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EXTREME POINTS OF UNIT BALLS IN LIPSCHITZ FUNCTION SPACES

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ABSTRACT. We give a new characterization of the set ext $(B_{X^\#})$ of all extreme points of the unit ball $B_{X^\#}$ in the Banach space $X^\#$ of all Lipschitz functions on a metric space X. This result is applied to get a total variation characterization of ext $(B_{X^\#})$ in the particular case when X is a convex subset of a Banach space.

Let $0 \in X$ be an arbitrarily chosen point of a metric space $X = (X, d)$ which consists of at least two distinct points. Following Lindenstrauss [3] denote by $X^\#$ the Banach space of all functions $f : X \to \mathcal{R}$ such that $f(0) = 0$ and

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in X, x \neq y \right\} < \infty.$$

In other words, the Banach space $X^\#$ consists of all real-valued Lipschitz functions defined on X, which are equal zero at the distinguished point 0. In the following, we always assume that the distinguished point 0 is equal to the origin of the Banach space E, whenever X is a subset of E containing the origin of E.

In the study of geometric Banach space theory and its various applications it is important to have a good characterization of the extreme points of unit balls. The investigation of the set of all extreme points $\mathrm{ext}(B_{X^\#})$ of the unit ball $B_{X^\#}$ of $X^\#$ has been originated by Rolewicz [4] who has proved the following theorem.

Theorem A. *Let f be a function in $[0,1]^\#$ with $\|f\| = 1$. Then $f \in \mathrm{ext}(B_{[0,1]^\#})$ if and only if $|f'(x)| = 1$ a.e. on $[0,1]$.*

Moreover, he has shown in [5] that a similar result cannot hold for the space $X = [0,1] \times [0,1]$ with Euclidean metric. Next, Cobzas [1] has characterized the extreme points in $X^\#$ for a rather restricted class of metric spaces X. Recently, Farmer [2] has presented a new characterization of the set $\mathrm{ext}(B_{X^\#})$ without any additional restrictions on X. More precisely, he proved the following theorem.

Theorem B. *Let X be a metric space, and let f be a function in $X^\#$ with the norm $\|f\| = 1$. Then $f \in \mathrm{ext}(B_{X^\#})$ if and only if* (i) *$\epsilon^f_{x,y} = 0$ for all $x, y \in X$,*

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where

$$\epsilon_{x,y}^{f} = \inf \Big\{ \epsilon > 0 : d\left(x_{i-1}, x_i\right) - \epsilon_i \leq \left|f(x_{i-1}) - f(x_i)\right| \;\; (i = 1, ..., n), \qquad x_0 = x, \; x_n = y, \; \sum_{i=1}^{n} \epsilon_i \leq \epsilon \Big\}$$

with the infimum taken over all finite sequences $\epsilon_1, ..., \epsilon_n > 0$ *and* $x_1, ..., x_{n-1} \in X$ *satisfying the above inequalities.*

Moreover, he noted that condition (i) is equivalent to the condition

(ii) $\epsilon_{x,0}^{f} = 0$ for every $x \in X$,

which is an immediate consequence of the triangle inequality

$$\epsilon_{x,y}^{f} \leq \epsilon_{x,z}^{f} + \epsilon_{z,y}^{f}; \; x, y, z \in X. \tag{1}$$

In this paper, we first apply Theorem B to derive a new characterization of $\operatorname{ext}(B_{X^\#})$. Next, we use this result to obtain the following

Theorem 1. *Let* X *be a convex subset of a normed linear space* $E = (E, \|\cdot\|)$, *and let* f *be a function in* $X^\#$ *such that* $\|f\| = 1$. *Then* $f \in \operatorname{ext}(B_{X^\#})$ *if and only if*

$$\text{(i)} \quad \inf \left\{ \sum_{i=1}^{n} \left(\|x_i - x_{i-1}\| - \int_0^1 \left| f'_{x_i, x_{i-1}}(t) \right| dt \right) : x_0 = x, \; x_n = y \right\} = 0$$

for all $x, y \in X$, *where the infimum is taken over all finite sequences* $x_1, ..., x_{n-1} \in X$, *and*

$$f_{x_i, x_{i-1}}(t) = f\left((1-t)\, x_{i-1} + t x_i\right), \; 0 \leq t \leq 1.$$

