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Extremal Properties of 0/1-Polytopes*

U. H. Kortenkamp,[1] J. Richter-Gebert,[1] A. Sarangarajan,[2†] and G. M. Ziegler[1]

[1]Department of Mathematics MA 6-1, Technische Universität Berlin,
Strasse des 17. Juni 136, 10623 Berlin, Germany
{hund, richter, ziegler}@math.tu-berlin.de

[2]Combinatorial Optimization, ZIB Berlin,
Heilbronner Strasse 10, 10711 Berlin, Germany
sarangarajan@zib-berlin.de

Abstract. We provide lower and upper bounds for the maximal number of facets of a d-dimensional 0/1-polytope, and for the maximal number of vertices that can appear in a two-dimensional projection ("shadow") of such a polytope.

1. Introduction

The combinatorics of 0/1-polytopes is at the core of many investigations of combinatorial optimization. In fact, the field of "polyhedral combinatorics" is concerned with classes of facets and other combinatorial structures of "special" 0/1-polytopes that are given as the convex hulls of the characteristic vectors of solutions of certain problem classes. In particular—just to mention one well-studied classical case—quite a lot is known about the facet structures of traveling salesman polytopes: see [7].

Much less is known about "general" 0/1-polytopes. However, it seems that the "special" polytopes of combinatorial optimization cannot be much simpler: so Billera and Sarangarajan [3] have recently demonstrated that in the very special class of asymmetric traveling salesman polytopes every 0/1-polytope is encountered as a face.

In the following, we discuss two classes of extremal problems for general 0/1-polytopes that arise from complexity considerations in combinatorial optimization.

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† Current address: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003, USA. sarang@lsa.math.umich.edu.

1.1. *The Maximal Number of Facets*

The first section of the Grötschel and Padberg chapter on “Polyhedral Computations” for the traveling salesman problem [7] is titled “1.1: The number of facets of TSP polytopes and algorithmic implications.” Grötschel and Padberg note that traveling salesman polytopes have “many” facets. To get a better notion of “many,” estimates on the numbers of facets of general 0/1-polytopes are required. Grötschel and Padberg use a very crude upper bound, namely, that a d-dimensional 0/1-polytope cannot have more than

$$f(d) \leq \binom{2^d}{d} \approx 2^{d^2}$$

facets, since every facet is determined by a set of d vertices. A much better bound was given by Bárány [13, Problem 0.15*]: $f(d) \leq d! + 2d$. Below—in Section 2—we slightly improve Bárány’s bound to

$$f(d) \leq d! - (d-1)! + 2(d-1)$$

for $d \geq 3$.

Still, all the lower bounds we can offer are singly exponential. While $f(d) \geq 2^d$ is easy to see (from the cross polytopes realized as 0/1-polytopes), we obtain

$$f(d) \geq (2.76)^d$$

for all sufficiently large d.

So, what does “many facets” mean? We take the (symmetric) traveling salesman polytopes Q_T^n as our benchmark, a polytope of dimension $d = n(n-3)/2$ with $(n-1)!/2$ vertices. For $n = 8$ this is a 20-dimensional polytope with 194,187 $\approx (1.8383)^{20}$ facets [5], while we can construct a polytope T_{20} of dimension 20 with as many as

$$f(20) \geq 690{,}953{,}796 \approx (2.76)^{20}$$

facets. Still, the upper bound we have gives

$$f(20) \leq 2{,}311{,}256{,}907{,}767{,}808{,}038 \approx (8.2)^{20}.$$

Similarly, in the case of 120 city problems, the TSP polytope Q_T^{120} has dimension $d = 7020$. The number of facets of this polytope is not known; Grötschel and Padberg note that a class of more than $2\cdot 10^{179} \approx (1.0606)^d$ facets (“comb constraints”) is known. At the same time, we can construct a 0/1-polytope T_{7020} of the same dimension that has more than $6\cdot 10^{3101} \approx (2.76)^d$ facets.

1.2. *The Size of a Two-Dimensional Shadow*

For any class of polytopes $\mathcal{P}$ we have the following quantities:

$M(\mathcal{P})$: The maximal number of vertices.

$H(\mathcal{P})$: The maximal number of vertices on a path that is strictly increasing with respect to a linear function (an *increasing* path).

$H_{\mathrm{sh}}(\mathcal{P})$: The maximal number of vertices on a two-dimensional projection ("shadow").

