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Extremal properties of the first eigenvalue of a class of elliptic eigenvalue problems

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Extremal Properties of the First Eigenvalue of a Class of Elliptic Eigenvalue Problems

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0. - Introduction

In this paper we will study the following

PROBLEM: Find

$$q \in B_A = \left\{ f : \left(\int_\Omega |f|^p k^2 \right)^{1/p} \leq A \right\}$$

maximizing the first eigenvalue λ_1 of

$$(0.1) \qquad \begin{cases} Eu + qu = \lambda hu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where
(i) Ω is an open, bounded and connected domain in R^n, $n \geq 1$.
(ii) $Eu = -\partial_i(a_{ij}\partial_j u)$ ($\sum$-convention)
is a symmetric uniformly elliptic operator with coefficients in $L^1(\Omega)$, that is

$$a_{ij} = a_{ji} \text{ for all } 1 \leq i, j \leq n, \quad \{a_{ij}\} \subset L^1(\Omega)$$

and there exists a $\nu > 0$ such that

$$a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \text{ for all } (x, \xi) \in \Omega \times R^n.$$

We will call ν the ellipticity constant of E.
(iii) $h \in L^\infty$, k is measurable, $h > 0$ and $k \geq 0$ a.e. in Ω.
(iv) $1 \leq p < \infty$ and $0 < A < \infty$.

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REMARK: The case $p = \infty$ has been excluded, since it has the trivial solution $q = A/k^2$ if we take $B_A = \{f : \|fk^2\|_\infty \le A\}$.

This problem has its origin in a question posed by A. Ramm [RA]. His formulation is a special case of our problem with

$$\Omega = [-L, L] \subset R^1, \quad E = -\Delta, \quad p = 1 \text{ and } k = h = 1.$$

The problem has been solved in this special case by M. Essén [ES] and by G. Talenti [TA]. E. Harrell [HA] has solved the problem when Ω is a bounded domain in R^n, however his proof is incorrect (cf. Example 5).

Essén obtained his solution using his earlier results (cf. [ES]). Harrell used perturbation theory of self-adjoint operators. However, the approach of this paper is closer to that of Talenti.

Let

$$R_q(u) = \frac{a(u,u) + \int_\Omega qu^2}{\int_\Omega u^2 h}$$

be the Rayleigh quotient, where

$$a(u,v) = \int_\Omega a_{ij}\partial_i u \partial_j v.$$

Then the problem can be rewritten as:

Find $\tilde{q}$ and $\tilde{u}$ such that $\tilde{q} \in B_A$, $\tilde{u} = 0$ on $\partial\Omega$ and

$$\sup_q \inf_u R_q(u) = \inf_u R_{\tilde{q}}(u) = R_{\tilde{q}}(\tilde{u}),$$

where the infimum and supremum are taken over the appropriate classes of functions.

The solution $\tilde{q}$ of this problem solves the original problem and $\tilde{u}$ is the first eigenfunction corresponding to $\tilde{q}$. We will say that $(\tilde{q}, \tilde{u})$ is an extremal couple.

Let us take $h = k = 1$ in the sequel for simplicity. The idea of Talenti is to estimate the Rayleigh quotient from above by a functional which is independent of q. To see this, we note that for any $q \in B_A$, Hölder's inequality gives

$$R_q(u) \le \frac{a(u,u) + A\|u\|_{2p'}^2}{\int_\Omega u^2} = J(u). \tag{0.2}$$

It will turn out that there exists a $\tilde{q} \in B_A$ such that

$$\inf_u R_{\tilde{q}}(u) = \inf_u J(u) \tag{0.3}$$

and both sides attain their minima for the same function $\tilde{u}$. Hence $(\tilde{q}, \tilde{u})$ is an extremal couple. It is remarkable that the minimizer $\tilde{u}$ of $R_{\tilde{q}}$ gives equality in the Hölder inequality (0.2), but that is the reason why this method works. The equality (0.3) turns out to be useful when proving other properties of the extremal couple.

The main results in this paper in the case $k = h = 1$ are collected in the following two theorems.

THEOREM I: *Let $p \in (1, \infty)$ be given and assume that $k = h = 1$. Then there exists an extremal couple $(\tilde{q}, \tilde{u})$ with the following properties:*

(i) *$\tilde{u}$ is the unique non-negative minimizer of J.*

(ii) *$\tilde{u}$ is the first eigenfunction of the eigenvalue problem*

$$\begin{cases} Eu + A\|u^2\|^{1-p'}|u|^{2(p'-1)}u = \mu u, \\ u = 0 \quad on\ \partial\Omega. \end{cases}$$

Furthermore, the first eigenvalue μ_1 is equal to both $J(\tilde{u})$ and the maximal eigenvalue λ_1 of (0.1) $\left(\frac{1}{p'} + \frac{1}{p}\right) = 1$.

(iii) $\tilde{q} = A|\tilde{u}|^{\frac{2p'}{p}} \|\tilde{u}^2\|_{p'}^{1-p'}$.

(iv) *$\tilde{q}$ is unique.*

(v) $\|\tilde{q}\|_\infty \leq \lambda_1$ *and* $\|\tilde{u}\|_\infty \leq \left(\dfrac{\lambda_1}{A}\right)^{\frac{p-1}{2}} \|\tilde{u}\|_{2p'} < \infty$.

(vi) *If $\{a_{ij}\} \subset L^\infty(\Omega)$ then $\tilde{q}$ and $\tilde{u}$ are locally Hölder continuous. Furthermore, if Ω has the exterior cone property then $\tilde{u}$ and $\tilde{q}$ are continuous in $\overline{\Omega}$.*

(vii) *If $\{a_{ij}\} \subset C^{m+1,\alpha}(\Omega)$, then all derivatives of order less than or equal to $m+2$ of $\tilde{u}$ and $\tilde{q}$ are locally Hölder continuous in Ω.*

(viii) *If $\{a_{ij}\} \subset C^{m+1,\alpha}(\overline{\Omega})$, $\partial\Omega \in C^{m+2,\alpha}$ and if either $m \leq \dfrac{p+1}{p-1}$ or $p = \dfrac{\ell+1}{\ell-1}$ for some positive integer ℓ, then $\tilde{u}$, $\tilde{q} \in C^{m+2,\beta}(\overline{\Omega})$ for some β.*

(ix) *$\tilde{q}$ and $\tilde{u}$ have the same symmetries as Ω and E.*

THEOREM II: *Let $p = 1$. Assume that $k = h = 1$ and $\{a_{ij}\} \subset L^\infty(\Omega)$. Then there exists an extremal couple $(\tilde{q}, \tilde{u})$ with the following properties:*

(i) *$\tilde{u}$ is the unique minimizer of J.*

(ii) $\tilde{u} \geq 0$ *and* $\|\tilde{u}\|_\infty = 1$.

(iii) *$\tilde{q} \leq \lambda_1 \chi\{I\}$ with equality in the interior of $I = \{x \in \Omega : \ \tilde{u}(x) = 1\}$. Here $\lambda_1 = J(\tilde{u})$ is the maximal first eigenvalue of* (0.1).

(iv) *If $\{a_{ij}\} \subset C^{0,1}(\Omega)$ then $\tilde{q} = \dfrac{A}{m(I)}\chi\{I\}$ and $\lambda_1 = \dfrac{A}{m(I)}$.*

(v) *$\tilde{q}$ is unique.*

(vi) *$\tilde{q}$ and $\tilde{u}$ have the same symmetries as Ω and E.*

Results similar to those mentioned above hold in the general case where k and h are non-constant (see Section 5).

In Section 6 we give some examples and point out some generalizations. In this section we also consider the problem of minimizing the first eigenvalue of (0.1) over B_A.

REMARK: After the completion of this manuscript I have received a preprint from E. Harrell and M. Ashbaugh [AH]. They have studied the problem of minimizing and maximizing the first eigenvalue of (0.1) over B_A when $p > 1$ and $k = h = 1$. Their paper also contains other related results.

1. - Some definitions and preliminary lemmas

In this section, we will construct a dense subspace of $H_0^1(\Omega)$ on which the bilinear form $a(u,v)$ is well defined and we will show some lemmas about functions in this subspace. These results will be used to prove some technical lemmas that will be needed later.

We have to define what we mean by a solution of (0.1). If $\{a_{ij}\} \subset L^\infty(\Omega)$ there is no problem with the following standard definition

$$u \in H_0^1(\Omega): \ a(u,v) + \int_\Omega quv = \int_\Omega \lambda uv, \quad \forall v \in H_0^1(\Omega). \tag{1.1}$$

However, if the coefficients $\{a_{ij}\}$ are supposed to be in $L^1(\Omega)$ only, then the bilinear form $a(u,v)$ is not well-defined on the whole of $H_0^1(\Omega)$. In this case we have to find an appropriate definition of a solution of (0.1).

The bilinear form $a(u,v)$ is well-defined on $H_0^{1,\infty}(\Omega)$ and $(a, H_0^{1,\infty}(\Omega))$ is a pre-Hilbert space. The completion of this space is a Hilbert space $(\tilde{a}, H_a(\Omega))$ with the following properties:

(1.2) $H_0^{1,\infty}(\Omega)$ is dense in $H_a(\Omega)$. Hence $C_0^1(\Omega)$ is also dense in $H_a(\Omega)$.

(1.3) If $u, v \in H_0^{1,\infty}(\Omega)$ then $\tilde{a}(u,v) = a(u,v)$ and $\tilde{a}(u,u) \geq \nu \|u\|^2_{H_0^1}$, where ν is the ellipticity constant.

Using this we can define a map

$$j: \ H_a(\Omega) \to H_0^1(\Omega)$$

as follows:

Take $u \in H_a(\Omega)$ and $\{u_n\} \subset H_0^{1,\infty}(\Omega)$ converging to u in $H_a(\Omega)$. From (1.3) above it follows that

$$u_n \to w \text{ in } H_0^1(\Omega).$$

We define $j(u) = w$.

Clearly j is a well defined continuous linear map. Furthermore, $j(\varphi) = \varphi$, whenever $\varphi \in H_0^{1,\infty}(\Omega)$.

LEMMA 1: *If $u_n \rightharpoonup u$ in $H_a(\Omega)$ and $j(u_n) \rightharpoonup v$ in $H_0^1(\Omega)$, then $j(u) = v$.*

REMARK: If follows that j is sequentially continuous w.r.t. the weak topologies.

PROOF. Let $j^* : H_0^1(\Omega) \to H_a(\Omega)$ be the Hilbert space adjoint of j defined by the relation

$$\tilde{a}(j^*(w), u) = (w, j(u))_{H_0^1} \text{ for all } u \in H_a(\Omega) \text{ and all } w \in H_0^1(\Omega).$$

From the assumptions, we get that for all $w \in H_0^1(\Omega)$,

$$\lim_{n\to\infty} \tilde{a}(j^*(w), u_n) = \tilde{a}(j^*(w), u),$$
$$\lim_{n\to\infty} (w, j(u_n))_{H_0^1} = (w, v)_{H_0^1}.$$

Therefore $(w, j(u))_{H_0^1} = (w, v)_{H_0^1}$ for all $w \in H_0^1(\Omega)$. Hence $j(u) = v$.

LEMMA 2: *The map j is injective.*

PROOF. Take $u \in H_a(\Omega)$ and $\{u_k\} \subset H_0^{1,\infty}(\Omega)$ such that $j(u) = 0$ and $u_k \to u$ in $H_a(\Omega)$. Since $u_k \to 0$ in $H_0^1(\Omega)$ we can assume that $\partial_i u_k \to 0$ a.e. in Ω for $i = 1, \ldots, n$. From Fatou's lemma, we see that

$$\tilde{a}(u_k, u_k) = a(u_k, u_k) \leq \varliminf_{m\to\infty} a(u_k - u_m, u_k - u_m).$$

The right-hand side tends to zero as $k \to \infty$. Hence $u = 0$.

From Lemma 2, it follows that we can use j to imbed $H_a(\Omega)$ as a dense subspace in $H_0^1(\Omega)$ and give $j(H_a(\Omega))$ the topology induced by j. From the open mapping theorem we get

$$j(H_a(\Omega)) = H_0^1(\Omega) \text{ algebraically } \Leftrightarrow \exists\Lambda \text{ such that}$$
$$a(u, u) \leq \Lambda \|u\|_{H_0^1}^2, \quad \forall u \in H_0^1(\Omega).$$

From now on we will always regard $H_a(\Omega)$ as a subspace of $H_0^1(\Omega)$.

DEFINITION: *If the operator E corresponds to the bilinear form $a(\cdot,\cdot)$ we will say that u is a solution of* (0.1) *if*

$$u \in H_a(\Omega) : \ \tilde{a}(u, v) + \int_\Omega quv = \int_\Omega \lambda uv, \quad \forall v \in H_a(\Omega).$$

REMARK: This definition is consistent with (1.1), since $H_a(\Omega) = H_0^1(\Omega)$ whenever $\{a_{ij}\} \subset L^\infty(\Omega)$. We also see that the problem of finding the first eigenvalue of (0.1) is equivalent to the problem of minimizing the Rayleigh quotient, provided $q \in L^1_{\text{loc}}(\Omega)$.

The method used to construct $H_a(\Omega)$ above is closely related to the standard procedure used to define a self-adjoint operator using bilinear forms. Actually, it turns out that E can be represented as a self-adjoint operator with domain $D(E)$ dense in $H_a(\Omega)$. For further information, see [WE], Chapter 5.

For functions in $H_0^1(\Omega)$ we have the following well-known facts.

LEMMA 3: *If $u \in H^1(\Omega)$, then $u^+ \in H^1(\Omega)$ and*

$$\partial_j u^+ = \begin{cases} \partial_j u & \text{on } \{x: \ u(x) > 0\} \\ 0 & \text{elsewhere.} \end{cases}$$

If $u \in H_0^1(\Omega)$ and $\{u_m\}$ is a sequence in $H_0^1(\Omega)$ converging to u, then
(i) $\min(u,n) \in H_0^1(\Omega)$ *for* $n = 0,1,2,\ldots$ *and* $\min(u,n) \to u$ *in* $H_0^1(\Omega)$ *as* $n \to \infty$.
(ii) $\min(u_m,n) \to \min(u,n)$ *in* $H_0^1(\Omega)$ *as* $m \to \infty$, *for* $n = 0,1,2,\ldots$.
(iii) $|u_m| \in H_0^1(\Omega)$ *and* $|u_m| \to |u|$ *in* $H_0^1(\Omega)$ *as* $m \to \infty$.

A proof of the first statement in the lemma can be found in [GT], Chapter 7. The other statements follow easily from the first one.

LEMMA 4: *If $u \in H_a(\Omega)$ and $n = 0,1,2,\ldots$, then*
(i) $\min(u,n) \in H_a(\Omega)$ *and* $\min(u,n) \to u$ *in* $H_a(\Omega)$ *as* $n \to \infty$.
(ii) *If* $\xi = u - \min(u,n)$ *then* $\tilde{a}(u,\xi) = \tilde{a}(\xi,\xi)$.
(iii) $|u| \in H_a(\Omega)$ *and* $\tilde{a}(|u|,|u|) = \tilde{a}(u,u)$.

PROOF: Take $\{u_j\} \subset H_0^{1,\infty}(\Omega)$ so that $u_j \to u$ in $H_a(\Omega)$ as $j \to \infty$. Then $\min(u_j,n) \in H_0^{1,\infty}(\Omega)$ and Lemma 3 above gives

$$\|\min(u_j,n)\|_{H_a}^2 = a(\min(u_j,n),\min(u_j,n)) \le a(u_j,u_j) = \|u_j\|_{H_a}^2.$$

The right-hand side is bounded and we know that $\min(u_j,n) \to \min(u,n)$ in $H_0^1(\Omega)$. Hence a compactness argument and Lemma 1 yield

$$\min(u_j,n) \rightharpoonup \min(u,n) \text{ in } H_a(\Omega) \text{ as } j \to \infty.$$

Furthermore,

$$\|\min(u,n)\|_{H_a} \le \|u\|_{H_a}$$

since the weak limit must be in the convex set

$$\left\{ w: \ \|w\|_{H_a} \le \lim_{j\to\infty} \|u_j\|_{H_a} \right\}.$$

The same argument as above gives

$$\min(u,n) \rightharpoonup u \text{ in } H_a(\Omega) \text{ as } n \to \infty \text{ and}$$
$$\lim_{n\to\infty} \|\min(u,n)\|_{H_a} \le \|u\|_{H_a}.$$

Now (i) follows from the fact that the norm is lower semicontinuous w.r.t. the weak topology.

To prove (ii) take $\{u_j\}$ as above and define $\xi_j = u_j - \min(u_j, n)$. Then $a(u_j, \xi_j) = a(\xi_j, \xi_j)$ and $\xi_j \to \xi$ in $H_a(\Omega)$ by (i). This proves (ii).

The last statement (iii) is proved using the same argument as above.

From now on we will always write $a(u, v)$ *instead of* $\tilde{a}(u, v)$.