For this purpose, let

$$\langle x, y \rangle = \{ z \in X : d(x, y) = d(x, z) + d(z, y) \}$$

be the *metric interval* with endpoints $x, y \in X$. Additionally, let $(x_i)_0^n$ be a *metric subdivision* of $\langle x, y \rangle$ with $x \neq y$, i.e., let $x_0 = x, \; x_n = y, \; x_i \in \langle x, y \rangle, \; x_i \neq x_j$ for $i \neq j$, and

$$d(x, y) = \sum_{i=1}^{n} d(x_{i-1}, x_i). \tag{2}$$

Then we define

$$\rho_f(x, y) = \inf \left\{ \|f\|\, d(x, y) - \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| \right\}, \tag{3}$$

where the infimum is taken over all finite metric subdivisions $(x_i)_0^n$ of the interval $\langle x, y \rangle$. Additionally, we put $\rho_f(x, x) = 0$. Since points $x_0 = x$ and $x_1 = y$ form a subdivision of $\langle x, y \rangle$, it follows from (3) that

$$\rho_f(x, y) \leq \|f\|\, d(x, y) - \left| f(x) - f(y) \right| \tag{4}$$

for all $x, y \in X$. Further, we have

$$\rho_f(x, y) \leq \|f\|\, d(x, y) - \sum_{i=1}^{n+m} \left| f(x_i) - f(x_{i-1}) \right|$$

for all metric subdivisions $(x_i)_0^n$ of $\langle x,z\rangle$ and $(x_i)_{n+1}^{n+m}$ of $\langle z,y\rangle$, where $z\in\langle x,y\rangle$. Hence one can take the first infimum over $x_1,...,x_{n-1}$ and the second over $x_{n+2},...,$ x_{n+m-1} to get

$$\rho_f(x,y)\le\rho_f(x,z)+\rho_f(z,y),\ z\in\langle x,y\rangle. \tag{5}$$

In general, ρ_f does not satisfy the triangle inequality. For example, let $\|x\|_p$ $(1<p<\infty)$ denote l^p- norm of $x=(x_1,x_2)\in X=\mathcal{R}^2$. Then we have

$$\rho_f(x,y)=\|x-y\|_p-|x_1-y_1|$$

for the function $f(x)=x_1$. Hence we get

$$1=\rho_f(x,y)>\rho_f(x,z)+\rho_f(z,y)=2^{\frac{1}{p}}-1,$$

whenever $x=(0,0)$, $y=(0,1)$ and $z=(1,0)$.

In view of this example, we define

$$\sigma_f(x,y)=\inf\{\rho_f(x,z_1)+\rho_f(z_1,z_2)+...+\rho_f(z_n,y):\ z_1,...,z_n\in X,\ n\in\mathcal{N}\} \tag{6}$$

for all $x,y\in X$ and $f\in X^{\#}$. Clearly, σ_f is a symmetric function such that $\sigma_f(x,x)=0$ and

$$0\le\sigma_f\le\rho_f. \tag{7}$$

In particular, this together with (4) gives

$$|f(x)-f(y)|\le\|f\|\,d(x,y)-\sigma_f(x,y);\ x,y\in X. \tag{8}$$

Further, taking the infimum over $(z_i)_1^n$ and $(y_i)_1^m$ of the right-hand side of the inequality

$$\begin{aligned}\sigma_f(x,y)\le{}&[\rho_f(x,z_1)+\rho_f(z_1,z_2)+\cdots+\rho_f(z_n,z)]\\&+[\rho_f(z,y_1)+\rho_f(y_1,y_2)+...+\rho_f(y_m,y)],\end{aligned}$$