For the class $\mathcal{P}_d$ of all d-dimensional 0/1-polytopes we have

$$\tfrac{1}{2}H_{\mathrm{sh}}(\mathcal{P}_d) + 1 \leq H(\mathcal{P}_d) \leq M(\mathcal{P}_d) = 2^d.$$

(For the class $\mathcal{P}(d, n)$ of d-dimensional polytopes with at most n facets the corresponding hierarchy was analyzed in [1].)

In Section 3 we give exponential (lower and upper) bounds for the quantity $H_{\mathrm{sh}}(\mathcal{P}_d)$. The motivation for this study comes from linear programming. The number of nondegenerate pivots that the simplex algorithm with the shadow boundary (or Gass–Saaty) pivot rule [4] can take on a 0/1 problem is bounded by $\lceil \frac{1}{2}H_{\mathrm{sh}}(\mathcal{P}_d)\rceil$ from below and $H_{\mathrm{sh}}(\mathcal{P}_d) - 1$ from above. This is one less than the maximal number of different basic solutions (i.e., vertices of the polytope) that the algorithm may visit. (However, since 0/1-polytopes are in general very degenerate, this does not bound the maximal number of pivots, or of basic solutions encountered.)

Is there any polynomial augmentation method on 0/1-polytopes? This is of interest since edge paths of polynomial length can be constructed from any augmentation oracle (i.e., a subroutine that provides a "better" vertex for any nonoptimal input, as in [11]) that would output only augmentation vectors that correspond to edges. Is there *any* strategy that on a 0/1-polytope would need only a polynomial number of nondegenerate pivots?

2. The Maximal Number of Facets

Let $f(d)$ be the largest number of facets of a d-dimensional 0/1-polytope. It is easy to see that it is sufficient to consider d-dimensional 0/1-polytopes that are subsets of $\mathbb{R}^d$. We call a 0/1-polytope $P \subseteq \mathbb{R}^d$ *centered* if $(\frac{1}{2}, \dots, \frac{1}{2})$ is in its interior. Let $f'(d)$ be the largest number of facets of a centered d-dimensional 0/1-polytope; we have $f'(d) \leq f(d)$ for all d, by definition. For small dimensions we have the following values (derived below):

$$\begin{aligned} f'(d) = f(d) = \ & 2^d \qquad \text{for} \quad d \leq 4,\\ 40 \leq f'(5) \leq f(5) \leq \ & 104,\\ 121 \leq f'(6) \leq f(6) \leq \ & 610. \end{aligned}$$

We use the following "direct sum" construction.

Proposition 2.1. *For* $i \in \{1, 2\}$ *let* $P_i = \mathrm{conv}(V_i) \subseteq \mathbb{R}^{d_i}$ *be* d_i*-dimensional centered* 0/1*-polytopes. Then there is a centered* $(d_1 + d_2)$*-dimensional* 0/1*-polytope, denoted*

$$P_1 \oplus P_2 := \mathrm{conv}(V_1) \oplus \mathrm{conv}(V_2) \subseteq \mathbb{R}^{d_1+d_2},$$

called the direct sum *of* V_1 *and* V_2, *that has*

$$f_{d_1+d_2-1}(P_1 \oplus P_2) = f_{d_1-1}(P_1) \cdot f_{d_2-1}(P_2)$$

facets.

Proof. We use the embedded 0/1-cubes

$$\operatorname{conv}(\{x \in \{0,1\}^{d_1+d_2} \colon x_1 = x_2 = \cdots = x_{d_1} = x_{d_1+1}\}) =: C'_{d_2} \cong C_{d_2},$$
$$\operatorname{conv}(\{x \in \{0,1\}^{d_1+d_2} \colon 1 - x_{d_1} = x_{d_1+1} = \cdots = x_{d_1+d_2}\}) =: C'_{d_1} \cong C_{d_1}$$

in the $(d_1 + d_2)$-dimensional 0/1-cube that are positioned in two orthogonal affine subspaces of $\mathbb{R}^{d_1+d_2}$ which intersect in $(\frac{1}{2}, \dots, \frac{1}{2})$. Lifting P_1 and P_2 to 0/1-subpolytopes of C'_{d_1} (resp. C'_{d_2}) we obtain the usual "free sum" construction for polytopes (see [8] and [9]) as a construction for centered 0/1-polytopes. □

