theta(n^{\lceil d/2 \rceil}), \qquad f_{d+1}^d(\mathcal{SI}_k^{d+1}, n) = \Theta(n^{\lceil d/2 \rceil}).$$

Proof. Clearly, we have

$$\Omega(n^{\lceil d/2 \rceil}) \le f_{d+1}^d(\mathcal{I}_k^{d+1}, n) \le f_{d+1}^d(\mathcal{SI}_k^{d+1}, n),$$

where the first inequality follows from the fact that there are triangulations of size $\Omega(n^{\lceil d/2 \rceil})$ with n vertices in $\Re^d$. Consider, e.g., the vertical projection of the lower part of any cyclic polytope of n vertices in $\Re^{d+1}$.

To see that $f_{d+1}^d(\mathcal{SI}_k^{d+1}, n) \le O(n^{\lceil d/2 \rceil})$, we set up a charging scheme. Let us regard $\Re^{d-1}$ as the coordinate hyperplane in $\Re^d$ spanned by the first $d-1$ axes, and let X_d denote the last coordinate axis. Suppose that X_d is vertical. Fix a geometric hypergraph

$H^d_{d+1} = (V, E)$ which has no k edges with a common interior point and whose n vertices are in general position. For any ℓ-dimensional simplex Δ induced by V, where $\ell \le \lfloor (d-1)/2 \rfloor$, let $E_\Delta \subseteq E$ denote the set of all edges of H^d_{d+1} that contain Δ on their boundaries. It follows from the condition on H^d_{d+1} that the infinite vertical cylinder $\Delta + X_d$ based on Δ intersects the interior of at most $2(k-1)$ elements of E_Δ. Let us charge Δ one unit for each of these edges. Since the total number of ℓ-simplices with $\ell \le \lfloor (d-1)/2 \rfloor$ is at most $\lceil d/2 \rceil n^{\lceil d/2 \rceil}$, it remains to show that every edge $e \in E$ has been charged for. Indeed, by Radon's theorem [23], the vertex set of the orthogonal projection of e into $\Re^{d-1}$ can be partitioned into two parts, S_1 and S_2, such that their convex hulls cross each other and $|S_1| + |S_2| = d+1$. Suppose without loss of generality that $|S_1| \le \lfloor (d+1)/2 \rfloor$. Then the convex hull of S_1 is an ℓ-dimensional simplex Δ_1 for some $\ell \le \lfloor (d-1)/2 \rfloor$, and we had to charge Δ_1 for e. $\square$

Theorem 2.2. *Let E be any set of d-dimensional simplices induced by an n-element point set $V \subseteq \Re^d$. If E has no two crossing elements, then $|E| = O(n^d)$, and this bound is asymptotically tight. In notation,*

$$f^d_{d+1}(\mathcal{C}^{d+1}_2, n) = \Theta(n^d).$$

Proof. To prove the lower bound, consider a geometric hypergraph consisting of all d-dimensional simplices induced by V that contain a given vertex $v \in V$.

Next we establish the upper bound. If E has no two simplices having a nontrivial intersection, then $|E| \le O(n^{\lceil d/2 \rceil})$, by the previous theorem. Otherwise, choose two d-simplices $\Delta_1, \Delta_2 \in E$ whose intersection is nontrivial. It is easy to show (see, e.g., [8] and [9]) that there exist an ℓ_1-face Δ'_1 of Δ_1 and an ℓ_2-face Δ'_2 of Δ_2 with $\ell_1 + \ell_2 = d$ such that Δ'_1 and Δ'_2 are crossing.

Assume first that there is an edge $e \in E$ which is vertex disjoint from Δ'_1 and contains Δ'_2 as a face. Then every edge $f \in E$ that contains Δ'_1 as a face must share at least one vertex with e. The number of such simplices f is at most $(d+1)\binom{n}{d-\ell_1-1}$. Let us remove all of them from E.

In the second case, every edge $e \in E$ that contains Δ'_2 as a face shares a vertex with Δ'_1. Obviously, the number of such simplices e is at most $(\ell_1 + 1)\binom{n}{d-\ell_2-1}$. Remove all of them from E.

