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Extremal solutions for some periodic fractional differential equations

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Abstract

By using the lower and upper solution method, the existence of an iterative solution for a class of fractional periodic boundary value problems,

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad t \in (0, h),$$
$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = h^{1-\alpha}u(h),$$

is discussed, where $0 < h < +\infty$, $f \in C([0,h] \times R, R)$, $D_{0+}^{\alpha}u(t)$ is the Riemann-Liouville fractional derivative, $0 < \alpha < 1$. Different from other well-known results, a new condition on the nonlinear term is given to guarantee the equivalence between the solution of the periodic boundary value problem and the fixed point of the corresponding operator. Moreover, the existence of extremal solutions for the problem is given.

MSC: 34B15; 34A08

Keywords: fractional periodic boundary value problem; extremal solution; existence

1 Introduction

Differential equations of fractional order have played a significant role in engineering, science, and pure and applied mathematics in recent years. Some researchers paid attention to the existence results of the solution of the periodic boundary value problem for fractional differential equations, such as [1–17]. Some recent contributions to the theory of fractional differential equations initial value problems can be found in [4, 9].

In [4], by using the fixed point theorem of Schaeffer and the Banach contraction principle, Belmekki *et al.* obtained the Green's function and gave some existence results for the nonlinear fractional periodic problem

$$D_{0+}^{\alpha}u(t) - \lambda u(t) = f(t, u(t)), \quad t \in (0,1] \ (0 < \alpha < 1),$$
$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1),$$

where $f : [0,1] \times R \to R$ is continuous and the following assumptions hold:

(1) there exists a constant $M > 0$ such that

$$|f(t,u)| \le M, \quad \text{for each } t \in (0,1), u \in R,$$

(2) there exists a constant $k > 0$ such that

$$|f(t,u) - f(t,v)| \leq k|u - v|, \quad \text{for each } t \in (0,1), u, v \in R.$$

The above conditions (see Lemma 4.2 of [4]) are very strong.

In [13], Wei *et al.* discussed the properties of the well-known Mittag-Leffler function, and consider the existence and uniqueness of the solution of the periodic boundary value problem for a fractional differential equation involving a Riemann-Liouville fractional derivative

$$\begin{aligned} &D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad t \in (0,T)\ (0 < \alpha < 1), \\ &t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=T}, \end{aligned}$$

by using the monotone iterative method. In this result, the bounded demand of f in [13] and the monotone demand of f in [9] were removed. However, the application of Lemma 1.1 in the proof of Theorem 3.1 was not correct, due to $\sigma(\eta)(t) \notin C[0,T]$. In other words, the definition of operator A may be not appropriate. Consequently, while the uniqueness result was correct, the existence of an extremal result was maybe wrong.

In [14], Wei and Dong studied the existence of solutions of the following periodic boundary value problem:

$$\begin{aligned} &D_{0+}^{2\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha}u(t)), \quad t \in (0,T)\ (0 < \alpha < 1), \\ &\lim_{t\to 0} t^{1-\alpha}u(t) = \lim_{t\to T} t^{1-\alpha}u(t), \\ &\lim_{t\to 0} t^{1-\alpha}D_{0+}^{\alpha}u(t) = \lim_{t\to T} t^{1-\alpha}D_{0+}^{\alpha}u(t), \end{aligned}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $D_{0+}^{2\alpha}u = D_{0+}^{\alpha}(D_{0+}^{\alpha}u)$ is the sequential Riemann-Liouville fractional derivative, $0 < T < \infty$, and f defined on $[0,T] \times R^2$ is continuous. The methods used in [14] are monotone iterative techniques and the Schauder fixed point theorem under the assumptions that there the upper and lower solutions exist.

In this paper, we will focus our attention on the following problem:

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad t \in (0,h), \tag{1.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = h^{1-\alpha}u(h), \tag{1.2}$$

where $f \in C([0,h] \times R, R)$, $D_{0+}^{\alpha}u(t)$ is the Riemann-Liouville fractional derivative, $0 < \alpha < 1$. The existence of the solution is obtained by the use of the upper and lower solution method which has been used by authors to deal with the fractional initial value problems [2].