Starting from $C_1 = [0,1] \subseteq \mathbb{R}$ and $f'(1) = f(1) = 2$ we thus obtain a d-dimensional 0/1-polytope

$$C_d^{\Delta\prime} := C_1 \oplus C_1 \oplus \cdots \oplus C_1$$

with 2^d facets that realizes the d-dimensional cross polytope as a 0/1-polytope:

$$C_d^{\Delta} \cong \operatorname{conv}(\{e_1, \dots, e_d, \mathbf{1} - e_1, \dots, \mathbf{1} - e_d\})$$
$$= \operatorname{conv}\left(\left\{\sum_{i \in A} e_i \colon |A| \in \{1, d-1\}\right\}\right).$$

This yields

$$f(d) \geq f'(d) \geq 2^d$$

for all d. The fact that equality $f(d) = 2^d$ holds for $d \leq 4$ is checked by complete enumeration. Such an enumeration (not complete) also provided the example that proves $f'(5) \geq 40$, here given as PORTA-input:

```
DIM = 5
CONV˙SECTION

0 0 0 0 0

1 0 1 0 0
1 1 1 0 0
0 1 0 1 0
0 1 1 1 0
0 0 1 0 1
0 0 1 1 1
1 0 0 1 0
1 0 0 1 1
0 1 0 0 1
1 1 0 0 1

1 1 1 1 1

END
```

For $d = 6, 7,$ and 8 the polytopes with the most facets that we know are obtained by the following construction:

$$S_d := \left\{ \sum_{i \in A} e_i \;\middle|\; \begin{array}{l} \text{either } |A| \in \{1, d-1\}, \\ \text{or } |A| > 0 \text{ is even and } A \subseteq \{1, 2, \dots, \lfloor d/2 \rfloor\}, \\ \\ \text{or } |A| > 0 \text{ is even and } A \subseteq \{\lfloor d/2 \rfloor + 1, \dots, d\}. \end{array} \right\}$$

By computing the convex hull of S_{10}, which is indeed 10-dimensional, we find 10,829 $\approx (2.531971631)^{10}$ facets.

In higher dimensions, $d \geq 9$, both R. Seidel and one of the referees noted that it is better to take a "random" polytope. By computing the convex hull of a set of 88 random 0/1-vectors in $\mathbb{R}^d$ we found a centered polytope R_{10} having 26,286 $\approx (2.7667661)^{10}$ facets. For the coordinates and data of our best examples of 0/1-polytopes with many facets, including R_{10}, we refer to [10].

Taking an appropriate direct sum

$$T_d := \bigoplus_{\lfloor d/10 \rfloor} R_{10} \oplus \bigoplus_{d \bmod 10} C_1$$

we obtain the following.

Corollary 2.2. *For $d \geq 0$ we have*

$$f(d) \geq f'(d) \geq (26{,}286)^{\lfloor d/10 \rfloor} 2^{d \bmod 10}.$$

Thus $f(d) > (2.76)^d$ for all sufficiently large d.

Upper bounds for $f(d)$ can be obtained from a volume argument due to Bárány [13, p. 25, Problem 0.15*] that we slightly refine with

Theorem 2.3. *The maximal number of facets $f(d)$ of a d-dimensional 0/1-polytope satisfies*

$$f(d) \leq d! - (d-1)! + 2(d-1) \qquad \textit{for} \quad d \geq 3.$$

Proof. Let P be a d-dimensional 0/1-polytope. We can obtain $\operatorname{conv}(\{0,1\}^d)$ from P by successive addition of 0/1-vertices, thus destroying all but the "trivial" facets of P. However, whenever a facet F_i of P is "destroyed" a cone over F_i is added. This cone is a d-dimensional 0/1-polytope, whence its volume is at least $1/d!$. Since the process stops at the d-dimensional 0/1-cube with $2d$ facets and volume 1, we get

$$f_{d-1}(P) \leq 2d + d!\,(1 - \operatorname{vol}(P)). \tag{1}$$

On the other hand, P can be triangulated without new vertices, say into t simplices of dimension d. Each of these simplices has volume at least $1/d!$, hence

$$t \leq d!\operatorname{vol}(P).$$

Each simplex has $d+1$ facets. The dual graph of the triangulation is connected; it has t nodes, hence at least $t-1$ edges. From this we get that at least $2(t-1)$ simplex facets are between simplices, so at most $t(d+1) - 2(t-1)$ simplex facets are in the surface of P. Since each facet of P is a union of simplex facets, we obtain

$$f_{d-1}(P) \leq t(d-1) + 2$$

and hence

$$f_{d-1}(P) \leq 2 + (d-1)d!\operatorname{vol}(P). \tag{2}$$

Addition of (2) to the $(d-1)$-fold of (1) cancels the summands that involve the volume; we obtain

$$d f_{d-1}(P) \leq 2 + 2d(d-1) + (d-1)d!.$$

Division by d and rounding down the right-hand side (since the left-hand side is integral) yields the result. $\square$