By the Rellich compactness theorem we know that the inclusion map

$$i : \; H_0^1(\Omega) \to L^2(\Omega)$$

is compact. Hence the inclusion map

$$i' : \; H_a(\Omega) \to L^2(\Omega)$$

is also compact.

PROPOSITION 5: *If* $q \geq 0$ *then both* R_q *and* J, *as defined in Section 0, attain their minima in* $H_a(\Omega)$. *Furthermore, we can assume that the minimizers are non-negative.*

REMARK: The condition $q \geq 0$ is always satisfied in the interesting cases, since the function q maximizing the first eigenvalue of (0.1) can be assumed to be non-negative.

PROOF. This proof is standard and we only prove the existence of a minimizer of J. Let $\{u_n\} \subset H_a(\Omega)$ be a minimizing sequence normalized so that $\|u_n\|_2 = 1$. Then $\{u_n\}$ is bounded in $H_a(\Omega)$ and in $L^{2p'}(\Omega)$. Therefore, assuming a subsequence has been selected, we have $u_n \overset{*}{\rightharpoonup} \tilde{u}$ in $H_a(\Omega)$ and in $L^{2p'}(\Omega)$. Furthermore, we have strong convergence in $L^2(\Omega)$. A semicontinuity argument yields that $J(\tilde{u}) \leq \underline{\lim}_{n\to\infty} J(u_n)$. Hence $\tilde{u} \in H_a(\Omega)$ is a minimizer. Finally, Lemma 4 shows that $J(\tilde{u}) = J(|\tilde{u}|)$, so the minimizer can be assumed to be non-negative.

The following lemma will be useful later on.

LEMMA 6: *Let* $\tilde{u}$ *be a non-negative minimizer of* J *and put* $\lambda = J(\tilde{u})$. *Assume that there exists a non-negative function* $\tilde{q} \in B_A$ $(k = 1)$ *such that*

$$\text{ess supp } \tilde{q} \subset \text{ess supp } \tilde{u} \textit{ and } E\tilde{u} + \tilde{q}\tilde{u} = \lambda\tilde{u}.$$

Then $\inf_{u \in H_a(\Omega)} R_{\tilde{q}}(u) = R_{\tilde{q}}(\tilde{u}) = \lambda$, *and* $(\tilde{q}, \tilde{u})$ *is an extremal couple.*

REMARK: The existence of a function $\tilde{q}$ with the properties in the lemma will be proved later.

PROOF. We can assume that $||\tilde{u}||_{2p'} = 1$. Let v be a non-negative minimizer of $R_{\tilde{q}}$ and assume that $R_{\tilde{q}}(v) = \lambda' < \lambda$. Then we get

$$E\tilde{u} + \tilde{q}\tilde{u} = \lambda\tilde{u},$$
$$Ev + \tilde{q}v = \lambda' v.$$

This shows that $(\lambda - \lambda')\int_\Omega \tilde{u}v = 0$. Hence

$$\tilde{u}v = 0 \text{ and } \tilde{q}v = 0 \text{ a.e. in } \Omega. \tag{1.4}$$

Put $v_n = \dfrac{1}{n}\min(v, n)$ and

$$\lambda_n = \frac{a(v_n, v_n)}{\int_\Omega v_n^2}.$$

From Lemma 4 we have $nv_n \to v$ in $H_a(\Omega)$ which shows that $\lambda_n \to \lambda'$ as $n \to \infty$. We note that $\int_\Omega v_n\tilde{u} = 0$ and that $a(\tilde{u}, v_n) = 0$.

If $p = 1$ we get, using $||\tilde{u} + v_n||_\infty = ||\tilde{u}||_\infty = 1$, that

$$J(\tilde{u} + v_n) = \frac{\lambda\int_\Omega \tilde{u}^2 + \lambda_n\int_\Omega v_n^2}{\int_\Omega \tilde{u}^2 + \int_\Omega v_n^2} < \lambda$$

if n is large enough. This is a contradiction and hence $\lambda = \lambda'$ if $p = 1$.

If $p > 1$, we argue in the following way. We choose n so large that $\lambda_n < \lambda$ and consider

$$J(\tilde{u} + \varepsilon v_n) = \frac{\lambda\int_\Omega \tilde{u}^2 + \varepsilon^2\lambda_n\int_\Omega v_n^2 + A(||\tilde{u} + \varepsilon v_n||_{2p'}^2 - ||\tilde{u}||_{2p'}^2)}{\int_\Omega \tilde{u}^2 + \varepsilon^2\int_\Omega v_n^2} =$$
$$= \lambda - \varepsilon^2(\lambda - \lambda_n)\frac{\int_\Omega v_n^2}{\int_\Omega \tilde{u}^2} + o(\varepsilon^2) < \lambda,$$

where the last relation holds if ε is small enough.

REMARK: If $\{a_{ij}\} \subset L^\infty(\Omega)$, then (1.4) gives us $Ev = \lambda' v \geq 0$. We can use the strong maximum principle for weak solutions [GT] to conclude that $v > 0$ a.e. in Ω. In this case, the proof is less complicated.

Finally, we have the following approximation lemma.

LEMMA 7: *Let* $\{a_{ij}\} \subset L^p(\Omega)$ *where* $1 \leq p \leq \infty$. *Then there exists* $\{a_{ij}^\varepsilon\} \subset C^\infty(\overline{\Omega})$ *with the following properties:*

(i) *They fulfil the uniform ellipticity condition with the same constant as* $\{a_{ij}\}$.
(ii) $a^{\varepsilon}_{ij} \to a_{ij}$ *in* L^q *as* $\varepsilon \to 0$, *where* $q = p$ *if* $p < \infty$ *and* q *can be any number in* $[1, \infty)$ *if* $p = \infty$.

PROOF. Extend $\{a_{ij}\}$ to functions on R^n by letting $a_{ij} = \nu$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$ outside Ω. Let φ_ε be a smooth non-negative approximative identity. Then $a^{\varepsilon}_{ij} = a_{ij} * \varphi_\varepsilon$ has the desired properties.

2. - Solution of the problem when $1 < p < \infty$ and $k = h = 1$

Because the problem is easier to solve when $p > 1$ we will deal with this case first to demonstrate the use of the lemmas in Section 1. We believe, however, that the problem is more interesting when $p = 1$.

If $u \in H_a(\Omega) \cap L^{2p'}(\Omega)$ then J is Gateaux-differentiable at u and we have with $\varphi \in C^1_0(\Omega)$

$$J'_\varphi(u) = \frac{2}{\int\limits_\Omega u^2}\left(a(u,\varphi) + A\|u^2\|^{1-p'}_{p'} \int\limits_\Omega |u|^{2(p'-1)} u\varphi - J(u)\int\limits_\Omega u\varphi\right).$$

Hence the non-negative minimizer $\tilde{u}$ solves the equation

$$u \in H_a(\Omega) : \ Eu + \tilde{q}u = \lambda u$$

where

$$\lambda = J(\tilde{u}) \text{ and } \tilde{q} = A\|\tilde{u}^2\|^{1-p'}_{p'}\tilde{u}^{2(p'-1)}.$$

A direct calculation yields $\|\tilde{q}\|_p = A$ and hence $\tilde{q} \in B_A$. Now, Lemma 6 shows that $(\tilde{q}, \tilde{u})$ is an extremal couple and $\lambda_1 = J(\tilde{u})$ is the maximal first eigenvalue. Note that $\tilde{u}$ is the first eigenfunction of the eigenvalue problem

$$u \in H_a(\Omega) : \ Eu + A\|u^2\|^{1-p'}_{p'}|u|^{2(p'-1)}u = \mu u \tag{2.1}$$

and the first eigenvalue μ_1 is equal to λ_1.

Thus we have proved the following result.

THEOREM 8: *Let* $1 < p < \infty$. *Then there exists an extremal couple* $(\tilde{q}, \tilde{u})$ *which solves the problem and which has the following properties:*
(i) $\tilde{u}$ *is a non-negative minimizer of* J *and the first eigenfunction of* (2.1) *with eigenvalue* $J(\tilde{u})$.
(ii) $\tilde{q} = A\tilde{u}^{\frac{2p'}{p}}\|\tilde{u}^2\|^{1-p'}_{p'}$
(iii) $R_{\tilde{q}}(\tilde{u}) = J(\tilde{u}) = \lambda_1$. *Here* λ_1 *is the maximal first eigenvalue.*

COROLLARY 9: *We have the following estimates:*

$$\|\tilde{u}\|_\infty \leq \left(\frac{\lambda_1}{A}\right)^{\frac{p-1}{2}} \|\tilde{u}\|_{2p'} < \infty,$$
$$\|\tilde{q}\|_\infty \leq \lambda_1.$$

PROOF. We can assume that $\tilde{u} \geq 0$ and $\|\tilde{u}\|_{2p'} = 1$. Put $c = \left(\frac{\lambda_1}{A}\right)^{\frac{p-1}{2}}$ and $\xi = \tilde{u} - \min(\tilde{u}, c)$. Then $\xi \in H_a(\Omega)$ and $\xi \geq 0$. Lemma 4 (ii) and Theorem 8 give

$$a(\xi,\xi) = a(\tilde{u},\xi) = \int_\Omega (\lambda_1 - A|\tilde{u}|^{2(p'-1)})\tilde{u}\xi \leq 0,$$

since the integrand is negative if $\xi > 0$. Hence $\xi = 0$, that is $\tilde{u} \leq c$. The estimate for $\tilde{q}$ follows directly from the estimate of $\tilde{u}$ and Theorem 8.

From Theorem 8 and Corollary 9 we see that $E\tilde{u} \in L^\infty(\Omega)$. Hence we can apply the regularity theory for elliptic operators with bounded measurable coefficients.

COROLLARY 10: *If $\{a_{ij}\} \subset L^\infty(\Omega)$ then $\tilde{u}$ and hence also $\tilde{q}$ is locally Hölder continuous in Ω. If the boundary $\partial\Omega$ satisfies the (uniform) exterior cone condition, then both $\tilde{u}$ and $\tilde{q}$ are in $C(\overline{\Omega})$ $(C^\alpha(\overline{\Omega}))$.*

For a proof of this regularity result see for example [GT], Chapter 8.

REMARK: Ω is said to satisfy the exterior cone condition if for every $x \in \partial\Omega$ there exists a finite circular cone V_x with vertex x such that $\overline{\Omega} \cap V_x = x$. For the uniform exterior cone condition we also need that the cones V_x are all congruent to some fix cone.

Higher regularity can now be deduced using the regularity theory for P.D.E.'s with Hölder continuous coefficients.

Write (2.1) as $Eu = f(u)$ and assume that $\{a_{ij}\} \subset L^\infty(\Omega)$ and that $\tilde{u}$ is a non-negative solution normalized so that $\|\tilde{u}\|_{2p'} = 1$. Since $f(\tilde{u}) \geq 0$ by Corollary 9, the strong maximum principle shows that $\tilde{u}$ does not have any interior minima. Hence $\tilde{u} > 0$ in Ω. Using this fact, it is easy to verify that $f(\tilde{u}) \in C^{m,\alpha}(\Omega)$ is $\tilde{u} \in C^{m,\alpha}(\Omega)$. If we replace Ω by $\overline{\Omega}$, we can still deduce the following result.

If $\tilde{u} \in C^{m,\alpha}(\overline{\Omega})$ and if either $m \leq \frac{p+1}{p-1}$ or $p = \frac{\ell+1}{\ell-1}$ for some positive integer ℓ, then $f(\tilde{u}) \in C^{m,\beta}(\overline{\Omega})$ for some β.

Now, we can apply the regularity theorems in [GT], Chapter 6.

COROLLARY 11: *If $\{\partial^\gamma a_{ij}\}$ are locally Hölder continuous for all $|\gamma| \leq m+1$ then $\partial^\gamma \tilde{u}$ and $\partial^\gamma \tilde{q}$ are locally Hölder continuous for all $|\gamma| \leq m+2$. If*

$\{a_{ij}\} \subset C^{m+1,\alpha}(\overline{\Omega})$, $\partial\Omega \in C^{m+2,\alpha}$, *and if either* $m \leq \frac{p+1}{p-1}$ *or* $p = \frac{\ell+1}{\ell-1}$ *for some integer* ℓ, *then* $\tilde{u}, \tilde{q} \in C^{m+2,\beta}(\overline{\Omega})$ *for some* β.

REMARK: In the discussion above we saw that if $\{a_{ij}\} \subset L^\infty(\Omega)$ then both $\tilde{u}$ and $\tilde{q}$ are strictly positive in Ω.

3. - Solution of the problem when $p = 1$ and $k = h = 1$

Since J is not Gateaux-differentiable for $p = 1$, we have to use a method different from the one given in Section 2. We shall transform the problem into a variational inequality and use the regularity theory which is available here.

Define $K = \{u \in H_a(\Omega) : \ \|u\|_\infty \leq 1\}$ and

$$T(u) = \frac{a(u,u) + A}{\int\limits_\Omega u^2}.$$

Then

$$\inf_{u \in H_a(\Omega)} J(u) = \inf_{u \in K} T(u).$$

If $\tilde{u}$ is a minimizer of J normalized so that $\|\tilde{u}\|_\infty = 1$, then $\tilde{u}$ is also a minimizer of T over K.

PROPOSITION 12: *If $\tilde{u}$ is a minimizer of J such that $\|\tilde{u}\|_\infty = 1$, then $\tilde{u}$ solves the variational inequality*

$$u \in K : \ a(u, v-u) - J(\tilde{u}) \int\limits_\Omega u(v-u) \geq 0, \quad \forall v \in K, \tag{3.1}$$

where $K = \{u \in H_a(\Omega) : \ |u| \leq 1\}$.

PROOF. Take $v \in K$ and $t \in (0,1)$, then $\tilde{u} + t(v - \tilde{u}) \in K$ by convexity and we get

$$\begin{aligned} 0 &\leq T(\tilde{u} + t(v-\tilde{u})) - T(\tilde{u}) = \\ &= \frac{2t}{\int\limits_\Omega \tilde{u}^2}\Big(a(\tilde{u}, v-\tilde{u}) - T(\tilde{u}) \int\limits_\Omega \tilde{u}(v-\tilde{u})\Big) + o(t). \end{aligned}$$

The result follows.

REMARK: If we take $\tilde{u}$ to be a non-negative minimizer then Proposition 12 holds with $K = \{u \in H_a(\Omega) : \ u \leq 1\}$.

The operator in the variational inequality (3.1) is not monotone, but it is still possible to apply the standard technique to obtain the following regularity result.

THEOREM 13: *If* $\partial\Omega \in C^2$, $\{a_{ij}\} \subset C^{0,1}(\overline{\Omega})$ *and* u *is a non-negative solution of* (3.1), *then*

$$u \in H^{2,s}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}), \quad \alpha = 1 - \frac{n}{s} \text{ for all } s \in (n, \infty).$$

The proof will be given in the Appendix.

Let us assume that the conditions in Theorem 13 hold and let $\tilde{u}$ be a non-negative minimizer of J normalized so that $\|\tilde{u}\|_\infty = 1$. Define the coincidence set as $I = \{x \in \Omega : \ \tilde{u}(x) = 1\}$, which is obviously closed. Since $\tilde{u} \in H^{2,s}(\Omega)$, a variation outside I in the variational inequality (3.1) yields $E\tilde{u} = \lambda\tilde{u}$ pointwise a.e. in $\Omega \setminus I$, where $\lambda = J(\tilde{u})$. In the coincidence set I, we can apply Lemma 7.7 [GT] twice to obtain $E\tilde{u} = 0$ pointwise a.e. in I. Hence $E\tilde{u} + \lambda_{\chi\{I\}}\tilde{u} = \lambda\tilde{u}$ holds a.e. in Ω. If we multiply this equation with $\tilde{u}$ and integrate we get

$$\lambda = J(\tilde{u}) = \frac{a(\tilde{u}, \tilde{u}) + \lambda m(I)}{\int\limits_\Omega \tilde{u}^2}.$$

This shows that $\lambda m(I) = A$. Thus it follows from Lemma 6 that $(\tilde{q}, \tilde{u})$ is an extremal couple, with $\tilde{q} = \frac{A}{m(I)}\chi\{I\}$ and maximal eigenvalue $\lambda_1 = \frac{A}{m(I)}$. However, the conditions on $\{a_{ij}\}$ and $\partial\Omega$ can be relaxed.

First we will prove a result when $\{a_{ij}\} \subset C^{0,1}(\Omega)$, then this result will be extended to coefficients $\{a_{ij}\} \subset L^\infty(\Omega)$.

THEOREM 14: *If* $\{a_{ij}\} \subset C^{0,1}(\Omega)$ *then there exists an extremal couple* $(\tilde{q}, \tilde{u})$ *which solves the problem and which has the following properties*

(i) $\tilde{u}$ *is a minimizer of* J.

(ii) $\tilde{u} \geq 0$ *and* $\|\tilde{u}\|_\infty = 1$.

(iii) $\tilde{q} = \frac{A}{m(I)}\chi\{I\}$, *where* $I = \{x \in \Omega : \ \tilde{u}(x) = 1\}$.

