EXTREMAL PROPERTIES OF EXTREME VALUE DISTRIBUTIONS

BY SIGEITI MORIGUTI

University of North Carolina

Summary. The upper and lower bounds for the expectation, the coefficient of variation, and the variance of the largest member of a sample from a symmetric population are discussed. The upper bound for the expectation (Table 1, Fig. 1), the lower bound for the C.V. (Table 2, Fig. 4) and the lower bound for the variance (Fig. 7) are actually achieved for the corresponding particular population distributions (Figs. 2, 3, 5, 6, equation (5.1)). The rest of the bounds are not actually achieved but approached as the limits, for example, for the three-point distribution (Section 3) by letting p tend to zero.

1. Introduction. The sampling distribution of the largest or the smallest member of a sample has been studied by several authors; Tippett [1] and de Finetti [2] considered a sample from a normal population, Olds [3] from a rectangular population. The case of a very large sample was treated by Dodd [4], Fisher and Tippett [5], and Gumbel [6], each for a certain class of population distributions.

Here we consider the upper and lower bounds for the expectation, the coefficient of variation, and the variance of the extreme member of a sample from a symmetrically distributed population with a finite variance. To be specific, we will discuss only the largest member and take the mean of the population equal to zero. These conventions do not imply any essential restriction.

2. Notations and formulas. Let the cumulative distribution function (cdf) of the population be denoted by $F(x)$; then the cdf of the largest member x_n of a sample of size n is given by $\{F(x)\}^n$. Hence the expectation of the largest member can be expressed by

$$E(x_n) = \int_{-\infty}^{\infty} xn\{F(x)\}^{n-1}\, dF(x). \tag{2.1}$$

Now we consider the inverse function $x(F)$ of $F(x)$, with an obvious additional definition at points of discontinuity, if any, of $F(x)$. Thus (2.1) can also be written as

$$E(x_n) = \int_0^1 x(F)nF^{n-1}\, dF. \tag{2.2}$$

Because of symmetry, $x(F) = -x(1 - F)$ holds almost everywhere, whence

$$E(x_n) = \int_{\frac{1}{2}}^1 x(F)n\{F^{n-1} - (1 - F)^{n-1}\}\, dF. \tag{2.3}$$

Similarly, we get as the variance

$$V(x_n) = \int_{\frac{1}{2}}^{1} \{x(F)\}^2 n\{F^{n-1} + (1 - F)^{n-1}\}\, dF - \{E(x_n)\}^2. \tag{2.4}$$

The population variance is of course given by

$$\sigma^2 = 2 \int_{\frac{1}{2}}^{1} \{x(F)\}^2\, dF. \tag{2.5}$$

3. Bounds for the expectation of the largest member. In Schwarz's inequality

$$\left(\int_a^b f(F)g(F)\, dF\right)^2 \leqq \int_a^b \{f(F)\}^2\, dF \int_a^b \{g(F)\}^2\, dF, \tag{3.1}$$

putting $a = \frac{1}{2}$, $b = 1$, $f(F) \equiv x(F)$, $g(F) \equiv n\{F^{n-1} - (1 - F)^{n-1}\}$, we get a formula which means, in view of (2.3) and (2.5), that

$$E(x_n) \leqq \frac{\sigma}{\sqrt{2}}\, n \left(\int_{\frac{1}{2}}^{1} \{F^{n-1} - (1 - F)^{n-1}\}^2\, dF\right)^{\frac{1}{2}}, \tag{3.2}$$

equality being satisfied if and only if $f(F) = \text{const.} \cdot g(F)$, that is,

$$x(F) = \text{const.} \cdot \{F^{n-1} - (1 - F)^{n-1}\}. \tag{3.3}$$

Therefore the expectation of the largest member has the right-hand side of (3.2) as an upper bound, which is actually achieved for a particular type of population distribution given by (3.3).

The integral in (3.2) can easily be evaluated as follows:

$$\begin{aligned} &\int_{\frac{1}{2}}^{1} \{F^{n-1} - (1 - F)^{n-1}\}^2\, dF \\ &= \frac{1}{2} \int_0^1 [F^{2n-2} + (1 - F)^{2n-2} - 2F^{n-1}(1 - F)^{n-1}]\, dF \\ &= \frac{1}{2}\left[\frac{1}{2n - 1} + \frac{1}{2n - 1} - 2\mathrm{B}(n, n)\right] = \frac{1}{2n - 1} - \mathrm{B}(n, n), \end{aligned} \tag{3.4}$$

where the Beta function of equal integral arguments can also be expressed as

$$\mathrm{B}(n, n) = \frac{1}{(2n - 1)C_{n-1}^{2n-2}}. \tag{3.5}$$

Thus the upper bound for $E(x_n)$ is given by

$$E(x_n) \leqq \frac{n}{\sqrt{2(2n - 1)}} \left(1 - \frac{1}{C_{n-1}^{2n-2}}\right)^{\frac{1}{2}} \sigma. \tag{3.6}$$

The numerical value of the coefficient is calculated for various sample sizes and compared with the values of $E(x_n)/\sigma$ for normal and rectangular populations in Table 1 and Fig. 1. It is to be noted that the value for a normal population is remarkably close to the upper bound if $n \leqq 7$. The cumulative distribution curve and frequency curve of the extremal distribution (3.3) is illustrated in Figs. 2 and 3 for several values of sample size n.