3. The Complexity of Two-Dimensional Shadows

The fact that $H(\mathcal{P}_d)$, the maximal number of vertices on an increasing path, is exponential already follows from the fact that there are 0/1-polytopes with exponentially many vertices, such that any two vertices are adjacent. So, for any generic linear function there is an increasing path through all the vertices. For an example "occurring in nature" (where the natural place for polytopes is combinatorial optimization) put $P := \operatorname{conv}(V) \subseteq \mathbb{R}^{k^2}$, with

$$V := \{xx^t\colon\ x \in \{0,1\}^k,\ x_k = 1\}.$$

This yields the *boolean quadric polytope* or *cut polytope* P of dimension $d < k^2$ with 2^{k-1} vertices, of which any two are adjacent [2]. In fact, for any $yy^t, zz^t \in V$ we can find a linear function $x \mapsto a^t x$ such that $a^t y = a^t z = 0$, but $a^t x \neq 0$ for any $x \in \{0,1\}^k$ with $x_k = 1$ and $x \neq y, z$. The scalar product with aa^t defines a linear function on P, where

$$\langle aa^t, xx^t \rangle := \sum_{i=1}^{k}\sum_{j=1}^{k}(aa^t)_{ij}(xx^t)_{ij} = \left(\sum_{i=1}^{k} a_i x_i\right)\left(\sum_{j=1}^{k} a_j x_j\right) = (a^t x)^2 \geq 0,$$

with equality if and only if $x = y$ or $x = z$. This gives us $H(\mathcal{P}_d) \geq 2^{\sqrt{d}}$. See below for an improvement that yields a genuinely exponential lower bound.

3.1. *A Lower Bound for $H_{\mathrm{sh}}(\mathcal{P}_d)$*

We give a proof for a lower bound on the maximal number of extremal vertices in the two-dimensional shadow of a 0/1-polytope. It relies on a special projection of the d-cube C_d onto a regular grid. We choose a suitable subset of the projected points that lies in convex position.

We consider the following projection $\pi\colon \mathbb{R}^d \to \mathbb{R}^2$ for $d = 3k$ and k a positive integer: The first k coordinates x_i are projected to $(2^{i-1}, 0)$ for $1 \leq i \leq k$. The remaining $2k$ coordinates x_i are projected to $(0, 2^{i-(k+1)})$ for $k+1 \leq i \leq d$. By π we obtain a bijection between the vertices of the d-cube and the vertices of a $2^k \times 2^{2k}$ integer grid G.

Now we take the subset of vertices of C_d which corresponds to the subset S of the grid with

$$S = \{(i, i^2) \mid 0 \leq i \leq 2^k - 1\} \cup \{(2^k - 1 - i, 2^{2k} - 3 - i^2) \mid 0 \leq i \leq 2^k - 1\} \subseteq G.$$

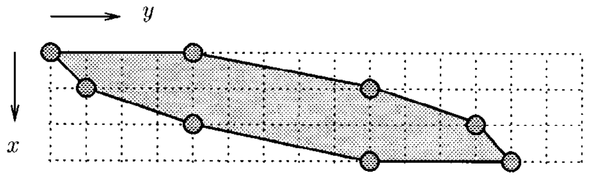

Fig. 1. The subset P of the $2^2 \times 2^4$ grid.

S is the set of grid points of a standard parabola, together with a rotated copy (see Fig. 1).

It is obvious that this subset yields a projection with all vertices being extremal, and since $|P| = 2 \cdot 2^k$ we have a lower bound for the maximal number of extremal vertices in the two-dimensional projection of a d-dimensional 0/1-polytope $H_{\text{sh}}(\mathcal{P}_d) \geq 2^{k+1}$. This bound may be refined either by using the slightly less growing convex function $i \mapsto \binom{i}{2}$ instead of the parabola, or by simply using the fact that the least significant bit (LSB) in the bit representation of i (resp. i^2) is the same and the second LSB of i^2 is always 0, which we can use to squeeze the number of bits required to represent the parabola and its mirror image, given $d \geq 4$. This yields