We continue this procedure until there remain no nontrivial intersections in E. At this point, E has at most $O(n^{\lceil d/2 \rceil})$ elements, and the total number of simplices that have been removed is at most

$$\binom{n}{\ell_1+1}(d+1)\binom{n}{d-\ell_1-1} + \binom{n}{\ell_2+1}(\ell_1+1)\binom{n}{d-\ell_2-1} = O(n^d). \quad \square$$

3. $(d-1)$-Simplices in d-Space

Theorem 3.1. *Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \Re^d$ such that E has no k members with pairwise nontrivial* *intersections* $(d, k > 1)$. *Then, for* $k = 2$ *and* 3, *we have* $|E| = O(n^{d-1})$. *Otherwise,* $|E| = O(n^{d-1}\log^{2k-6} n)$. *In notation,*

$$f_d^d(\mathcal{I}_k^d, n) = \begin{cases} O(n^{d-1}) & \text{if } k = 2, 3; \\ O(n^{d-1}\log^{2k-6} n) & \text{otherwise.} \end{cases}$$

This result is asymptotically tight if $d, k \leq 3$.

Proof. For $d = 2$, the assertion is true, by the results of [19] and [1]. Assume that $d \geq 3$. For any $(d-3)$-simplex Δ induced by V, let E_Δ denote the family of all members of E that contain Δ as a face. Pick any point p_Δ in the relative interior of Δ, and let F_Δ denote the three-dimensional flat orthogonal to Δ and passing through p_Δ.

Every $e \in E_\Delta$ meets F_Δ in a polygon, whose two sides incident to p_Δ are the intersections of F_Δ with the two $(d-2)$-faces of e containing Δ. Thus, the total number of sides incident to p_Δ that occur in some $e \cap F_\Delta$ $(e \in E_\Delta)$ is at most $n - d + 2 < n$. Take a small two-dimensional sphere $S^2 \subseteq F_\Delta$ centered at p_Δ. The intersections of S^2 with the elements of E_Δ form the edge set of a graph with at most n vertices. It follows from the properties of E that this graph has no k pairwise crossing edges, so, by the planar results, its number of edges, $|E_\Delta|$, satisfies

$$|E_\Delta| = \begin{cases} O(n) & \text{if } k = 2, 3; \\ O(n\log^{2k-6} n) & \text{otherwise.} \end{cases}$$

Summing over all $(d-3)$-simplices Δ induced by V, we obtain $\binom{d}{2}|E| = \sum_\Delta |E_\Delta|$, and hence the upper bound.

To show that the result is tight for $d = 3$, $k = 2$, consider a nested sequence of $n/2$ pyramids based on the same two-dimensional convex $n/2$-gon. These pyramids have a total of $n^2/4$ triangular faces, no two of which have a nontrivial intersection. □

It is an outstanding open problem to decide whether the order of magnitude of the above bound can be improved, e.g., for $d = 4$, $k = 2$. However, modifying the procedure described in the proof Theorem 2.2, one can show that the following related result is asymptotically tight. The details are left to the reader.

Theorem 3.2. *Let* E *be a family of* $(d-1)$*-dimensional simplices induced by an* n*-element point set* $V \subseteq \Re^d$. *If* E *has no two crossing members, then* $|E| = O(n^{d-1})$, *and this bound cannot be improved. In notation,*

$$f_d^d(\mathcal{C}_2^d, n) = \Theta(n^{d-1}).$$

The results in the next section enable us to establish the following generalization of Theorem 3.2.

Theorem 3.3. *Let* E *be a family of* $(d-1)$*-dimensional simplices induced by an* n*-element point set* $V \subseteq \Re^d$, *where* $d, k > 1$. *If* E *has no* k *pairwise crossing members, then* $|E| = O(n^{d-(1/d)^{k-2}})$. *In notation,*

$$f_d^d(\mathcal{C}_k^d, n) = O(n^{d-(1/d)^{k-2}}).$$

4. *k*-Tuples of Strongly Crossing Simplices

Given any d-dimensional geometric r-hypergraph $H_r^d = (V, E)$ with n vertices, let $x_k(H_r^d)$ denote the number of *strongly crossing* k-tuples of edges. Using our notations, $|E| > f_r^d(\mathcal{SC}_k^r, n)$ obviously implies that $x_k(H_r^d) > 0$. Define

$$x_{k,r}^d(n, e) = \min_{H_r^d} x_k(H_r^d),$$

where the minimum is taken over all $H_r^d = (V, E)$ with $|V| = n$ vertices and $|E| = e$ edges (simplices).

