FORMALIZED MATHEMATICS
Volume 5, Number 1, 1996
Warsaw University - Białystok

# Extremal Properties of Vertices on Special Polygons, Part I

Yatsuka Nakamura
Shinshu University
Nagano

Czesław Byliński
Warsaw University
Białystok

**Summary.** First, extremal properties of endpoints of line segments in n-dimensional Euclidean space are discussed. Some topological properties of line segments are also discussed. Secondly, extremal properties of vertices of special polygons which consist of horizontal and vertical line segments in 2-dimensional Euclidean space, are also derived.

MML Identifier: SPPOL_1.

The terminology and notation used in this paper are introduced in the following articles: [18], [2], [12], [17], [21], [19], [22], [6], [15], [10], [16], [1], [7], [3], [5], [13], [4], [8], [20], [9], [14], and [11].

## 1. Preliminaries

One can prove the following propositions:

(1) For every finite sequence $f$ holds $f$ is trivial iff $\operatorname{len} f < 2$.

(2) For every finite set $A$ holds $A$ is trivial iff $\operatorname{card} A < 2$.

(3) For every set $A$ holds $A$ is non trivial iff there exist arbitrary $a_1$, $a_2$ such that $a_1 \in A$ and $a_2 \in A$ and $a_1 \neq a_2$.

(4) Let $D$ be a non empty set and let $A$ be a subset of $D$. Then $A$ is non trivial if and only if there exist elements $d_1$, $d_2$ of $D$ such that $d_1 \in A$ and $d_2 \in A$ and $d_1 \neq d_2$.

We follow a convention: $n$, $i$, $k$, $m$ will denote natural numbers and $r$, $r_1$, $r_2$, $s$, $s_1$, $s_2$ will denote real numbers.

Next we state a number of propositions:

(5) If $n \leq k$, then $n - 1 \leq k$ and $n - 1 < k$ and $n \leq k + 1$ and $n < k + 1$.




ISSN 0777–4028

(6) If $n < k$, then $n - 1 \leq k$ and $n - 1 < k$ and $n + 1 \leq k$ and $n \leq k - 1$ and $n \leq k + 1$ and $n < k + 1$.

(7) If $1 \leq k - m$ and $k - m \leq n$, then $k - m \in \operatorname{Seg} n$ and $k - m$ is a natural number.

(8) If $r_1 \geq 0$ and $r_2 \geq 0$ and $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.

(9) If $r_1 \leq 0$ and $r_2 \leq 0$ and $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.

(10) If $0 \leq r_1$ and $r_1 \leq 1$ and $0 \leq r_2$ and $r_2 \leq 1$ and $r_1 \cdot r_2 = 1$, then $r_1 = 1$ and $r_2 = 1$.

(11) If $r_1 \geq 0$ and $r_2 \geq 0$ and $s_1 \geq 0$ and $s_2 \geq 0$ and $r_1 \cdot s_1 + r_2 \cdot s_2 = 0$, then $r_1 = 0$ or $s_1 = 0$ but $r_2 = 0$ or $s_2 = 0$.

(12) If $0 \leq r$ and $r \leq 1$ and $s_1 \geq 0$ and $s_2 \geq 0$ and $r \cdot s_1 + (1 - r) \cdot s_2 = 0$, then $r = 0$ and $s_2 = 0$ or $r = 1$ and $s_1 = 0$ or $s_1 = 0$ and $s_2 = 0$.

(13) If $r < r_1$ and $r < r_2$, then $r < \min(r_1, r_2)$.

(14) If $r > r_1$ and $r > r_2$, then $r > \max(r_1, r_2)$.

In this article we present several logical schemes. The scheme *FinSeqFam* deals with a non empty set $\mathcal{A}$, a finite sequence $\mathcal{B}$ of elements of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(\mathcal{B}, i) : i \in \operatorname{dom} \mathcal{B} \ \wedge \ \mathcal{P}[i]\}$ is finite

for all values of the parameters.

The scheme *FinSeqFam'* concerns a non empty set $\mathcal{A}$, a finite sequence $\mathcal{B}$ of elements of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(\mathcal{B}, i) : 1 \leq i \ \wedge \ i \leq \operatorname{len} \mathcal{B} \ \wedge \ \mathcal{P}[i]\}$ is finite

for all values of the parameters.