The remainder of this paper is as follows. In Section 2, we recall some notions and the theory of the fractional calculus. Section 3 is devoted to the study of the existence of a solution utilizing the method of upper and lower solutions. The existence of extremal solutions is given. An example is given to illustrate the main result.

2 Preliminaries

Given $0 \leq a < b < +\infty$ and $r > 0$, define

$$C_r[a,b] = \{u \mid u \in C(a,b], (t-a)^r u(t) \in C[a,b]\}.$$

Clearly, $C_r[a,b]$ is a linear space with the normal multiplication and addition. Given $u \in C_r[a,b]$, define

$$\|u\| = \max_{t\in[a,b]} (t-a)^r |u(t)|,$$

then $(C_r[a,b], \|\cdot\|)$ is a Banach space.

Lemma 2.1 ([13]) *For $0 < \alpha \leq 1$, $\lambda \geq 0$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ satisfies*

$$0 \leq E_{\alpha,\alpha}(-\lambda t^\alpha) < \frac{1}{\Gamma(\alpha)}, \quad t \in (0,\infty).$$

Lemma 2.2 *The linear periodic problem*

$$D_{0+}^{\alpha} u(t) + \lambda u(t) = q(t), \tag{2.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha} u(t) = h^{1-\alpha} u(h), \tag{2.2}$$

where $\lambda \geq 0$ is a constant and $q \in L(0,h)$, has the following integral representation of the solution:

$$u(t) = \Gamma(\alpha)u(h)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)q(s)\,ds.$$

Proof According to [8], for every initial condition

$$\lim_{t\to 0+} t^{1-\alpha}u(t) = u_0$$

the unique solution of equation (2.1) is given by

$$u(t) = \Gamma(\alpha)u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)q(s)\,ds.$$

Specially, choose u_0 as

$$u_0 = \frac{h^{1-\alpha}\int_0^h (h-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(h-s)^\alpha)q(s)\,ds}{1-\Gamma(\alpha)E_{\alpha,\alpha}(-\lambda h^\alpha)},$$

then $u(t)$ satisfies the periodic boundary condition (2.2). That is to say that the linear periodic problem (2.1), (2.2) has the following integral representation of the solution:

$$u(t) = \Gamma(\alpha)h^{1-\alpha}u(h)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)q(s)\,ds.$$

The proof is complete. □

Lemma 2.3 ([18]) *Suppose that E is an ordered Banach space, $x_0, y_0 \in E$, $x_0 \le y_0$, $D = [x_0, y_0]$, $T : D \to E$ is an increasing completely continuous operator and $x_0 \le Tx_0$, $y_0 \ge Ty_0$. Then the operator T has a minimal fixed point x^* and a maximal fixed point y^*. If we let*

$$x_n = Tx_{n-1}, \qquad y_n = Ty_{n-1}, \quad n = 1, 2, 3, \ldots,$$

then

$$x_0 \le x_1 \le x_2 \le \cdots \le x_n \le \cdots \le y_n \le \cdots \le y_2 \le y_1 \le y_0,$$
$$x_n \to x^*, \qquad y_n \to y^*.$$

Definition 2.1 A function $v(t) \in C_{1-\alpha}[0, h]$ is called a lower solution of problem (1.1), (1.2), if it satisfies

$$D_{0+}^{\alpha} v(t) \le f\big(t, v(t)\big), \quad t \in (0, h), \tag{2.3}$$

$$\lim_{t \to 0^+} t^{1-\alpha} v(t) \le h^{1-\alpha} v(h). \tag{2.4}$$

Definition 2.2 A function $w(t) \in C_{1-\alpha}[0, h]$ is called an upper solution of problem (1.1), (1.2), if it satisfies

$$D_{0+}^{\alpha} w(t) \ge f\big(t, w(t)\big), \quad t \in (0, h), \tag{2.5}$$

$$\lim_{t \to 0^+} t^{1-\alpha} w(t) \ge h^{1-\alpha} w(h). \tag{2.6}$$

3 The main results

The following assumptions will be used in this section:

