Extreme Point Results for Robust Stabilization of Interval Plants with First Order Compensators

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***Abstract*—It has recently been shown that a first-order compensator robustly stabilizes an interval plant family if and only if it stabilizes all of the *extreme plants*. That is, if the plant is described by an mth order numerator and a monic nth order denominator with coefficients lying in prescribed intervals, it is necessary and sufficient to stabilize the set of 2^{m+n+1} extreme plants. These extreme plants are obtained by considering all possible combinations for the extreme values of the numerator and denominator coefficients. In this paper, we prove a stronger result. Namely, it is necessary and sufficient to stabilize only *sixteen* of the extreme plants. These sixteen plants are generated using the Kharitonov polynomials associated with the numerator and denominator. Furthermore, when additional *a priori* information about the compensator is specified (sign of the gain and signs and relative magnitudes of the pole and zero), then in some cases, it is necessary and sufficient to stabilize *eight* critical plants while in other cases, it is necessary and sufficient to stabilize *twelve* critical plants.**

I. Introduction

The seminal theorem of Kharitonov [1] has sparked a whole new line of research (see [2] and [3] for reviews) dealing with questions of the following sort: Given a family of polynomials, under what conditions does stability of a "small" finite subset of "extreme" members of this family imply stability of the entire family? Such extreme point results make it possible to develop a number of computationally tractable methods to solve a variety of robust stability analysis problems for feedback control systems.

In this paper, the so-called *interval plant paradigm* is considered, i.e., the uncertainty in the plant is manifested via *a priori* interval bounds for each numerator and denominator coefficient. It should be noted that in applications, the assumption of interval bounds is inherently conservative because each uncertain parameter typically enters into more than one plant coefficient. Nevertheless, it is argued that development of results at the level of interval plant models is an essential step in establishing a more comprehensive theory. Further motivation for the interval plant paradigm is provided by the fact that in many applications, the dependence of plant coefficients on uncertain parameters is not identified accurately enough to merit working at a finer level of detail—for all practical purposes, one might as well assume interval bounds.

Given the power of existing extreme point results in an analysis context, it is natural to ask whether the theory can be extended to a synthesis context. In this regard, few extreme point results are available in the literature. Some notable exceptions include the work of Ghosh [4] where it is shown that a pure gain compensator $C(s) = K$ stabilizes the entire interval plant family if and only if it stabilizes a distinguished set of eight of the extreme plants.

Motivated by the desire to deal with more practical controllers, Hollot and Yang [7] consider the same setup as Ghosh but allow the controller to be first order. Subsequently, they prove that to robustly stabilize the entire family, it is necessary and sufficient to stabilize the set of extreme plants. These extreme plants are obtained by taking all possible combinations of extreme values of the plant numerator coefficients with extreme values of the plant denominator coefficients. Hence, the number of extreme plants can be quite large, i.e., if the plant numerator has degree m and the plant denominator is monic with degree n, the number of extreme plants N_{ext} can be as high as

$$N_{\text{ext}} = 2^{m+n+1}.$$

Given the fact that all extreme plants must be considered, one often uses the jargon *weak Kharitonov-like result* to describe this work.

In contrast, the focal point of this paper is the issue of *strong Kharitonov-like results*. Namely, we prove that for an interval plant with a first-order compensator, it is necessary and sufficient to stabilize only *sixteen* extreme plants in order to stabilize the entire family. These sixteen extreme plants are obtained exactly as in [5] (see also the subsequent paper [6]) where a robust version of the small gain theorem is given. Namely, we take each of the four Kharitonov polynomials for the numerator in combination with each of the four Kharitonov polynomials for the denominator. We use the words "strong Kharitonov-like results" because the number of extreme plants exploited (sixteen in our case) is indepen-

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dent of the numerator and denominator degrees m and n, respectively.

This sixteen plants result can be strengthened as follows: When the sign of the gain and signs and relative magnitudes of the pole and zero of the compensator are specified, then, in some cases, it is necessary and sufficient to stabilize a critical subset of eight plants while in other cases, twelve critical plants are required.