Theorem 3.1. *The maximal number of extremal vertices $H_{\text{sh}}(\mathcal{P}_d)$ of the two-dimensional shadow of a d-dimensional 0/1-polytope is bounded from below for $d \geq 4$ by*

$$2^{\lfloor (d+5)/3 \rfloor} \leq H_{\text{sh}}(\mathcal{P}_d).$$

We would like to mention a rather similar, although more indirect, method to show asymptotically the same lower bound. For this, project C_d for $d = 2k$ to a regular $2^k \times 2^k$ grid with projection vectors $(2^i, 0)$ and $(0, 2^i)$ for $i = 0, \ldots, k-1$. Using Satz 4.1.9 of

Table 1. A comparison of the lower bound valid for all d, an explicit construction given by the projection vectors $(2^i, 2^{d-i-1})$ for $i = 0, \ldots, d-1$ that we could only calculate up to $d = 12$, and the upper bound as given by Corollary 3.3, where the minimization step was done explicitly.

Dimension d	Lower bound (for all d, Theorem 3.1)	Construction (for small d only)	Upper bound (Corollary 3.3)
1	—	2	4
2	—	4	6
3	—	6	10
4	8	10	16
5	8	14	24
6	8	18	38
7	16	22	58
8	16	32	88
9	16	42	138
10	32	52	210
11	32	66	320
12	32	82	500

[12] we find a convex polygon with $(12/(2\pi)^{2/3})n^{2/3} + O(n^{1/3}\log n)$ extremal vertices on the grid, where $n = 2^k$.

However, comparison of the explicit calculations for certain grid sizes as worked out by Thiele with the bound given by Theorem 3.1 shows no substantial difference, while there are constructions that yield much better lower bounds (see Table 1).

The same technique as shown above can be used to prove a truly exponential lower bound for $H(\mathcal{P}_d)$, as was pointed out by one referee: Take a projection of the $(d = 10k)$-dimensional cube to the $2^k \times 2^{2k} \times 2^{3k} \times 2^{4k}$ integer grid and choose 2^k vertices on the grid which are the vertices of a cyclic 4-polytope. The convex hull of the preimages of these is a 2-neighborly 0/1-polytope, and so $H(\mathcal{P}_d) > 2^{d/10}$.

3.2. *An Upper Bound for* $H_{\text{sh}}(\mathcal{P}_d)$

We derive upper bounds for $H_{\text{sh}}(\mathcal{P}_d)$ by relating this to a problem on set systems.

A collection of sets $\mathcal{S} \subseteq 2^{[d]}$ is said to have property (SYM) if the pairs $(A\backslash B, B\backslash A)$ are distinct for all $A, B \in \mathcal{S}$ with $A \neq B$. We define

$$X(d) = \max\{|\mathcal{S}|\colon \mathcal{S} \subseteq 2^{[d]} \text{ satisfies (SYM)}\}.$$

We note that the projection of a d-dimensional 0/1-polytope is described by a collection of d points $\mathcal{P} = \{p_1, \ldots, p_d\}$ in the plane. If p_i is the image of the unit vector $e_i \in \mathbb{R}^d$, then the image of a general 0/1-vector with support S is $\mathcal{P}(S) = \sum_{i\in S} p_i$. This defines a collection of at most 2^d points

$$2^{\mathcal{P}} := \{\mathcal{P}(S) \mid S \subseteq [d]\}.$$

If $g(\mathcal{P}, d)$ is the largest number of points in $2^{\mathcal{P}}$ in convex position, then

$$H_{\text{sh}}(\mathcal{P}_d) = \max_{\mathcal{P}} g(\mathcal{P}, d).$$

For subsets $S_1, S_2 \subseteq [d]$, the vector joining $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ is

$$\mathcal{P}(S_2) - \mathcal{P}(S_1) = \mathcal{P}(S_2\backslash S_1) - \mathcal{P}(S_1\backslash S_2)$$

which corresponds to the ordered pair $(S_2\backslash S_1, S_1\backslash S_2)$, and at most two copies of such a vector can appear in any (strictly convex) polygon with vertices in $2^{\mathcal{P}}$. In fact, by discarding half the vertices of the polygon, we ensure that each vector joining pairs of vertices appears exactly once. Then the subsets corresponding to the vertices of the polygon satisfy (SYM). We have thus shown that the functions $H_{\text{sh}}(\mathcal{P}_d)$ and $X(d)$ are related by

$$\frac{H_{\text{sh}}(\mathcal{P}_d)}{2} \leq X(d).$$

Thus it suffices to find an upper bound for $X(d)$ in order to bound $H_{\text{sh}}(\mathcal{P}_d)$. We first establish the following simple bound for $X(d)$: If $\mathcal{S} \subseteq 2^{[d]}$ satisfies (SYM) and $|\mathcal{S}| = k$, then $k(k-1) \leq 3^d$, since the total number of disjoint pairs of subsets (A, B) in $[d]$ is 3^d. Hence $X(d) \leq 2\,3^{d/2}$. We improve this bound in the following result.