(iv) *The maximal first eigenvalue* $\lambda_1 = \frac{A}{m(I)}$.

(v) $E\tilde{u} + \tilde{q}\tilde{u} = \lambda_1\tilde{u}$ *holds pointwise a.e. in* Ω.

(vi) $R_{\tilde{q}}(\tilde{u}) = J(\tilde{u})$.

PROOF. According to Theorem 13 and the discussion above, the theorem is true if we assume also that $\partial\Omega \in C^2$ and $\{a_{ij}\} \subset C^{0,1}(\overline{\Omega})$. To remove these restrictions, let $\{\Omega_n\}$ be smooth open domains in Ω with the following properties

$$\Omega_n \subset\subset \Omega \text{ and } \Omega_n \subset \Omega_{n+1}, \quad n = 1, 2, \ldots, \tag{3.2a}$$

$$\forall K \subset\subset \Omega, \quad \exists N \text{ such that } K \subset \Omega_N. \tag{3.2b}$$

Such domains can easily be constructed with the aid of the regularized distance [ST] and Sard's Theorem [NI].

Let (u_n, λ_n, q_n) be the solution of the problem with Ω replaced by Ω_n, and assume that $u_n \geq 0$ and $\|u_n\|_\infty = 1$. Here λ_n is the corresponding maximal first eigenvalue. Since $\partial\Omega_n \in C^\infty$ and $\{a_{ij}\} \subset C^{0,1}(\overline{\Omega}_n)$ we know that this solution exists and that it has the properties given in the theorem. Furthermore, we can extend u_n and q_n to functions in $H_0^1(\Omega)$ and $L^\infty(\Omega)$ respectively, by letting them be zero outside Ω_n.

It is easy to see that $\lambda_n \searrow \lambda$ as $n \to \infty$, $\{q_n\}$ is bounded in $L^\infty(\Omega)$ and that $a(u_n, u_n)$ is bounded. Hence we have, assuming a subsequence has been chosen,

$$\begin{aligned} &q_n \overset{*}{\rightharpoonup} q \text{ in } L^\infty(\Omega),\\ &u_n \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ and}\\ &u_n \to u \text{ in } L^2(\Omega) \text{ and pointwise a.e. in } \Omega. \end{aligned}$$

Using this we get

$$0 \leq u \leq 1 \text{ and } 0 \leq q \leq \lambda \text{ a.e. in } \Omega \tag{3.3}$$

$$\int_\Omega q = \lim_{n\to\infty} \int_\Omega q_n = A, \quad \int_\Omega qu^2 = \lim_{n\to\infty} \int_\Omega q_n u_n^2 = A. \tag{3.4}$$

Now, take $\varphi \in C_0^1(\Omega)$ and by (3.2b) we can choose n so that supp $\varphi \subset \Omega_n$. Then

$$\int_\Omega a_{ij}\partial_i u_n \partial_j \varphi + \int_\Omega q_n u_n \varphi = \lambda_n \int_\Omega u_n \varphi.$$

Letting $n \to \infty$ we obtain

$$a(u, \varphi) + \int_\Omega qu\varphi = \lambda \int_\Omega u\varphi, \quad \forall \varphi \in C_0^1(\Omega). \tag{3.5}$$

By continuity, this equation holds also for all $\varphi \in H_0^1(\Omega)$. If we put $\varphi = u$ in (3.5) it is easy to establish the following equalities

$$\lambda = R_q(u) = J(u) = \lim_{n\to\infty} \inf_{v\in H_0^1(\Omega_n)} J(v) = \inf_{v\in H_0^1(\Omega)} J(v).$$

Therefore, we conclude that u is a minimizer of J.

From (3.3) and (3.4) above we also get

$$q(1-u^2) \geq 0 \text{ and } \int_\Omega q(1-u^2) = 0,$$

showing that

$$q(1-u^2) = 0 \text{ a.e. in } \Omega.$$

Therefore q has its essential support in $I = \{x \in \Omega : u(x) = 1\}$. Hence Lemma 6 shows that (q, u) is an extremal couple, and $\lambda_1 = \lambda$ is the maximal eigenvalue.

Since $Eu \in L^\infty(\Omega)$ and $\{a_{ij}\} \subset C^{0,1}(\Omega)$ we find that $u \in H^2_{\text{loc}}(\Omega)$, [GT], Chapter 8. Therefore $Eu = 0$ a.e. in I and by (3.5) $Eu + qu = \lambda u$ holds pointwise a.e. in Ω. Hence $q = \lambda_1 \chi_{\{I\}}$ and by (3.4) $\lambda_1 = \dfrac{A}{m(I)}$.

We can use an argument similar to the one given above to obtain the following result.

THEOREM 15: *If $\{a_{ij}\} \subset L^\infty(\Omega)$ then there exists an extremal couple $(\tilde{q}, \tilde{u})$ which solves the problem and which has the properties:*

(i) *$\tilde{u}$ is a minimizer of $J(u)$.*

(ii) *$\tilde{u} \geq 0$ and $\|\tilde{u}\|_\infty = 1$.*

(iii) *$\tilde{q} \leq \lambda_1 \chi_{\{I\}}$ with equality in the interior of $I = \{x \in \Omega : \tilde{u}(x) = 1\}$.*

(iv) *The maximal first eigenvalue $\lambda_1 = J(\tilde{u})$.*

REMARK: Since $E\tilde{u} = \lambda_1 \tilde{u} - \tilde{q}\tilde{u} \in L^\infty(\Omega)$, $\tilde{u}$ is locally Hölder continuous in Ω and hence the coincidence set I is relatively closed in Ω. If Ω satisfies the exterior cone condition, then $u \in C(\overline{\Omega})$ and I is compact in Ω.

PROOF. Let $\{a^\varepsilon_{ij}\} \subset C^\infty(\overline{\Omega})$ be the sequence approximating $\{a_{ij}\}$ constructed in Lemma 7 and let $(u_\varepsilon, q_\varepsilon, \lambda_\varepsilon)$ be the solution of the problem with $\{a_{ij}\}$ replaced by $\{a^\varepsilon_{ij}\}$. As before, we assume u_ε to be non-negative and $\|u_\varepsilon\|_\infty = 1$. A semicontinuity argument yields

$$\lambda = \overline{\lim_{\varepsilon \to 0}} \lambda_\varepsilon \leq \inf_{u \in H^1_0(\Omega)} J(u). \tag{3.6}$$

For all $\varepsilon > 0$, $\{a^\varepsilon_{ij}\}$ has the same ellipticity constant as $\{a_{ij}\}$. This shows that $\{u_\varepsilon\}$ is bounded in $H^1_0(\Omega)$. Choosing a subsequence and arguing as in the proof of Theorem 14, we see that

$$\begin{aligned}
&\lambda_{\varepsilon'} \to \lambda \\
&q_{\varepsilon'} \xrightarrow{*} q \text{ in } L^\infty(\Omega), \\
&u_{\varepsilon'} \rightharpoonup u \text{ in } H^1_0(\Omega) \text{ and} \\
&u_{\varepsilon'} \to u \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega.
\end{aligned}$$

This implies as before

$$\begin{aligned}
&0 \leq q \leq \lambda \text{ and } 0 \leq u \leq 1 \text{ a.e. in } \Omega, \\
&\int_\Omega q = \int_\Omega qu^2 = A \text{ and } q(u^2 - 1) = 0 \text{ a.e. in } \Omega, \text{ and}
\end{aligned}$$

$$a(u, v) + \int_\Omega quv = \lambda \int_\Omega uv, \quad \forall v \in H^1_0(\Omega) \tag{3.7}$$

We conclude that q has its essential support in $\{x : \ u(x) = 1\}$. Hence $\|u\|_\infty = 1$. Combining this with (3.7), we obtain $\lambda = R_q(u) = J(u)$. It follows from (3.6) that u is a minimizer of J. From Lemma 6, we see that (q, u) is an extremal couple and $\lambda_1 = \lambda$ is the extremal first eigenvalue.

4. - Uniqueness and other properties of the extremal couple $(\tilde{q}, \tilde{u})$

In this section we will discuss some properties of the extremal couple constructed in Section 2 and 3. Unless stated otherwise, we will always assume that $(\tilde{q}, \tilde{u})$ is one of the extremal couples constructed in Theorem 8, 14 or 15 respectively.

We recall that the following equalities hold for $(\tilde{q}, \tilde{u})$

$$\inf_{u \in H_a(\Omega)} R_{\tilde{q}}(u) = R_{\tilde{q}}(\tilde{u}) = J(\tilde{u}) = \inf_{u \in H_a(\Omega)} J(u).$$

Using these relations we can give a simple proof of the following uniqueness theorem.

THEOREM 16: *The function $\tilde{q}$ is a unique maximizer of the first eigenvalue of* (0.1).

PROOF. Suppose that $q \in B_A$ is another maximizer, which can be assumed to be non-negative. Then we have

$$\inf_{u \in H_a(\Omega)} R_q(u) = J(\tilde{u}).$$

But since $R_q(\tilde{u}) \leq J(\tilde{u})$ whenever $q \in B_A$, $\tilde{u}$ is a minimizer of R_q. Hence $\tilde{u}$ solves the two equations

$$E\tilde{u} + \tilde{q}\tilde{u} = \lambda\tilde{u},$$
$$E\tilde{u} + q\tilde{u} = \lambda\tilde{u},$$

which shows that $q = \tilde{q}$ a.e. on the essential support of $\tilde{u}$. But we know that ess supp $\tilde{q} \subset$ ess supp $\tilde{u}$ and $\|\tilde{q}\|_p = A$. Hence $\tilde{q} = q$ a.e. in Ω.

An immediate consequence is

COROLLARY 17: *If $g : R^n \to R^n$ is a linear transformation such that both Ω and E are invariant under g, that is*

$$g(\Omega) = \Omega \text{ and } a_{ij} = g_{ik}(a_{k\ell} \circ g)g_{j\ell}$$

where g is represented by the matrix (g_{ij}). Then $\tilde{q}$ is invariant under g, $\tilde{q} = \tilde{q} \circ g$.

PROOF. A change of coordinates gives

$$R_{\tilde{q}}(u) = R_{\tilde{q}\circ g}(u \circ g).$$

Thus both $\tilde{q}$ and $\tilde{q} \circ g$ are maximizers, and the result follows from uniqueness.

We have a similar result for $\tilde{u}$.

PROPOSITION 18: *Assume that either*

$$p = 1 \textit{ and } \{a_{ij}\} \subset L^\infty(\Omega)$$

or

$$1 < p < \infty \textit{ and } \{a_{ij}\} \subset L^1(\Omega).$$

Then $\tilde{u}$ is the unique non-negative minimizer of J.

REMARK: Assume that $\{a_{ij}\} \subset L^\infty(\Omega)$. Since $E\tilde{u} \geq 0$, the strong maximum principle shows that $\tilde{u} > 0$ in Ω. Hence $\tilde{u}$ is a unique minimizer of J.

PROOF. If $1 < p < \infty$ the result follows from Theorem 8 and Theorem 16. We only need to prove the result for $p = 1$.

Suppose that v is another minimizer of J. We can assume that $v \geq 0$ and $||v||_\infty = 1$. A direct calculation, using the properties of $\tilde{u}$ and $\tilde{q}$ yields

$$J(\tilde{u}) = \lambda \leq R_{\tilde{q}}(\tilde{u} + v) = \lambda + \frac{\int_\Omega \tilde{q}v^2 - A}{\int_\Omega (\tilde{u} + v)^2}$$

showing that $\int_\Omega \tilde{q}v^2 \geq A$. But

$$\int_\Omega \tilde{q}v^2 \leq \int_\Omega \tilde{q} = A.$$

Hence $\int_\Omega \tilde{q}v^2 = A$, $R_{\tilde{q}}(v) = J(v) = \lambda$ and ess supp $\tilde{q} \subset \{x : \ v(x) = 1\}$. Using this we find that $R_{\tilde{q}}(\tilde{u} - v) = \lambda$, that is $\tilde{u} - v$ is a minimizer of $R_{\tilde{q}}$. But then $w = |\tilde{u} - v|$ is also a minimizer and $w = 0$ on the essential support of $\tilde{q}$ since both v and $\tilde{u}$ are equal to one on the same set.

We conclude that $Ew = \lambda w$ in Ω, $w \geq 0$ and $w = 0$ on ess supp $\tilde{q}$. But the strong maximum principle [GT] then shows that $w = 0$. Hence $\tilde{u} = v$.

COROLLARY 19: *Assume that g has the same properties as in Corollary 17 and that the assumptions in Proposition 18 hold. Then $\tilde{u} = \tilde{u} \circ g$.*

PROOF. This is an immediate consequence of the uniqueness and the fact that $J(\tilde{u}) = J(\tilde{u} \circ g)$.

We have seen that $E\tilde{u} \geq 0$ and that $E\tilde{u} \in L^\infty(\Omega)$. If $\{a_{ij}\} \subset L^\infty(\Omega)$ then $\tilde{u}$ is locally Hölder continuous and the strong maximum principle yields the following result.

PROPOSITION 20: *If $\{a_{ij}\} \subset L^\infty(\Omega)$ then the level sets $\Omega_c = \{\tilde{u} \geq c\}$ and $\Omega_c' = \{\tilde{q} \geq c\}$ have the properties that $R^n \setminus \Omega_c$ and $R^n \setminus \Omega_c'$ have no components contained in Ω.*

Using spherical symmetrization we can obtain an estimate from below of the maximal first eigenvalue.

If u is a measurable function defined on Ω then the spherical decreasing symmetrization u^* defined on Ω^* is given by

(i) Ω^* is a ball with center at the origin and with the same volume as Ω.

(ii) $u^*(x) = \sup\{t \geq 0 : \ \lambda(t) \geq \omega_n |x|^n\}$ for $x \in \Omega^*$, where $\lambda(t) = m(\{x \in \Omega : |u(x)| \geq t\})$ and ω_n is the volume of the unit ball in R^n.

The spherical increasing symmetrization u^{**} defined on Ω^* is defined as follows:

(i) If $u \in L^\infty(\Omega)$ then $u^{**} = \|u\|_\infty - (\|u\|_\infty - |u|)^*$.

(ii) If u is measurable then $u^{**} = \lim_{n \to \infty} (\min(n, |u|))^{**}$.

We will use the following properties:

(4.1a) $|u|$, u^* and u^{**} are equimeasurable.

(4.1b) If $u \in H_0^1(\Omega)$ then $u^* \in H_0^1(\Omega^*)$ and $\int_{\Omega^*} |\nabla u^*|^2 \leq \int_\Omega |\nabla u|^2$.

(4.1c) $\int_{\Omega^*} u^* v^{**} \leq \int_\Omega |uv| \leq \int_{\Omega^*} u^* v^*$, whenever u and v are measurable.

Note that the first inequality in (4.1c) follows easily from the second one. For a proof of these statements see [HL].

Define $\lambda(\Omega, E) = \inf_{u \in H_a(\Omega)} J(u)$, where we have introduced the domain Ω and the operator E as parameters.

PROPOSITION 21: *We have*

$$\lambda(\Omega^*, -\delta\Delta) \leq \lambda(\Omega, E),$$

where Δ is the Laplace operator and δ is the ellipticity constant of E. If E has constant coefficients then δ can be chosen as the geometric mean of the eigenvalues of the matrix $\{a_{ij}\}$, that is $\delta^n = \mu_1 \mu_2 \cdot \ldots \cdot \mu_n$, where $\{\mu_\ell\}$ are the eigenvalues.

PROOF. This is an easy consequence of the properties of spherical symmetrization given in (4.1). We have

$$J(u) \geq \frac{\nu \int_\Omega |\nabla u|^2 + A\|u\|_{2p'}^2}{\int_\Omega u^2} \geq \frac{\nu \int_{\Omega^*} |\nabla u^*|^2 + A\|u^*\|_{2p'}^2}{\int_{\Omega^*} u^{*2}}.$$

If E has constant coefficients, then we first make an orthogonal transformation so that we can assume that $a_{ij} = \text{diag}(\mu_1, \dots \mu_n)$. Then we make the transformation $y_i = \sqrt{\frac{\delta}{\mu_i}} x_i$, $i = 1, \dots, n$. This gives us a linear map g such that

$$a(u, u) = \int_\Omega a_{ij}\partial_i u \partial_j u = \delta \int_{g(\Omega)} \nabla u' \cdot \nabla u' |\det\, g|^{-1}, \quad u' = u \circ g^{-1}.$$

If we take $\delta = (\mu_1 \cdot \ldots \cdot \mu_n)^{1/n}$ then the map g preserves volume, and we get

$$\lambda(\Omega, E) = \lambda(g(\Omega), -\delta\Delta) \geq \lambda(\Omega^*, -\delta\Delta).$$

REMARK: (i) If $(\tilde{q}, \tilde{u})$ is an extremal couple, then the corresponding maximal eigenvalue λ_1 is given by $\lambda(\Omega, E)$ and it can be estimated from below by $\lambda(\Omega^*, -\delta\Delta)$ which is easier to calculate. Actually, the problem with $\Omega = \Omega^*$ and $E = -\delta\Delta$ is a one-dimensional problem since we have rotational symmetry. At the end of this section we discuss this problem for $p = 1$.