Before proceeding with the formal development, one final point should be noted. A major benefit associated with the extreme point results above is that it becomes possible to carry out a control synthesis via computer graphics. To illustrate, suppose one wants to construct a robust stabilizing PI controller $C(s) = K_1 + K_2/s$. Then, to determine if appropriate gains K_1 and K_2 exist, one feasible approach would be as follows: First, set up sixteen Routh tables—one for each extreme plant with compensator $C(s)$. Noting that the first column entries of these tables will be functions of K_1 and K_2, the positivity requirement for stability leads to a set of inequalities. Since only two parameters K_1 and K_2 are involved, the satisfaction set for each of these Routh table inequalities is easily graphed. Then, it follows that a necessary and sufficient condition for the existence of a robust stabilizing controller is nonemptiness of the intersection of all the satisfaction sets so obtained. Moreover, any point (K_1^*, K_2^*) in this intersection is associated with a robust stabilizing PI controller, i.e., the intersection region completely characterizes the set of robust PI stabilizers.

The paper is organized as follows: In Section II, we introduce the necessary notation and definitions for robust stabilization of interval plants. In Section III, we state the main results of the paper and in Section IV we present a numerical example illustrating their application. Finally, in Section V we provide conclusions.

II. Notation and Definitions

We first introduce the notation needed in the sequel. Consider a strictly proper interval plant family $\mathscr{P}$ consisting of all plants of the form

$$P(s,q,r) = \frac{N(s,q)}{D(s,r)} \tag{1}$$

where the numerator and denominator polynomials are of the form

$$N(s,q) = q_m s^m + q_{m-1}s^{m-1} + \cdots + q_1 s + q_0; \tag{2}$$

$$D(s,r) = s^n + r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 \tag{3}$$

and where vectors q and r lie in given rectangles Q and R, respectively, i.e.,

$$q \in Q = \{q: q_i^- \le q_i \le q_i^+ \quad \text{for } i = 0,1,\cdots,m\} \tag{4}$$

and

$$r \in R = \{r: r_i^- \le r_i \le r_i^+ \quad \text{for } i = 0,1,\cdots,n-1\} \tag{5}$$

where the bounds q_i^-, q_i^+, r_i^-, and r_i^+ are specified *a priori*. The resulting interval polynomial families for the numerator and denominator are denoted by

$$\mathscr{N} \doteq \{N(\cdot,q): q \in Q\};$$
$$\mathscr{D} \doteq \{D(\cdot,r): r \in R\}. \tag{6}$$

To stabilize the interval plant family $\mathscr{P}$, we consider a proper first-order compensator of the form

$$C(s) = K\frac{s-z}{s-p}. \tag{7}$$

We say that this compensator $C(s)$ *robustly stabilizes* the interval plant family $\mathscr{P}$ if, for all $q \in Q$ and all $r \in R$, the resulting closed-loop polynomial

$$\Delta(s,q,r) = K(s-z)N(s,q) + (s-p)D(s,r) \tag{8}$$

has all its roots in the strict left half plane; that is $\Delta(s,q,r)$ is Hurwitz. This being the case, $C(s)$ is said to be a *robust stabilizer* and the closed-loop system is said to be *robustly stable*.

Next, we introduce the Kharitonov polynomials for the numerator and denominator of the plant. Namely, for the numerator, let

$$N_1(s) \doteq q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \cdots;$$
$$N_2(s) \doteq q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots;$$
$$N_3(s) \doteq q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \cdots;$$
$$N_4(s) \doteq q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \cdots; \tag{9}$$

and for the denominator, let

$$D_1(s) \doteq r_0^+ + r_1^+ s + r_2^- s^2 + r_3^- s^3 + r_4^+ s^4 + r_5^+ s^5 + \cdots;$$
$$D_2(s) \doteq r_0^- + r_1^- s + r_2^+ s^2 + r_3^+ s^3 + r_4^- s^4 + r_5^- s^5 + \cdots;$$
$$D_3(s) \doteq r_0^- + r_1^+ s + r_2^+ s^2 + r_3^- s^3 + r_4^- s^4 + r_5^+ s^5 + \cdots;$$
$$D_4(s) \doteq r_0^+ + r_1^- s + r_2^- s^2 + r_3^+ s^3 + r_4^+ s^4 + r_5^- s^5 + \cdots. \tag{10}$$

By taking all combinations of the $N_i(s)$ and $D_k(s)$, we obtain the *sixteen Kharitonov plants*

$$P_{ik}(s) \doteq \frac{N_i(s)}{D_k(s)} \tag{11}$$

for $i, k = 1,2,3,4$. For these extreme plants, when we say "$C(s)$ stabilizes $P_{ik}(s)$," the understanding is that the closed-loop polynomial

$$\Delta_{ik}(s) = K(s-z)N_i(s) + (s-p)D_k(s) \tag{12}$$

is Hurwitz.