The following theorem provides us with a recipe of how to give a lower bound on the *number* of crossing k-tuples of edges, if we know how many edges are necessary to guarantee the existence of *one* such k-tuple. This result generalizes Theorem 1.1. (See also [8], [9], [16], and [21], for related problems and results.)

Theorem 4.1. *Assume that* $f_r^d(\mathcal{SC}_k^r, n) < c_1\binom{n}{r}/n^\delta$ *and that* $e \geq (c_1 + 1)\binom{n}{r}/n^\delta$ *for suitable constants* c_1 *and* $0 \leq \delta \leq 1$. *Then there exists* $c_2 > 0$ *such that*

$$x_{k,r}^d(n, e) > c_2\binom{n}{kr}e^\gamma \Big/ \binom{n}{r}^\gamma,$$

where $\gamma = 1 + (k-1)r/\delta$.

Proof. By induction on n. Let $H = H_r^d = (V, E)$ be a d-dimensional geometric r-hypergraph with n vertices and e edges. Suppose further that H has the smallest possible number of crossing k-tuples, i.e., $x_k(H) = x_{k,r}^d(n, e) = x(n, e)$. We can also assume that $n > kr$, for otherwise the assertion is trivial.

First we consider the range $(c_1+1)\binom{n}{r}/n^\delta \leq e \leq (c_1+r+1)\binom{n}{r}/n^\delta$. It follows from the assumptions that if H has more than $c_1\binom{n}{r}/n^\delta$ edges, then any additional edge will participate in a new k-tuple of strongly crossing edges. Hence, the number of crossing k-tuples is at least

$$e - c_1\frac{\binom{n}{r}}{n^\delta} \geq \frac{\binom{n}{r}}{n^\delta} > c\binom{n}{kr}\frac{e^\gamma}{\binom{n}{r}^\gamma},$$

as long as c is sufficiently small.

Next we assume that $e > (c_1+r+1)\binom{n}{r}/n^\delta$. For any point $p \in V$, let H_p denote the geometric hypergraph obtained from H by removing p together with all edges ($(r-1)$-simplices) that contain p as a vertex. The number of edges of H_p is denoted by e_p. If we sum over all p the number of crossing k-tuples of edges in H_p, then every crossing k-tuple of H will be counted exactly $n - kr$ times. Therefore,

$$(n-kr)\cdot x(n, e) = (n-kr)\cdot x_k(H) = \sum_{p\in V} x_k(H_p) \geq \sum_{p\in V} x(n-1, e_p).$$

Note that $\binom{n-1}{r-1} \leq r\binom{n}{r}/n^\delta$, because $\delta \leq 1$. Since p can be a vertex of at most $\binom{n-1}{r-1}$ edges of H, we have

$$e_p > (c_1+r+1)\frac{\binom{n}{r}}{n^\delta} - \binom{n-1}{r-1} \geq (c_1+1)\frac{\binom{n}{r}}{n^\delta} \geq (c_1+1)\frac{\binom{n-1}{r}}{(n-1)^\delta}.$$

Thus, we can apply the induction hypothesis to every H_p to obtain

$$(n - kr) \cdot x(n, e) > c \frac{\binom{n-1}{kr}}{\binom{n-1}{r}^{\gamma}} \sum_{p \in V} e_p^{\gamma}.$$

Obviously, $\sum_{p \in V} e_p = (n - r)e$. Using the fact that $\gamma > 1$, by Jensen's inequality we have $\sum_{p \in V} e_p^{\gamma} \geq n((n - r)e/n)^{\gamma}$. This finally yields

$$x_{k,r}^d(n, e) = x(n, e) > c \cdot \frac{n\binom{n-1}{kr}}{n - kr} \cdot \frac{(\frac{n-r}{n})^{\gamma}}{\binom{n-1}{r}^{\gamma}} \cdot e^{\gamma} = c\binom{n}{kr}\frac{e^{\gamma}}{\binom{n}{r}^{\gamma}}. \qquad \square$$

Now we are ready to apply the last—somewhat technical—result.

Theorem 4.2. *Let* $x_{2,d}^d(n, e)$ *denote the minimum number of crossing pairs in any e-element set of* $(d - 1)$*-simplices induced by n points in* $\Re^d$ *in general position. Then, for every n and* $e \geq cn^{d-1}$*, we have*

$$c_1 \frac{e^{d+1}}{n^{d(d-1)}} < x_{2,d}^d(n, e) < c_2 \frac{e^{2+1/\lfloor d/2 \rfloor}}{n^{d/\lfloor d/2 \rfloor}}.$$

Proof. By Theorem 3.2, we can apply Theorem 4.1 with $k = 2$, $r = d$, and $\delta = 1$, and the lower bound immediately follows.