we derive

$$\sigma_f(x,y)\le\sigma_f(x,z)+\sigma_f(z,y),$$

and therefore

$$|\sigma_f(x,y)-\sigma_f(x,z)|\le\sigma_f(y,z) \tag{9}$$

for all $x,y,z\in X$. Note also that

$$\sigma_f\le\mu\le\rho_f\Longrightarrow\sigma_f=\mu, \tag{10}$$

whenever the function $\mu:X\times X\to\mathcal{R}$ satisfies the triangle inequality on X. Indeed, note that

$$\begin{aligned}\rho_f(x,z_1)+\rho_f(z_1,z_2)+...+\rho_f(z_n,y)\ge\mu(x,z_1)+\mu(z_1,z_2)\\+...+\mu(z_n,y)\ge\mu(x,y)\ge\sigma_f(x,y),\end{aligned}$$

and take the infimum over $(z_i)_1^n$ to get $\sigma_f=\mu$.

Theorem 2. *Let X be a metric space, and let f be a function in $X^{\#}$ with the norm $\|f\|=1$. Then $f\in\operatorname{ext}(B_{X^{\#}})$ if and only if*

(i) $\sigma_f(x,y)=0$ *for all* $x,y\in X$.

Proof. Suppose first that $\sigma_f(x,y)=0$ for all $x,y\in X$. Moreover, take an arbitrary $\epsilon>\rho_f(x,y)$. Then it follows from (2) − (3) that there exists a metric subdivision $(x_i)_0^n$ of $\langle x,y\rangle$ for which

$$d(x,y)-\epsilon=\left(\sum_{i=1}^n d_i\right)-\epsilon<\sum_{i=1}^n c_i, \tag{11}$$

where

$$d_i=d(x_{i-1},x_i)>0 \text{ and } c_i=|f(x_i)-f(x_{i-1})|.$$

Since $\|f\|=1$, we have $c_i\le d_i$. Moreover, by (11) one can find n numbers e_i $(i=1,...,n)$ such that $0\le e_i<c_i$ (if $c_i>0$), $e_i=0$ (if $c_i=0$), and

$$\left(\sum_{i=1}^n d_i\right)-\epsilon=\sum_{i=1}^n e_i.$$

Now denote $\epsilon_i=d_i-e_i$. Then we have $\epsilon_i>0$, $\sum_{i=1}^n\epsilon_i=\epsilon$, and $c_i\ge e_i=d_i-\epsilon_i$, i.e.,

$$d(x_{i-1},x_i)-\epsilon_i\le|f(x_i)-f(x_{i-1})|\ \ (i=1,...,n).$$

Hence it follows from the definition of $\epsilon_{x,y}^f$ that $\epsilon_{x,y}^f\le\epsilon$. Since $\epsilon>\rho_f(x,y)$ was arbitrary, we conclude that

$$0=\sigma_f(x,y)\le\epsilon_{x,y}^f\le\rho_f(x,y)$$

for all $x,y\in X$. This in conjunction with (1) enables to apply (10) in order to get $\epsilon_{x,y}^f=\sigma_f(x,y)=0$. Thus Theorem B yields $f\in\operatorname{ext}(B_{X^\#})$, which completes the proof of necessity.

For the proof of sufficiency, suppose that there exist $f\in X^\#$ and $z\in X$ for which $\|f\|=1$ and $Y=\{y:\sigma_f(z,y)>0\}\neq\emptyset$. Then the triangle inequality and symmetry of σ_f yield

$$\sigma_f(x,y)=\sigma_f(z,y) \tag{12}$$

for all $x\in X\setminus Y$ and $y\in Y$. This together with (8) and (9) enables to repeat mutatis mutandis Farmer's proof [2] of sufficiency of Theorem B, with $\epsilon_{x,y}^f$ replaced by $\sigma_f(x,y)$, in order to show that $f\notin\operatorname{ext}(B_{X^\#})$. □

From now on, we will assume that X is a convex subset of a normed linear space $(E,\|\cdot\|)$. In this case, we define

$$\hat\rho_f(x,y)=\inf\left\{\|f\|\,\|x-y\|-\sum_{i=1}^n|f(x_i)-f(x_{i-1})|\right\}, \tag{13}$$

where the infimum is taken only over all finite subdivisions $(x_i)_0^n$ of the form

$$x_i=(1-t_i)x+t_iy\ \ (0=t_0<t_1<...<t_n=1).$$

It is clear that (2) holds for these *algebraic subdivisions* of the *algebraic interval*

$$[x,y]=\{(1-t)x+ty:0\le t\le1\},$$

and that $[x,y]=\langle x,y\rangle$ and $\hat\rho_f(x,y)=\rho_f(x,y)$ for all $x,y\in X$, whenever E is a strictly convex space. In general, we have only $\rho_f\le\hat\rho_f$.