III. Main Results

In this section, we state the main results of the paper. The proofs of the following theorems are given in Appendices A and B.

Theorem 1: A first-order compensator $C(s)$ robustly stabilizes the interval plant family $\mathscr{P}$ if and only if it stabilizes all of the sixteen Kharitonov plants $P_{ik}(s)$; $i, k = 1, 2, 3, 4$. □

Relations with the Robust Small Gain Theorem: In this section, we show how one specializes Theorem 1 to obtain a version of the so-called robust small gain theorem; see [5] and [6] for embellishments. An example is also provided to show that a "reverse implication" does not hold, i.e., the robust small gain theorem does *not* imply Theorem 1.

To state the robust small gain theorem in the context of this paper, we let $\|\cdot\|_\infty$ denote the H_∞ norm and assume that the interval plant family $\mathscr{P}$ is robustly stable, i.e., the denominator $D(s, r)$ is stable for all $r \in R$. Now, this theorem states that

$$\|P(s, q, r)\|_\infty < 1$$

for all $q \in Q$ and $r \in R$ if and only if

$$\|P_{ik}(s)\|_\infty < 1$$

for $i, k = 1, 2, 3, 4$.

To see how this result is a special case of Theorem 1, we exploit a well-known fact (for example, see [8]). If $P(s)$ is proper rational and stable, then $\|P(s)\|_\infty < 1$ only if every first-order compensator of the form

$$C(s) = \pm \frac{s - a}{s + a}$$

with $a \in [0, \infty]$ stabilizes $P(s)$. Using this fact and invoking Theorem 1 with $K = \pm 1$, $z = a$, and $p = -a$, the robust small gain theorem follows immediately.

We now argue that Theorem 1 is *not* a special case of the robust small gain theorem, i.e., Theorem 1 is more general. To this end, we consider the robustly stable interval plant family $\mathscr{P}$ described by

$$P(s, q, r) = \frac{1}{s^2 + 0.1s + r}$$

where $r \in [0.5, 1]$. From Theorem 1, it follows that the first-order compensator

$$C(s) = 3.33 \frac{s + 3}{s + 10}$$

robustly stabilizes the interval plant family $\mathscr{P}$. However, for $r = 0.5$,

$$\|C(s)P(s, q, r)\|_\infty \approx 14.5.$$

Hence, no conclusion can be drawn from the robust small gain theorem, even though Theorem 1 does show that this interval plant can be robustly stabilized with a first-order compensator.

As mentioned in the Introduction, Theorem 1 can be strengthened when additional *a priori* information about the compensator is specified. Such a strengthening is illustrated in the theorem below for the case when the compensator is stable minimum phase with positive gain—either lead or lag. We see that it is necessary and sufficient to stabilize only eight plants.

For other combinations of the pole, zero, and gain, results similar to Theorem 2 can readily be given. All of the various possibilities are summarized in Appendix C in Tables I and II. Notice that when p and z have the same sign, we have an eight-plant result, whereas twelve plants are required when p and z have opposite sign.

Theorem 2: Consider the first-order compensator $C(s)$ as in (7) with $K > 0$, $z < 0$, and $p < 0$. Then if $p < z$ (lead compensator), $C(s)$ robustly stabilizes $\mathscr{P}$ if and only if it stabilizes the eight plants $P_{11}(s)$, $P_{13}(s)$, $P_{22}(s)$, $P_{24}(s)$, $P_{32}(s)$, $P_{33}(s)$, $P_{41}(s)$, $P_{44}(s)$. If $p > z$ (lag compensator), $C(s)$ robustly stabilizes $\mathscr{P}$ if and only if it stabilizes the eight plants $P_{11}(s)$, $P_{14}(s)$, $P_{22}(s)$, $P_{23}(s)$, $P_{31}(s)$, $P_{33}(s)$, $P_{42}(s)$, $P_{44}(s)$. □

Further Extensions: It is straightforward to extend Theorem 1 so as to accommodate a cascade of integrators in the compensator. That is, considering a compensator of the form

$$C(s) = K \frac{s - z}{s^k(s - p)} \tag{13}$$

a nearly identical line of proof can be used to establish that $C(s)$ robustly stabilizes $\mathscr{P}$ if and only if it stabilizes the sixteen Kharitonov plants.