(S1) $f : [0, h] \times R \to R$ is continuous and there exist constants $A, B \ge 0$ and $0 < r_1 \le 1 < r_2 < 1/(1-\alpha)$ such that for $t \in [0, h]$

$$\big|f(t, u) - f(t, v)\big| \le A|u - v|^{r_1} + B|u - v|^{r_2}, \quad u, v \in R. \tag{3.1}$$

Theorem 3.1 *Suppose* (S1) *holds. Then u solves problem* (1.1), (1.2) *if and only if it is a fixed point of the operator $T_\lambda : C_{1-\alpha}[0, h] \to C_{1-\alpha}[0, h]$ defined by*

$$(T_\lambda u)(t) = \Gamma(\alpha) h^{1-\alpha} u(h) t^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda t^\alpha\big) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda (t-s)^\alpha\big)\big[f\big(s, u(s)\big) + \lambda u(s)\big]\, ds,$$

where $\lambda \ge 0$ is a constant.

Proof First of all, we show that the operator T_λ is well defined. Clearly $t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \in C_{1-\alpha}[0, h]$, so it is enough to prove that for every $u \in C_{1-\alpha}[0, h]$, the function

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda (t-s)^\alpha\big)\big[f\big(s, u(s)\big) + \lambda u(s)\big]\, ds$$

belongs to $C_{1-\alpha}[0,h]$. Taking into account that f is continuous on $[0,h]\times R$, for $u\in C_{1-\alpha}[0,h]$, we have

$$\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl[f\bigl(s,u(s)\bigr)+\lambda u(s)\bigr]\,ds\in C(0,h].$$

On the other hand, under the condition (S1), we have

$$\bigl|f(t,u)\bigr|\le A|u|^{r_1}+B|u|^{r_2}+C,$$

where $C=\max_{t\in[0,h]}f(t,0)$.

By Lemma 2.1, for $u\in C_{1-\alpha}[0,h]$, we have

$$\begin{aligned}
&\left|t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl[f\bigl(s,u(s)\bigr)+\lambda u(s)\bigr]\,ds\right|\\
&\quad\le t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl|f\bigl(s,u(s)\bigr)+\lambda u(s)\bigr|\,ds\\
&\quad\le t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl(A|u|^{r_1}+\lambda|u|+B|u|^{r_2}+C\bigr)\,ds\\
&\quad\le t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl\{As^{(\alpha-1)r_1}\bigl[s^{1-\alpha}\bigl|u(s)\bigr|\bigr]^{r_1}\\
&\qquad+\lambda s^{\alpha-1}s^{1-\alpha}\bigl|u(s)\bigr|+Bs^{(\alpha-1)r_2}\bigl[s^{1-\alpha}\bigl|u(s)\bigr|\bigr]^{r_2}+C\bigr\}\,ds\\
&\quad\le \frac{A\|u\|^{r_1}t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{(\alpha-1)r_1}\,ds+\frac{\lambda\|u\|t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,ds\\
&\qquad+\frac{B\|u\|^{r_2}t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{(\alpha-1)r_2}\,ds+\frac{Ct}{\Gamma(\alpha+1)}\\
&\quad\le A\|u\|^{r_1}\frac{\Gamma((\alpha-1)r_1+1)}{\Gamma((\alpha-1)r_1+\alpha+1)}t^{(\alpha-1)r_1+\alpha+1-\alpha}+\lambda\|u\|\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^\alpha\\
&\qquad+B\|u\|^{r_2}\frac{\Gamma((\alpha-1)r_2+1)}{\Gamma((\alpha-1)r_2+\alpha+1)}t^{(\alpha-1)r_2+\alpha+1-\alpha}+\frac{Ct}{\Gamma(\alpha+1)}\\
&\quad\le \frac{\Gamma[(\alpha-1)r_1+1]\cdot A\cdot t^{(\alpha-1)r_1+1}}{\Gamma[(\alpha-1)r_1+\alpha+1]}\|u\|^{r_1}+\lambda\|u\|\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^\alpha\\
&\qquad+\frac{\Gamma[(\alpha-1)r_2+1]\cdot B\cdot t^{(\alpha-1)r_2+1}}{\Gamma[(\alpha-1)r_2+\alpha+1]}\|u\|^{r_2}+\frac{Ct}{\Gamma(\alpha+1)}.
\end{aligned}$$

