Extreme neutron stars from Extended Theories of Gravity

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We discuss neutron stars with strong magnetic mean fields in the framework of Extended Theories of Gravity. In particular, we take into account models derived from $f(R)$ and $f(\mathcal{G})$ extensions of General Relativity where functions of the Ricci curvature invariant R and the Gauss-Bonnet invariant $\mathcal{G}$ are respectively considered. Dense matter in magnetic mean field, generated by magnetic properties of particles, is described by assuming a model with three meson fields and baryons octet. As result, the considerable increasing of maximal mass of neutron stars can be achieved by cubic corrections in $f(R)$ gravity. In principle, massive stars with $M > 4M_{\odot}$ can be obtained. On the other hand, stable stars with high strangeness fraction (with central densities $\rho_c \sim 1.5 - 2.0$ GeV/fm^3) are possible considering quadratic corrections of $f(\mathcal{G})$ gravity. The magnetic field strength in the star center is of order $6 - 8 \times 10^{18}$ G. In general, we can say that other branches of massive neutron stars are possible considering the extra pressure contributions coming from gravity extensions. Such a feature can constitute both a probe for alternative theories and a way out to address anomalous self-gravitating compact systems.

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I. INTRODUCTION

The discovery of the pulsar PSR J1614-2230 [1] led to several interpretative problems on the physics of neutron stars and to the possibility that the standard theory of such objects could be revised including also the other anomalous objects that observations are revealing [2, 3]. In particular, for a realistic description of nuclear matter, one needs to account for the appearance of exotic particle at densities $\sim 5 - 8 \times 10^{14}$ g/cm^3 [4]. Despite of this requirement, the equation of state (EoS) considerably softens ($M_{max} \sim 1.5 - 1.6M_{\odot}$) (see [5–9]) and therefore the maximal neutron star mass results reduced. Some approaches have been recently proposed to solve the problem [10–20], however there is no final agreement on the solution of the puzzle. In particular, the required maximal mass ($\sim 2M_{\odot}$) can be obtained for hyperon EoS from more complex model of strong interaction than simple $\sigma\omega\omega$-model with realistic hyperon-meson couplings. However, it is worth noticing that one cannot derive a reliable M-R relation for neutron stars from observations because there are no precise radius measurements for any stars [21].

On the other hand, the maximal limit of neutron star mass can increases considerably due to strong magnetic field inside the star. The observations of gamma-ray repeaters and anomalous X-ray pulsars may indicate the existence of magnetic fields of the order 10^{15} G on the stellar surface. In the center of the star, the magnetic field can exceed 10^{18} G.

Realistic EoS with hyperons and quarks in presence of strong magnetic field are considered in [22–25]. The maximal mass of neutron star can exceed, in these cases, $3M_{\odot}$ for $B_c \sim 3.3 \times 10^{18}$ G.

Therefore it seems that the existence of neutron stars with masses exceeding considerably two solar mass (without strong magnetic field) is impossible in the framework of General Relativity (GR). Despite of this theoretical constraint, it is interesting to note that there are several observational indications in favor of maximal masses that exceed this limit. In fact, masses of B1957+20 and 4U 1700-377 are estimated as $\sim 2.4M_{\odot}$ [26, 27] while for PSR J1748-2021B , the mass limit reaches $M \sim 2.7M_{\odot}$ [28].

In principle, observational data on neutron stars (mainly the mass-radius $M - R$ relation) can be used to investigate possible deviations from GR as probe for alternative gravity theories.

Alternatives to GR have been developed in order to solve several shortcomings related to the ultraviolet and infrared behaviors of the gravitational field as formulated in the Einstein theory [29]. In particular, Extended Theories of Gravity could successfully address the recently established phenomenon of the accelerated expansion of the universe.

This fact is confirmed by observations of type Ia supernovae [30–32], by microwave background radiation anisotropies

[33], by cosmic shear through gravitational weak leasing surveys [34] and by data coming from Lyman alpha forest absorption lines [35]. This acceleration should occur thanks to the so called *dark energy*, a cosmic fluid with negative pressure according to the standard approach. In the framework of GR, the simplest explanation is given by the ΛCDM model where the Einstein Cosmological Constant and the dark matter supply almost the 95% of the cosmic budget. However, up to now, no final answer to the question of what are the fundamental constituents of such dark ingredients has been definitely found.

Alternative approaches to the resolution of dark side puzzle (both dark matter and dark energy) require to modify gravity (for details see [29, 36–40] and references therein). Main advantages of such approaches are the possible unification of dark energy and early-time inflation [42] and the explanation of cosmic structures without new exotic material ingredients [41]. In principle, these theories can satisfactorily reproduce the data of astronomical observations and therefore one cannot distinguish if the dark side issues can be addressed by geometry (l.h.s. of field equations) or by matter fluids (r.h.s. of field equations).

Addressing exotic compact objects by modified gravity could give, in principle, new signatures in favor (or in disfavor) of possible extensions of GR. Some models of $f(R)$ gravity can be considered as unreliable because stable star configurations do not always exist [43–50]. However the existence of stable star configurations can be achieved in certain cases due to the so-called *Chameleon Mechanism* [51, 52] or may depend on the chosen EoS. Furthermore, strong gravitational regimes could be considered if one assume GR as the weak field limit of some more complicated effective gravitational theory [53]. In this sense, higher-order corrections and extensions could be "detected" as the mechanism capable of producing anomalous neutron stars. In other words, the interior of a self-gravitating compact object could behave, in broad sense, like the early inflationary universe where strong gravitational fields were present.

As discussed in [54], the proposition of validity of GR as the only theory capable of describing neutron stars is rather an extrapolation because the strength of gravity within a star is orders and orders of magnitude larger than the gravitational strengths probed in the Solar System weak field limit tests.