It is obvious that the expectation of the largest member has the lower bound zero. However it may be of some interest to see that this lower bound can be approached as closely as one desires. One of the simplest ways is to consider the three-point distribution, such as the values a, 0 and $-a$ occurring with probabilities p, $1 - 2p$, and p, respectively. If we make p approach zero for a fixed sample size n, the ratio $E(x_n)/\sigma$ also approaches zero, because in this case

TABLE 1

Expectation of the largest member in the unit of σ, $E(x_n)/\sigma$

Sample size n	Upper bound	For normal distribution*	For rectangular distribution
2	.5774	.5642	.5774
3	.8660	.8463	.8660
4	1.0420	1.0294	1.0392
5	1.1701	1.1630	1.1547
6	1.2767	1.2672	1.2372
7	1.3721	1.3522	1.2990
8	1.4604	1.4236	1.3472
9	1.5434	1.4850	1.3856
10	1.6222	1.5388	1.4171
11	1.6974	1.5864	1.4434
12	1.7693	1.6292	1.4656
13	1.8385	1.6680	1.4846
14	1.9052	1.7034	1.5011
15	1.9696	1.7359	1.5155
16	2.0320	1.7660	1.5283
17	2.0926	1.7939	1.5396
18	2.1514	1.8200	1.5497
19	2.2087	1.8450	1.5588
20	2.2645	1.8673	1.5671

* From [9], p. 165.

(3.7) $$E(x_n) = nap + O(p^2),$$

(3.8) $$\sigma^2 = 2a^2p.$$

4. Bounds for the coefficient of variation of the largest member of a sample. Putting in (3.1) $a = \frac{1}{2}$, $b = 1$, and

$$(4.1)\qquad \begin{aligned} f(F) &\equiv x(F)\sqrt{n}\{F^{n-1} + (1-F)^{n-1}\}^{\frac{1}{2}},\\ g(F) &\equiv \frac{\sqrt{n}\{F^{n-1} - (1-F)^{n-1}\}}{\{F^{n-1} + (1-F)^{n-1}\}^{\frac{1}{2}}}, \end{aligned}$$

we get a formula which means, in view of (2.3) and (2.4), that

$$(4.2)\qquad \frac{V(x_n)}{E(x_n)^2} \geqq \frac{1}{M_n} - 1,$$

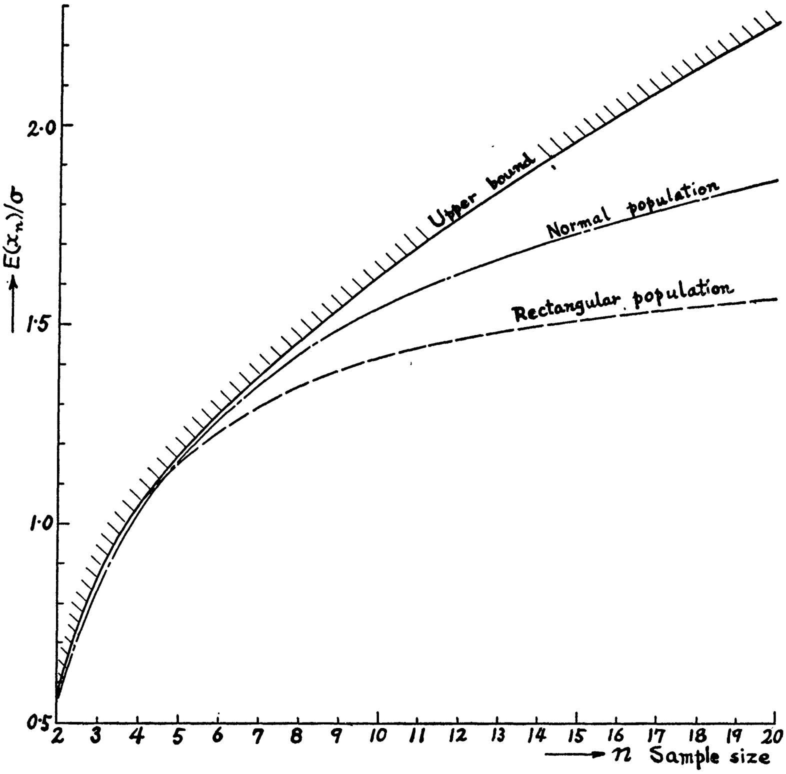

FIG. 1. Expectation of the largest member

where

$$(4.3)\qquad M_n \equiv \int_{\frac{1}{2}}^{1} \frac{n\{F^{n-1} - (1-F)^{n-1}\}^2}{F^{n-1} + (1-F)^{n-1}}\,dF.$$