It is also worth noting that simplifications of the results in this paper arise when the interval plant family $\mathscr{P}$ is of sufficiently low order. For example, with lead compensator $C(s)$ as in Theorem 2, if all members of $\mathscr{P}$ have denominator degree less than four, only the two extreme plants $P_{24}(s)$ and $P_{44}(s)$ need to be stabilized.

IV. Numerical Example

In this section, we illustrate the application of Theorem 1 via computer graphics. To this end, we consider the model given in [9] for an experimental oblique wing aircraft. In the absence of perturbations, the aircraft transfer function is

$$P(s) = \frac{64s + 128}{s^4 + 3.7s^3 + 65.6s^2 + 32s}.$$

Now, for robust stabilization purposes, we replace $P(s)$ by the interval plant family $\mathscr{P}$ described by

$$P(s, q, r) = \frac{q_1 s + q_0}{s^4 + r_3 s^3 + r_2 s^2 + r_1 s + r_0}.$$

For illustrative purposes, we consider parameter uncertainties $q_0 \in [90, 166]$, $q_1 \in [54, 74]$, $r_0 \in [-0.1, 0.1]$, $r_1 \in [30.1, 33.9]$, $r_2 \in [50.4, 80.8]$, and $r_3 \in [2.8, 4.6]$.

For the family $\mathscr{P}$ above, the objective is to determine if a robust stabilizing PI compensator

$$C(s) = K_1 + \frac{K_2}{s}$$

exists. This being the case, we also want to construct an appropriate compensator.

Indeed, in view of Theorem 1 (also recall the discussion in the Introduction), we first set up sixteen Routh tables parametrically in K_1 and K_2—one for each of the sixteen Kharitonov plants with compensator $C(s)$. For example, using the Kharitonov polynomials for $N_1(s)$ and $D_2(s)$, we obtain the array

s^5	1	80.8	$-\gamma + 74K_2$
s^4	4.6	$30.1 + 74K_1$	$166K_2$
s^3	$74.3 - 16.1K_1$	$-\gamma + 37.9K_2$	0
s^2	α_1/α_2	$166K_2$	0
s^1	β_1/β_2	0	0
s^0	$166K_2$	0	0

where

$$\alpha_1 = 2236 + 5777.2K_1 + 174.3K_2 - 1191.4K_1^2;$$
$$\alpha_2 = 74.3 - 16.1K_1;$$
$$\beta_1 \approx -223.68 + 371,445.56K_1 - 202,102.06K_1^2 - 197,772.4K_1^3 - 831,660.75K_2 + 413,304.94K_1K_2 - 88,182.92K_1^2K_2 + 14,229.93K_2^2;$$
$$\beta_2 = \alpha_1$$

and

$$\gamma = 0.1 - 166K_1.$$

Using the sixteen Routh tables, we now enforce positivity for each of the first columns. This leads to inequalities involving K_1 and K_2, e.g., for the Routh table for the Kharitonov plant $P_{12}(s)$ above, stability forces

$$K_1 < 4.6;$$
$$K_2 > 0;$$
$$\alpha_1/\alpha_2 > 0;$$
$$\beta_1/\beta_2 > 0.$$

It is now straightforward to display the satisfaction set for the "Routh inequalities" above, i.e., any one of a wide variety of two variable graphics routines can be used. In Fig. 1, we show the computed set of stabilizing pairs (K_1, K_2) over the range

$$0 < K_1 < 1.6; \quad 0 < K_2 < 2.$$

For implementation purposes, one can use any (K_1, K_2) combination in this set to generate a stabilizing controller. For example, a specific robust PI stabilizing controller is

$$C(s) = 0.9 + \frac{0.2}{s}.$$

V. Conclusion

The extreme point results presented in this paper suggest an important open research problem: Suppose that we remain within the realm of interval plants but we allow the compensator to be more general, i.e., we no longer restrict $C(s)$ to be first order. Then, it is of interest to give conditions under which stabilization of some distinguished subset of the extreme plants implies stabilization of the entire interval family. In this regard, note that some sort of condition must be imposed because of the counterexamples given in Hollot and Yang [7].