Neutron star models with quadratic corrections like $f(R) = R + \alpha R^2$ gravity are considered in [55–58, 60, 61]. If also magnetic fields are assumed, more realistic models can be considered [62, 63]). In the case of these quadratic corrections, the possible increasing (or decreasing) of maximal star mass in comparison with GR is negligible for realistic values of the parameter α. Realistic models addressing the observed models can be achieved by considering also cubic corrections in the Ricci scalar R [57, 58].

In this paper, we consider the possibility of existence of neutron stars with extreme properties in the framework of Extended Theories of Gravity. In particular we will take into account generalizations of the Einstein gravity containing functions of the Gauss-Bonnet invariant $f(\mathcal{G})$ and we will compare them with $f(R)$ gravity. In fact, involving the Gauss-Bonnet invariant into the gravitational action is a useful approach to cure shortcomings related to $f(R)$ as, for example, the presence of ghosts. Furthermore, some functions of the topological invariant $\mathcal{G}$ are related to conserved quantities (see also [64]). This fact seems to contribute to make stable self-gravitating systems, as we will see below.

The paper is organized as follows. In Section II, we derive the field equations for $f(\mathcal{G})$ and $f(R)$ gravity and the corresponding modified Tolman-Oppenheimer-Volkoff (TOV) equations. In Section III is devoted to the discussion of relativistic mean field theory for dense matter. Here we show how electromagnetic properties of particles give rise to the magnetic mean field of the neutron star and how such a field contributes to the EoS.

In Section IV, modified TOV equations are numerically solved for realistic EoS for matter in strong magnetic field. In particular, we investigate the cubic corrections for $f(R)$ gravity and quadratic corrections for $f(\mathcal{G})$ gravity. The results are compared to GR. Conclusions and outlook are given in Section V.

II. TOLMAN-OPPENHEIMER-VOLKOFF EQUATIONS IN EXTENDED GRAVITY

Let us consider now corrections to the GR where higher order curvature invariants are added to the standard Hilbert-Einstein action. As discussed above, such an approach is useful to address several issues related to dark components. In our case, we will show that such corrections results as further effective pressure in the TOV equations. We will start with dealing with $f(\mathcal{G})$ gravity and compare results with $f(R)$ gravity.

The action for $R + f(\mathcal{G})$ gravity is

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + f(\mathcal{G})\right] + S_m \tag{1}$$

As standard in the Einstein gravity, by working with the commutative connections in a Riemann spacetime, R is the Ricci scalar, and the function $f(\mathcal{G})$ corresponds to a generic globally differentiable function of the Gauss-Bonnet topological invariant $\mathcal{G}$. We add also the matter action S_m which induces the energy momentum tensor $T_{\mu\nu}$. We assume $\kappa^2 = 8\pi G/c^4$, where G is the standard Newtonian gravitational coupling. In metric formalism and by taking the metric as the dynamical variable of the model, the field equations are [65] (see also refs. [66])

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+8\Big[R_{\mu\rho\nu\sigma}+R_{\rho\nu}g_{\sigma\mu}-R_{\rho\sigma}g_{\nu\mu}-R_{\mu\nu}g_{\sigma\rho}+R_{\mu\sigma}g_{\nu\rho}+ \tag{2}$$

$$+\frac{R}{2}\left(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho}\right)\Big]\nabla^\rho\nabla^\sigma f_{\mathcal{G}}+\left(f_{\mathcal{G}}\mathcal{G}-f\right)g_{\mu\nu}=\kappa^2T_{\mu\nu},$$

where $f_{\mathcal{G}}=df(\mathcal{G})/d\mathcal{G}$ and the Gauss-Bonnet invariant is defined as

$$\mathcal{G}=R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\,, \tag{3}$$

where $R_{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}$ are the Ricci and Riemann tensors, respectively. We adopted the signature for the Riemannian metric as $(-+++)$. It is $\nabla_\mu V_\nu=\partial_\mu V_\nu-\Gamma^\lambda_{\mu\nu}V_\lambda$ and $R^\sigma_{\ \mu\nu\rho}=\partial_\nu\Gamma^\sigma_{\mu\rho}-\partial_\rho\Gamma^\sigma_{\mu\nu}+\Gamma^\omega_{\mu\rho}\Gamma^\sigma_{\omega\nu}-\Gamma^\omega_{\mu\nu}\Gamma^\sigma_{\omega\rho}$ for the covariant derivative and the Riemann tensor, respectively.

Let us suppose now that the metric is spherically symmetric with coordinates $x^\mu=(ct,r,\theta,\varphi)$ in the following form:

$$ds^2=-c^2e^{2\phi}dt^2+e^{2\lambda}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{4}$$

We assume that the interior of star is filled with a perfect fluid with energy-momentum tensor of the form $T_{\mu\nu}=\mathrm{diag}(e^{2\phi}\rho c^2,e^{2\lambda}p,r^2p,r^2\sin^2\theta p)$, where ρ is the matter density and p the pressure. The tt and rr components of field equations (2) read:

$$-\frac{1}{r^2}(2r\lambda'+e^{2\lambda}-1)+8e^{-2\lambda}\Big[f_{\mathcal{G}\mathcal{G}}(\mathcal{G}''-2\lambda'\mathcal{G}')+f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G}')^2\Big]\Big[\frac{1-e^{2\lambda}}{r^2}-2(\phi''+\phi'^2)\Big]+(f_{\mathcal{G}}\mathcal{G}-f)e^{2\lambda}=-\kappa^2\rho e^{2\lambda}c^2. \tag{5}$$

$$\frac{1}{r^2}(2r\phi'-e^{2\lambda}+1)+(f_{\mathcal{G}}\mathcal{G}-f)e^{2\lambda}=\kappa^2pe^{2\lambda}. \tag{6}$$

The trace of (2) gives

$$R+8G_{\rho\sigma}\nabla^\rho\nabla^\sigma f_{\mathcal{G}}-4(f_{\mathcal{G}}\mathcal{G}-f)=\kappa^2(\rho c^2-3p). \tag{7}$$

The hydrostatic continuity equation follows from the contracted Bianchi identities $\nabla_\mu T^\mu_\nu$ for $\nu=r$. It is

$$\frac{dp}{dr}=-(p+\rho c^2)\phi'. \tag{8}$$

The continuty equation is identically satisfied for $\nu=t$. Note that if $f(\mathcal{G})=\mathcal{G}$, Eqs.(5) and (6) reduce trivially to the GR field equations being the linear $\mathcal{G}$ a conserved topological invariant resulting null when integrated in a 4D-spacetime.