To prove the upper bound, we exhibit a geometric hypergraph $H_d^d = (V, E)$ with n vertices and e edges, in which every $\lfloor d/2 \rfloor$-dimensional face of every edge $\Delta \in E$ crosses at most $c' \cdot e^{1+1/\lfloor d/2 \rfloor}/n^{d/\lfloor d/2 \rfloor}$ other elements of E. This is indeed sufficient, because there are only $O(e)$ such faces, and if $\Delta_1, \Delta_2 \in E$ cross, then one of them will always be crossed by a $\lfloor d/2 \rfloor$-dimensional face of the other.

Consider two points p and p' of the d-dimensional *moment curve* $M(t) = (t, t^2, \ldots, t^d)$, $t \in \Re$. We say that *p precedes* p' (in notation, $p \prec p'$) if $p = M(t)$, $p' = M(t')$ for some $t < t'$. The following elementary properties can be easily deduced from the fact that every hyperplane intersects the moment curve in at most d points. To simplify the presentation, let $u = \lfloor d/2 \rfloor$ and $v = \lceil d/2 \rceil$.

Claim 1. *Let* $p_1 \prec p_2 \prec \cdots \prec p_{u+1}$ *and* $q_1 \prec q_2 \prec \cdots \prec q_{v+1}$ *be distinct points on the moment curve, which form a u-simplex* σ *and a v-simplex* τ*, respectively. Then* σ *and* τ *cross each other if and only if the points* p_i *and* q_j *interleave, i.e., every interval* $q_j q_{j+1}$ *contains exactly one* p_i.

Claim 2. *Let* σ *and* Δ *be a u-dimensional simplex and a* $(d-1)$*-dimensional simplex, respectively, all of whose vertices are on the moment curve. If* σ *and* Δ *are crossing, then* σ *must also cross some v-dimensional face of* Δ.

We only prove the second claim. Color the vertices of σ and Δ by red and blue, respectively. If σ does not cross any v-face of Δ, then by Claim 1 there is no sequence of length $d + 2$ with alternating colors. Thus, the set of all vertices can be partitioned into at most $d+1$ monochromatic intervals. This implies that the red and blue points (and their convex hulls) can be separated by a hyperplane passing through any d points of the moment curve separating the monochromatic intervals. This proves Claim 2. (Note that the statement does not remain true if we drop the assumption that all vertices of the simplices are taken from the moment curve.)

Let us define a geometric hypergraph H_d^d on the vertex set $V = \{p_i = M(i) \mid 1 \le i \le n\}$, as follows. Choose d points $p_{i_1} \prec p_{i_2} \prec \cdots \prec p_{i_d}$ from V such that the first v points $p_{i_1} \prec p_{i_2} \prec \cdots \prec p_{i_v}$ are selected arbitrarily, and all other gaps $i_{v+1} - i_v, i_{v+2} - i_{v+1}, \ldots, i_d - i_{d-1} \le k$. Set $k = \Theta(e^{1/u}/n^{v/u})$, so that the total number of such sequences is roughly $e = \Theta(n^v k^u)$. Let E, the edge set of H_d^d, be defined as the collection of all $(d-1)$-simplices induced by these sequences.

We have to show that any u-face $\sigma = p_{l_1} p_{l_2} \cdots p_{l_{u+1}}$ $(l_1 < l_2 < \cdots < l_{u+1})$ of any edge of H_d^d crosses at most $O(e^{1+1/u}/n^{d/u})$ other edges $\Delta \in E$. By Claim 2, such an edge Δ must have a v-face $\tau = p_{m_1} p_{m_2} \cdots p_{m_{v+1}}$ $(m_1 < m_2 < \cdots < m_{v+1})$ which is crossed by σ. By Claim 1, the l's and m's must interleave. It follows from the way how Δ was created that at least one of the gaps between two consecutive m's is at most k. Consequently, any $\Delta \in E$ that crosses σ must have two consecutive vertices whose distance is at most k and the interval determined by them contains a vertex of σ. This immediately implies that for each vertex $p_l \in \sigma$, there are at most $n^{v-1}k^{u+1}$ edges that intersect σ and contain p_l in one of their intervals of size at most k. Since σ has only $u+1$ vertices, we obtain that the number of edges of H_d^d crossing σ cannot exceed $O(n^{v-1}k^{u+1}) = O(ke/n) = O(e^{1+1/u}/n^{d/u})$, as desired. □

Note that the upper bound in Theorem 4.2 beats the bound $c_2(e^{2d-1}/n^d)^{1/(d-1)}$ obtained by taking n/ν disjoint copies of the complete d-uniform hypergraph on (roughly) ν vertices, where $\nu = (e/n)^{1/(d-1)}$. Actually, the two bounds are of the same order of magnitude when the hypergraph is *dense*, i.e., $e = \Omega(n^d)$.