If $\hat\sigma_f(x,y)$ is defined by formula (6) with ρ_f replaced by $\hat\rho_f$, then $\sigma_f\le\hat\sigma_f$. By the same arguments as above, one can also prove that $\hat\rho_f$ and $\hat\sigma_f$ satisfy inequality

(4) and the triangle inequality, respectively. In particular, by using (2) and (4) we obtain

$$\hat{\sigma}_f(x,y) \le \hat{\rho}_f(x_0,x_1) + ... + \hat{\rho}_f(x_{n-1},x_n) \le \|f\| \|x-y\| - \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

for all metric subdivisions $(x_i)_0^n$ of $\langle x, y\rangle$. Hence we derive $\hat{\sigma}_f \le \rho_f$. Therefore, one can apply (10) with $\mu = \hat{\sigma}_f$ in order to get $\hat{\sigma}_f = \sigma_f$.

Lemma 1. *Let X be a convex subset of a normed linear space $E = (E, \|\cdot\|)$, and let $f \in X^{\#}$. Then we have*

$$\hat{\rho}_f(x,y) = \|f\| \|x-y\| - V_0^1(f_{x,y}) = \|f\| \|x-y\| - \int_0^1 \left| f'_{x,y}(t) \right| dt,$$

where $V_0^1(f_{x,y})$ denotes the total variation of the function $f_{x,y}$ defined by

$$f_{x,y}(t) = f((1-t)x + ty) \ \ (0 \le t \le 1).$$

Proof. By (13) we obtain

$$\begin{aligned} \hat{\rho}_f(x,y) &= \|f\| \|x-y\| - \sup\left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\} \\ &= \|f\| \|x-y\| - V_0^1(f_{x,y}), \end{aligned} \tag{14}$$

where the supremum is taken over all finite algebraic subdivisions $(x_i)_0^n$ of $[x,y]$. Since $f \in X^{\#}$, we have

$$|f_{x,y}(t) - f_{x,y}(s)| \le \|f\| \|x-y\| |t-s| \ \ (0 \le t, s \le 1).$$

Hence the derivative $f'_{x,y}(t)$ exists almost everywhere on $[0,1]$, and the function $t \to f'_{x,y}(t)$ is integrable. Moreover, we have

$$V_0^1(f_{x,y}) = \int_0^1 \left| f'_{x,y}(t) \right| dt.$$

This in conjunction with (14) completes the proof. □

In view of the fact that $\hat{\sigma}_f = \sigma_f$, Theorem 1 is an immediate consequence of Lemma 1 and Theorem 2. Moreover, it follows from the triangle inequality for σ_f that Theorems 1 and 2 remain true, whenever we put either $x = 0$ or $y = 0$ into them. In particular, if the interval $X = [0,1]$ is equipped with the metric $d(x,y) = |x-y|$, then Lemma 1 yields

$$\rho_f(x,y) = \hat{\rho}_f(x,y) = |x-y| \left(1 - \int_0^1 |f'(s)| \, ds \right)$$

for all $x, y \in [0,1]$. On the other hand, by (5) and (7) one can apply (10) with $\mu = \rho_f$ to get $\sigma_f = \rho_f$. Hence Theorem A follows directly from Theorem 2.