That is to say that

$$\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl[f\bigl(s,u(s)\bigr)+\lambda u(s)\bigr]\,ds\in C_{1-\alpha}[0,h].$$

The above inequalities and the assumption $0<r_1\le 1<r_2<1/(1-\alpha)$ imply that

$$\lim_{t\to 0+}t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl[f\bigl(s,u(s)\bigr)+\lambda u(s)\bigr]\,ds=0.$$

Combining with the fact that $\lim_{t\to 0+}E_{\alpha,\alpha}(-\lambda t^\alpha)=E_{\alpha,\alpha}(0)=1/\Gamma(\alpha)$ yields

$$\lim_{t\to 0+}t^{1-\alpha}(T_\lambda u)(t)=h^{1-\alpha}u(h).$$

The above arguments combined with Lemma 2.2 imply that the fixed point of the operator T_λ solves the periodic boundary value problem (1.1), (1.2), and *vice versa*. The proof is complete. □

In the following, we consider the compactness of the set of the space $C_r[0,h]$.

Let $F \subset C_r[0,h]$ and $E = \{g(t) = t^r h(t) \mid h(t) \in F\}$, then $E \subset C[0,h]$. It is clear that F is a bounded set of $C_r[0,h]$ if and only if E is a bounded set of $C[0,h]$.

Therefore, to prove that $F \subset C_r[0,h]$ is a compact set, it is enough to prove that $E \subset C[0,h]$ is a bounded and equicontinuous set.

Theorem 3.2 *Suppose* (S1) *holds. Then the operator* $T_\lambda : C_{1-\alpha}[0,h] \to C_{1-\alpha}[0,h]$ *is completely continuous.*

Proof Given $u_n \to u \in C_{1-\alpha}[0,h]$, with the definition of T_λ, the condition (S1), and Lemma 2.1, one has

$$\begin{aligned}
&\|T_\lambda u_n - T_\lambda u\| \\
&\quad = \bigl\| t^{1-\alpha}(T_\lambda u_n - T_\lambda u)\bigr\|_\infty \\
&\quad = \max_{0\le t\le h}\biggl\{ \bigl|\Gamma(\alpha)h^{1-\alpha}E_{\alpha,\alpha}\bigl(-\lambda t^\alpha\bigr)\bigl[u_n(h) - u(h)\bigr]\bigr| \\
&\qquad + \biggl| t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl[f(s,u_n) - f(s,u) + \lambda(u_n - u)\bigr]\,ds\biggr|\biggr\} \\
&\quad \le \frac{1}{\Gamma(\alpha)}\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\bigl[A|u_n-u|^{r_1} + B|u_n-u|^{r_2} + \lambda|u_n-u|\bigr]\,ds \\
&\qquad + \|u_n - u\| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\biggl[A\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\cdot s^{-r_1(1-\alpha)}\cdot s^{r_1(1-\alpha)}\cdot|u_n-u|^{r_1}\,ds \\
&\qquad + \lambda\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\cdot s^{-(1-\alpha)}\cdot s^{(1-\alpha)}\cdot|u_n-u|\,ds \\
&\qquad + B\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\cdot s^{-r_2(1-\alpha)}\cdot s^{r_2(1-\alpha)}\cdot|u_n-u|^{r_2}\,ds\biggr] + \|u_n-u\| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\biggl[A\|u_n-u\|^{r_1}\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\cdot s^{-r_1(1-\alpha)}\,ds \\
&\qquad + \lambda\|u_n-u\|\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\cdot s^{-(1-\alpha)}\,ds \\
&\qquad + B\|u_n-u\|^{r_2}\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\cdot s^{-r_2(1-\alpha)}\,ds\biggr] + \|u_n-u\| \\
&\quad \le \frac{A\|u_n-u\|^{r_1}\Gamma[1-r_1(1-\alpha)]}{\Gamma[1-r_1(1-\alpha)+\alpha]}h^{1-r_1(1-\alpha)} + \frac{\lambda\|u_n-u\|\Gamma[\alpha]}{\Gamma[2\alpha]}h^\alpha \\
&\qquad + \frac{B\|u_n-u\|^{r_2}\Gamma[1-r_2(1-\alpha)]}{\Gamma[1-r_2(1-\alpha)+\alpha]}h^{1-r_2(1-\alpha)} + \|u_n-u\| \\
&\quad \to 0 \quad (n\to\infty).
\end{aligned}$$