Next we state several propositions:

(15) For all elements $x_1$, $x_2$, $x_3$ of $\mathcal{R}^n$ holds $|x_1 - x_2| - |x_2 - x_3| \leq |x_1 - x_3|$.

(16) For all elements $x_1$, $x_2$, $x_3$ of $\mathcal{R}^n$ holds $|x_2 - x_1| - |x_2 - x_3| \leq |x_3 - x_1|$.

(17) Every point of $\mathcal{E}^n_{\mathrm{T}}$ is an element of $\mathcal{R}^n$ and a point of $\mathcal{E}^n$.

(18) Every point of $\mathcal{E}^n$ is an element of $\mathcal{R}^n$ and a point of $\mathcal{E}^n_{\mathrm{T}}$.

(19) Every element of $\mathcal{R}^n$ is a point of $\mathcal{E}^n$ and a point of $\mathcal{E}^n_{\mathrm{T}}$.

## 2. Properties of line segments

In the sequel $p$, $p_1$, $p_2$, $q_1$, $q_2$ will denote points of $\mathcal{E}^n_{\mathrm{T}}$.

One can prove the following propositions:

(20) For all points $u_1$, $u_2$ of $\mathcal{E}^n$ and for all elements $v_1$, $v_2$ of $\mathcal{R}^n$ such that $v_1 = u_1$ and $v_2 = u_2$ holds $\rho(u_1, u_2) = |v_1 - v_2|$.

(21) For all $p$, $p_1$, $p_2$ such that $p \in \mathcal{L}(p_1, p_2)$ there exists $r$ such that $0 \leq r$ and $r \leq 1$ and $p = (1 - r) \cdot p_1 + r \cdot p_2$.

(22) For all $p_1$, $p_2$, $r$ such that $0 \leq r$ and $r \leq 1$ holds $(1 - r) \cdot p_1 + r \cdot p_2 \in \mathcal{L}(p_1, p_2)$.

(23) Given $p_1$, $p_2$ and let $P$ be a non empty subset of $\mathcal{E}^n_{\rm T}$. Suppose $P$ is closed and $P \subseteq \mathcal{L}(p_1, p_2)$. Then there exists $s$ such that $(1-s)\cdot p_1 + s \cdot p_2 \in P$ and for every $r$ such that $0 \leq r$ and $r \leq 1$ and $(1-r)\cdot p_1 + r\cdot p_2 \in P$ holds $s \leq r$.

(24) For all $p_1$, $p_2$, $q_1$, $q_2$ such that $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$ and $p_1 \in \mathcal{L}(q_1, q_2)$ holds $p_1 = q_1$ or $p_1 = q_2$.

(25) For all $p_1$, $p_2$, $q_1$, $q_2$ such that $\mathcal{L}(p_1, p_2) = \mathcal{L}(q_1, q_2)$ holds $p_1 = q_1$ and $p_2 = q_2$ or $p_1 = q_2$ and $p_2 = q_1$.

(26) $\mathcal{E}^n_{\rm T}$ is a $\rm T_2$ space.

(27) $\{p\}$ is closed.

(28) $\mathcal{L}(p_1, p_2)$ is compact.

(29) $\mathcal{L}(p_1, p_2)$ is closed.

Let us consider $n$, $p$ and let $P$ be a subset of $\mathcal{E}^n_{\rm T}$. We say that $p$ is extremal in $P$ if and only if:

(Def.1) $p \in P$ and for all $p_1$, $p_2$ such that $p \in \mathcal{L}(p_1, p_2)$ and $\mathcal{L}(p_1, p_2) \subseteq P$ holds $p = p_1$ or $p = p_2$.

We now state several propositions:

(30) For all subsets $P$, $Q$ of $\mathcal{E}^n_{\rm T}$ such that $p$ is extremal in $P$ and $Q \subseteq P$ and $p \in Q$ holds $p$ is extremal in $Q$.

(31) $p$ is extremal in $\{p\}$.