The equality in (4.2) is satisfied if and only if $f = \text{const.}\cdot g$, i.e.

$$(4.4)\qquad x(F) = \text{const.} \cdot \frac{F^{n-1} - (1-F)^{n-1}}{F^{n-1} + (1-F)^{n-1}}.$$

Therefore the coefficient of variation of the largest member has $\sqrt{(1/M_n) - 1}$ as a lower bound which is actually achieved for a particular type of population distribution given by (4.4).

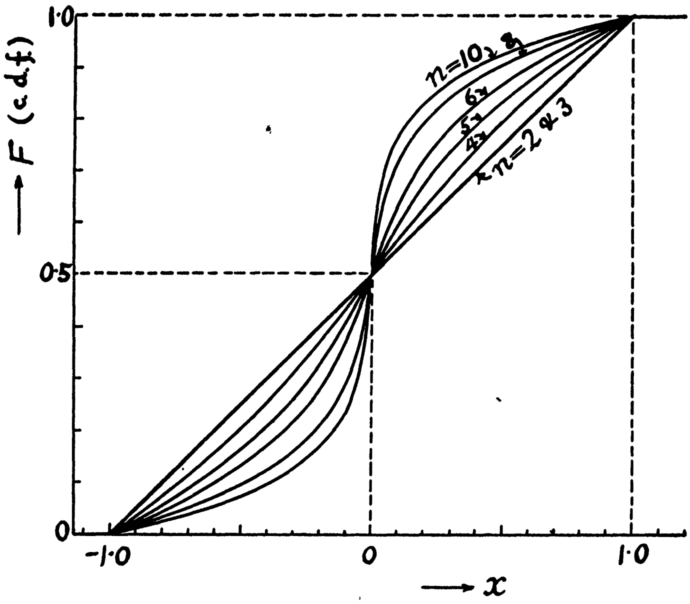

Fig. 2. Extremal distribution (cdf) for $E(x_n)$

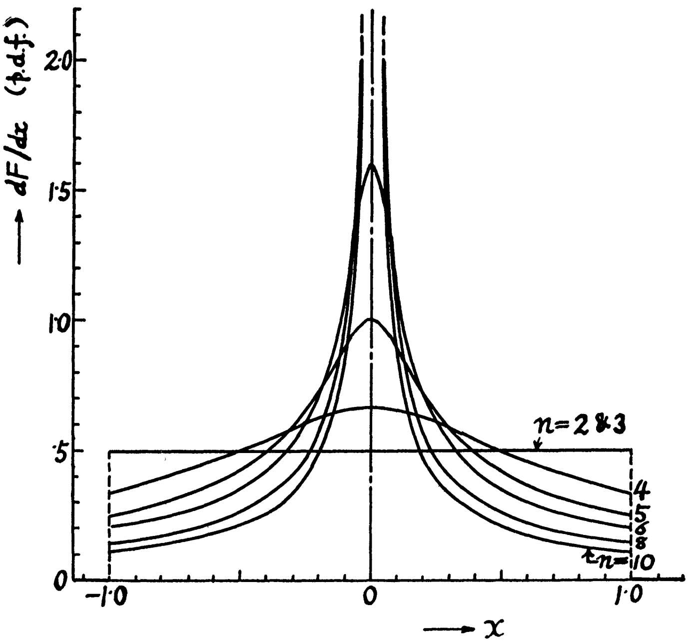

Fig. 3. Extremal distribution (pdf) for $E(x_n)$

The integral (4.3) can be evaluated by an elementary method of quadrature. To show the results for small values of n,

$$M_2 = \frac{1}{3} = 0.33333,$$

$$M_3 = 3 - \frac{3}{4}\pi = 0.64381,$$

$$M_4 = \frac{23}{15} - \frac{32\pi}{81\sqrt{3}} = 0.81677,$$

$$M_5 = -\frac{55}{3} + \frac{35\sqrt{2}}{4}\pi - \frac{25}{4}\pi = 0.90695,$$

$$M_6 = -\frac{6}{7} + 0.6\pi\{(0.704 + 0.8\sqrt{0.8}) \cdot\sqrt{5 - 2\sqrt{5}} + 2(0.704 - 0.8\sqrt{0.8})\sqrt{5 + 2\sqrt{5}}\}$$

$$= 0.95300.$$

TABLE 2

Coefficient of variation of the largest member

Sample size n	Lower bound	For normal population*	For rectangular population
2	1.4142	1.4634	1.4142
3	.7438	.8838	.7746
4	.4737	.6812	.5443
5	.3203	.5752	.4226
6	.2221	.5089	.3464

* Cf. [7].