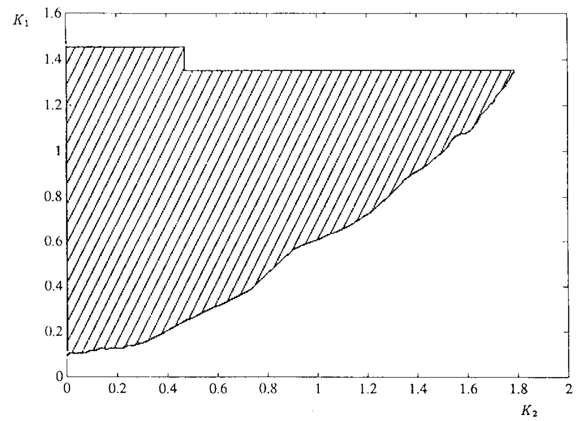

Fig. 1. Set of robust PI stabilizers.

If it turns out that more general classes of compensators lend themselves to extreme point results, then the issue of "computability" of a robust stabilizing controller becomes paramount. That is, since the number of parameters entering $C(s)$ can be significant, the two-dimensional graphics approach described in this paper will no longer work. Said another way, although one can still use a finite number of Routh tables to generate inequality constraints on the compensator parameters, the finding of a feasible point may amount to solving a difficult nonlinear program.

Appendix A
Proof of Theorem 1

The proof of Theorem 1 is accomplished with the aid of two lemmas. The first of these lemmas is a known result relating real and complex Hurwitz polynomials.

Lemma 1: (See [10, p. 61] for proof) Consider a real coefficient polynomial $p(s)$ expressed as

$$p(s) = f(s^2) + sg(s^2)$$

where $f(\cdot)$ and $g(\cdot)$ are also polynomials and assume that $p(s)$ has positive coefficients. Then, the following three statements are equivalent.

1) The real coefficient polynomial $p(s)$ is Hurwitz.

2) The complex coefficient polynomial

$$\tilde{p}_1(s) = f(js) + jg(js) \tag{14}$$

is Hurwitz.

3) The complex coefficient polynomial

$$\tilde{p}_2(s) = f(-js) + sg(-js) \tag{15}$$

is Hurwitz. □

The next lemma is a minor extension of Lemma 2.1 in [7]. It deals with the special case when uncertain parameters only enter into two consecutive coefficients in an affine linear manner. This extension is proven using Lemma 1 and value set geometry arguments which generalize those given in [11].

Lemma 2: Let $p_0(s)$ be a polynomial of degree n and suppose $p_1(s)$ is a polynomial of degree $k < n-1$ containing only all even or all odd powers of s. Then, given any $\alpha, \beta \in \boldsymbol{R}$, the polynomial

$$p(s,\lambda) = p_0(s) + \lambda(\alpha s + \beta) p_1(s) \tag{16}$$

is Hurwitz for all $\lambda \in [0,1]$ if and only if $p(s,0)$ and $p(s,1)$ are Hurwitz.

Proof: Since necessity is trivial, we proceed to establish sufficiency. Indeed, we assume that $p(s,0)$ and $p(s,1)$ are Hurwitz and without loss of generality, we make the following assumptions:

1) α and β are both nonzero; otherwise λ enters into only all even order terms or all odd order terms and sufficiency follows immediately from the result of Bialas and Garloff [12].

2) $p_1(s)$ has only even powers of s. In this case, the proof to follow uses only relation (14) and can be simply modified to handle the odd power case using relation (15).

3) $p(s,0)$ and $p(s,1)$ both have all positive coefficients; there is no loss of generality here because $p(s,0)$ and $p(s,1)$ are both Hurwitz and have the same coefficient of s^n.