(32) $p_1$ is extremal in $\mathcal{L}(p_1, p_2)$.

(33) $p_2$ is extremal in $\mathcal{L}(p_1, p_2)$.

(34) If $p$ is extremal in $\mathcal{L}(p_1, p_2)$, then $p = p_1$ or $p = p_2$.

## 3. Alternating special sequences

We follow the rules: $P$, $Q$ will be subsets of $\mathcal{E}^2_{\rm T}$, $f$, $f_1$, $f_2$ will be finite sequences of elements of the carrier of $\mathcal{E}^2_{\rm T}$, and $p$, $p_1$, $p_2$, $p_3$, $q$ will be points of $\mathcal{E}^2_{\rm T}$.

The following proposition is true

(35) For all $p_1$, $p_2$ such that $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ there exists $p$ such that $p \in \mathcal{L}(p_1, p_2)$ and $p_{\mathbf{1}} \neq (p_1)_{\mathbf{1}}$ and $p_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (p_1)_{\mathbf{2}}$ and $p_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

Let us consider $P$. We say that $P$ is horizontal if and only if:

(Def.2) For all $p$, $q$ such that $p \in P$ and $q \in P$ holds $p_{\mathbf{2}} = q_{\mathbf{2}}$.

We say that $P$ is vertical if and only if:

(Def.3) For all $p$, $q$ such that $p \in P$ and $q \in P$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$.

Let us observe that every subset of $\mathcal{E}^2_{\rm T}$ which is non trivial and horizontal is also non vertical and every subset of $\mathcal{E}^2_{\rm T}$ which is non trivial and vertical is also non horizontal.

Next we state a number of propositions:

(36) $p_{\mathbf{2}} = q_{\mathbf{2}}$ iff $\mathcal{L}(p, q)$ is horizontal.

(37) $p_{\mathbf{1}} = q_{\mathbf{1}}$ iff $\mathcal{L}(p, q)$ is vertical.

(38) If $p_1 \in \mathcal{L}(p, q)$ and $p_2 \in \mathcal{L}(p, q)$ and $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$, then $\mathcal{L}(p, q)$ is horizontal.

(39) If $p_1 \in \mathcal{L}(p, q)$ and $p_2 \in \mathcal{L}(p, q)$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$, then $\mathcal{L}(p, q)$ is vertical.

(40) $\mathcal{L}(f, i)$ is closed.

(41) If $f$ is special, then $\mathcal{L}(f, i)$ is vertical or $\mathcal{L}(f, i)$ is horizontal.

(42) If $f$ is one-to-one and $1 \leq i$ and $i+1 \leq \operatorname{len} f$, then $\mathcal{L}(f, i)$ is non trivial.

(43) If $f$ is one-to-one and $1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $\mathcal{L}(f, i)$ is vertical, then $\mathcal{L}(f, i)$ is non horizontal.

(44) For every $f$ holds $\{\mathcal{L}(f, i) : 1 \leq i \ \wedge \ i \leq \operatorname{len} f\}$ is finite.

(45) For every $f$ holds $\{\mathcal{L}(f, i) : 1 \leq i \ \wedge \ i+1 \leq \operatorname{len} f\}$ is finite.

(46) For every $f$ holds $\{\mathcal{L}(f, i) : 1 \leq i \ \wedge \ i \leq \operatorname{len} f\}$ is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$.

(47) For every $f$ holds $\{\mathcal{L}(f, i) : 1 \leq i \ \wedge \ i+1 \leq \operatorname{len} f\}$ is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$.

(48) For every $f$ such that $Q = \bigcup\{\mathcal{L}(f, i) : 1 \leq i \ \wedge \ i+1 \leq \operatorname{len} f\}$ holds $Q$ is closed.

(49) $\widetilde{\mathcal{L}}(f)$ is closed.

A finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is alternating if:

(Def.4) For every $i$ such that $1 \leq i$ and $i+2 \leq \operatorname{len} \mathrm{it}$ holds $(\pi_i \mathrm{it})_{\mathbf{1}} \neq (\pi_{i+2} \mathrm{it})_{\mathbf{1}}$ and $(\pi_i \mathrm{it})_{\mathbf{2}} \neq (\pi_{i+2} \mathrm{it})_{\mathbf{2}}$.