Now we are in a position to establish Theorem 3.3 by induction on k. According to Theorem 3.2, the statement is true for $k=2$. It follows from Theorem 4.2 that if H_d^d has n vertices and $e > cn^{d-1}$ edges, then it has an edge Δ crossing at least $c_1 e^d/n^{d(d-1)}$ other edges. We may assume that there are no $k-1$ pairwise crossing elements among the edges that cross Δ, for otherwise H_d^d would contain k pairwise crossing edges. Therefore, we can use the induction hypothesis for $k-1$ to conclude that $c_1 e^d/n^{d(d-1)} = O(n^{d-(1/d)^{k-3}})$, whence $e = O(n^{d-(1/d)^{k-2}})$.

5. Related Problems

Lower-Dimensional Simplices. So far, most of our results concerned full-dimensional simplices or $(d-1)$-simplices in $\Re^d$. Now we apply a theorem of Vrećica and Živaljević [28] to deduce a result on geometric hypergraphs with lower-dimensional edges.

Consider a *complete* r*-partite geometric hypergraph* K_r^d, whose vertex set is the union of r disjoint sets $V_1, V_2, \ldots, V_r$ of size ℓ each, and whose edge set consists of all $(r-1)$-dimensional simplices that have a vertex in each V_i. Generalizing a result of [29], Vrećica and Živaljević [28] have shown that if $\ell \ge 2p-1$ for some prime $p \le d/(d-r+1)$, then K_r^d contains p strongly crossing edges. Note that for $r < \lceil d/2 \rceil + 1$, we have $p \leq 1$, which is impossible.

Proposition 5.1. *Let $f_r^d(\mathcal{SC}_k^r, n)$ denote the maximum number of edges that a d-dimensional geometric r-hypergraph of n vertices can have without containing k strongly crossing edges. Suppose that there is a prime p such that $k \leq p \leq d/(d-r+1)$. Then*

$$\Omega(n^{r-1}) \leq f_r^d(\mathcal{SC}_k^r, n) \leq O(n^{r-\delta}),$$

where $\delta = 1/(2p-1)^{r-1}$.

Proof. By the above-mentioned result, if H_r^d does not have k strongly crossing edges, then it cannot contain a complete r-partite subhypergraph with more than $2p-1$ vertices in each of its classes. A result of Erdős [10] now implies that $|E| < n^{r-1/(2p-1)^{r-1}}$. The lower bound follows by taking an r-hypergraph consisting of all $\binom{n-1}{r-1}$ simplices containing a fixed vertex. $\square$

We conjecture that the lower bound in Proposition 5.1 is asymptotically tight.

Crossing Many Simplices by a Flat. In [16], we posed the following question. What is the largest number $g = g_{k,r}^d(n, e)$ such that for any d-dimensional geometric r-hypergraph with n vertices and e edges, one can find a k-flat crossing at least g edges?

Applying the results of the previous sections, we can obtain the following two bounds.

Proposition 5.2. $g_{k,d}^d(n, e) = \Omega(e^d/n^{d(d-1)})$ *for $k \geq \lfloor d/2 \rfloor$.*

Proof. Let H_d^d be a d-dimensional geometric d-hypergraph with n vertices and $e > cn^{d-1}$ edges. By Theorem 4.2, $x_{2,d}^d(n, e) > c_1 \cdot e^{d+1}/n^{d(d-1)}$. Let $\{\sigma_1, \sigma_2\}$ be a crossing pair of edges in H_d^d. Then either σ_1 or σ_2 has a $\lfloor d/2 \rfloor$-face σ that crosses the other edge [8]. Therefore, at least one edge of H_d^d has a $\lfloor d/2 \rfloor$-face crossing at least $x_{2,d}^d(n, e)/(\binom{d}{\lfloor d/2 \rfloor} e) = \Omega(e^d/n^{d(d-1)})$ other edges. $\square$