As the sample size n increases, the evaluation of M_n by quadrature becomes more and more laborious. Numerical integration would be preferable for larger values of n. In this case, however, we can derive (see Appendix 1 for the derivation) an asymptotic formula of M_n for large n

$$M_n = 1 - \frac{\pi}{2^n}\left[1 + O\left(\frac{1}{n}\right)\right] \tag{4.5}$$

which happens to be a fairly close approximation even for as small a value of n as six, where this formula gives 0.95091. Using these results, we compare the lower bound with the value of the C.V. of the largest member for a normal population and a rectangular population, as in Table 2 and Fig. 4.

It is interesting to observe that the C.V. of the largest member of a sample from a two-point population, such as values 1 and -1 each occurring with probability 1/2, behaves asymptotically similarly to the lower bound except for a numerical factor $\sqrt{\pi}/2$. In fact, we can easily derive in this case the following formulas

$$E(x_n) = 1 - \frac{1}{2^{n-1}}, \qquad V(x_n) = \frac{1}{2^{n-2}} - \frac{1}{2^{2n-2}}, \tag{4.6}$$

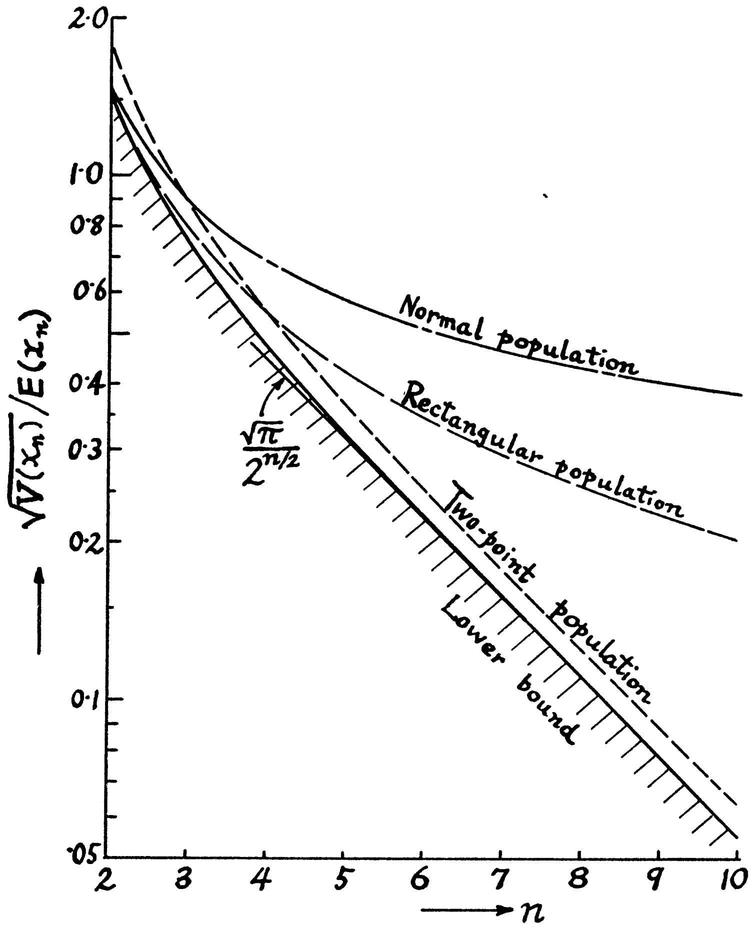

FIG. 4. Coefficient of variation of the largest member

$$\frac{\sqrt{V(x_n)}}{E(x_n)} = \frac{\sqrt{(2^n - 1)}}{2^{n-1} - 1} \approx \frac{1}{2^{\frac{1}{2}n-1}}. \tag{4.7}$$

This similarity in the asymptotic behavior may be taken to be the reflection of the similarity in the population distribution, which is seen in comparing Figs. 5 and 6 with the corresponding graphs for the two-point distribution.