Let us now replace the metric function with the following expression in terms of the gravitational mass function $M=M(r)$:

$$e^{-2\lambda}=1-\frac{2GM}{c^2r}\Longrightarrow\frac{GdM}{c^2dr}=\frac{1}{2}\left[1-e^{-2\lambda}(1-2r\lambda')\right]. \tag{9}$$

The aim is to rewrite Eqs. (5) and (6) in terms of $\frac{dp}{dr},\frac{dM}{dr}$ and ρ in a dimensionless form. For this purpose, we introduce the following set of dimensionless parameters

$$M\to mM_\odot,\quad r\to r_gr,\quad \rho\to\frac{\rho M_\odot}{r_g^3},\quad p\to\frac{pM_\odot c^2}{r_g^3},\quad G\to\frac{G}{r_g^4}\,. \tag{10}$$

Here $r_g=GM_\odot/c^2$ corresponds to one half of the gravitational radius of the Sun. We also introduce the dimensionless function $f\to r_g^{-2}f$. From now on, we will use dimensionless parameters. The continuity equation becomes the following

$$\frac{dp}{dr}=-(p+\rho)\phi'. \tag{11}$$

Eq. (9) becomes

$$\frac{d\lambda}{dr}=\frac{m}{r}\left(\frac{1-\frac{r}{m}\frac{dm}{dr}}{2m-r}\right). \tag{12}$$

Using Eqs. (11) and (12) in Eq. (6), we obtain

$$\frac{2}{r^2}\left(\frac{r-2m}{p+\rho}\right)\frac{dp}{dr}+\frac{2m}{r^2}+8\pi p-(\mathcal{G}f_{\mathcal{G}}-f)=0. \tag{13}$$

For the (tt) equation, we get

$$\frac{2}{r^2}\frac{dm}{dr}-8\left(1-\frac{2m}{r}\right)^2\left[f_{\mathcal{G}\mathcal{G}}\left(\mathcal{G}''-\frac{2m}{r}\left(\frac{1-r\frac{dm}{dr}}{2m-r}\right)\mathcal{G}'\right)+r_g^2 f_{\mathcal{G}\mathcal{G}\mathcal{G}}\mathcal{G}'^2\right]\times \tag{14}$$

$$\times\left[-\left(\frac{2m/r^3}{1-\frac{2m}{r}}\right)+2\frac{d}{dr}\left(\frac{\frac{dp}{dr}}{p+\rho}\right)-2\left(\frac{\frac{dp}{dr}}{p+\rho}\right)^2\right]-(f_{\mathcal{G}}\mathcal{G}-f)=8\pi\rho.$$

Our aim here is to solve the system of the differential Eqs. (7), (13), (14) numerically. For solving, we use the perturbative approach described in [55] for $f(R)$ gravity (for independent derivation of TOV equations in $F(\mathcal{G})$ gravity, see [67]). In the framework of perturbative approach, terms containing $f(\mathcal{G})$ and its derivatives are assumed to be of first order in the small parameter α (i.e. $f(\mathcal{G})\sim\alpha h(\mathcal{G})$), so all such terms should be evaluated at $\mathcal{O}(\alpha)$ order. Therefore one needs to calculate the Gauss-Bonnet invariant at zero order. We have, for $\mathcal{G}$, the following equation

$$-e^{4\lambda}\mathcal{G}=8\frac{\left(\phi''+\phi'^2-\lambda'\phi'\right)\left(e^{2\lambda}-1\right)+2\phi'\lambda'}{r^2}. \tag{15}$$

At zero order, the λ and ϕ functions are determined by the equations which follow from the standard TOV equations in GR, that is

$$\lambda'^{(0)}=\frac{1}{r}\frac{m^{(0)}-4\pi r^3\rho^{(0)}}{2m^{(0)}-r}, \tag{16}$$

$$\phi'^{(0)}=\frac{1}{r}\frac{m^{(0)}+4\pi r^3p^{(0)}}{r-2m^{(0)}}. \tag{17}$$

In order to calculate second derivatives for λ and ϕ at zero order, one needs the TOV equations again. Finally we have for $\mathcal{G}^{(0)}$:

$$\mathcal{G}^{(0)}=\frac{48\ m^{(0)2}}{r^6}-\frac{128\pi m^{(0)}\rho^{(0)}}{r^3}-256\pi^2\rho^{(0)}p^{(0)}. \tag{18}$$

At first order in α, one obtains the following system

$$\frac{dm}{dr}=4\pi\rho r^2+4\alpha r^2\left[h^{(0)}_{\mathcal{G}\mathcal{G}}\left(\mathcal{G}''^{(0)}+\frac{2(m^{(0)}/r-4\pi r^2\rho^{(0)})}{r-2m^{(0)}}\mathcal{G}'^{(0)}\right)+h^{(0)}_{\mathcal{G}\mathcal{G}\mathcal{G}}\mathcal{G}'^{(0)2}\right]\times \tag{19}$$

$$\times\left[\frac{2m^{(0)}}{r^3}-\frac{2m^{(0)2}}{r^4}-8\pi(\rho^{(0)}+p^{(0)})+\frac{8\pi m(\rho^{(0)}+3p^{(0)})}{r}-32\pi^2r^2\rho p\right]+\frac{1}{2}\alpha r^2(h^{(0)}_{\mathcal{G}}\mathcal{G}^{(0)}-h^{(0)}),$$

$$\left(\frac{r-2m}{p+\rho}\right)\frac{dp}{dr}=-\frac{m}{r}-4\pi pr^2+\frac{1}{2}\alpha r^2(h^{(0)}_{\mathcal{G}}\mathcal{G}^{(0)}-h^{(0)}). \tag{20}$$