Proposition 5.3. $g_{k,r}^d(n, e) = \Omega(e^{3^{(r-1)}r}/n^{r(3^{(r-1)}r-1)})$ *for $\lfloor d/2 \rfloor \leq k \leq r$, $\lceil d/2 \rceil + 1 \leq r$.*

Proof. By Proposition 5.1, $f_r^d(\mathcal{C}_2^r, n) = O(n^{r-\delta})$ holds with $\delta = (1/3)^{r-1}$. Plugging this into Theorem 4.1, we obtain $x_{2,r}^d(n, e) > c\binom{n}{2r}e^\gamma/\binom{n}{r}^\gamma$, where $\gamma = 1 + 3^{r-1}r$. Just like in the proof of the previous proposition, we can argue that there exists a $\lfloor d/2 \rfloor$-simplex that crosses at least $\Omega(x_{2,r}^d(n, e)/e)$ simplices. $\square$

Ramsey-Type Questions. Let us color with two colors all $(r-1)$-dimensional simplices induced by n points in general position in $\Re^d$. Is it true that one of the color classes necessarily contains certain special subconfigurations, provided that n is large enough? If the answer is in the affirmative, then we can ask for the smallest n for which this will occur. A variety of questions of this type are discussed in [11], in the planar case. Some of the results can be generalized to higher dimensions.

Theorem 5.1. *Let us color with two colors all $(d-1)$-dimensional simplices induced by $(d+1)n-1$ points in general position in $\Re^d$. Then one can always find n disjoint simplices of the same color. This result cannot be improved.*

Proof. Let P be a set of $(d+1)n-1$ points in general position in $\Re^d$, and fix a 2-coloring of all $(d-1)$-simplices induced by P. An i-element subset of P is called an *i-set* if it can be obtained by intersecting P with an open half-space. It is easy to see that any two i-sets Q and Q' can be connected by a chain of i-sets, $Q = Q_0, Q_1, Q_2, \ldots, Q_s = Q'$, whose any two consecutive members have symmetric difference 2. Indeed, let H and H' be two oriented hyperplanes, each passing through precisely one point p (resp. p') $\in P$, such that the intersection of P with the open positive half-spaces determined by them is Q and Q', respectively. We can assume without loss of generality that $H \cap H'$ is a $(d-2)$-flat F and that every $(d-2)$-flat parallel to F contains at most one point of P. Rotate H in the clockwise direction around the $(d-2)$-flat through p parallel to F until it hits another point $q \in P$, and then continue the rotation around the $(d-2)$-flat through q parallel to F, etc. During this procedure, whenever H passes through precisely one point of P, the points lying on its positive side form a new i-set Q_k.

Assume that we have already established the theorem for every integer smaller than n, and that n is even. Consider a oriented hyperplane H passing through precisely one point of P and dividing the remaining points into two equal halves Q and Q', where Q lies in the positive half-space bounded by H. By the induction hypothesis, one can find $n/2$ pairwise disjoint monochromatic $(d-1)$-simplices both in Q and in Q'. If they are of the same color, we are done. So we can assume that they are of different colors, say red and blue. Connect Q and Q' by a chain of $(d+1)n/2-1$-sets, as described above, and let $H = H_0, H_1, H_2, \ldots, H_s = -H$ denote some corresponding positions of the rotating (oriented) hyperplane. Then we can find a t such that Q_t has $n/2$ disjoint red simplices but Q_{t+1} does not, i.e., it has $n/2$ disjoint blue simplices. Notice that now the $((d+1)n/2-1)$-sets lying in the negative half-spaces bounded by H_t and H_{t+1} must be identical. This set also contains a family of $n/2$ disjoint monochromatic simplices, which can be augmented either by the red simplices of Q_t or by the blue simplices of Q_{t+1} to a family of n disjoint simplices of the same color. The case when n is odd is somewhat more complicated, but it can be treated by a slight modification of the planar proof in [11]. Note, however, that the above argument already shows that the theorem is true for every n that is a power of 2.

To show that the result is best possible, let $P = P_1 \cup P_2$, where $|P_1| = dn-1$ and $|P_2| = n-1$. Color all $(d-1)$-simplices in P_1 red and all other $(d-1)$-simplices in P blue. Obviously, there are no n pairwise disjoint monochromatic simplices. □

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