Theorem 3.2.

$$X(d) \le (1+\sqrt{3})\, 2^{d(\log 3/\log 6)}.$$

Corollary 3.3.

$$H_{\mathrm{sh}}(\mathcal{P}_d) \le 2(1+\sqrt{3})\, 2^{d(\log 3/\log 6)}.$$

Proof. Let $\mathcal{S} \subseteq 2^{[d]}$ be a collection of sets satisfying (SYM). For a k-subset $T \subseteq [d]$, let $N(T)$ be the number of pairs (A, B) with $A, B \in \mathcal{S}$ and $A \backslash B, B \backslash A \subseteq T$. Let $\bar{T} = [d] \backslash T$ be the complement of T and let $m = 2^{d-k}$. We count $N(T)$ by partitioning $\mathcal{S}$ into subcollections $\mathcal{S}_1, \ldots, \mathcal{S}_m$ in such a way that $A, B \in \mathcal{S}_i$ if and only if $A \cap \bar{T} = B \cap \bar{T}$. Thus $A, B \in \mathcal{S}_i$ implies that $A \backslash B, B \backslash A \subseteq T$. If $|\mathcal{S}_i| = d_i$, then

$$d_1(d_1-1) + \cdots + d_m(d_m-1) = N(T) \le 3^k,$$

since the number of disjoint pairs of subsets in T is at most 3^k. Thus, using the arithmetic-geometric mean inequality twice, we get

$$\begin{aligned}
|\mathcal{S}| &= d_1 + \cdots + d_m \\
&= (d_1 - \tfrac{1}{2}) + \cdots + (d_m - \tfrac{1}{2}) + \frac{m}{2} \\
&= m\left(\tfrac{1}{2} + \frac{(d_1-\frac{1}{2}) + \cdots + (d_m - \frac{1}{2})}{m}\right) \\
&\le m\left(\tfrac{1}{2} + \sqrt{\frac{(d_1-\frac{1}{2})^2 + \cdots + (d_m - \frac{1}{2})^2}{m}}\right) \\
&= \frac{m}{2} + \sqrt{m}\sqrt{d_1(d_1-1) + \cdots + d_m(d_m-1) + \frac{m}{4}} \\
&\le \frac{m}{2} + \sqrt{m}\sqrt{3^k + \frac{m}{4}} \\
&\le m + \sqrt{3^k m} \\
&= 2^{d-k} + \sqrt{2^{d-k}3^k}.
\end{aligned}$$

The right-hand side is minimized when $3^k = 2^{d-k}$. Hence choosing $k = \lceil d \log 2/\log 6 \rceil$ we get

$$|\mathcal{S}| \le (1+\sqrt{3})\, 2^{d \log 3/\log 6}$$

as desired. □

We conjecture that $X(d) = 2^{(1/2+o(1))d}$. A lower bound of the order of $2^{d/2}$ can be constructed for $X(d)$ by relating this problem to the existence of Sidon sets in the following sense. A *Sidon set* is a set of integers such that all pairs have distinct sums. By associating a set $S \subseteq [d]$ with the number $1 + \sum_{i \in S} 2^{i-1}$, we get a one-to-one correspondence between the subsets of $[d]$ and the elements of $[2^d]$. Then, given a Sidon subset of $[2^d]$, the corresponding collection of sets in $[d]$ satisfy (SYM). A Sidon subset of $[2^d]$ of size $2^{d/2} - c2^{5d/16}$ has been constructed in [6]. While the lower bound for $X(d)$ does not reveal any further information on the shadow vertex problem, it is of interest in its own right.

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Note added in proof, January 10, 1997. Very recent improvements on the estimates for the maximal numbers of facets $f(d)$ include

$$\begin{aligned} f(5) &= 40 && \text{(Oswin Aichholzer)}, \\ (3.26)^d &\le f(d) && \text{for large } d \qquad \text{(Komei Fukuda } et\ al.\text{), and} \\ f(d) &\le 6.4d!/\sqrt{d} && \text{for large } d \qquad \text{(Günter Rote).} \end{aligned}$$

See [10] for latest results.