Finally, we present another application of Theorem 2 which shows that the set $\text{ext}(B_{X^{\#}})$ of all extreme points of the unit ball $B_{X^{\#}}$ of $X^{\#}$ is quite rich, whenever

X is a normed linear space. For this purpose, denote by X^* the dual space of X, and note that

$$\sigma_f(x, x+\alpha z) = \sigma_f(0, \alpha z) = |\alpha| \, \sigma_f(0, z) \ \ (\alpha \in \mathcal{R}; \ x, z \in X) \tag{15}$$

for every functional $f \in X^*$. To prove these identities, we need only to change variables $z_k \to z_k + x$ $(z_k \to \alpha z_k)$ in the definition of $\hat{\sigma}_f = \sigma_f$ applied to $y = x + \alpha z$ $(y = \alpha z$, respectively), and use the identity

$$\hat{\rho}_f(x, y) = \|f\| \, \|x-y\| - |f(x-y)| \ \ (x, y \in X),$$

which is a direct consequence of Lemma 1 and linearity of f. Since σ_f satisfies the triangle inequality, it follows from (15) that

$$\sigma_f(0, z_1+z_2) \le \sigma_f(0, z_1) + \sigma_f(z_1, z_1+z_2) = \sigma_f(0, z_1) + \sigma_f(0, z_2).$$

This in conjunction with (15) means that the function $z \to \sigma_f(0, z)$ $(z \in X)$ is a seminorm on X.

Theorem 3. *Let X be a normed linear space. Then we have*

$$\operatorname{ext}(B_{X^\#}) \cap X^* = \operatorname{ext}(B_{X^*}).$$

Proof. In view of definition of extreme points, we directly have

$$\operatorname{ext}(B_{X^\#}) \cap X^* \subset \operatorname{ext}(B_{X^*}).$$

Conversely, let a functional $f \in X^*$ be such that $\|f\| = 1$ and $f \notin \operatorname{ext}(B_{X^\#})$. We need only to prove that $f \notin \operatorname{ext}(B_{X^*})$. By Theorem 2 the set

$$Y = \{y : \sigma_f(0, y) > 0\}$$

is nonempty. Moreover, it follows from (15) that the set $X \setminus Y$ is a linear subspace of X which, in view of (12), has the property

$$\sigma_f(x, y) = \sigma_f(0, y) \ \ (x \in X \setminus Y, \ y \in Y). \tag{16}$$

Now take a point $y_0 \in Y$, and denote by X_0 the linear subspace spanned by y_0 and $X \setminus Y$. Next, define the linear functional g on X_0 by the formula

$$g(x + \alpha y_0) = \alpha \sigma_f(0, y_0) \ \ (x \in X \setminus Y, \ \alpha \in \mathcal{R}).$$

Then it follows from (15) and (16) that

$$|g(x+\alpha y_0)| = \sigma_f(x, x+\alpha y_0) = \sigma_f(0, x+\alpha y_0) \ \ (x \in X \setminus Y, \ \alpha \in \mathcal{R}), \tag{17}$$

whenever $x + \alpha y_0 \in Y$. Otherwise, if $x + \alpha y_0 \notin Y$ then $\alpha = 0$ and (17) is obvious. Since the function $z \to \sigma_f(0, z)$ is a seminorm on X and g satisfies condition (17) on X_0, it follows from the Hahn-Banach theorem that the functional $g : X_0 \to \mathcal{R}$ has an extension to the whole space X, which satisfies the inequality

$$|g(z)| \le \sigma_f(0, z), \ z \in X.$$

Consequently, one can apply (4) and (7) to get

$$|g(z)| \le \|z\| - |f(z)|, \ z \in X.$$

Thus $g \in X^*$ and $f_k \in X^*$ $(k = 1, 2)$, where functionals f_k are defined by

$$f_k(z) = f(z) + (-1)^k g(z).$$

Therefore, we obtain

$$|f_k(z)| \le |f(z)| + |g(z)| \le \|z\|$$

for every $z \in X$. Hence we have $\|f_k\| \leq 1$ $(k = 1, 2)$ and $f_k(y_0) \neq f(y_0)$, which in conjunction with the identity $f = (f_1 + f_2)/2$ shows that $f \notin \text{ext}(B_{X^*})$. Thus the proof is completed. □

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