Writing

$$p_0(s) = f(s^2) + sg(s^2);$$
$$p_1(s) = h(s^2)$$

where $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ are polynomials, we obtain

$$\begin{aligned} p(s,\lambda) &= f(s^2) + sg(s^2) + \lambda(\alpha s + \beta) h(s^2) \\ &= \left[f(s^2) + \lambda\beta h(s^2) \right] + s\left[g(s^2) + \lambda\alpha h(s^2) \right]. \end{aligned}$$

In view of 3) above, $p(s,\lambda)$ has positive coefficients for all $\lambda \in [0,1]$. Therefore, by Lemma 1, it suffices to prove that the complex coefficient polynomial

$$\begin{aligned} \tilde{p}(s,\lambda) &= \left[f(js) + \lambda\beta h(js) \right] + j\left[g(js) + \lambda\alpha h(js) \right] \\ &= \left[f(js) + jg(js) \right] + \lambda(\beta + \alpha j) h(js) \end{aligned}$$

is Hurwitz for all $\lambda \in [0,1]$. In this regard, we already know from Lemma 1 that $\tilde{p}(s,0)$ and $\tilde{p}(s,1)$ are Hurwitz.

The proof is now completed by contradiction. If $\tilde{p}(s,\tilde{\lambda})$ is not Hurwitz for some $\tilde{\lambda} \in [0,1]$, then continuous root dependence on λ dictates that there exists some $\omega^* \in \boldsymbol{R}$ and $\lambda^* \in (0,1)$ such that

$$\tilde{p}(j\omega^*, \lambda^*) = 0. \tag{17}$$

We first rule out the possibilities that $\tilde{p}(j\omega^*, 0) = 0$ or $\tilde{p}(j\omega^*, 1) = 0$ because this would contradict Hurwitzness of $\tilde{p}(s,0)$ and $\tilde{p}(s,1)$. We also rule out the possibility that $\tilde{p}(s,0)$ and $\tilde{p}(s,1)$ are equal because of the fact that $\tilde{p}(j\omega^*, \lambda^*) = 0$ with λ^* neither zero nor unity.

Hence, it follows that for each ω in some neighborhood Ω of ω^*, the value set

$$\tilde{p}(j\omega, [0,1]) \doteq \{ \tilde{p}(j\omega, \lambda) : \lambda \in [0,1] \}$$

is a line segment in the complex plane with end points $\tilde{p}(j\omega, 0)$ and $\tilde{p}(j\omega, 1)$ and constant slope $\tilde{m}(\omega) = \alpha/\beta$.

To complete the proof, note that the origin lies on the relative interior of the line segment $\tilde{p}(j\omega^*, [0,1])$. Moreover, since $\tilde{p}(s,0)$ and $\tilde{p}(s,1)$ are Hurwitz, the angles $\measuredangle\tilde{p}(j\omega, 0)$ and $\measuredangle\tilde{p}(j\omega, 1)$ are strictly increasing functions of ω. Using this fact in conjunction with the fact that the origin lies on $\tilde{p}(j\omega^*, [0,1])$, it follows that for $\omega > \omega^*$ with $|\omega - \omega^*|$ sufficiently small, the value set has slope $\tilde{m}(\omega) > \tilde{m}(\omega^*)$. This, however, contradicts the constancy of $\tilde{m}(\omega)$ for $\omega \in \Omega$. The proof of the lemma is now complete. □

Preliminaries for Proof of Theorem 1: In the argument to follow, all sets in the complex plane $\boldsymbol{C}$ are convex polygons. If $Z \subseteq \boldsymbol{C}$ is a convex polygon, then $\mathscr{E}[Z]$ will denote the edges of Z; if $Z_1, Z_2 \subseteq \boldsymbol{C}$ are convex polygons, then the direct sum

$$Z_1 + Z_2 \doteq \{ z_1 + z_2 : z_1 \in Z_1; z_2 \in Z_2 \}$$

is a convex polygon. Likewise, multiplication of a complex number z_0 by a convex polygon Z will be defined in the obvious way, i.e.,

$$z_0 Z \doteq \{ z_0 z : z \in Z \}$$

is a convex polygon.

The proof of Theorem 1 involves some routine manipulations of convex polygons such as set addition and linear transformation; for further elaboration, see Rockafellar [13] where algebraic operations on convex sets are discussed in detail. In this regard, we draw attention to the following fact. Suppose that Z_1 and Z_2 are polygons in $\boldsymbol{R}^2$ and $T_i: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is affine linear for $i = 1,2$. Then, it follows that $T_1 Z_1 + T_2 Z_2$ is also a polygon with edge points which come from the edges of Z_1 and the edges of Z_2. In the proof of Theorem 1, we exploit a version of this result which is embodied below.