Such modified TOV equations can be compared to the corresponding equations coming from $f(R)$ corrections. In this case, the action is modified as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2}+f(R)\right]+S_m\,. \tag{21}$$

As before, one can assume a perturbative approach considering a perturbation parameter α, that is $f(R)=\alpha h(R)$. The resulting modified TOV system results (see [57] for details)

$$\frac{dm}{dr} = 4\pi\rho r^2 - \alpha r^2\left[4\pi\rho^{(0)}h_R^{(0)} - \frac{1}{4}\left(h_R^{(0)}R^{(0)} - h^{(0)}\right)\right] + \tag{22}$$

$$+\frac{1}{2}\alpha\left[h_{RR}^{(0)}\left(R'^{(0)}\left(2r-3m^{(0)}-4\pi\rho^{(0)}r^3\right)+r(r-2m^{(0)})R''^{(0)}\right)+r(r-2m^{(0)})h_{RRR}^{(0)}R'^{(0)2}\right],$$

$$\left(\frac{r-2m}{p+\rho}\right)\frac{dp}{dr} = -\frac{m}{r} - 4\pi p r^2 + \alpha r^2\left[4\pi p^{(0)}h_R^{(0)} + \frac{1}{4}\left(h_R^{(0)}R^{(0)} - h^{(0)}\right)\right] + \alpha\left(r - \frac{3}{2}m^{(0)} + 2\pi p^{(0)}r^3\right)h_{RR}^{(0)}R'^{(0)}. \tag{23}$$

The Ricci scalar at zero order is $R^{(0)} = 8\pi(\rho^{(0)} - 3p^{(0)})$.

For the numerical integration of Eqs. (19), (20) and Eqs. (22), (23), one needs to add an EoS of the form $p = p(\rho)$.

III. RELATIVISTIC MEAN FIELD THEORY FOR DENSE MATTER

A realistic EoS can be considered assuming dense matter in presence of a strong magnetic field generated by the electromagnetic properties of particles. In our perturbative approach, we do not need axially symmetric solutions for metric in order to consider a magnetic field coupled to the star angular momentum. We are assuming here how the TOV equations are modified by the geometric terms coming from extended gravity models and how the effective magnetic field, derived from the mean quantum properties of particles, contributes to the dynamics of the self-gravitating system. Furthermore, it is worth stressing another important point. We are developing our considerations in the Jordan frame where matter is minimally coupled with respect to the geometry. In this case, we can easily control the relativistic EoS that we are adopting and directly confront it with the GR counterpart (see [59] for further details). This is not the case if we were adopting the Einstein frame. In that case, matter results non-minimally coupled with geometry and the meaning of EoS parameters could result of difficult interpretation.

As a first step, let us briefly describe the relativistic mean field theory for dense matter. We follow the same treatment of magnetic field for magnetic neutron stars in GR (see [22, 68–70]) since the addition of purely gravitational (geometric) terms does not change the electromagnetic dynamics of system. One considers the nuclear matter consisting of baryon octet ($b = p,\ n,\ \Lambda,\ \Sigma^{0,\pm},\ \Xi^{0,-}$) interacting with magnetic field (in general case) and scalar σ, isoscalar-vector ω_μ and isovector-vector ρ meson fields and leptons ($l = e^-,\ \mu^-$). The relativistic Lagrangian is [71]

$$\mathcal{L} = \sum_b \bar{\psi}_b\left[\gamma_\mu(i\partial^\mu - q_b A^\mu - g_{\omega b}\omega^\mu - \frac{1}{2}g_{\rho b}\tau\cdot\rho^\mu) - (m_b - g_{\sigma b}\sigma)\right]\psi_b + \sum_l \bar{\psi}_l\left[\gamma_\mu(i\partial^\mu - q_l A^\mu) - m_l\right]\psi_l \tag{24}$$

$$+\frac{1}{2}\left[-(\partial_\mu\sigma)^2 - m_\sigma^2\sigma^2\right] - V(\sigma) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\omega^2\omega^2 + \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{4}\rho_{\mu\nu}\rho^{\mu\nu} + \frac{1}{2}m_\rho^2\rho_\mu^2.$$

Here $\omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, $\rho_{\mu\nu} = \partial_\mu\rho_\nu - \partial_\nu\rho_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the mesonic and electromagnetic field strength tensors. For simplicity, one neglects the anomalous magnetic momenta of particles and consider frozen-field configurations of electromagnetic field. For density-dependent couplings $g_{b\sigma}$, $g_{b\omega}$ and $g_{b\rho}$, we use the parameterization by [71]. For expectation values of meson fields, one can obtain the following equations of motion

$$m_\sigma^2\sigma = \sum_b g_{\sigma b}n_b^s, \quad m_\omega^2\omega_0 = \sum_b g_{\omega b}n_b, \quad m_\omega^2\rho_{03} = \sum_b g_{\rho b}n_b. \tag{25}$$

Here the scalar and vector baryon number densities are n_b^s and n_b correspondingly. These quantities are defined as [72]

$$n_b^s = \frac{m_b^{*2}}{2\pi^2}\left(E_f^b k_f^b - m_b^{*2}\ln\left|\frac{k_f^b + E_f^b}{m_b^*}\right|\right), \quad n_b = \frac{1}{3\pi^2}k_f^{b3} \tag{26}$$