One can prove the following propositions:

(50) If $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $(\pi_i f)_{\mathbf{1}} = (\pi_{i+1} f)_{\mathbf{1}}$, then $(\pi_{i+1} f)_{\mathbf{2}} = (\pi_{i+2} f)_{\mathbf{2}}$.

(51) If $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $(\pi_i f)_{\mathbf{2}} = (\pi_{i+1} f)_{\mathbf{2}}$, then $(\pi_{i+1} f)_{\mathbf{1}} = (\pi_{i+2} f)_{\mathbf{1}}$.

(52) Suppose $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $p_1 = \pi_i f$ and $p_2 = \pi_{i+1} f$ and $p_3 = \pi_{i+2} f$. Then $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

(53) Suppose $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $p_1 = \pi_i f$ and $p_2 = \pi_{i+1} f$ and $p_3 = \pi_{i+2} f$. Then $(p_2)_{\mathbf{1}} = (p_3)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ or $(p_2)_{\mathbf{2}} = (p_3)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$.

(54) If $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$, then $\mathcal{L}(\pi_i f, \pi_{i+2} f) \not\subseteq \mathcal{L}(f, i) \cup \mathcal{L}(f, i+1)$.

(55) If $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $\mathcal{L}(f, i)$ is vertical, then $\mathcal{L}(f, i+1)$ is horizontal.

(56) If $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $\mathcal{L}(f, i)$ is horizontal, then $\mathcal{L}(f, i+1)$ is vertical.

(57) Suppose $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$. Then $\mathcal{L}(f,i)$ is vertical and $\mathcal{L}(f,i+1)$ is horizontal or $\mathcal{L}(f,i)$ is horizontal and $\mathcal{L}(f,i+1)$ is vertical.

(58) Suppose $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $\pi_{i+1} f \in \mathcal{L}(p,q)$ and $\mathcal{L}(p,q) \subseteq \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$. Then $\pi_{i+1} f = p$ or $\pi_{i+1} f = q$.

(59) If $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$, then $\pi_{i+1} f$ is extremal in $\mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$.

(60) Let $u$ be a point of $\mathcal{E}^2$. Suppose $f$ is special and alternating and $1 \leq i$ and $i+2 \leq \operatorname{len} f$ and $u = \pi_{i+1} f$ and $\pi_{i+1} f \in \mathcal{L}(p,q)$ and $\pi_{i+1} f \neq q$ and $p \notin \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$. Given $s$. If $s > 0$, then there exists $p_3$ such that $p_3 \notin \mathcal{L}(f,i) \cup \mathcal{L}(f,i+1)$ and $p_3 \in \mathcal{L}(p,q)$ and $p_3 \in \operatorname{Ball}(u,s)$.

Let us consider $f_1$, $f_2$, $P$. We say that $f_1$ and $f_2$ are generators of $P$ if and only if the conditions (Def.5) are satisfied.

(Def.5) (i) $f_1$ is alternating,
(ii) $f_2$ is alternating,
(iii) $\pi_1 f_1 = \pi_1 f_2$,
(iv) $\pi_{\operatorname{len} f_1} f_1 = \pi_{\operatorname{len} f_2} f_2$,
(v) $\langle \pi_2 f_1, \pi_1 f_1, \pi_2 f_2 \rangle$ is alternating,
(vi) $\langle \pi_{\operatorname{len} f_1 - 1} f_1, \pi_{\operatorname{len} f_1} f_1, \pi_{\operatorname{len} f_2 - 1} f_2 \rangle$ is alternating,
(vii) $\pi_1 f_1 \neq \pi_{\operatorname{len} f_1} f_1$,
(viii) $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{\pi_1 f_1, \pi_{\operatorname{len} f_1} f_1\}$, and
(ix) $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$.

Next we state the proposition

(61) If $f_1$ and $f_2$ are generators of $P$ and $1 < i$ and $i < \operatorname{len} f_1$, then $\pi_i f_1$ is extremal in $P$.

## References


[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[14] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[17] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[20] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

*Received May 11, 1994*