(ii) If $E = -\Delta$ then we see that among all domains Ω with given fixed volume, the ball minimizes the maximal first eigenvalue $\lambda_1 = \lambda(\Omega)$.

When $p > 1$ we have derived an explicit equation for the extremal couple $(\tilde{q}, \tilde{u})$ in Section 2. To conclude this section we will discuss the problem of evaluating $\tilde{q}$ when $p = 1$.

Let us assume that $\{a_{ij}\}$ and $\partial\Omega$ are smooth. Then $\tilde{u} \in C^{1,\alpha}(\overline{\Omega})$ and $\tilde{u}$ solves the following free boundary problem

$$\text{(4.2)} \qquad \begin{cases} Eu = \lambda u \text{ in } \Omega \setminus I, \\ u = 0 \text{ on } \partial\Omega, \\ u = 1 \text{ and } \nabla u = 0 \text{ on } \partial I, \\ 0 \leq u < 1 \text{ in } \Omega \setminus I, \\ \lambda m(I) = A, \end{cases}$$

where λ, u and I are unknown. This problem is difficult. However, if we have symmetry, we can use Corollaries 17 and 19 to simplify (4.2).

Let Ω be a ball with radius R centered at the origin. Let $Eu = -\partial_i(f\partial_i u)$ where f is smooth, invariant under rotations and $f \geq \nu > 0$ in Ω. Since E and Ω are invariant under rotations, we conclude that both $\tilde{q}$ and $\tilde{u}$ are spherically symmetric. From Proposition 20 it follows that $I = \{x : \tilde{u}(x) = 1\}$ is a ball with center at the origin. Let ρ denote the radius of I. Thus, (4.2) can in this case be rewritten as the following one-dimensional problem:

$$\begin{cases} (f(r)u')' + \dfrac{n-1}{r} f(r)u' + \lambda u = 0 \text{ if } \rho < r < R, \\ u(R) = 0, \quad u(\rho) = 1 \text{ and } u'(\rho) = 0, \\ 0 < u < 1 \text{ if } \rho < r < R, \\ \omega_n \rho^n \lambda = A, \end{cases}$$

where λ, u and ρ are unknown.

If $f(r) \equiv \delta$ then the solution u can be expressed in terms of Bessel functions. In this case ρ and λ can easily be evaluated numerically. This gives us an estimate of $\lambda(\Omega^*, -\delta\Delta)$ in Proposition 21.

5. - Extensions to weighted L^p-balls

In this section we will discuss the general problem where h and k are non-constant. We will use the same technique as in Chapter 2, 3 and 4. There are minor changes in many of the proofs and all details will not be given. We will use the same numbering of the theorems, corollaries and lemmas as for the special case $k = h = 1$ but with primes.

As mentioned in the introduction we will always assume that

(i) h and k are measurable and $h \in L^\infty(\Omega)$,

(ii) $h > 0$ and $k \geq 0$ a.e. in Ω.

Instead of the estimate (0.2) of the Rayleigh quotient given in Section 0 we now have

$$R_q(u) = \frac{a(u,u) + \int_\Omega qu^2}{\int_\Omega u^2 h} \leq \frac{a(u,u) + A \left\| \frac{u}{k^{1/p}} \right\|_{2p'}^2}{\int_\Omega u^2 h} = J(u), \tag{5.1}$$

whenever $q \in B_A = \left\{ f : \left(\int_\Omega |f|^p k^2 \right)^{1/p} \leq A \right\}$. The estimate (5.1) is obtained using Hölder's inequality. Since k might be zero we have to use the following conventions

$$0 \cdot \infty = 0 \quad \text{and} \quad \frac{0}{0} = 0$$

to interpret (5.1) and we will do so in the rest of this section.

The Rayleigh quotient R_q is Gateaux-differentiable in the directions of functions in $C_0^1(\Omega)$ only if $q \in L^1_{\text{loc}}(\Omega)$. However, if we assume that

$$k^{-1} \in L^{\frac{2p'}{p}}_{\text{loc}}(\Omega), \tag{5.2}$$

then Hölder's inequality gives $B_A \subset L^1_{\text{loc}}(\Omega)$. The integrability condition (5.2) turns out to be natural as we will see later. At the end of this section we will study this problem, when (5.2) does not hold.

The results in Section 1 hold with the following changes in Proposition 5 and Lemma 6.

PROPOSITION 5′: *If* $q \geq 0$ *and if there exists a* $u \in H_a(\Omega)$ *such that* $J(u) < \infty$ $(R_q(u) < \infty)$, *then there exists a minimizer of* J (R_q). *Furthermore, the minimizer can be assumed to be non-negative.*

REMARK: If (5.2) holds and $q \in B_A$ then the conclusion of the theorem holds.

LEMMA 6′: *Let* $\tilde{u}$ *be a non-negative minimizer of* J *and put* $\lambda = J(\tilde{u})$. *Assume that one of the following two conditions hold*

$$\text{(I)} \qquad k^{-1} \in L^{\frac{2p'}{p}}(\Omega)$$

$$\text{(II)} \qquad \{a_{ij}\} \subset L^{\infty}(\Omega) \text{ and } k^{-1} \in L^{\frac{2p'}{p}}_{\text{loc}}(\Omega).$$

If there exists a non-negative $\tilde{q} \in B_A$ *such that*

$$\text{ess supp } \tilde{q} \subset \text{ess supp } \tilde{u},$$
$$E\tilde{u} + \tilde{q}\tilde{u} = \lambda h \tilde{u},$$

then

$$\inf_{u \in H_a(\Omega)} R_{\tilde{q}}(u) = R_{\tilde{q}}(\tilde{u}) = J(\tilde{u}),$$

that is $(\tilde{q}, \tilde{u})$ *is an extremal couple.*

PROOF. If condition (I) holds we can use the same argument as in the proof of Lemma 6.

Under condition (II) we can argue as in the proof of Lemma 6 to obtain (1.4). The result follows from the strong maximum principle (cf. the remark after Lemma 6).

The case $1 < p < \infty$

The same argument as in the proof of Theorem 8 can be used to obtain the following result.

THEOREM 8′: *Let* $1 < p < \infty$ *and assume that one of the conditions* (I) *and* (II) *in Lemma* 6′ *hold. Then there exists an extremal couple* $(\tilde{q}, \tilde{u})$ *which solves the problem and which has the following properties:*

(i) $\tilde{u}$ *is a minimizer of* J *and the maximal first eigenvalue* $\lambda_1 = J(\tilde{u})$.

(ii) $\tilde{u}$ *is the first eigenfunction of*

$$v \in H_a(\Omega): \; Ev + A \left\| \frac{v^2}{k^{2/p}} \right\|_{p'}^{1-p'} \left| \frac{v}{k} \right|^{2p'/p} v = \mu h v$$

with eigenvalue $\mu_1 = \lambda_1$.

(iii) $\tilde{q} = A \left\| \frac{\tilde{u}^2}{k^{2/p}} \right\|_{p'}^{1-p'} \left| \frac{\tilde{u}}{k} \right|^{2p'/p}$

(iv) $R_{\tilde{q}}(\tilde{u}) = J(\tilde{u})$.

If $1 < p < \infty$ then the existence of a maximizer $q \in B_A$ can also be proved using the following standard argument.

Let $\lambda(q) = \inf\limits_{u \in C_0^1(\Omega)} R_q(u)$ and let $\{q_n\} \subset B_A$ be a maximizing sequence. Since $\{q_n\}$ is bounded in $L^p(\Omega, k^2 \mathrm{d}x)$ we can assume that $q_n \rightharpoonup \overline{q}$ in $L^p(\Omega, k^2 \mathrm{d}x)$ and by convexity $\overline{q} \in B_A$.

Consider the functional on $L^p(\Omega, k^2 \mathrm{d}x)$ given by

$$q \to \int_\Omega q u^2 = \int_\Omega q \left(\frac{u}{k}\right)^2 k^2,$$

where $u \in C_0^1(\Omega)$. This is continuous w.r.t. the weak topology if and only if $\left(\frac{u}{k}\right)^2 \in L^{p'}(\Omega, k^2 \mathrm{d}x)$. The last condition holds for all $u \in C_0^1(\Omega)$ if and only if $k^{-1} \in L^{2p'/p}_{\mathrm{loc}}(\Omega)$. Hence if $k^{-1} \in L^{2p'/p}_{\mathrm{loc}}(\Omega)$ then the map $B_A \ni q \to \lambda(q)$ is upper semicontinuous w.r.t. the weak topology and we find that $\overline{q}$ is a maximizer.

It is interesting to see that condition (5.2) turns up again.

To prove that $\tilde{u}$ in Theorem 8$'$ is bounded we notice that $E\tilde{u} \leq \lambda h \tilde{u}$ and Theorem A3 in the Appendix gives us

$$\|\tilde{u}\|_\infty \leq C \|\tilde{u}\|_2 < \infty.$$

The same technique as in the proof of Corollary 9 can be used to obtain a more precise estimate.

COROLLARY 9$'$: *We have the following estimates*

$$\|\tilde{u}\|_\infty \leq \left(\frac{\lambda_1}{A}\right)^{\frac{p-1}{2}} \left\| k h^{\frac{p-1}{2}} \right\|_\infty \left\| \frac{\tilde{u}}{k^{1/p}} \right\|_{2p'},$$
$$\tilde{q} \leq D k^{2(1-p')},$$

where D is a constant. Furthermore, $\tilde{u} \in L^\infty(\Omega)$.

To prove regularity results for $\tilde{u}$ and $\tilde{q}$, we have to impose further conditions on k and h. There are no problems to obtain these results so we leave it to the reader.

The case $p = 1$

In this case we have a nice geometric interpretation. It will turn out that the minimizer $\tilde{u}$ solves a variational inequality with the weight function k as an obstacle.

PROPOSITION 12′: *Assume that* $\tilde{u}$ *is a minimizer of* J, *normalized so that* $\left\|\dfrac{\tilde{u}}{k}\right\|_\infty = 1$. *Then*

(i) $\tilde{u}$ *solves the variational inequality*

$$u \in K: \ a(u, v-u) - \lambda \int_\Omega hu(v-u) \geq 0, \quad \forall v \in K, \tag{5.3}$$

where $\lambda = J(\tilde{u})$ *and* $K = \{u \in H_a(\Omega): \ |u| \leq k\}$.

(ii) $\|\tilde{u}\|_\infty \leq C(n, \lambda\|h\|_\infty, \nu, m(\Omega))\|\tilde{u}\|_2 < \infty$.

(iii) *If* $\tilde{u}$ *is non-negative, then* $\tilde{u}$ *solves the variational inequality* (5.3) *with* $K = \{u \in H_a(\Omega): \ u \leq k\}$ *and* $\tilde{u}$ *satisfies* $E\tilde{u} \leq \lambda h\tilde{u}$.

(iv) *Assume that there exists a measurable function* $\overline{k}$ *such that* $\overline{k} \leq k$ *a.e. in* Ω *and* $\overline{k} = k$ *a.e. in* $\{x \in \Omega: \ \overline{k}(x) \leq \|\tilde{u}\|_\infty\}$. *Let* $\overline{J}$ *be the functional* J *with* k *replaced by* $\overline{k}$. *Then* $\tilde{u}$ *is a minimizer of* $\overline{J}$. *Furthermore,* (i) *and* (iii) *hold with* k *replaced by* $\overline{k}$.

PROOF. The proof of (i) is the same as before and will be omitted.

Put $f = \lambda h\tilde{u}$ and assume that $\tilde{u}$ is non-negative. Then $\tilde{u}$ is the unique solution of

$$u \in K: \ a(u, v-u) - \int_\Omega f(v-u) \geq 0, \quad \forall v \in K, \tag{5.4}$$

where K is the convex set given in (i). The uniqueness follows since the operator in (5.4) is strictly monotone.

It is easy to see that (5.4) is equivalent to the problem of finding the minimizer of

$$F(u) = a(u,u) - 2\int_\Omega fu, \quad u \in K.$$

Let u_δ be the unique solution of (5.4) with

$$K = K_\delta = \{u \in H_a(\Omega): \ |u| \leq k + \delta\} \quad \text{where } 0 < \delta < 1.$$

Then u_δ also minimizes F over K_δ, and u_δ is non-negative since $F(|u_\delta|) \leq F(u_\delta)$.

The family $\{u_\delta\}$ is bounded in $H_a(\Omega)$. Hence, for a subsequence we get $u_\delta \rightharpoonup \overline{u}$ in $H_a(\Omega)$ and $u_\delta \to \overline{u}$ in $L^2(\Omega)$ and a.e. in Ω. Thus, $0 \leq \overline{u} \leq k$.

We have

$$a(u_\delta, v-u_\delta) - \int_\Omega f(v-u_\delta) \geq 0, \quad \forall v \in K_\delta. \tag{5.5}$$

If we let δ tend to zero in (5.5) and note that $\varliminf_{\delta\to 0} a(u_\delta, u_\delta) \geq a(\overline{u}, \overline{u})$, we obtain

$$a(\overline{u}, v - \overline{u}) - \int_\Omega f(v - \overline{u}) \geq 0, \quad \forall v \in K.$$

Hence $\overline{u}$ solves (5.4) and $\overline{u} = \tilde{u}$ by uniqueness.

To prove (iii), take $\varphi \in C_0^1(\Omega)$ such that $\varphi \geq 0$, then $v = u_\delta - \varepsilon\varphi \in K_\delta$ if $\varepsilon > 0$ is small enough. Hence (5.5) gives

$$a(u_\delta, \varphi) - \int_\Omega f\varphi \leq 0, \quad \forall \varphi \in C_0^1(\Omega), \quad \varphi \geq 0.$$

Letting δ tend to zero yields $E\tilde{u} \leq \lambda h \tilde{u}$.

Take $v \in H_a(\Omega)$ such that $v \leq k$. We have the decomposition $v = v^+ - v^-$, where $v^+ \in K$ and $v^- \in H_a(\Omega)$. Thus

$$a(\tilde{u}, v - \tilde{u}) - \int_\Omega f(v - \tilde{u}) =$$

$$= \left[a(\tilde{u}, v^+ - \tilde{u}) - \int_\Omega f(v^+ - \tilde{u})) - (a(\tilde{u}, v^-) - \int_\Omega f v^- \right] \geq 0.$$

This proves (iii).

If $\tilde{u}$ is a minimizer of J, then $|\tilde{u}|$ is also a minimizer. Now (iii) gives $E(|\tilde{u}|) \leq \lambda h |\tilde{u}|$ and Theorem A3 in the Appendix yields (ii).

If $\overline{k}$ is as in (iv) then it is clear that $\tilde{u}$ is a minimizer of $\overline{J}$ since $J(u) \leq \overline{J}(u)$ and $J(\tilde{u}) = \overline{J}(\tilde{u})$. We can repeat the argument above to show that (i) and (iii) hold with k replaced by $\overline{k}$.

The following regularity result is proved in the Appendix.

THEOREM 13$'$: *Assume that $\{a_{ij}\} \subset C_0^{0,1}(\overline{\Omega})$, $\partial\Omega \in C^2$, $k \in H^1(\Omega)$ and that $E(k)$ is a Radon measure such that $E(k)^- \in L^s(\Omega)$ for some $s > n$. Let $\tilde{u}$ be a non-negative solution of the variational inequality* (5.3). *Then*

$$\tilde{u} \in H^{2,s}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \text{ where } \alpha = 1 - \frac{n}{s}.$$

Let us assume that the conditions in Theorem 13$'$ hold and that k is continuous a.e. in Ω. Define the coincidence set as $I = \{x \in \Omega : \ \tilde{u} = k\}$. A variation of (5.3) in $\Omega \setminus I$ yields $E\tilde{u} = \lambda h \tilde{u}$ pointwise a.e. in $\Omega \setminus I$, where $\lambda = J(\tilde{u})$. Thus

$$E\tilde{u} + \tilde{q}\tilde{u} = \lambda h \tilde{u} \text{ holds pointwise a.e. in } \Omega,$$

where $\tilde{q} = \left(\lambda h - \frac{E(\tilde{u})}{k}\right)\chi\{I\}$. From Proposition 12′ it follows that $\tilde{q}$ is non-negative. If we multiply the equation by $\tilde{u}$ and integrate, we find that

$$J(\tilde{u}) = R_{\tilde{q}}(\tilde{u}) = \frac{a(\tilde{u},\tilde{u}) + \int\limits_{\Omega} \tilde{q}\tilde{u}^2}{\int\limits_{\Omega} \tilde{u}^2 h}.$$

Thus

$$\int\limits_{\Omega} \tilde{q}k^2 = \int\limits_{\Omega} \tilde{q}\tilde{u}^2 = A,$$

which shows that $\tilde{q} \in B_A$. Hence, if we also assume that $k^{-1} \in L^\infty_{\text{loc}}(\Omega)$, Lemma 6′ shows that $(\tilde{q}, \tilde{u})$ is an extremal couple.