Model	n_s (fm^{-3})	$g_{\sigma N}/m_\sigma$ (fm)	$g_{\omega N}/m_\omega$ (fm)	$g_{\rho N}/m_\rho$ (fm)	b	c
TW	0.153	3.84901	3.34919	1.89354	0	0
GM1	0.153	3.434	2.674	2.100	0.002947	-0.001070
GM2	0.153	3.025	2.195	2.189	0.003487	0.01328
GM3	0.153	3.151	2.195	2.189	0.008659	-0.002421

TABLE I: The nucleon-meson couplings and parameters of scalar field potential for some models [5, 71]. The nuclear saturation density n_s is also given.

for neutral baryons and

$$n_b^s = \frac{|q_b|Bm_b^*}{2\pi^2}\sum_\nu g_\nu \ln\left|\frac{k_{f,\nu}^b + E_f^b}{\sqrt{m_b^{*2} + 2\nu|q_b|B}}\right|, \quad n_b = \frac{|q_b|B}{2\pi^2}\sum_\nu g_\nu k_{f,\nu}^b \tag{27}$$

for charged baryons (or leptons). Here $m_b^* = m_b - g_{\sigma b}\sigma$ is the so called "effective" mass for baryons, E_f^b, $k_{f,\nu}^b$ are the Fermi energy and momentum defined as $E_f^b = (k_f^2 + m_b^{*2} + 2\nu|q_b|B)^{1/2}$ for charged particles and $E_f^b = (k_f^2 + m_b^{*2})^{1/2}$ for neutral particles. The summation over $\nu = n + 1/2 - sgn(q)s/2$ ends at value ν_{max} at which the square of Fermi momenta is still positive. A summary of nucleon-meson couplings and scalar field parameters for some models in literature is given in Table I.

For hyperon-meson couplings one can assume fixed fractions of nucleon-meson couplings, i.e. $g_{iH} = x_{iH}g_{iN}$, where $x_{\sigma H} = x_{\rho H} = 0.600$, $x_{\omega H} = 0.653$, $x_{\rho H} = 0.6$ (see [22]).

Baryon number conservation, charge neutrality and β-equilibrium conditions allow to obtain the EoS. Matter energy density is defined as

$$\epsilon_m = \sum_{b_n}\epsilon_b^n + \sum_{b_c}\epsilon_b^c + \sum_l \epsilon_l + \frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_\omega^2\omega^2 + \frac{1}{2}m_\rho^2\rho_0^2 + U(\sigma). \tag{28}$$

Here

$$\epsilon_b^n = \frac{1}{4\pi^2}\left[k_f^b(E_f^b)^3 - \frac{1}{2}m_b^*\left(m_b^*k_f^bE_f^b + m_b^{*3}\ln\left|\frac{k_f^b + E_f^b}{m_b^*}\right|\right)\right].$$

is the energy density for neutral baryons and

$$\epsilon_b^c = \frac{|q_b|B}{4\pi^2}\sum_\nu g_\nu\left(k_{f,\nu}^bE_f^b + (m_b^{*2} + 2\nu|q_b|B)\ln\left|\frac{k_{f,\nu}^b + E_f^b}{\sqrt{m_b^{*2} + 2\nu|q_b|B}}\right|\right)$$

is the energy density for neutral baryons. For energy density of leptons, one needs to change $m_b^* \to m_l$ in the last equation.

The pressure of dense matter is given by

$$p = \sum_{b_n}p_b^n + \sum_{b_c}p_b^c + \sum_l p_l - \frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_\omega^2\omega^2 + \frac{1}{2}m_\rho^2\rho_0^2 - U(\sigma) + \Sigma_0^R, \tag{29}$$

where

$$p_b^n = \frac{1}{12\pi^2}\left[k_{f3}^bE_f^b - \frac{3}{2}m_b^*\left(m_b^*k_f^bE_f^b - m_b^{*3}\ln\left|\frac{k_f^b + E_f^b}{m_b^*}\right|\right)\right]$$

is the pressure for neutral baryons and

$$p_n^c = \frac{|q_b|B}{12\pi^2}\sum_\nu g_\nu\left(k_{f,\nu}^bE_f^b - (m_b^{*2} + 2\nu|q_b|B)\ln\left|\frac{k_{f,\nu}^b + E_f^b}{\sqrt{m_b^{*2} + 2\nu|q_b|B}}\right|\right)$$

is the pressure for charged baryons. The rearrangement of self-energy terms is defined by

$$\Sigma_0^R = -\frac{\partial \ln g_{\sigma N}}{\partial n}m_\sigma^2\sigma^2 + \frac{\partial \ln g_{\omega N}}{\partial n}m_\omega^2\omega_0^2 + \frac{\partial \ln g_{\rho N}}{\partial n}m_\rho^2\rho_0^2. \tag{30}$$

B_0, 10^5	β, $10^2 r_g^4$	$M_{max}/M_\odot$	R, km	E_c, GeV/fm^3	B_c, 10^{18} G
	0	2.73	12.44	0.93	4.19
2	−0.50	3.54	12.60	0.79	3.67
	−0.75	3.92	12.63	0.78	3.61
	0	2.98	13.78	0.76	3.86
3	−0.50	3.90	13.86	0.69	3.56
	−0.75	4.16	13.99	0.65	3.38
	0	3.15	14.34	0.68	3.69
4	−0.50	4.03	14.47	0.63	3.45
	−0.75	4.62	14.59	0.59	3.32

TABLE II: Compact star properties (maximal mass and corresponding radius) using TW model for cubic $f(R)$ gravity for certain values of α (in units of r_g^4) for fast varying magnetic field. The magnetic field B_c and energy density E_c in center are given.