There is no finite upper bound for the coefficient of variation of the largest member. It can be proved, for instance, by observing the behavior in the case of the three-point distribution mentioned in the previous section when p approaches zero for fixed n. In fact, in this case, it is easy to show that

$$V(x_n) = na^2p + O(p^2),$$

$$\text{(4.8)} \qquad \frac{\sqrt{V(x_n)}}{E(x_n)} = \frac{1}{\sqrt{np}} + O(\sqrt{p}) \to \infty.$$

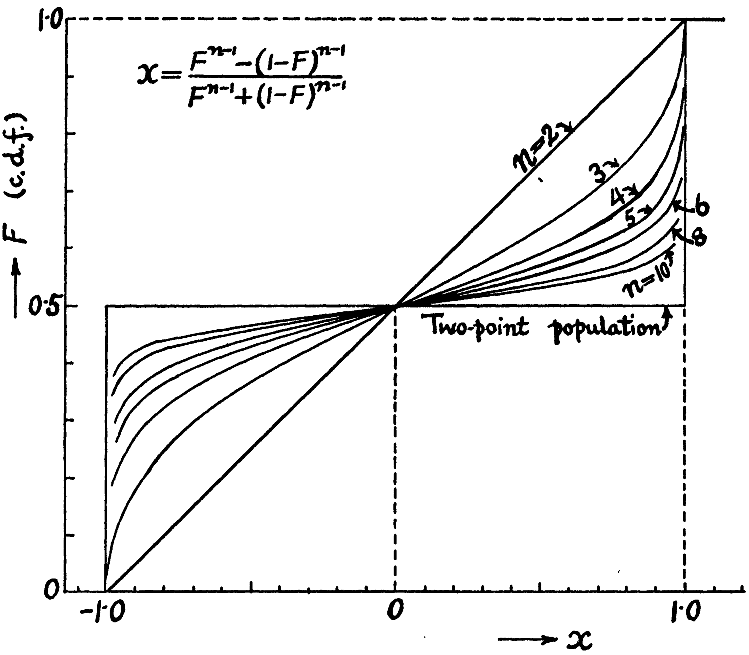

FIG. 5. Extremal distribution (cdf) for C.V.(x_n)

5. Bounds for the variance of the largest member.[1] As we shall prove, $V(x_n)$ has a lower bound $\lambda_n \sigma^2$, which is actually achieved for a particular type of population distribution given, when F is not 0 or 1, by

$$\text{(5.1)} \qquad x = \text{const.} \cdot \frac{n\{F^{n-1} - (1 - F)^{n-1}\}}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda_n},$$

where λ_n is the only root of the equation[2]

$$\text{(5.2)} \qquad M_n(\lambda) \equiv \int_{\frac{1}{2}}^{1} \frac{n^2\{F^{n-2} - (1 - F)^{n-1}\}^2}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda}\, dF = 1$$

in the interval $0 \leqq \lambda \leqq n/2^{n-1}$.

[1] A heuristic derivation of the formulas (5.1) and (5.2) is given in Appendix 2.
[2] The notation is such that $M_n(0)$ equals M_n as previously defined.

First, in order to prove that there exists one and only one root of (5.2) in the stated interval, it is sufficient to show that

$$M_n(0) < 1, \qquad M_n(n/2^{n-1}) > 1, \tag{5.3}$$

and to note that $M_n(\lambda)$ is a monotone increasing continuous function of λ in the interval. Since

$$\int_{\frac{1}{2}}^{1} n\{F^{n-1} + (1 - F)^{n-1}\}\, dF = 1, \tag{5.4}$$

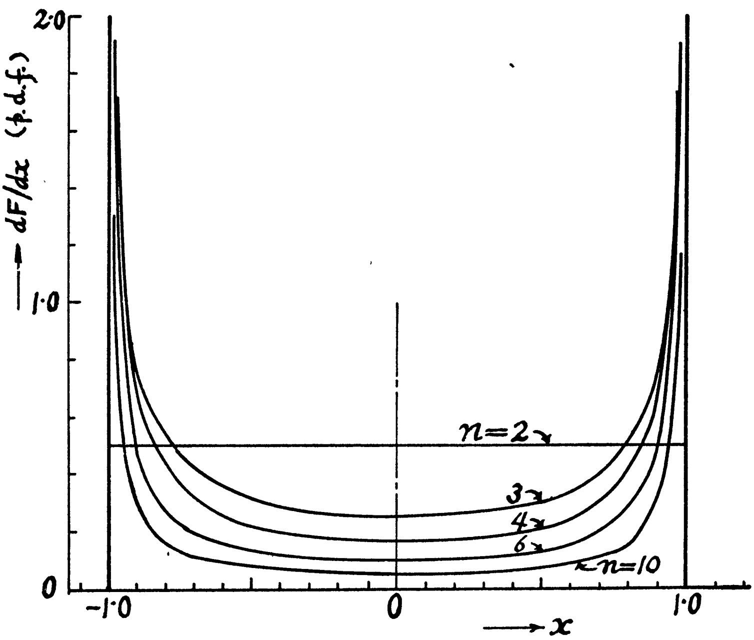

FIG. 6. Extremal distribution (pdf) for C.V.(x_n)

we have, for any λ in the interval,

$$\begin{aligned} 1 - M_n(\lambda) &= \int_{\frac{1}{2}}^{1} \frac{n^2\{F^{n-1} + (1-F)^{n-1}\}^2 - 2\lambda n\{F^{n-1} + (1-F)^{n-1}\} - n^2\{F^{n-1} - (1-F)^{n-1}\}^2}{n\{F^{n-1} + (1-F)^{n-1}\} - 2\lambda}\, dF \\ &= \int_{\frac{1}{2}}^{1} \frac{4n^2F^{n-1}(1-F)^{n-1} - 2\lambda n\{F^{n-1} + (1-F)^{n-1}\}}{n\{F^{n-1} + (1-F)^{n-1}\} - 2\lambda}\, dF. \end{aligned} \tag{5.5}$$