That is to say that T_λ is continuous.

Suppose that $F \subset C_{1-\alpha}[0,h]$ is a bounded set and there is a positive constant M such that $\|u\| \leq M$ for $u \in F$. The proof process of Theorem 3.1 shows that $T_\lambda(F) \subset C_{1-\alpha}[0,h]$ is bounded.

We omit the proof details of the equicontinuity of $T(F)$ here and refer the reader to [2] for a similar details. The proof is complete. □

Theorem 3.3 *Assume* (S1) *hold and* $v, w \in C_{1-\alpha}[0,h]$ *are lower and upper solutions of problem* (1.1), (1.2), *respectively, such that*

$$v(t) \leq w(t), \quad 0 \leq t \leq h. \tag{3.2}$$

Moreover, $f : [0,h] \times R \to R$ *satisfies*

$$f(t,x) - f(t,y) + \lambda(x-y) \geq 0, \quad \text{for } v \leq y \leq x \leq w. \tag{3.3}$$

Then the fractional periodic boundary value problem (1.1), (1.2) *has a minimal solution* x^* *and a maximal solution* y^* *such that*

$$x^* = \lim_{n\to\infty} T_\lambda^n v, \qquad y^* = \lim_{n\to\infty} T_\lambda^n w.$$

Proof Clearly, if the functions v, w are lower and upper solutions (or strict) of problem (1.1), (1.2), then there are $v \leq T_\lambda v$, $w \geq T_\lambda w$ (or the inequality is strict). In fact, by the definition of the lower solution, there exist $q(t) \geq 0$ and $\epsilon \geq 0$ such that

$$\begin{aligned} &D_{0+}^\alpha v(t) = f\bigl(t, v(t)\bigr) - q(t), \quad t \in (0,h), \\ &\lim_{t\to 0^+} t^{1-\alpha} v(t) = h^{1-\alpha} v(h) - \epsilon. \end{aligned}$$

By the use of Theorem 3.1 and Lemma 2.1, one has

$$\begin{aligned} v(t) &= \Gamma(\alpha)\bigl(h^{1-\alpha} v(h) - \epsilon\bigr) t^{\alpha-1} E_{\alpha,\alpha}\bigl(-\lambda t^\alpha\bigr) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl[f\bigl(s, v(s)\bigr) + \lambda v(s) - q(s)\bigr]\, ds \\ &\leq (T_\lambda v)(t). \end{aligned}$$

Similarly, we have $w \geq T_\lambda w$.

By condition (3.3) and Theorem 3.2, the operator $T_\lambda : C_{1-\alpha}[0,h] \to C_{1-\alpha}[0,h]$ is an increasing completely continuous operator. Setting $D := [v,w]$, by the use of Lemma 2.3, the existence of x^*, y^* is obtained. The proof is complete. □

Remark 3.1 The main result is a consequence of the classical monotone iterative technique [19, 20]. However, the periodic condition is not the same.