Basic Fact: Given convex polygons $Z_1, Z_2 \subseteq \boldsymbol{C}$ and fixed complex numbers z_1 and z_2, it follows that

$$\mathscr{E}[z_1 Z_1 + z_2 Z_2] \subseteq z_1 \mathscr{E}[Z_1] + z_2 \mathscr{E}[Z_2].$$

Proof of Theorem 1: Since necessity is trivial, we proceed directly with sufficiency. Indeed, we assume that $C(s)$ stabilizes the sixteen Kharitonov plants and must show that $C(s)$ robustly stabilizes $\mathscr{P}$. Since $\Delta(s,q,r)$ has degree $n+1$ for all $q \in Q$ and $r \in R$, this requirement is equivalent to a "zero exclusion" from the *value set*

$$\Delta(j\omega, Q, R) \doteq \{ \Delta(j\omega, q, r) : q \in Q; r \in R \}$$

at all frequencies $\omega \in \boldsymbol{R}$, e.g., see [2].

We claim that the simple linear coefficient dependencies on q and r imply that $\Delta(j\omega, Q, R)$ is a convex polygon in the complex plane. To see this, observe that we have a direct sum decomposition

$$\Delta(j\omega, Q, R) = K(j\omega - z) N(j\omega, Q) + (j\omega - p) D(j\omega, R)$$

where

$$N(j\omega, Q) \doteq \{ N(j\omega, q) : q \in Q \};$$
$$D(j\omega, R) \doteq \{ D(j\omega, r) : r \in R \}$$

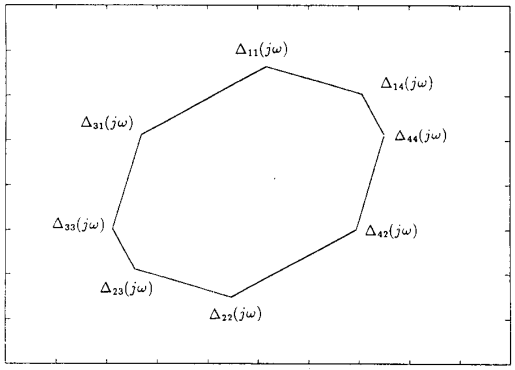

Fig. 2. Value set of $\Delta(j\omega, Q, R)$.

are the so-called *Kharitonov rectangles* with vertices (clockwise from northeast) $N_1(j\omega)$, $N_4(j\omega)$, $N_2(j\omega)$, and $N_3(j\omega)$ for $N(j\omega, Q)$ and $D_1(j\omega)$, $D_4(j\omega)$, $D_2(j\omega)$, and $D_3(j\omega)$ for $D(j\omega, Q)$. It is now apparent that $\Delta(j\omega, Q, R)$ is a convex polygon described by its convex hull (see Fig. 2)

$$\Delta(j\omega, Q, R) = \text{conv}\{K(j\omega - z)N_{i_1}(j\omega) + (j\omega - p)D_{k_1}(j\omega);\, i_1, k_1 = 1,2,3,4\}.$$

Equivalently

$$\Delta(j\omega, Q, R) = \text{conv}\{\Delta_{i_1, k_1}(j\omega):\, i_1, k_1 = 1,2,3,4\}.$$

The remainder of the proof involves a contradiction argument. To this end, assume that at some frequency $\hat{\omega} \in \boldsymbol{R}$

$$0 \in \Delta(j\hat{\omega}, Q, R).$$

Then using the fact that $0 \notin \Delta(j\omega, Q, R)$ for $|\omega|$ sufficiently large (domination by s^n term), it follows that

$$0 \in \mathscr{E}[\Delta(j\omega^*, Q, R)] \tag{18}$$

for some $\omega^* \in \boldsymbol{R}$. Now, invoking the "basic fact" preceding this proof, we have

$$0 \in K(j\omega^* - z)\mathscr{E}[N(j\omega^*, Q)] + (j\omega^* - p)\mathscr{E}[D(j\omega^*, R)].$$