On the other hand, it is obvious that $F^{n-1} + (1 - F)^{n-1}$ assumes a minimum value $1/2^{n-2}$, and $F^{n-1}(1 - F)^{n-1}$ a maximum value $1/2^{2n-2}$ at $F = 1/2$. Therefore, in the interval[3] $1/2 < F < 1$, the denominator of the integrand is always positive, the numerator being always positive for $\lambda = 0$ and always negative for $\lambda = n/2^{n-2}$. Hence we get (5.3). The above mentioned nature of $M_n(\lambda)$ is also obvious (cf. the definition (5.2) and the above statement about the denominator).

Next, again in the Schwarz's inequality, let us put $a = 1/2$, $b = 1$ and

$$f(F) \equiv x(F)[n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda_n]^{\frac{1}{2}}, \tag{5.6}$$

$$g(F) \equiv \frac{n\{F^{n-1} - (1 - F)^{n-1}\}}{[n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda_n]^{\frac{1}{2}}}. \tag{5.7}$$

Then we obtain a formula which means, in view of (2.3), (2.4), (2.5) and $M_n(\lambda_n) = 1$, that

$$V(x_n) \geqq \lambda_n \sigma^2, \tag{5.8}$$

equality being satisfied if and only if $f = \text{const.}\cdot g$, i.e. (5.1) holds. Thus the statement at the beginning of this section has been proved.

The numerical evaluation of λ_n requires a little more effort than the evaluation of M_n in the previous section, as the former requires solution of a transcendental equation after an integration. For instance,[4] for $n = 3$, λ_3 can be obtained by solving

$$\tan^{-1} \frac{1}{\sqrt{1 - \frac{4}{3}\lambda}} = \frac{2}{3\sqrt{1 - \frac{4}{3}\lambda}},$$

as $\lambda_3 = .394$. For $n = 4$, we have to solve

$$\sqrt{\frac{1 - 2\lambda}{3}} \tan^{-1} \sqrt{\frac{3}{1 - 2\lambda}} = 1 - \frac{88 - 5\lambda}{10(4 + \lambda)^2}$$

to get $\lambda_4 = .209$. Moreover, when $n \geqq 7$, the quadrature itself is tedious. For large n, however, an asymptotic formula is again available as shown in Appendix 3. It is closely related to (4.5), and takes the form

$$\lambda_n = \frac{\pi}{2^n}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{5.9}$$

Again it is fairly close even if n is small.

The general picture is seen in Fig. 7, in which the lower bound of $\sqrt{V(x_n)}/\sigma$ is shown together with the value for normal [7], rectangular, and two-point distributions.

As for the upper bound, it is easy to see, from (2.4) and (2.5), that

$$V(x_n) < n \int_{\frac{1}{2}}^{1} x(F)^2\, dF = \tfrac{1}{2} n\sigma^2, \tag{5.10}$$

[3] The suspicion about the singularity which might occur in the case of $\lambda = n/2^{n-2}$ at $F = \frac{1}{2}$ is dissolved if we note that the numerator also has a zero of the second order at $F = \frac{1}{2}$.

[4] For $n = 2$, (5.1) reduces to a rectangular distribution, for which no more calculation is necessary.

for $F^{n-1} + (1 - F)^{n-1}$ is a monotone increasing function taking the value unity at the end $F = 1$ of the interval. The value $n/2$ of the ratio $V(x_n)/\sigma^2$ can be