At the end of this section we will discuss the problem when $k^{-1} \notin L^\infty_{\text{loc}}(\Omega)$.

THEOREM 14′: *Let $p = 1$ and $\{a_{ij}\} \subset C^{0,1}(\Omega)$. Assume that one of the conditions* (I) *and* (II) *below hold.*

(I) *$k \in H^1(\Omega)$, k is continuous a.e. in Ω, $k^{-1} \in L^\infty_{\text{loc}}(\Omega)$ and $E(k)$ is a Radon measure such that*

$$E(k)^- \in L^s(\Omega) \text{ for some } s > n.$$

(II) *There exists a sequence of functions $\{k_n\}$ such that for each n, $k_n \le k$ a.e. in Ω, $k_n = k$ a.e. in $\{x : k_n(x) \le n\}$ and the assumptions in* (I) *holds for each k_n.*

Then there exists an extremal couple which solves the problem and which has the properties:

(i) *$\tilde{u}$ is a non-negative minimizer of J, normalized so that* $\left\|\frac{\tilde{u}}{k}\right\|_\infty = 1$

(ii) $\tilde{u} \in L^\infty(\Omega)$.

(iii) $\tilde{q} = \left(\lambda_1 h - \frac{E(\tilde{u})}{k}\right)\chi\{I\} \le \left(\lambda_1 h - \frac{E(k)}{k}\right)^+ \chi\{I\}$ *and if $k \in H^{2,1}(\Omega)$, then*

$$\tilde{q} = \left(\lambda_1 h - \frac{E(k)}{k}\right)\chi\{I\}.$$

Here $I = \{x \in \Omega : \tilde{u} = k\}$ and $\lambda_1 = J(\tilde{u})$ is the maximal first eigenvalue.

(iv) *$E\tilde{u} + \tilde{q}\tilde{u} = \lambda_1 h\tilde{u}$ holds pointwise a.e. in Ω,*

(v) $R_{\tilde{q}}(\tilde{u}) = J(\tilde{u})$.

PROOF: First we conclude the proof of the theorem under the extra conditions that k satisfies condition (I), $\{a_{ij}\} \subset C^{0,1}(\overline{\Omega})$ and $\partial\Omega \in C^2$.

From the discussion preceding the theorem it is clear that (i), (ii), (iv) and (v) hold. In (iii) the inequality

$$\tilde{q} \leq \left(\lambda h - \frac{E(k)}{k}\right)^{+} \chi\{I\}$$

follows from Corollary A2 in the Appendix. The equality $\tilde{q} = \left(\lambda h - \frac{E(k)}{k}\right) \chi\{I\}$ follows since $E(k) = E(\tilde{u})$ a.e. on I if $\tilde{u}$ and $k \in H^{2,1}(\Omega)$, [GT], Chapter 7.

Under condition (II), the results follow easily from Proposition 12′ (iv).

To finish the proof, we have to remove the smoothness conditions on $\{a_{ij}\}$ and $\partial\Omega$. To do this, we use the same method as in Theorem 14. We will not give the proof here.

THEOREM 15′: *Let $p = 1$ and $\{a_{ij}\} \subset L^{\infty}(\Omega)$. Assume that one of the conditions* (I) *and* (II) *below hold.*

(I) *There exists an $s > n$ such that $k \in H^{1,s}(\Omega)$, $k^{-1} \in L^{\infty}_{\text{loc}}(\Omega)$ and $E(k)$ is a Radon measure with negative part $E(k)^{-} \in L^{s}(\Omega)$.*

(II) *There exists a sequence of functions $\{k_n\}$ such that for each n, $k_n \leq k$ a.e. in Ω, $k_n = k$ a.e. in $\{x : k_n(x) \leq n\}$ and the assumptions in (I) holds for each k_n.*

Then there exists an extremal couple $(\tilde{q}, \tilde{u})$ which solves the problem and which has the properties

(i) *$\tilde{u}$ is a non-negative minimizer of J, normalized so that* $\left\|\frac{\tilde{u}}{k}\right\|_{\infty} = 1$.

(ii) *$\tilde{u} \in L^{\infty}(\Omega)$ and $\tilde{u}$ is locally Hölder continuous.*

(iii) $\tilde{q} \leq \left(\lambda_1 h - \frac{E(k)}{k}\right)^{+} \chi\{I\}$ *with equality in the interior of I. Here $I = \{x : \tilde{u}(x) = k(x)\}$ and $\lambda_1 = J(\tilde{u})$ is the maximal first eigenvalue.*

(iv) $R_{\tilde{q}}(\tilde{u}) = J(\tilde{u})$.

REMARK: k is continuous in Ω since $k \in H^{1,s}(\Omega)$ where $s > n$, [GT], Chapter 7.

PROOF: The proof is almost the same as the proof of Theorem 15. But some new complications occur.

First we prove the theorem under condition (I). Let $\{a^{\varepsilon}_{ij}\}$ be the sequence approximating $\{a_{ij}\}$ constructed in Lemma 7. Define k_{ε} as the solution of

$$w - k \in H^1_0(\Omega) : \; E_{\varepsilon}(w) = E(k),$$

where E_{ε} is the operator corresponding to $\{a^{\varepsilon}_{ij}\}$. We have $k_{\varepsilon} \in C(\Omega)$ since $E(k) \in H^{-1,s}(\Omega)$ where $s > n$, [GT], Theorem 8.24.

From [GT], Theorem 8.16, we get the following estimate

$$\|k_{\varepsilon} - k\|_{\infty} \leq \frac{c}{\nu}\|E(k) - E_{\varepsilon}(k)\|_{H^{-1,s}}, \tag{5.6}$$

where c does not depend on ε. To see this we note that

$$k - k_\varepsilon \in H_0^1(\Omega),$$
$$E_\varepsilon(k_\varepsilon - k) = E(k) - E_\varepsilon(k)$$

and that E_ε has the same ellipticity constant as E.

The right hand side of (5.6) tends to zero as $\varepsilon \to 0$. Hence $k_\varepsilon \to k$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$.

We can add a constant $o(\varepsilon)$ to k_ε so that $k_\varepsilon > k$ a.e. in Ω. Then k_ε fulfils the conditions in Theorem 14′. Thus, we have a solution $(q_\varepsilon, u_\varepsilon, \lambda_\varepsilon)$ of the problem with k and $\{a_{ij}\}$ replaced by k^ε and $\{a_{ij}^\varepsilon\}$. Let u_ε be non-negative and normalized so that $\left\| \frac{u_\varepsilon}{k_\varepsilon} \right\|_\infty = 1$.

A semicontinuity argument yields

$$\lambda = \overline{\lim_{\varepsilon \to 0}} \lambda_\varepsilon \le \inf_u \frac{a(u,u) + A \left\| \frac{u}{k} \right\|_\infty^2}{\int_\Omega u^2}.$$

Theorem 14′ (iii) gives

$$0 \le q_\varepsilon \le \left(\lambda_\varepsilon h - \frac{E(k)}{k_\varepsilon} \right)^+,$$

since $E_\varepsilon(k_\varepsilon) = E(k)$. This shows that $\{q_\varepsilon\}$ is bounded in $L^s(\Omega')$ for every $\Omega' \subset\subset \Omega$. It is also easy to see that $\{u_\varepsilon\}$ is bounded in $H_0^1(\Omega)$. Hence, for a subsequence, we have

$$\lambda_\varepsilon \to \lambda,$$
$$q_\varepsilon \overset{*}{\rightharpoonup} q \text{ in } L^s(\Omega') \text{ for every } \Omega' \subset\subset \Omega,$$
$$u_\varepsilon \rightharpoonup u \text{ in } H_0^1(\Omega),$$
$$u_\varepsilon \to u \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega.$$

The statement about the convergence of q_ε involves a diagonalization procedure.

It is easy to establish the following

$$0 \le u \le k \text{ and } 0 \le q \le \left(\lambda h - \frac{E(k)}{k} \right)^+ \text{ a.e. in } \Omega, \tag{5.7}$$

$$\int_\Omega qk^2 = \int_\Omega qu^2 = A \tag{5.8}$$

and

$$a(u,v)+\int_{\Omega} quv=\lambda\int_{\Omega} huv,\quad \forall v\in H_0^1(\Omega). \tag{5.9}$$

Let us prove (5.8). Theorem 14′ yields

$$\int_{\Omega} q_\varepsilon u_\varepsilon^2=\int_{\Omega} q_\varepsilon k_\varepsilon^2=A,$$

since $u_\varepsilon=k_\varepsilon$ on the essential support of q_ε. From the above we see that $q_\varepsilon k_\varepsilon^2$ is bounded from above by $2(\lambda hk^2-kE(k))^+\in L^1(\Omega)$, if ε is small enough. Hence, letting ε tend to zero yields (5.8). The proofs of (5.7) and (5.9) are also easy and we omit the details.

From (5.8) and (5.9) we find that $\lambda=R_q(u)=J(u)$. Hence u is a minimizer of J. Furthermore (5.7) and (5.8) show that

$$\text{ess supp } q\subset I=\{x\in\Omega:\ u(x)=k(x)\}.$$

Now we can apply Lemma 6′ to conclude that (q,u) is an extremal couple.

From the above it is also clear that (i), (iii) and (iv) hold.

To finish the proof of (ii) we only have to note that $Eu\in L^s(\Omega)$ which follows from the fact that $qu\in L^s(\Omega)$ and $u\in L^\infty(\Omega)$. Thus, the local Hölder continuity of u follows from [GT], Chapter 8.

To prove the theorem under condition (II) we argue as follows. Let $\tilde u$ be a minimizer of J. Then $\|\tilde u\|_\infty<\infty$ by Proposition 12′. For each $n>\|\tilde u\|_\infty$, we get a solution $(\tilde q_n,\tilde u_n)$ to the problem with k replaced by k_n. Furthermore, all these solutions have the maximal first eigenvalue $\lambda_1=J(\tilde u)$ by Proposition 12′ and $\{\tilde u_n\}$ is bounded in $H_0^1(\Omega)$. Proposition 12′ also yields that $\{\tilde u_n\}$ is bounded in $L^\infty(\Omega)$. Hence, $\{x:\ \tilde u_n(x)=k_n(x)\}=\{x:\ \tilde u_n(x)=k(x)\}$ if n is large enough and we get

$$\int_{\Omega}\tilde q_n k_n^2=\int_{\Omega}\tilde q_n k^2=A$$

That is $(\tilde q_n,\tilde u_n)$ is an extremal couple and it is clear that properties (i) - (iv) hold.

Often the family $\{k_n\}$ of functions in Theorems 14′ and 15′ can be chosen as $k_n=\min(k,n)$. To see this, let us assume that $\{a_{ij}\}$ and k are smooth functions. Let $\varphi\in C_0^1(\Omega)$ be non-negative. We have

$$E(k_n)(\varphi)=\int_{\{x:\,k<n\}} a_{ij}\partial_i k\partial_j\varphi=$$

$$=\int_{\partial\{x:\,k<n\}} a_{ij}\partial_i k\varphi \mathrm{d}\sigma_j-\int_{\{x:\,k<n\}}\partial_j(a_{ij}\partial_i k)\varphi,$$

provided Gauss Theorem holds for the set $\{x : \ k < n\}$. Since $\overline{\nabla} k$ and $d\overline{\sigma}$ have the same direction, it follows that the contribution from the truncation is a positive measure and thus it does not affect $E(k)^-$. This technique is used in Chapter 6, Example 3.

For certain functions k we can use the following method to obtain the family $\{k_n\}$.

Let $\Omega_n = \{x \ k(x) > n\}$. Take n fix and assume that there exists an N such that $\Omega_N \subset\subset \Omega_n$. Take $\varphi \in C_0^\infty(\Omega_n)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on Ω_N. Put $k_n = (1-\varphi)k + \varphi n$. Then $k_n = k$ on $\{x : \ k_n(x) \leq n\}$ and on Ω we have

$$k_n \leq (1-\varphi)k + \varphi k = k.$$

This gives us a family $\{k_n\}$ with the desired properties.

Uniqueness and other properties of the extremal couple $(\tilde{q}, \tilde{u})$

In this section we will extend the results obtained in Section 4. Throughout we will assume that $(\tilde{q}, \tilde{u})$ is one of the extremal couples constructed in Theorems 8′, 14′ and 15′ respectively.

The following four results are proved as before.

PROPOSITION 16′: *The function $\tilde{q}$ is a unique maximizer of the first eigenvalue of* (0.1).

COROLLARY 17′: *If $g : R^n \to R^n$ is a linear transformation such that Ω, k, h and E are invariant under g, that is*

$$\begin{aligned} &g(\Omega) = \Omega \\ &k = k \circ g, \quad h = h \circ g \ \textit{and} \\ &a_{ij} = g_{ik}(a_{k\ell} \circ g) g_{j\ell}, \end{aligned}$$

where g is represented by the matrix (g_{ij}). Then $\tilde{q}$ is invariant under g.

PROPOSITION 18′: *Assume that the conditions in either of Theorems* 8′, 14′ *or* 15′ *hold. Then $\tilde{u}$ is the unique non-negative minimizer of J.*

COROLLARY 19′: *Assume that g has the same properties as in Corollary* 17′ *and that the assumptions in Proposition* 18′ *hold. Then $\tilde{u}$ is invariant under g, that is $\tilde{u} = \tilde{u} \circ g$.*

Proposition 20 has no natural counterpart here.

Define

$$\lambda(\Omega, E, h, k) = \inf_{u \in H_a(\Omega)} J(u),$$

where we have introduced Ω, E, h and k as parameters.

PROPOSITION 21′: *We have*

$$\lambda(\Omega^*, -\delta\Delta, h^*, k^*) \leq \lambda(\Omega, E, h, k)$$

where Δ *is the Laplace operator and* δ *is the ellipticity constant of* E. *If* E *has constant coefficients, then* δ *can be chosen as the geometric mean of the eigenvalues of the matrix* $\{a_{ij}\}$. *Here* * *is the spherical symmetrization defined in Section* 4.

PROOF. If we note that $(k^{-1})^{**} = (k^*)^{-1}$, the result follows easily from the properties of spherical symmetrization and the proof of Proposition 21 given in Chapter 4.

Since the problem of finding $\lambda(\Omega^*, -\delta\Delta, h^*, k^*)$ is one-dimensional and therefore easier to solve, we can use Proposition 21′ to obtain an estimate from below of the maximal first eigenvalue.

If Ω is a ball, $E = -\Delta$ and $h = h^*$, then Proposition 21′ shows:

> Among all admissible equimeasurable weight functions k,
> k^* gives the smallest maximal first eigenvalue.

The same result holds with h and k interchanged. We can use similar arguments to obtain related results.

The singular case where $p = 1$ and $k^{-1} \notin L^\infty_{\mathrm{loc}}(\Omega)$

In the argument after Theorem 13′ we arrived at the following conclusions. Assume that the conditions in Theorem 13′ hold and that k is continuous a.e. in Ω. Then the non-negative minimizer $\tilde{u}$ of J satisfies

$$E\tilde{u} + \tilde{q}\tilde{u} = \lambda h\tilde{u} \text{ a.e. in } \Omega, \tag{5.10}$$

where

$$\begin{aligned}
&0 \leq \tilde{q}k = (\lambda hk - E\tilde{u})\chi\{I\},\\
&\lambda = J(\tilde{u}),\\
&I = \{x \in \Omega : \ \tilde{u}(x) = k(x)\} \text{ and}\\
&\int_\Omega \tilde{q}k^2 = A, \text{ that is } \tilde{q} \in B_A.
\end{aligned}$$

Nothing in the result above is changed if we redefine $\tilde{q}$ to be ∞ on the set $\{x \in \Omega : \ k(x) = 0\}$. Actually, the zero set of k is negligable in our problem, since it is obvious that the maximizing function q can be assumed to be ∞ there. Thus, the corresponding eigenfunction is zero a.e. on this set.

However, if $k^{-1} \notin L^\infty_{\mathrm{loc}}(\Omega)$ we can not apply Lemma 6′ to conclude that $(\tilde{q}, \tilde{u})$ is an extremal couple. But we have the following result.

THEOREM 22: *Let* $p = 1$ *and* $\{a_{ij}\} \subset L^\infty(\Omega)$. *Assume that one of the conditions* (I) *and* (II) *below hold.*

(I) *There exists an* $s > n$ *such that* $k \in H^{1,s}(\Omega) \cap H^{2,1}(\Omega)$ *and* $E(k)$ *is a Radon measure with negative part* $E(k)^- \in L^s(\Omega)$.

(II) *There exists a sequence of functions* $\{k_n\}$ *such that for each* n, $k_n \le k$ *a.e. in* Ω, $k_n = k$ *a.e. in* $\{x :\ k_n(x) \le n\}$ *and the assumptions in* (I) *hold for each* k_n.

Then

$$\sup_{q \in B_A} \inf_{u \in H_0^1(\Omega)} R_q(u) = \inf_{u \in H_0^1(\Omega)} J(u). \tag{5.11}$$

There exists a couple $(\tilde{q}, \tilde{u})$ *with the following properties:*

(i) $\tilde{u}$ *is a non-negative minimizer of* J.