B_0, 10^5	β, $10^2 r_g^4$	M_{max}, $M_\odot$	R, km	E_c, GeV/fm^3	B_c, 10^{18} G
	0	2.80	13.99	0.79	3.49
2	−0.5	3.21	13.68	0.86	3.73
	0	3.21	15.67	0.63	3.29
3	−0.5	3.55	15.52	0.65	3.39
	0	3.53	17.16	0.52	3.04
4	−0.75	4.01	16.93	0.54	3.13

TABLE III: Compact star properties using TW model for cubic $f(R)$ gravity for certain values of β for slowly varying magnetic field.

For effective EoS, one needs also to account for the energy density and pressure generated from the magnetic field. One obtain

$$\epsilon_f = \frac{B^2}{8\pi}, \quad p_f = \frac{B^2}{8\pi}. \tag{31}$$

Clearly, the magnetic field depends only on the baryon density n. We use a simple parameterization (see, for example, [22, 25])

$$B = B_s + B_0\left(1 - \exp\left(-\beta(n/n_s)^\gamma\right)\right), \tag{32}$$

where B_s is the magnetic field on star surface (10^{15} G). For parameters γ and β one can realistically assume the values $\gamma = 2$, $\beta = 0.05$ (slowly varying field) and $\gamma = 3$, $\beta = 0.02$ (fast varying field). The value $B_0 = B_c B_0^*$ where $B_c = 4.414 \times 10^{13}$ G is the critical field for electrons.

IV. RESULTS

The modified TOV equations, equipped with the above EoS (we use TW parametrization), can be used to construct self-consistent models for extreme neutron stars. Here we will develop two specific cases where cubic corrections for $f(R)$ gravity and quadratic corrections for $f(\mathcal{G})$ gravity are adopted. The general result is that stable configurations for extremely massive neutron stars with strong magnetic fields can be achieved.

A. The case of $f(R)$ gravity.

Although the possible existence of supermassive neutron stars is discussed in [63], in the present paper this question is considered in more detail. We consider more variants of magnetic field profile (various B_0). Furthermore, we

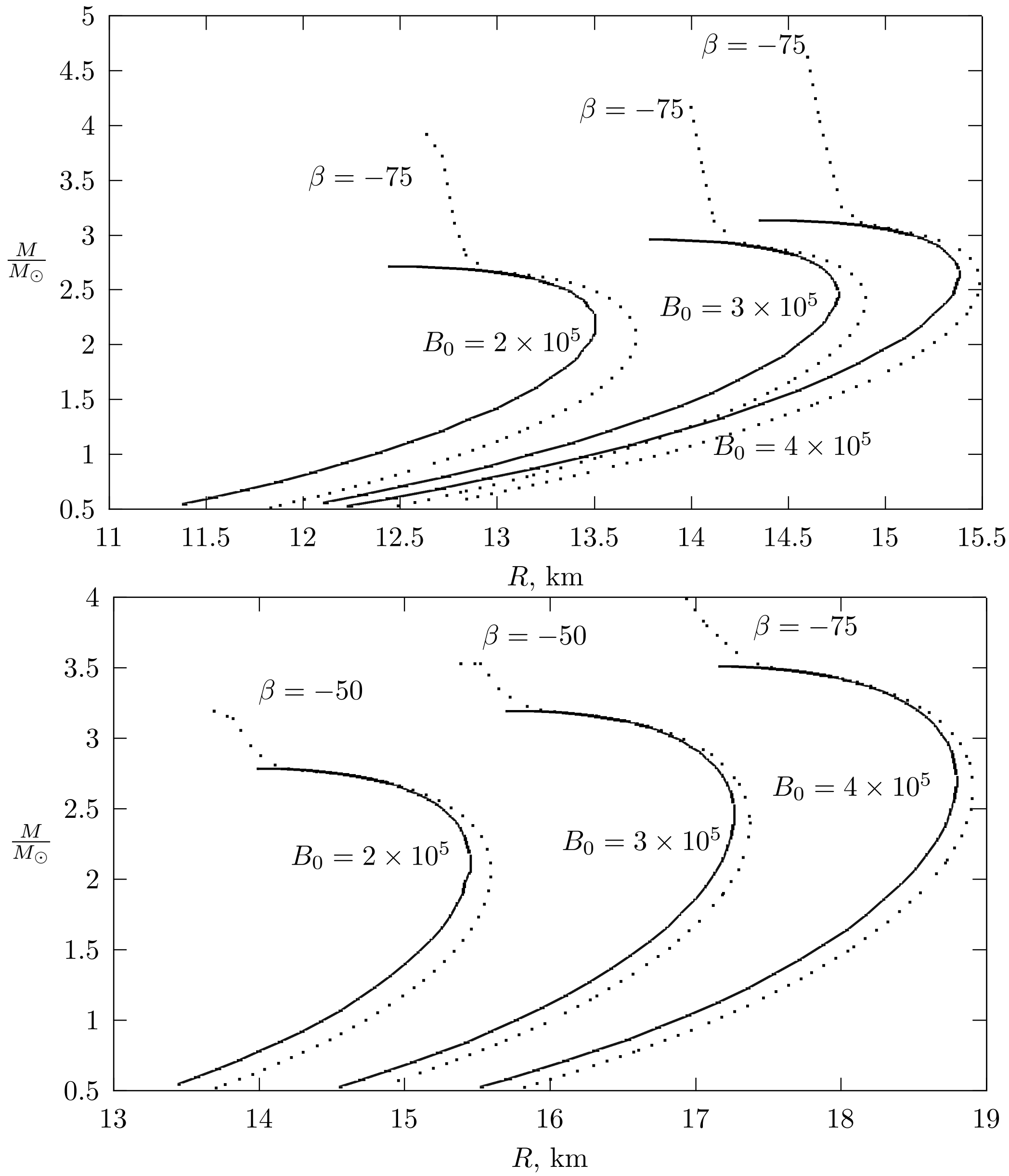

FIG. 1: The mass-radius diagram in model $f(R)=\beta R^3$ (dotted lines) and in General Relativity (solid lines) for fast (upper panel) and slowly (lower panel) varying field.