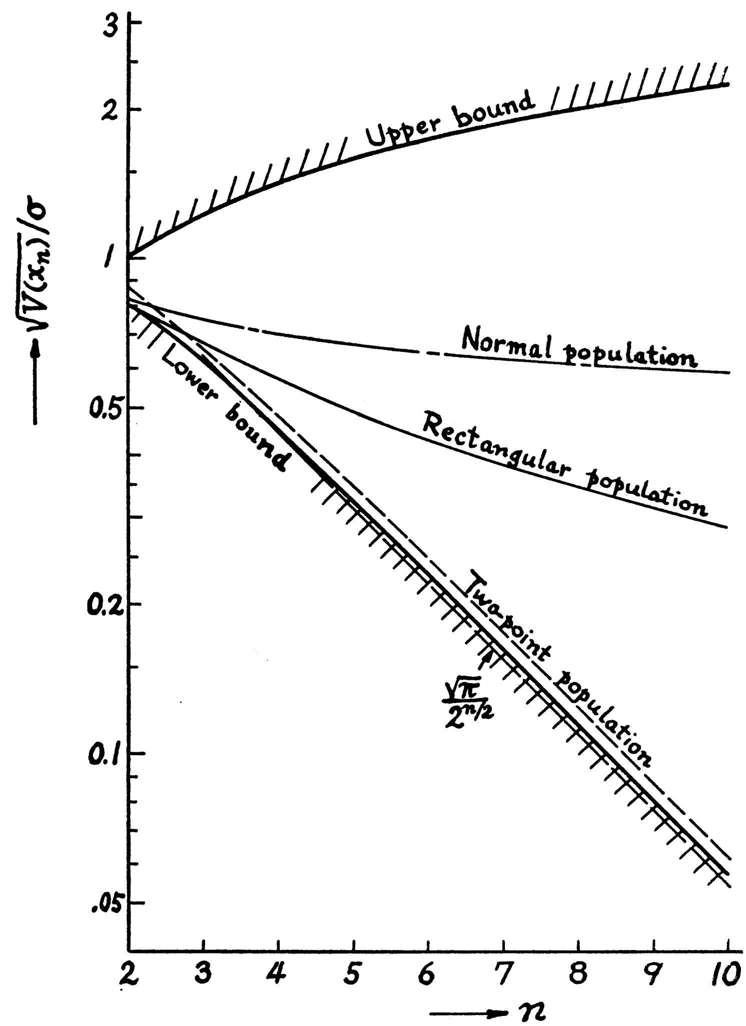

FIG. 7. Standard deviation of the largest member

approached as closely as desired, for example, for the three-point distribution (Section 3) by letting p be sufficiently small. (See (3.8) and (4.8).)

6. Final remarks and acknowledgement. We considered the upper and lower bounds for the expectation, the coefficient of variation, and the variance of the largest member of a sample from a symmetric population. The upper bound for the expectation and the lower bound for the C.V. or the variance are actually achieved for particular distributions, which we may call "extremal distributions". These distributions as well as the values of the corresponding bounds were first obtained, as illustrated in Appendix 2, by applying the techniques of the Calculus of Variations. The same methods can be applied also to the distribution of the range[5] of the sample and some other useful statistics.

The writer is indebted to Professor Harold Hotelling for his suggestions which induced him to undertake this study and for his kind guidance and encouragement in the course of study.

APPENDICES

1. Asymptotic formula for M_n. Putting $\lambda = 0$ in (5.5), we get

$$1 - M_n = \int_{\frac{1}{2}}^{1} \frac{4nF^{n-1}(1 - F)^{n-1}}{F^{n-1} + (1 - F)^{n-1}} \, dF.$$

With the change of variable $t = 2F - 1$, this integral becomes

$$1 - M_n = \frac{1}{2^{n-1}} \int_0^1 \frac{n(1 - t^2)^{n-1}}{(1 + t)^{n-1} + (1 - t)^{n-1}} \, dt.$$

When n increases indefinitely,

$$(1 + t)^{n-1} = e^{nt}\left[1 + O\left(\frac{1}{n}\right)\right],$$

$$(1 - t)^{n-1} = e^{-nt}\left[1 + O\left(\frac{1}{n}\right)\right];$$

therefore,

$$(1 - t^2)^{n-1} = 1 + O\left(\frac{1}{n}\right).$$

Thus,

$$\begin{aligned} 1 - M_n &= \frac{1}{2^{n-2}} \int_0^1 \frac{n}{e^{nt} + e^{-nt}} \left[1 + O\left(\frac{1}{n}\right)\right] dt \\ &= \frac{1}{2^{n-2}} \int_0^1 \frac{ne^{nt}\, dt}{e^{2nt} + 1} \left[1 + O\left(\frac{1}{n}\right)\right] \\ &= \frac{1}{2^{n-2}} (\tan^{-1} e^{nt}) \Big|_0^1 \left[1 + O\left(\frac{1}{n}\right)\right] \\ &= \frac{1}{2^{n-2}} \left(\tan^{-1} e^n - \frac{\pi}{4}\right) \left[1 + O\left(\frac{1}{n}\right)\right]. \end{aligned}$$

---

[5] Thanks are due to Professor Olds at Carnegie Institute of Technology for calling the author's attention to R. L. Plackett's paper [8] which derived essentially the same result as given in Section 3 of the present paper by a somewhat different approach.

But

$$\tan^{-1} e^n = \frac{\pi}{2} + o\left(\frac{1}{n}\right).$$

Therefore,

$$1 - M_n = \frac{\pi}{2^n}\left[1 + O\left(\frac{1}{n}\right)\right].$$

This is (4.5).