(ii) $E\tilde{u} + \tilde{q}\tilde{u} = \lambda h \tilde{u}$, *where* $\lambda = J(\tilde{u})$.

(iii) $\tilde{u} \in L^\infty(\Omega)$ *and* $\tilde{u}$ *is locally Hölder continuous in* Ω.

(iv) $0 \le \tilde{q}k \le (\lambda hk - E(k))^+ \chi\{I\}$, *with equality in the interior of* $I = \{x : \tilde{u}(x) = k(x)\}$.

(v) $R_{\tilde{q}}(\tilde{u}) = J(\tilde{u})$.

If we also have $\{a_{ij}\} \subset C^{0,1}(\Omega)$, *then*

(vi) $\tilde{q}k = (\lambda hk - E(k))\chi\{I\}$,

(vii) $E\tilde{u} + \tilde{q}\tilde{u} = \lambda h\tilde{u}$ *holds pointwise a.e. in* Ω.

If the supremum in (5.11) *is attained at* $q \in B_A$, *then* $q = \tilde{q}$ *a.e.* $(k^2 \mathrm{d}x)$ *and* $(\tilde{q}, \tilde{u})$ *is an extremal couple.*

PROOF. First we prove the theorem under condition (I).

Let (q_δ, u_δ) be the solution of the problem with $k_\delta = k + \delta$, $0 < \delta < 1$. The existence of a solution is guaranteed by Theorem 15′. Assume that each u_δ is non-negative and normalized so that $\left\| \dfrac{u_\delta}{k_\delta} \right\|_\infty = 1$.

Let J_δ be the functional J with k replaced by k_δ. We have the following facts

$$\lambda_\delta = R_{q_\delta}(u_\delta) = J_\delta(u_\delta) \le \sup_{q \in B_A} \inf_{u \in H_0^1(\Omega)} R_q(u) \le \inf_{u \in H_0^1(\Omega)} J(u), \tag{5.12}$$

$$\lambda_\delta \nearrow \lambda \text{ as } \delta \searrow 0,$$

$$0 \le q_\delta k_\delta \le (\lambda h(k+\delta) - E(k))^+, \tag{5.13}$$

$$\{u_\delta\} \text{ is bounded in } H_0^1(\Omega).$$

Here (5.13) follows from Theorem 15′ and the other statements are obvious.

For a subsequence we have

$$u_\delta \rightharpoonup \tilde{u} \text{ in } H_0^1(\Omega), \quad u_\delta \to \tilde{u} \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega,$$

$$q_\delta k_\delta \overset{*}{\rightharpoonup} \tilde{q}k \text{ in } L^s(\Omega), \tag{5.14}$$

$$0 \le \tilde{u} \le k \text{ and } 0 \le \tilde{q}k \le (\lambda hk - E(k))^+, \tag{5.15}$$

(5.16) $$\int_\Omega \tilde{q}k^2 = \int_\Omega \tilde{q}\tilde{u}^2 = A,$$

(5.17) $$\text{ess supp } \tilde{q} \subset \{x : \ k(x) = \tilde{u}(x)\},$$

(5.18) $$a(\tilde{u}, v) + \int_\Omega \tilde{q}\tilde{u}v = \int_\Omega h\tilde{u}v, \quad \forall v \in H_0^1(\Omega).$$

We will only prove (5.14) and (5.18). The other statements follow easily from the properties of (q_δ, u_δ) given in Theorem 15$'$.

To prove (5.14) we first note that $\{q_\delta k_\delta\}$ is bounded in $L^s(\Omega)$ by (5.13). Hence, for a subsequence we have

$$q_\delta k_\delta \overset{*}{\rightharpoonup} f \text{ in } L^s(\Omega)$$

But (5.13) also gives that

$$0 \leq q_\delta k_\delta \leq \lambda h \delta \text{ on } \{x : \ k(x) = 0\},$$

since $k \in H^{2,1}(\Omega)$. Thus $f = 0$ on the zero set of k, and (5.14) follows with $\tilde{q} = \frac{f}{k}$ on $\{x : k(x) > 0\}$ and $\tilde{q} = \infty$ elsewere.

Take $\varphi \in C_0^1(\Omega)$, then

$$a(u_\delta, \varphi) + \int_\Omega q_\delta u_\delta \varphi = \lambda_\delta \int_\Omega h u_\delta \varphi$$

by Theorem 15$'$. If we let δ tend to zero we obtain

(5.19) $$a(\tilde{u}, \varphi) + \int_\Omega \tilde{q}\tilde{u}\varphi = \lambda \int_\Omega h\tilde{u}\varphi.$$

The only non-trivial term is $\int_\Omega q_\delta u_\delta \varphi$. We argue as follows. Write

$$\int_\Omega q_\delta u_\delta \varphi = \int_{\{x:\, k(x)>0\}} q_\delta u_\delta \varphi + \int_{\{x:\, k(x)=0\}} q_\delta u_\delta \varphi = I_1 + I_2,$$

then $I_2 \to 0$ by (5.13). If we note that $\left|\frac{u_\delta}{k_\delta}\right| \leq 1$ and $\frac{u_\delta}{k_\delta} \to \frac{\tilde{u}}{k}$ a.e. on $\{x : k(x) > 0\}$, then dominated convergence yields

$$I_1 = \int_{\{x:k(x)>0\}} q_\delta k_\delta \frac{u_\delta}{k_\delta} \varphi \rightharpoonup \int_{\{x:\, k(x)>0\}} \tilde{q}k \frac{\tilde{u}}{k} \varphi = \int_\Omega \tilde{q}\tilde{u}\varphi.$$

This shows (5.19), and (5.18) follows by continuity.

If we put $v = \tilde{u}$ in (5.18) and use (5.16) we find that $\lambda = R_{\tilde{q}}(\tilde{u}) = J(\tilde{u})$. Then (5.11) follows from (5.12) and $\tilde{u}$ is a minimizer of J.

From the above, it follows that $E\tilde{u} \in L^s(\Omega)$ where $s > n$. Hence $\tilde{u}$ is locally Hölder continuous. If $\{a_{ij}\} \subset C^{0,1}(\Omega)$ then $\tilde{u} \in H^2_{\text{loc}}(\Omega)$ and $E\tilde{u}+\tilde{q}\tilde{u} = \lambda h\tilde{u}$ holds pointwise a.e. in Ω.

We have proved the existence of a couple $(\tilde{q}, \tilde{u})$ satisfying (i) - (vii) above. To see that this is the only possible maximizer we argue as follows.

Assume that the supremum in (5.11) is attained at $q \in B_A$. Since $R_q(\tilde{u}) \leq J(\tilde{u})$, it follows that $\tilde{u}$ is a minimizer of R_q. Clearly $q \in L^1_{\text{loc}}(\{x : k(x) > 0\})$ and we obtain

$$a(\tilde{u}, \varphi) + \int_{\Omega} q\tilde{u}\varphi = \lambda \int_{\Omega} h\tilde{u}\varphi, \quad \forall \varphi \in C^1_0(\{x : \ k(x) > 0\}).$$

Combining this equation with (ii) yields that $q = \tilde{q}$ a.e. on the essential support of $\tilde{u}$. But then (5.16) shows that $q = \tilde{q}$ a.e. on the support of k. Hence $\tilde{q}$ is the only possible maximizer $(k^2 \mathrm{d}x)$.

To extend the result to hold under condition (II) we argue as in the proof of Theorem 15'.

REMARK: The condition $k \in H^{2,1}(\Omega)$ was only needed to conclude that $E(k) = 0$ a.e. on $\{x : \ k(x) = 0\}$ (and (vi)). Clearly, this condition can be weakened.

This section will be concluded with a discussion whether there exists a maximizer $q \in B_A$ in (5.11) or not. We do not have a definite solution of this problem but we will give some partial answers.

Let $(\tilde{q}, \tilde{u})$ be the couple constructed in Theorem 22 and let v be a non-negative minimizer of $R_{\tilde{q}}$. Assume that there exists no maximizer of (5.11). Then $R_{\tilde{q}}(v) = \lambda' < \lambda = R_{\tilde{q}}(\tilde{u})$. It is clear that $R_{\tilde{q}}$ is Gateaux-differentiable in the direction of v at $\tilde{u}$ and viceversa. This gives us $(\lambda - \lambda') \int_{\Omega} \tilde{u}v = 0$. Hence

$$\tilde{u}v = \tilde{q}v = 0 \text{ a.e. in } \Omega.$$

Let $\{\Omega_j\}$ be the components of $\Omega \setminus \{x : \ k(x) = 0\}$ and let $\tilde{q}$ be ∞ on the zero set of k. It is clear that $\tilde{u}$ and v have their essential support in $\bigcup_j \Omega_j$ and that $Ev = \lambda' v$ holds in each component Ω_j. The strong maximum principle yields that for each Ω_j, either $\tilde{u} = 0$ $(v = 0)$ in Ω_j or $\tilde{u} > 0$ $(v > 0)$ in Ω_j. Thus we have the following corollaries.

COROLLARY 23: *Assume that the conditions in Theorem 22 hold and that either $\Omega \setminus \{x : \ k(x) = 0\}$ is connected or $\tilde{u}$ is non identically zero in any of the components $\{\Omega_j\}$, then there exists an extremal couple $(\tilde{q}, \tilde{u})$.*

The proof follows from the discussion above.

COROLLARY 24: *Assume that the conditions in Theorem 22 hold and that there exists a minimizer v of $R_{\tilde{q}}$ with the properties given above. If v is not identically zero in a component Ω_j of $\Omega \setminus \{k: \ k(x)=0\}$ and if $v|_{\Omega_j} \in H_0^1(\Omega_j)$, then there exists an extremal couple $(\tilde{q}, \tilde{u})$.*

PROOF. Assume that the conclusion is false. Take $\{w_n\} \subset C_0^1(\Omega_j)$ such that $w_n \to v$ and $w_n \geq 0$. Let $v_n = \varepsilon_n w_n$, where $\varepsilon_n > 0$ is small enough so that $\left\| \frac{v_n}{k} \right\|_\infty \leq 1$. We have

$$\lambda_n = \frac{a(v_n, v_n)}{\int\limits_\Omega v_n^2 h} \to \lambda' \text{ as } n \to \infty.$$

Hence

$$J(\tilde{u} + v_n) = \frac{\lambda \int\limits_\Omega h\tilde{u}^2 + \lambda_n \int\limits_\Omega h v_n^2}{\int\limits_\Omega h\tilde{u}^2 + \int\limits_\Omega h v_n^2} < \lambda$$

if n is large enough. This is a contradiction.

Assume that the components $\{\Omega_j\}$ satisfy $\mathrm{d}(\Omega_i, \Omega_j) > 0$ for all $i \neq j$, and that each $\partial\Omega_i$ satisfies some "minimal" smoothness conditions. Then it can be shown that the minimizer $v|_{\Omega_j} \in H_0^1(\Omega_j)$ for each component Ω_j, and hence there exists an extremal couple $(\tilde{q}, \tilde{u})$ by Corollary 24.

We do not have an example where (5.11) does not have a maximizer. But what can occur is the following.

Let Ω_1 and Ω_2 be two components such that $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$. Although v is zero a.e. on $\{x : k(x) = 0\}$ it is clear that we cannot conclude that $v \in H_0^1(\Omega_i)$ for $i = 1, 2$, and we might have $R_{\tilde{q}}(v) < R_{\tilde{q}}(u)$. The inequality in (5.11) is obtained by letting q tend to a measure with mass on $\partial\Omega_1 \cap \partial\Omega_2$.

6. - Examples and generalizations

In this section we will give some simple examples to illustrate the results and to indicate some generalizations.

Let $p = 1$, $\Omega = (-1, 1)$, $A = 1$, $E = -\Delta$, $h = 1$ and assume that k fulfils the conditions in Theorem 14′ or Theorem 22. Furthermore, let $(\tilde{q}, \tilde{u})$ be an extremal couple. We know that $\tilde{u} \in C^1(\overline{\Omega})$ (Theorem 13′). If k is C^2 on the coincidence set $I = \{x : \ k(x) = \tilde{u}(x)\}$, then $\tilde{u}$ solves the free boundary problem

$$\begin{cases} u'' + \lambda u = 0 \text{ in } \Omega \setminus I, \\ u(\pm 1) = 0, \quad 0 \leq u < k \text{ in } \Omega \setminus I, \\ u = k \text{ and } u' = k' \text{ on } \partial I, \\ \displaystyle\int\limits_I (\lambda k^2 + k'' k) = 1. \end{cases}$$

Here λ, I and u are the unknown. If we can solve this problem, we obtain the maximizing function $\tilde{q} = \left(\lambda_1 + \frac{k''}{k}\right)\chi\{I\}$.

EXAMPLE 1: If $k = 1$ we get $\tilde{q} = \lambda_1\chi\{[-\alpha, \alpha]\}$ where

$$\lambda_1 = \frac{\pi^2}{4}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{\pi^2}}\right)^2 \quad \text{and } \alpha = (2\lambda_1)^{-1}.$$

EXAMPLE 2: Take $k = (x - b)^2 + e$, $e \geq 0$. Then

$$\tilde{q} = \left(\lambda_1 + \frac{2}{(x-b)^2 + e}\right)\chi\{[\alpha, \beta]\}.$$

In the figures below we give some numerical solutions.

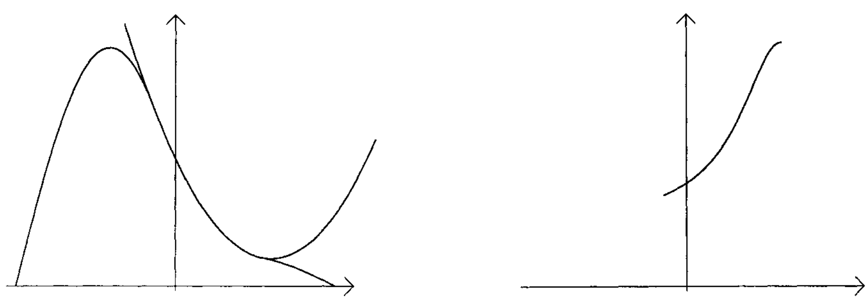

Fig. 1: $b = 0.6$, $e = 0, 1$, $\lambda \approx 7.13$ and $I \approx [-0.148, 0.548]$

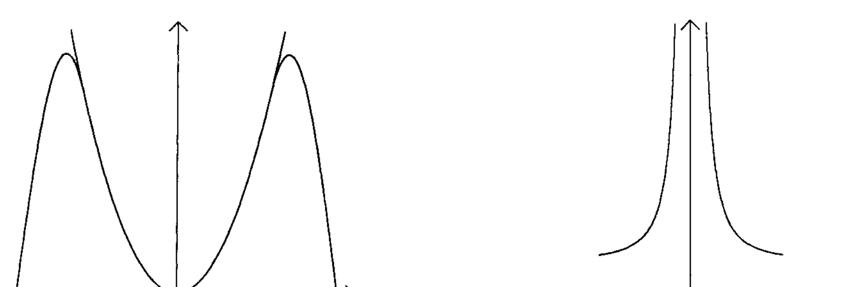

Fig. 2: $b = 0$, $e = 0$, $\lambda \approx 26.88$ and $I \approx [-0.585, 0.585]$

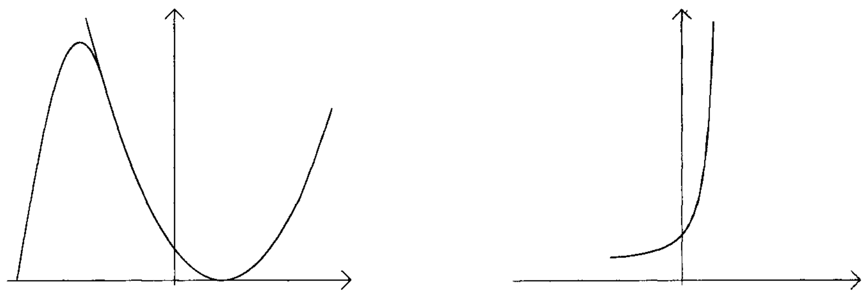

Fig. 3: $b = 0.3$, $e = 0$, $\lambda \approx 15.42$ and $I \approx [-0.448, 0.3]$

The example in Figure 2 is interesting. Here the eigenspace corresponding to the first eigenvalue is two-dimensional.

The solution $\tilde{u}$ in Figure 3 is zero in one of the components of $\Omega \backslash I$. But we can show that $(\tilde{q}, \tilde{u})$ is an extremal by using the corollaries of Theorem 22.

EXAMPLE 3: Let us consider a problem where we have to truncate the obstacle k. Take $k = \dfrac{1}{|x|} - 1$. As the family $\{k_n\}$ we take $k_n = \min(k, n)$. Clearly, this family satisfies the conditions of Theorem 14′. We get the following numerical solution.