considered also the speculative question about stars with masses $\sim 4M_\odot$ in f(R) gravity. For cubic corrections of the form $f(R)=\beta R^3$, massive neutron stars with $M>3M_\odot$ can exist (for $\beta<0$). The small parameter α is now $\alpha=\beta|R^{(0)3}_{max}|$. The parameter $R^{(0)}_{max}$ corresponds to the maximal value of the Ricci scalar. This value strictly depends on the EoS. We take into account the cases $B_0=2,\ 3,\ 4\times10^5$. The value of the parameter is $\alpha<4\times10^{-3}$ for realistic densities related to these cases. The properties of stars with maximal mass are given in Tables II and III and are compared to GR ($\beta=0$). The mass-radius relation is represented in Fig. 1. It is worth noticing that, although the maximal mass increases, the central density decreases (for fast varying field). In the case of fast varying magnetic field, the increasing of mass can exceed 1.5 solar masses for certain values of parameter β. The maximal possible mass is close to $4.5M_\odot$ for $B_0=4\times10^5$ and $\beta=-75$ (in units of r_g^6). The minimal value of parameter β, at which the increasing of maximal mass is $\sim 1M_\odot$, is $\beta\sim-60$ for these values of B_0. In the case of slowly varying field, the minimal value of β for $\Delta M\sim0.5M_\odot$ is $\beta\sim-60$ for $B_0=2-3\times10^5$ and $\beta\sim-75$ for $B_0=3-4\times10^5$.

The curvature in zeroth order $R^{(0)}$ is $\sim\rho^{(0)}-3p^{(0)}$. For large densities, in the case of magnetic field, the value $\rho^{(0)}-3p^{(0)}$ changes sign. For example, for fast varying magnetic field, the change of sign corresponds to $\rho\approx580$ MeV/fm^3 ($B_0=2\times10^5$), $\rho\approx440$ MeV/fm^3 ($B_0=3\times10^5$) and $\rho\approx400$ MeV/fm^3 ($B_0=4\times10^5$). At these densities,

B_0, 10^5	β, r_g^6	$M_{max}/M_\odot$	R, km	E_c, GeV/fm^3	B_c, 10^{18} G
2	0	2.73	12.44	0.93	4.19
	-0.05	2.74	12.51	2.03	7.29
	-0.1	2.74	12.53	1.43	5.81
3	0	2.98	13.78	0.76	3.86
	-0.05	2.99	13.76	2.11	7.92
	-0.1	2.99	13.79	1.58	6.57
4	0	3.15	14.34	0.68	3.69
	-0.05	3.16	14.36	2.15	8.75
	-0.1	3.16	14.42	1.64	6.92

TABLE IV: Compact star properties using TW model for quadratic $f(\mathcal{G})$ gravity for certain values of β (in units of r_g^6) for fast varying magnetic field.

the $M-R$ diagram begins to diverge from curve in GR. The "effective" density ($\frac{1}{4\pi r^2}\frac{dM}{dr}$) increases and we get larger mass than in GR at the same radii.

A consideration is in order at this point. In the cases we are dealing with, the particle Σ^- is the hyperon that contributes to the EoS. In this picture, this is the only strange particle contributing to the stars of maximal mass. For $B_0 = 2\times 10^5$ (slowly varying field) there are also Λ hyperons. In contrast to this, for $B_0 = 4\times 10^5$ (fast varying field), the maximal possible mass corresponds to densities where hyperons do not appear yet. It is interesting to see that this picture takes place both in GR and $f(R)$ gravity. This means that extreme neutron stars can be achieved without exotic particles out of the Standard Model.

Some considerations are needed on the validity of perturbative approach. For neutron stars, one can estimate curvature as 10^{-2} (in units of r_g^{-2}) and therefore one can conclude that the term βR^3 is comparable with R for $\beta \sim 10^4$. For β, we considered much smaller values. Of course the question arises why this lead to substantial departure from GR? The situation can be explained in the following way. For some EoS, we have the rapid growth of mass (without substantial change of radii) for some range of central densities. In GR, then we have decreasing stellar masses. But $f(R)$ terms lead to the change of density profile and then the mass increases. In fact the, "effective" EoS changes so that stable stars can exist at high central densities.

B. The case of $f(\mathcal{G})$ gravity

The case of Gauss-Bonnet quadratic corrections $f(\mathcal{G}) = \beta\mathcal{G}^2$ is interesting because allows to achieve stable neutron star configurations with high central densities for negative β. The situation is more clear from an intuitive point of view in comparison with $f(R)$ gravity: masses and radii of stars differ from GR insignificantly because the term $\sim G^2$ is small in comparison with R term. But the character of mass-density dependence changes: the mass begins to increase with the increasing of central density in a narrow range of densities. Although this increasing is small, $dM/d\rho_c > 0$ and stability of star configurations is achieved at high central densities.

Therefore the existence of stable neutron stars with extremely magnetic fields in the center (in comparison with GR) is possible in this model. Although the increasing of mass is negligible, the central energy density for stars with maximal mass is close to $1.5-2.0$ GeV/fm^3 (see Fig. 2 for mass-density relation in the case of fast varying field. For slowly varying field similar effects take place). The cores of such stars can contain considerable fraction of Σ^- and Λ hyperons. The corresponding field in the center of the star can exceed $7-8\times 10^{18}$ G (in GR, the maximal central field for these models is only $\sim 4.2\times 10^{18}$ G, see Table IV). It is interesting to note that similar effect takes place also in the case of quadratic $f(R)$ gravity (see [63]). It is worth noticing that, for $\beta <\sim -0.2$ and quadratic $f(\mathcal{G})$ corrections, there is no stable stars with extremely large central densities.