2. Derivation of (5.1) and (5.2). In order to minimize (2.4) under the condition that (2.5) is kept constant, we put the first variation of

$$\int_{\frac{1}{2}}^{1} x(F)^2 n\{F^{n-1} + (1 - F)^{n-1}\}\, dF - \{E(x_n)\}^2 - 2\lambda \int_{\frac{1}{2}}^{1} x(F)^2\, dF$$

equal to zero, of course taking account of (2.3). Thus we obtain as the characteristic equation

$$x(F)n\{F^{n-1} + (1 - F)^{n-1}\} - E(x_n)n\{F^{n-1} - (1 - F)^{n-1}\} - 2\lambda x(F) = 0,$$

which can easily be solved as

$$x(F) = \frac{E(x_n)n\{F^{n-1} - (1 - F)^{n-1}\}}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda}.$$

But this solution is eligible only if it satisfies (2.3), that is only if

$$E(x_n) = E(x_n) \int_{\frac{1}{2}}^{1} \frac{n^2\{F^{n-1} - (1 - F)^{n-1}\}^2}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda}\, dF.$$

As $E(x_n)$ cannot be zero except in the trivial case $x(F) \equiv 0$, λ must be a solution of (5.2). If there exists a solution λ_n as is actually the case, then

$$x = \text{const.} \cdot \frac{n\{F^{n-1} - (1 - F)^{n-1}\}}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda_n}$$

is eligible as a solution of the characteristic equation.

3. Asymptotic formula for λ_n. $M_n(\lambda)$ can be transformed as follows.

$$M_n(\lambda) = \int_{\frac{1}{2}}^{1} \frac{n^2[\{F^{n-1} + (1 - F)^{n-1}\} - 2(1 - F)^{n-1}]^2}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda}\, dF$$

$$= \int_{\frac{1}{2}}^{1} \left[n\{F^{n-1} + (1 - F)^{n-1}\} + 2\lambda - 4n(1 - F)^{n-1} + \frac{\{2\lambda - 2n(1 - F)^{n-1}\}^2}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda} \right] dF$$

$$= 1 + \lambda - \frac{4}{2^n} + \int_{\frac{1}{2}}^{1} \frac{\{2\lambda - 2n(1 - F)^{n-1}\}^2}{n\{F^{n-1} + (1 - F)^{n-1}\} - 2\lambda}\, dF.$$

Therefore, λ_n must satisfy the equation

$$\lambda_n = \frac{4}{2^n} - \int_{\frac{1}{2}}^{1} \frac{\{2\lambda_n - 2n(1-F)^{n-1}\}^2}{n\{F^{n-1} + (1-F)^{n-1}\} - 2\lambda_n} dF.$$

As the integral is positive, we get $\lambda_n < 4/2^n$. This inequality certifies that the last term of the denominator in the last integral, or in (5.5), can be neglected as of order $1/n$ times that of the first term. Therefore

$$\lambda_n = \int_{\frac{1}{2}}^{1} \frac{4nF^{n-1}(1-F)^{n-1}}{F^{n-1} + (1-F)^{n-1}} dF \left[1 + O\left(\frac{1}{n}\right)\right]$$

$$= (1 - M_n)\left[1 + O\left(\frac{1}{n}\right)\right]$$

$$= \frac{\pi}{2^n}\left[1 + O\left(\frac{1}{n}\right)\right].$$

REFERENCES

[1] L. H. C. Tippett, "On the extreme individuals and the range of samples taken from a normal population," *Biometrika*, Vol. 17 (1925), pp. 364–387.

[2] B. de Finetti, "Sulla legge di probabilità degli estremi," *Metron*, Vol. 9 (1932), pp. 127–138.

[3] E. G. Olds, "Distribution of greatest variates, least variates and intervals of variation in samples from a rectangular universe," *Bull. Am. Math. Soc.*, Vol. 41 (1935), pp. 297–304.

[4] E. L. Dodd, "The greatest and the least variate under general laws of error," *Trans. Am. Math. Soc.*, Vol. 25 (1923), pp. 525–539.

[5] R. A. Fisher and L. H. C. Tippett, "Limiting forms of the frequency distribution of the largest or smallest member of a sample," *Proc. Cambridge Philos. Soc.*, Vol. 24 (1928), pp. 180–190.

[6] E. J. Gumbel, "Les valeurs extrêmes des distributions statistiques," *Annales Institut Henri Poincaré*, Vol. 4 (1935), pp. 115–158.

[7] H. J. Godwin, "Some low moments of order statistics," *Annals of Math. Stat.*, Vol. 20 (1949), pp. 279–285.

[8] R. L. Plackett, "Limits of the ratio of mean range to standard deviation," *Biometrika*, Vol. 34 (1947), pp. 120–122.

[9] K. Pearson, *Tables for Statisticians and Biometricians*, Part II, 1st ed., Cambridge University Press, 1931.