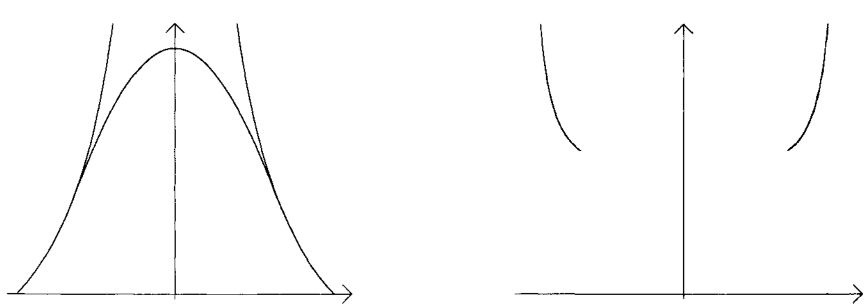

Fig. 4: $\lambda_1 \approx 3.31$ and $I \approx [-1, -0.647] \cup [0.647, 1]$

EXAMPLE 4: If we take $k = 1 \pm |x|$, then $E(k) = -k'' = \mp 2\delta$. For $k = 1 - |x|$ we have $E(k)^- = 0$ and we obtain the following solution.

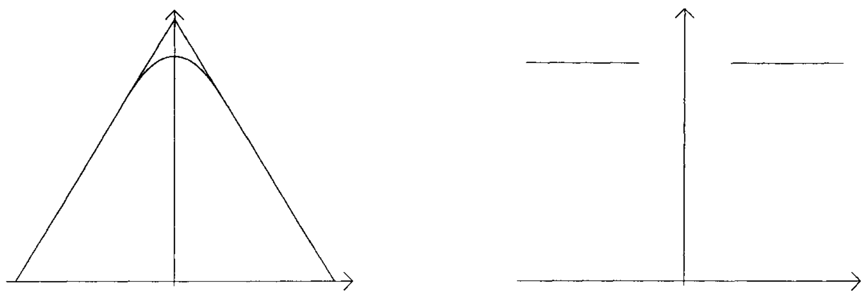

Fig. 5: $A = 1$, $\lambda_1 \approx 4.23$ and $I \approx [-1, -0.29] \cup [0.29, 1]$

If $k = 1 + |x|$, then $E(k)^- = 2\delta_0$ and the assumptions in the theorems do not hold. However, formally we have $\tilde{q} = (\lambda_1 + 2\delta_0)\chi\{[-\alpha, \alpha]\}$ if $A > 2$ and $\tilde{q} = A\delta_0$ if $A \leq 2$. The graph of $\tilde{u}$ is given below.

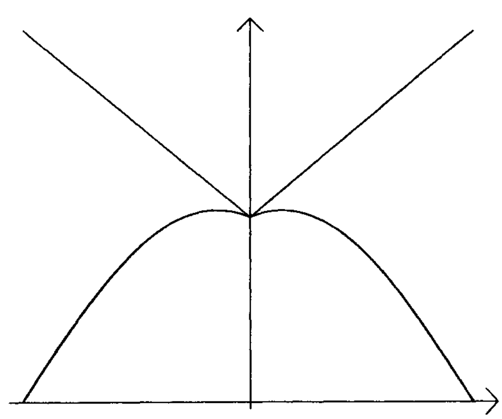

Fig. 6: $A = 1$ and $\lambda \approx 3.37$

It can be shown that $\tilde{q}$ is a maximizer of the first eigenvalue. Furthermore, there exists no maximizer of the first eigenvalue in $L^1(\Omega)$. Let us prove the last statement. Assume that $q \in L^1(\Omega)$ is a maximizer. Then we have $\inf_u R_q(u) = J(\tilde{u})$. But then $\tilde{u}$ is a minimizer of R_q. Hence $-\tilde{u}'' + q\tilde{u} = \lambda_1\tilde{u}$, and thus q cannot be in $L^1(\Omega)$.

EXAMPLE 5: In this paper we have only considered the eigenvalue problem with zero boundary data. However, the technique used in this paper can be used to study the problem when we have mixed boundary data. By mixed boundary data we mean the following.

Let $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ be a decomposition of the boundary. We take $u = 0$ on $\partial\Omega_1$ and $\partial_\nu u = 0$ on $\partial\Omega_2$, where ∂_ν is the normal derivative.

However, we have to be careful. For example, Theorems 13 (13′) do not always hold if we have mixed boundary data.

As an example, let us consider the following problem. Find

$$q \in B_A = \{f : \int_{-1}^{1} |f| \leq A\}$$

maximizing the first eigenvalue of

$$\begin{cases} -u'' + qu = \lambda u \\ u(-1)\sin(\alpha) + u'(-1)\cos\alpha = 0 \\ u(1)\sin(\beta) + u'(1)\cos\beta = 0. \end{cases}$$

We get the Rayleigh quotient

$$R_q(u) = \frac{a(u,u) + \int_{-1}^{1} qu^2 + u(1)^2 \tan\beta - \tilde{u}(-1)^2 \tan\alpha}{\int_{-1}^{1} u^2},$$

which is estimated from above by

$$J(u) = \frac{a(u,u) + A\|u\|_\infty^2 + u(1)^2\tan\beta - u(-1)^2\tan\alpha}{\int_{-1}^{1} u^2}.$$

If $\tan\alpha$ or $\tan\beta$ is unbounded, then the corresponding term is excluded in the formulas for R_q and J.

Let $\tilde{u}$ be a non-negative minimizer of J normalized so that $\|\tilde{u}\|_\infty = 1$. Then $\tilde{u}$ solves the variational inequality

$$\text{(6.1)} \qquad \begin{aligned} u \in K :\ a(u, v-u) - \lambda \int_{-1}^{1} u(v-u) + u(1)\tan\beta(v(1) - u(1)) - \\ -u(-1)\tan\alpha(v(-1) - u(-1)) \geq 0, \quad \forall v \in K. \end{aligned}$$

Here $K = \{v \in H^1(\Omega) :\ v \leq 1\}$ and $\lambda = J(\tilde{u})$. If $\tan\alpha$ or $\tan\beta$ is unbounded, then we impose zero boundary data at the corresponding boundary point.

Note that $\tilde{u}$ also solves the variational inequality above with $K = \{v \in H^1(\Omega) : |v| \leq 1 \text{ and } v(x) = \tilde{u}(x) \text{ for } x = \pm 1\}$. Thus, we can make a transformation in order to satisfy the hypothesis in Theorem A2. Hence

$$\tilde{u} \in H^{2,s}(\Omega) \cap C^{1,\alpha}(\Omega), \quad \alpha = 1 - \frac{n}{s} \text{ for all } s \in (n, \infty).$$

Thus, $\tilde{u}$ solves

$$-\tilde{u}'' + q\tilde{u} = \lambda\tilde{u}, \tag{6.2}$$

where $\tilde{q} = \lambda_{\chi\{I\}}$ and $I = \{x : \ \tilde{u}(x) = 1\}$.

A variation of (6.1) at the boundary yields

$$\begin{cases} \tilde{u}'(1) + \tilde{u}(1)\tan\beta \le 0, \\ \tilde{u}'(-1) + \tilde{u}(-1)\tan\alpha \ge 0, \end{cases} \tag{6.3}$$

with equality if the boundary point does not belong to the coincidence set I.

If we assume that $\tan\beta \ge 0$ and $\tan\alpha \le 0$, then it is easy to see that we have equality in (6.3). Hence, if we combine this fact with (6.2), we get

$$\int_{-1}^{1} \tilde{q} = \int_{-1}^{1} \tilde{q}\tilde{u}^2 = A.$$

This shows that $\tilde{q} \in B_A$.

Using the same method as before, we can show that $(\tilde{q}, \tilde{u})$ is an extremal couple. Thus, we have proved that the problem has an extremal couple, provided $\tan\beta \ge 0$ and $\tan\alpha \le 0$.

Now let us consider the case where $\tan\beta < 0$ and $u(-1) = 0$. In this case we do not have equality in the first inequality in (6.3). However, a simple calculation shows that we have the following *formal* solution.

Case I: If $A + \tan\beta > 0$ then $\tilde{q} = -\tan\beta\delta_1 + \lambda_1\chi\{[m, 1]\}$ where $m = \sqrt{c(c+4)} - c - 1$, $c = \dfrac{\pi^2}{8(A + \tan\beta)}$ and $\lambda_1 = \dfrac{A + \tan\beta}{1 - m}$.

Case II: If $-\frac{1}{2} < A + \tan\beta < 0$ then $\tilde{q} = A\delta_1$ and λ_1 is the first positive root of $\tan(2\sqrt{\lambda}) = \dfrac{-\sqrt{\lambda}}{A + \tan\beta}$.

Case III: If $A + \tan\beta = -\frac{1}{2}$ then $\tilde{q} = A\delta_1$ and $\lambda_1 = 0$.

Case IV: If $A + \tan\beta < -\frac{1}{2}$ then $\tilde{q} = A\delta_1$ and λ_1 is the negative root of $\tan h(\sqrt{-\lambda}2) = \dfrac{-\sqrt{-\lambda}}{A + \tan\beta}$.

The same argument as in Example 4 shows that there is no maximizer in B_A. This shows that Harrell's result [HA] is incorrect. He claims that whenever we have a self-adjoint realization of $-\Delta + q$, then the maximizer $\tilde{q}$ is the maximal eigenvalue times a characteristic function.

Clearly, the arguments above can be extended to the general case, where k and h are non-constant. However, we have to impose certain conditions on k. We give two numerical solutions of the problem in the figures below.

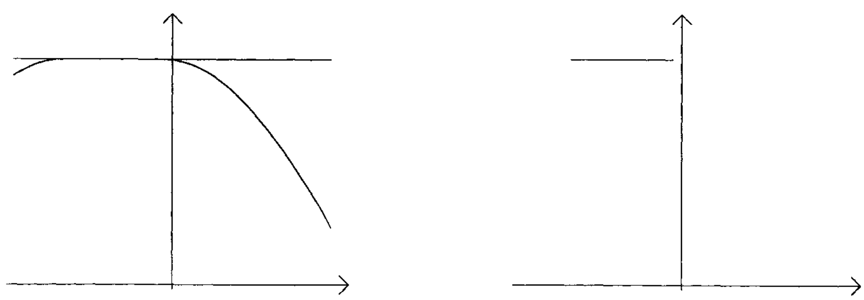

Fig. 7: $\alpha = -\frac{\pi}{10}$, $\beta = \frac{4\pi}{10}$, $\lambda_1 \approx 1.57$, $I \approx [-0.69, -0.05]$ and $A = 1$

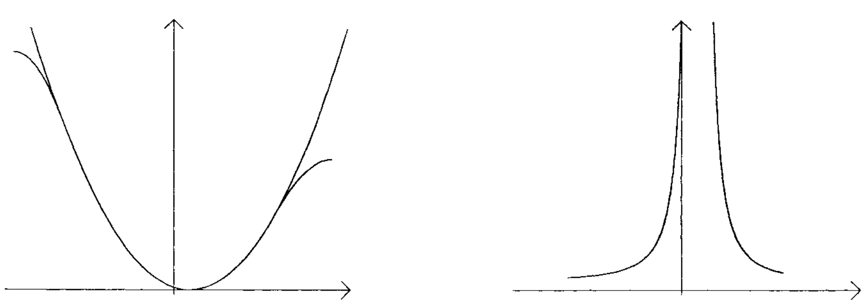

Fig. 8: $\alpha = \beta = 0$, $k = (x - 0.1)^2$, $\lambda_1 \approx 6.86$, $I \approx [-0.71, 0.63]$ and $A = 1$

If we take $B_A = \{f : \int_{-1}^{1} |f|^p \leq A^p\}$, where $1 < p < \infty$, then a solution can be obtained using the same technique as in Section 2.

EXAMPLE 6: In this paper we have assumed that Ω is connected. However, all results except Proposition 18 (18′) hold if this condition is excluded.

Let us consider a simple example. Take $k = h = 1$, $p = 1$, $E = -\Delta$ and $\Omega = (-1, 0) \cup (r, 1)$, where $0 < r < 1$. We can argue as before to obtain an extremal couple $(\tilde{q}, \tilde{u})$.

If

$$r \leq \left(1 + \frac{\pi^2}{2A}\right) - \sqrt{\frac{\pi^2}{2A}\left(2 + \frac{\pi^2}{2A}\right)}, \tag{6.4}$$

then

$$\lambda_1 = \frac{\pi^2}{(2-r)^2}\left(1 + \sqrt{1 + \frac{A(2-r)}{\pi^2}}\right)^2 \text{ and}$$

$$\tilde{q} = \lambda_1 \chi\{[\alpha, \beta] \cup [\gamma, \delta]\},$$

where $\alpha = \frac{\pi}{2\sqrt{\lambda}} - 1$, $\beta = -\frac{\pi}{2\sqrt{\lambda}}$, $\gamma = \frac{\pi}{2\sqrt{\lambda}} + r$ and $\delta = 1 - \frac{\pi}{\sqrt{\lambda}}$.
The solution is illustrated in the following figure.

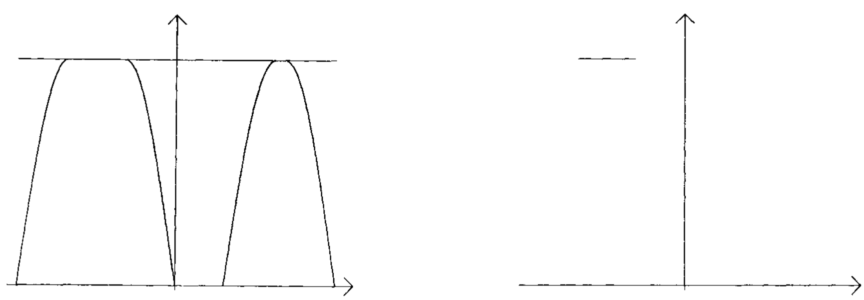

Fig. 9: $A = 10$, $r = 0.3$, $\lambda_1 \approx 24.0$ and $I \approx [-0.68, -0.32] \cup [0.62, 0.68]$

If (6.4) does not hold, then $\tilde{u} = 0$ in the component $(r, 1)$. In this case we obtain the same solution as in Example 1 in the component $(-1, 0)$.

This example also shows that Proposition 18 does not hold, if we have equality in (6.4). On the other hand, if we do not have equality in (6.4) then $\tilde{u}$ is the unique non-negative minimizer of J.

EXAMPLE 7: We have obtained several explicit solutions in the case where $p = 1$. In this example the case $p = 2$ will be considered.

Take $\Omega = (-1, 1)$, $k = h = 1$, $A = 1$, and $E = -\Delta$. In Section 2 we proved that the extremal couple $(\tilde{q}, \tilde{u})$ exists and is in $C^\infty(\overline{\Omega})$. Furthermore, $\tilde{q}$ and $\tilde{u}$ are even non-negative functions and $\tilde{u}$ is the first eigenfunction of

$$\begin{cases} -u'' + \|u^2\|_2^{-2}|u|^2 u = \mu u \text{ in } \Omega, \\ u(-1) = u(1) = 0. \end{cases}$$

A straightforward calculation yields

$$\begin{cases} \displaystyle\int_0^{\tilde{u}(x)} \frac{dv}{\sqrt{v^4 - \frac{4\mu}{3}v^2 + C}} = \frac{1}{2}(x+1), \quad -1 \leq x \leq 0, \\ C = -\tilde{u}^4(0) + \dfrac{4\mu}{3}\, \tilde{u}^2(0) \geq 0, \\ 2\displaystyle\int_{-1}^{0} \tilde{u}^4 = 1. \end{cases}$$

Since the level sets $\{x : \tilde{u}(x) \geq c\}$ are connected (Proposition 20), we also find that $\tilde{u}'(x) \leq 0$ for $-1 \leq x \leq 0$, with equality at $x = 0$.

The solution μ_1, $\tilde{u}$ of these equations gives us the first maximal eigenvalue $\lambda_1 = \mu_1$ and the maximizer $\tilde{q} = \tilde{u}^2$. Thus $\tilde{q}$ is the square of an elliptic function. However, it seems difficult to find explicit expressions for $\tilde{u}$ and μ_1.

EXAMPLE 8: Consider the problem of finding $q \in B_A$, *minimizing* the first eigenvalue of (0.1). This problem can be solved using the same technique as in this paper. We will take $k = h = 1$ for simplicity.

The Rayleigh quotient R_q is estimated from below by

$$J^-(u) = \frac{a(u,u) - A\|u\|_{2p'}^2}{\int_\Omega u^2}.$$

To determine whether J^- is bounded from below or not, we need the following result.

PROPOSITION 25: *If $p > \frac{n}{2}$ and Ω is bounded, then there exists a constant C such that for all $\varepsilon > 0$ there exists a μ_ε such that*

$$\|u\|_{2p'}^2 \leq C\left(\varepsilon\|u\|_{H_0^1}^2 + \mu_\varepsilon\|u\|_2^2\right), \quad \forall u \in H_0^1(\Omega). \tag{6.5}$$

Furthermore, the following shows that the result is sharp.