Using the orientation of the vertices of the rectangles $N(j\omega, Q)$ and $D(j\omega, R)$ above, it follows that

$$K(j\omega^* - z)[\alpha^* N_{i_1}(j\omega^*) + (1 - \alpha^*)N_{i_2}(j\omega^*)] + (j\omega^* - p)[\beta^* D_{k_1}(j\omega^*) + (1 - \beta^*)D_{k_2}(j\omega^*)] = 0 \tag{19}$$

for some $(\alpha^*, \beta^*) \in [0, 1] \times [0, 1]$ and some

$$(i_1, i_2) \in \{(1,3), (1,4), (2,3), (2,4)\};$$
$$(k_1, k_2) \in \{(1,3), (1,4), (2,3), (2,4)\}.$$

Without loss of generality, we simplify notation and henceforth consider

$$(i_1, i_2) = (1,3);$$
$$(k_1, k_2) = (2,4)$$

noting that the identical line of proof to follow can be used for the other (i_1, i_2) and (k_1, k_2) combinations.

Note that the quantity on the left-hand side of (19) is a particular evaluation of

$$\delta(s, a, b) \doteq K(s - z)[aN_1(s) + (1 - a)N_3(s)] + (s - p)[bD_2(s) + (1 - b)D_4(s)]$$

with $s = j\omega^*$, $a = \alpha^*$, and $b = \beta^*$. Hence, it follows from (19) that

$$0 \in \delta(j\omega^*, [0, 1], [0, 1]) \tag{20}$$

where

$$\delta(j\omega, [0, 1], [0, 1]) \doteq \{\delta(j\omega, a, b):\, a \in [0, 1];\, b \in [0, 1]\}$$

is a convex polygon. Since

$$\delta(j\tilde{\omega}, [0, 1], [0, 1]) \subset \Delta(j\tilde{\omega}, Q, R)$$

relations (18) and (20) imply that

$$0 \in \mathscr{E}[\delta(j\tilde{\omega}, [0, 1], [0, 1])] \tag{21}$$

for some $\tilde{\omega} \in \boldsymbol{R}$. Again recalling the "basic fact," the edges of $\delta(j\tilde{\omega}, [0, 1], [0, 1])$ are obtained from the edges of the rectangle $[0, 1] \times [0, 1]$. Without loss of generality, suppose that (21) is attained with $a = \tilde{\lambda} \in [0, 1]$ and $b = 1$; note that the other edge combinations for a and b are handled using a line of proof which is virtually identical to the one used for this case. Under these conditions, it follows that

$$\delta(j\tilde{\omega}, \tilde{\lambda}, 1) = 0.$$

Equivalently,

$$\Delta_{34}(j\tilde{\omega}) + \tilde{\lambda}K(j\tilde{\omega} - z)[N_1(j\tilde{\omega}) - N_3(j\tilde{\omega})] = 0. \tag{22}$$

The proof is now completed by making the following identifications with Lemma 2:

$$p_0(s) \sim \Delta_{34}(s);$$
$$p_1(s) \sim N_1(s) - N_3(s);$$
$$\lambda(\alpha s + \beta) \sim \lambda K(s - z).$$

Recalling that $C(s)$ stabilizes the sixteen Kharitonov plants, it is easy to verify that all preconditions of Lemma 2 are satisfied. In particular, notice that $N_1(s) - N_3(s)$ has only even powers of s and $p(s, 0) = \Delta_{34}(s)$ and $p(s, 1) = \Delta_{14}(s)$ are Hurwitz. Therefore, Lemma 2 dictates that $\delta(s, \lambda, 1)$ must be Hurwitz for all $\lambda \in [0, 1]$. On the other hand, (22) implies that $\delta(s, \tilde{\lambda}, 1)$ has a root $s = j\tilde{\omega}$ on the imaginary axis which contradicts its Hurwitzness. With this contradiction, the proof of the theorem is now complete. □

Appendix B

Proof of Theorem 2

Preliminaries for Proof of Theorem 2: If $\mathscr{R}$ is a rectangle in the complex plane with sides parallel to the real and imaginary axes, then we can unambiguously associate its vertices with compass directions, i.e., we use v^{NE} to denote its northeast vertex, v^{SE} to denote its southeast vertex, v^{SW}