Example 3.1 Consider the following periodic fractional boundary value problem:

$$D_{0+}^\alpha u(t) = f\bigl(t, u(t)\bigr), \quad t \in (0,h), \tag{3.4}$$

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = h^{1-\alpha}u(h), \tag{3.5}$$

where $\alpha = 0.3$, $h = 0.7$, $f(t,u) = \frac{t}{10}[1 + u(t)]$. Obviously, the function $f(t,u)$ satisfies condition (3.3) and (S1), $f(t,0) \geq 0$, and $f(t,0) \not\equiv 0$ for $t \in [0,h]$. Thus, $v(t) \equiv 0$ is a lower solution of problem (3.4), (3.5). Choose $u(t) = 2t^{\alpha-1}\operatorname{Cos}[2t] + t^{\alpha}$, one can check that $u \in C_{1-\alpha}[0,h]$ is an upper solution of problem (3.4), (3.5), and $v(t) \leq u(t)$ for $t \in [0,h]$. By the use of Theorem 3.3, problem (3.4), (3.5) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Ahmad, B, Nieto, JJ: Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions. Bound. Value Probl. **2009**, Article ID 708576 (2009)
2. Bai, Z: Monotone iterative method for a class of fractional differential equations. Electron. J. Differ. Equ. **2016**, 6 (2016)
3. Bai, Z: Theory and Applications of Fractional Differential Equation Boundary Value Problems. China Sci. Tech., Beijing (2013) (in Chinese)
4. Belmekki, M, Nieto, JJ, Lopez, RR: Existence of periodic solution for a nonlinear fractional differential equation. Bound. Value Probl. **2009**, Article ID 324561 (2009)
5. Deekshitulu, GVSR: Generalized monotone iterative technique for fractional R-L differential equations. Nonlinear Stud. **16**, 85-94 (2009)
6. Dong, X, Bai, Z, Zhang, W: Positive solutions for nonlinear eigenvalue problems with conformable fractional differential derivatives. J. Shandong Univ. Sci. Technol. Nat. Sci. **35**(3), 85-90 (2016) (in Chinese)
7. Jia, M, Liu, X: Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions. Appl. Math. Comput. **232**, 313-323 (2014)
8. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
9. Lakshmikantham, V, Vatsala, AS: Basic theory of fractional differential equations. Nonlinear Anal. **69**, 2677-2682 (2008)
10. Nieto, JJ: Maximum principles for fractional differential equations derived from Mittag-Leffler functions. Appl. Math. Lett. **23**, 1248-1251 (2010)
11. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives, Theory and Applications. Gordon & Breach, Amsterdam (1993)
12. Schneider, WR: Completely monotone generalized Mittag-Leffler functions. Expo. Math. **14**, 3-16 (1996)
13. Wei, Z, Dong, W, Che, J: Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative. Nonlinear Anal. **73**, 3232-3238 (2010)
14. Wei, Z, Dong, W: Periodic boundary value problems for Riemann-Liouville fractional differential equations. Electron. J. Qual. Theory Differ. Equ. **2011**, 87 (2011)
15. Yin, C, Chen, Y, Zhong, S: Fractional-order sliding mode based extremum seeking control of a class of nonlinear system. Automatica **50**, 3173-3181 (2014)
16. Wu, HH, Sun, SJ: Multiple positive solutions for a fourth order boundary value via variational method. J. Shandong Univ. Sci. Technol. Nat. Sci. **33**(2), 96-99 (2014) (in Chinese)
17. Zhou, Y: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
18. Guo, D, Sun, J, Liu, Z: Functional Methods in Nonlinear Ordinary Differential Equations. Shandong Sci. Tech., Jinan (1995) (in Chinese)
19. Ladde, GS, Lakshmikantham, V, Vatsala, AS: Monotone Iterative Techniques for Nonlinear Differential Equations. Pitman, Boston (1985)
20. Nieto, JJ: An abstract monotone iterative technique. Nonlinear Anal. TMA **28**, 1923-1933 (1997)