V. DISCUSSION AND CONCLUSIONS

We have considered neutron star models with strong magnetic fields in the framework of $f(R)$ and $f(\mathcal{G})$ gravity models. We have used a model with three meson fields for dense matter in strong magnetic field and coupled the corresponding EoS to the modified TOV equations. The reason to adopt such an approach is to investigate the

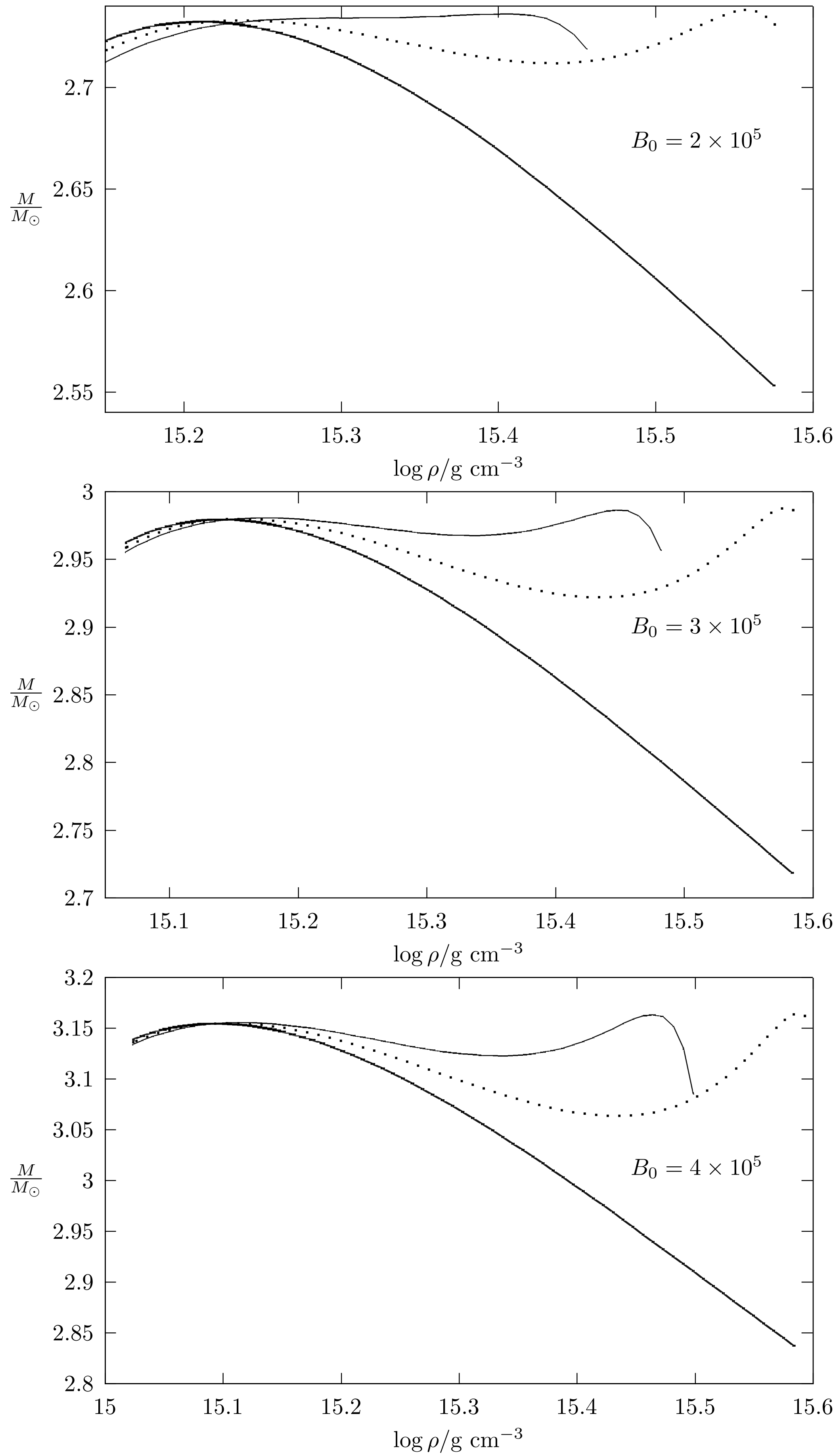

FIG. 2: The mass-density diagram in the vicinity of maximal mass for the model $f(\mathcal{G}) = \beta\mathcal{G}^2$ ($\beta = -0.05$ - dotted lines, $\beta = -0.1$ - thin lines) and for General Relativity (thick lines) for fast varying field. There is a second branch of stability ($dM/d\rho > 0$) corresponding to stars with large central densities. However, the maximal possible mass and radius are close to the values in General Relativity.

possibility of the existence of neutron stars with extreme features as high central densities and large masses as recently pointed out by observations of some peculiar objects.

Our considerations show that considerable increasing of mass can be achieved adopting cubic $f(R)$ gravity corrections. Thus, the possibility of supermassive ($M > 4M_{\odot}$) neutron stars with $R \sim 12 - 15$ km in modified gravity seems, in principle, realistic. If such stars will be explicitly observed, this could be considered as a clear signature that some self-gravitating systems can violate General Relativity constraints in favor of modified gravity.

On the other hand, quadratic $f(\mathcal{G})$ gravity corrections indicate that another interesting effect is possible: namely stable stars with central densities close to $\rho_c = 1.5 - 2.0$ GeV/fm^3 (and therefore with high strangeness fraction) can exist. The field strength in the center can exceed 8×10^{18} G. This limit cannot be achieved in General Relativity by using standard observed matter.

As a general remark, we can say that the puzzles related to the existence of extreme neutron stars could be realistically addressed by supposing the emergence of corrections and extensions to the General Relativity in the strong field regimes. In some sense, the mechanism could be similar to that supposed in the early-time inflation where higher-order curvature terms naturally emerge into dynamics. Beside the explanation of anomalous compact star, this approach could be considered as an independent probe for modified gravity with respect to the analogue descriptions invoked for dark matter and dark energy.

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