(i) *If $p < \frac{n}{2}$ or if $n = 2$ and $p = 1$, then* (6.5) *does not hold for any fix ε.*

(ii) *If $p = \frac{n}{2}$ and $n > 2$, then* (6.5) *does not hold uniformly in $\varepsilon > 0$.*

PROOF. Take $\frac{1}{r} = \frac{1}{2} - \frac{1}{n}$ (if $n = 2$, then let $r < \infty$). The Sobolev imbedding theorem yields

$$\|u\|_r \leq C\|u\|_{H_0^1}, \quad \forall u \in H_0^1(\Omega). \tag{6.6}$$

For $2p' < r$, we have the following interpolation inequality ([GT], Section 7.1)

$$\|u\|_{2p'}^2 \leq \varepsilon\|u\|_r^2 + \varepsilon^{-\mu}\|u\|_2^2.$$

Combining this inequality with (6.6) gives the first part of the proposition.

If $p < \frac{n}{2}$, then (i) follows by a dilatation argument. Actually (i) just states that the Sobolev imbedding theorem is sharp. If $n = 2$ and $p = 1$, then (i) follows from [AD], Example 5.26.

Clearly, (6.5) holds for some $\varepsilon > 0$, if $p = \frac{n}{2}$ and $n > 2$. However, if we assume that (6.5) holds uniformly in $\varepsilon > 0$, then it follows that the inclusion map

$$i: \ H_0^1(\Omega) \to L^{2p'}(\Omega), \quad \frac{1}{2p'} = \frac{1}{2} - \frac{1}{n},$$

is compact. But this is a contradiction (cf. [AD], Example 6.11).

THEOREM 26: *Let $\lambda_1(q)$ be the first eigenvalue of* (0.1), *where $k = h = 1$. Then the following hold:*

(I) *If $p > 1$ and $p > \frac{n}{2}$ then there exist an extremal couple $(\tilde{q}, \tilde{u})$ such that*

(i) *$\tilde{q}$ minimizes λ_1 over B_A, $\lambda_1(\tilde{q}) > -\infty$ and $\tilde{q} = -A|\tilde{u}|^{\frac{2p'}{p}}\|\tilde{u}^2\|_{p'}^{1-p'}$.*

(ii) *$\tilde{u}$ is a non-negative minimizer of J^- and $\tilde{u}$ is the first eigenfunction of*

$$u \in H_a(\Omega): \ Eu - A\|u^2\|_{p'}^{1-p'}|u|^{2(p'-1)}u = \mu u, \tag{6.7}$$

where the eigenvalue $\mu_1 = \lambda_1(\tilde{q})$.

Furthermore, if there exists an open set $U \subset \Omega$ such that $|a_{ij}|$ is bounded on U for $1 \leq i, j \leq n$ then:

(II) *If $p < \frac{n}{2}$ or $p = 1$ and $n = 2$, then there exist functions $\{q_n\} \subset B_A$ such that $\lambda_1(q_n) \to -\infty$ as $n \to \infty$.*

(III) *If $p = \frac{n}{2}$ and $n > 2$, then there exists a positiv constant A^* such that if $A < A^*$ then* (I) *holds and if $A > A^*$ then* (II) *holds. If $A = A^*$ none of the cases* (I) *and* (II) *need to hold.*

Part (I) and (II) in the theorem is proved using Proposition 25 and arguing as in Section 2. The only new difficulty in part (III) is to prove that there exists a minimizer of J^-. Since $H_0^1 \to L^{2p'}$ is not compact we cannot use the standard technique. However, the following lemma can be proved using a result of Brezis and Lieb [BL].

LEMMA 27: *If there exists a $\delta > 0$ such that $J_{A+\delta}^-$ is bounded from below in H_0^1, then J_A^- attains its minimum.*

The proof is omitted.

If $E = -\Delta$ in Theorem 26 part (III), then we can show the following results.

(i) $A^* = C^{-2}$, where C is the best constant in $\|u\|_{2p'} \leq C\|\nabla u\|_2$.

(ii) If $n > 4$, then $\inf\limits_{u \in H_0^1} J_{A^*}^-(u) = 0$ and the infimum is not attained. Thus in this case there is no minimizer of the first eigenvalue and $\inf\limits_{q \in B_{A^*}} \lambda_1(q) = 0$.

(iii) If $n \geq 4$ and $\Omega = \{x \in R^n : \ 1 < |x| < 2\}$ then the minimizer of J_A^- is nonradial if A is sufficiently close to A^*. This shows that the minimizer $\tilde{q}$ is not unique and does not have the same symmetries as E and Ω (compare with Theorem 16 and Corollary 17).

These results are closely related to a paper by Brezis and Nirenberg [BN] and the proof will be given in a later paper.

The minimizer in part (I) of the theorem is a bounded function and if $\{a_{ij}\} \subset L^\infty$ then the same holds for the minimizer in part (III). Thus in this case we can obtain the same regularity results for the extremal couple as in Section 2.

The theorem does not include the case $n = 1$, $p = 1$. However, Talenti [TA$'$] has showed that if $E = -\Delta$ and $\Omega = (-R, R)$, then $\tilde{q} = -A\delta_0$ minimizes the first eigenvalue over B_A.

If k and h are not assumed to be constant, we can obtain results similar to Theorem 26.

Appendix

Here we shall prove two results used in this paper. The first theorem concerns the regularity of solutions of the variational inequality.

$$\text{(A.1)} \qquad u \in K : \ a(u, v-u) - \lambda \int_\Omega hu(v-u) \geq 0, \quad \forall v \in K$$

$$\text{where } K = \{u \in H_0^1(\Omega) : \ |u| \leq k\}.$$

The second theorem states that if u is non-negative and satisfies

$$\text{(A.2)} \qquad u \in H_a(\Omega) : \ Eu - \lambda u \leq 0,$$

then u is bounded.

The following theorem is proved using the same technique as in [KS], Chapter 4.

THEOREM A1: *Let $\{a_{ij}\} \subset C^{0,1}(\overline{\Omega})$ and $\partial\Omega \in C^2$. Assume that $h \in L^\infty(\Omega)$, $h > 0$ a.e. in Ω, $k \in H^1(\Omega)$, $k \geq 0$ a.e. in Ω and $E(k)$ is a Radon measure* *such that* $E(k)^- \in L^s(\Omega)$ *for some* $s > n$. *Then the non-negative solution* $\tilde{u}$ *of* (A.1) *is in* $H^{2,s}(\Omega) \subset C^{1,\alpha}(\overline{\Omega})$, *where* $\alpha = 1 - \dfrac{n}{s}$.

REMARK: The theorem does not state that there exists a solution of (A.1).

PROOF. First, we recall that $\tilde{u} \in L^\infty(\Omega)$ and that $\tilde{u}$ also solves (A.1) if we take $K = \{u \in H_0^1(\Omega) : \ u \leq k\}$ (cf. Proposition 12′). Hence, $\tilde{u}$ is the *unique* solution of the variational inequality

$$u \in K : \ a(u, v - u) \geq \int_\Omega f(v - u), \quad \forall v \in K, \tag{A.3}$$

where $f = h\tilde{u}\lambda \in L^\infty(\Omega)$ and

$$K = \{u \in H_0^1(\Omega) : \ u \leq k\}.$$

The uniqueness follows since the operator in (A.3) is strictly monotone.

Consider the penalized problem

$$u \in H_0^1(\Omega) : \ Eu = -(E(k) - f)^- \theta_\varepsilon(k - u) + f, \tag{A.4}$$

where $\varepsilon > 0$ and

$$\theta_\varepsilon(t) = \begin{cases} 1 & \text{if } t < 0 \\ 1 - t/\varepsilon & \text{if } 0 \leq t \leq \varepsilon \\ 0 & \text{if } t > \varepsilon. \end{cases}$$

The operator $L : \ H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$< Lu, \xi >= a(u, \xi) + \int_\Omega ((E(k) - f)^- \theta_\varepsilon(k - u) - f)\xi,$$

is coercive, strictly monotone and continuous on finite dimensional subspaces. Hence, (A.4) has a unique solution u_ε, [KS], Chapter 3.

Our next step is to prove that $\{u_\varepsilon\} \subset K$. The non-negative function $\xi = \tilde{u}_\varepsilon - \min(u_\varepsilon, k) \in H_0^1(\Omega)$ and (A.4) yields

$$a(u_\varepsilon, \xi) = \int_\Omega (-(E(k) - f)^- \theta_\varepsilon(k - u_\varepsilon) + f)\xi.$$

By the assumptions, $E(k)$ is a Radon measure, thus

$$a(k, \xi) = \int_\Omega E(k)\xi.$$

If we subtract these two equations and note that

$$\partial_i \xi = \begin{cases} \partial_i(u_\varepsilon - k) \text{ a.e. on } \{x: \ u_\varepsilon(x) > k(x)\} \\ 0 \text{ elsewhere,} \end{cases}$$

we obtain

$$a(\xi, \xi) = \int_\Omega (f - E(k) - (E(k) - f)^-)\xi.$$

The right hand side is non-positive. Hence $\xi = 0$, that is $u_k \leq k$.

Applying the L^p estimate ([MO], Section 5.6) on solutions of (A.4) yields

$$\|u_\varepsilon\|_{H^{2,s}} \leq C(s, \Omega, E)(\|f\|_s + \|(E(k) - f)^-\|_s). \tag{A.5}$$

Hence $\{u_\varepsilon\}$ is bounded in $H^{2,s}(\Omega)$. For a subsequence, we have

$$\begin{aligned} &u_\varepsilon \rightharpoonup \overline{u} \text{ in } H^{2,s}(\Omega) \text{ and} \\ &u_\varepsilon \to \overline{u} \text{ in } H^{1,s}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}), \quad \alpha < 1 - \frac{n}{s}. \end{aligned}$$

If we combine Minty's lemma [KS] and (A.4) we obtain

$$a(v, v - u_\varepsilon) + \int_\Omega ((E(k) - f)^- \theta_\varepsilon(k - v) - f)(v - u_\varepsilon)) \geq 0, \quad \forall v \in H_0^1(\Omega).$$

Let us assume that $\operatorname{ess\,inf}_\Omega k > 0$. Then we can choose $\delta > 0$ and $v \leq k - \delta$ in the inequality above. Letting $\varepsilon \to 0$ yields

$$a(v, v - \overline{u}) - \int_\Omega f(v - \overline{u}) \geq 0, \quad \forall v \in H_0^1(\Omega), \quad v \leq k - \delta.$$

Finally, if we let $\delta \to 0$ and apply Minty's lemma again, we find that $\overline{u}$ solves (A.3). Hence, $\overline{u} = \tilde{u}$ by uniqueness and we get $\tilde{u} \in H^{2,s}(\Omega)$.

To finish the proof we have to remove the condition $\operatorname{ess\,inf} k > 0$. Let $\tilde{u}_\delta$ be the solution of (A.3), where we have replaced K by $K_\delta = \{u \in H_0^1(\Omega) : u \leq k + \delta\}$, $\delta > 0$. Clearly, the L^p estimate (A.5) holds for $\tilde{u}_\delta$. Hence $\{\tilde{u}_\delta\}$ is bounded in $H^{2,s}(\Omega)$. The same arguments as above yield that $\tilde{u}_\delta \rightharpoonup \tilde{u}$ in $H^{2,s}(\Omega)$ as $\delta \to 0$.

By the Sobolev imbedding theorem we have $H^{2,s}(\Omega) \subset C^{1,\alpha}(\overline{\Omega})$, where $\alpha = 1 - \frac{n}{s}$. This concludes the proof.

From (A.3) we obtain $\lambda h\tilde{u} - E(\tilde{u}) \geq 0$. Equation (A.4) yields $\lambda h\tilde{u} - E(u_\varepsilon) \leq (E(k) - \lambda h\tilde{u})^-$. Thus, if we let $\varepsilon \to 0$ we obtain

$$0 \leq \lambda h\tilde{u} - E(\tilde{u}) \leq (\lambda h\tilde{u} - E(k))^+.$$

Define the coincidence set I as the complement of $\{x \in \Omega : \exists \varphi \in C_0^1(\Omega)$ such that $\varphi(x) > 0$ and $\tilde{u} + \varphi \leq k\}$. A variation of (A.3) in $\Omega \setminus I$ yields $E\tilde{u} + \lambda h\tilde{u} = 0$ a.e. in $\Omega \setminus I$.

Thus, we have proved the following corollary of Theorem A1.

COROLLARY A2: *We have*

$$0 \leq \lambda h\tilde{u} - E(\tilde{u}) \leq (\lambda h\tilde{u} - E(k))^+ \chi\{I\}.$$

REMARK: If k is continuous, then

$$I = \{x \in \Omega : \ \tilde{u}(x) = k(x)\}.$$

THEOREM A3: *Assume that* $Ev = -\partial_i(a_{ij}\partial_j v)$, *where* $\{a_{ij}\} \subset L^1(\Omega)$ *and* $a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$ *for all* $(x, \xi) \in \Omega \times R^n$. *Let u be non-negative and satisfy* (A.2). *Then*

$$||u||_\infty \leq C(n, \lambda ||h||_\infty, \nu, m(\Omega)) ||u||_2.$$

If $\{a_{ij}\} \subset L^\infty(\Omega)$, then $H_a(\Omega) = H_0^1(\Omega)$ and the result is a special case of [GT], Theorem 8.15.

We will need the following lemma.

LEMMA A4: *If* $u \in H_a(\Omega)$, $F \in C^1(R^1)$, $F(0) = 0$ *and* $F' \in L^\infty(R^1)$, *then* $F(u) \in H_a(\Omega)$. *If also* $F' \geq 0$, *then* $a(F(u), u) \geq \nu \int_\Omega |\nabla u|^2 F'(u)$.

PROOF. If $H_a(\Omega) = H_0^1(\Omega)$, then the result follows from [GT], Section 7.4.

Take $u \in H_a(\Omega)$ and $\{u_n\} \subset C_0^1(\Omega)$ such that $u_n \to u$ in $H_a(\Omega)$. Then each $F(u_n) \in H_a(\Omega)$ and it is easy to verify that $F(u_n) \to F(u)$ in $H_0^1(\Omega)$. If we can show that $\{F(u_n)\}$ is bounded in $H_a(\Omega)$, then Lemma 1 yields that $F(u) \in H_a(\Omega)$.

The boundedness of $\{F(u)\}$ follows from the estimate

$$a(F(u_n), F(u_n)) = \int_\Omega F'(u_n)^2 a_{ij}\partial_i u_n \partial_j u_n \leq ||F'||_\infty^2 a(u_n, u_n).$$

To prove the last part of the lemma, we observe that

$$a(u_n, F(u_n)) = \int_\Omega F'(u_n) a_{ij}\partial_i u_n \partial_j u_n \geq \nu \int_\Omega |\nabla u_n|^2 F'(u_n).$$

Choose a subsequence such that $u_n \to u$ and $\nabla u_n \to \nabla u$ a.e. in Ω as $n \to \infty$. If we let $n \to \infty$ and use Fatou's lemma we finally obtain

$$a(u, F(u)) \geq \nu \int_\Omega F'(u)|\nabla u|^2.$$

PROOF OF THEOREM A3: Clearly, we can assume that $h = 1$. The proof is the same as that Theorem 8.15 [GT] and goes as follows.

Let N be a positive integer and take $\beta \geq 1$. Define $G \in C^1([0,\infty))$ by $G(s) = s^\beta$ if $0 \leq s \leq N$ and G is linear if $s > N$.

Using Lemma A4, we get

$$F(u) = \int_0^u |G'(s)|^2 \mathrm{d}s \in H_a(\Omega) \text{ and } \nu \int_\Omega |\nabla u|^2 F'(u) \leq \lambda \int_\Omega uF(u),$$

since u is non-negative and satisfies (A.2). If we also note that $F(s) \leq sF'(s)$ we obtain

$$\int_\Omega |\nabla u|^2 F'(u) \leq \frac{\lambda}{\nu} \int_\Omega u^2 F'(u).$$

Now, the Sobolev imbedding theorem and the definition of F yield

$$||G(u)||_{\frac{2n}{n-2}} \leq C(n, \lambda, \nu, m(\Omega))||G'(u)u||_2,$$

where we take $n = 2.1$ in the two-dimensional case.

Finally, letting $N \to \infty$ gives us

$$||u||_{\beta\chi 2} \leq (C\beta)^{1/\beta}||u||_{2\beta}, \tag{A.6}$$

where $\chi = \frac{n}{n-2} > 1$.

If we note that (A.6) holds for any $\beta \geq 1$, we get by induction with $\beta = \chi^m$, $m = 0, 1, \ldots$,

$$||u||_{2\chi^N} \leq \left(\prod_{m=0}^{N-1} (C\chi^m)^{\chi^{-m}}\right) ||u||_2 \leq C^\sigma \chi^\tau ||u||_2,$$

where $\sigma = \sum_{m=0}^{N-1} \chi^{-m}$ and $\tau = \sum_{m=0}^{N-1} m\chi^{-m}$. Hence, if $N \to \infty$ we get $||u||_\infty \leq C||u||_2$, where $C = C(n, \nu, \lambda, m(\Omega))$.

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