Extreme Value Distributions *

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1 Introduction

Extreme Value distributions arise as limiting distributions for maximums or minimums (*extreme values*) of a sample of independent, identically distributed random variables, as the sample size increases. Extreme Value Theory (EVT) is the theory of modelling and measuring events which occur with very small probability. This implies its usefulness in risk modelling as risky events per definition happen with low probability. Thus, these distributions are important in statistics. These models, along with the Generalized Extreme Value distribution, are widely used in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications, and many other industries dealing with extreme events. The class of Extreme Value Distributions (EVD's) essentially involves three types of extreme value distributions, types I, II and III, defined below.

Definition 1 (Extreme Value Distributions *for maxima*).
The following are the standard Extreme Value distribution functions:

(i) Gumbel (type I): $\Lambda(x) = \exp\{-\exp(-x)\}$, $x \in \mathbb{R}$;

(ii) Fréchet (type II): $\Phi_\alpha(x) = \begin{cases} 0, & x \le 0; \\ \exp\{-x^{-\alpha}\}, & x > 0,\ \alpha > 0; \end{cases}$

(iii) Weibull (type III): $\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x \le 0,\ \alpha > 0; \\ 1, & x > 0. \end{cases}$

The EVD families can be generalized with the incorporation of location (λ) and scale (δ) parameters, leading to

$$\Lambda(x;\lambda,\delta) = \Lambda((x-\lambda)/\delta),\ \Phi_\alpha(x;\lambda,\delta) = \Phi_\alpha((x-\lambda)/\delta),\ \Psi_\alpha(x;\lambda,\delta) = \Psi_\alpha((x-\lambda)/\delta), \quad \lambda \in \mathbb{R}, \delta > 0.$$

Among these three families of distribution functions, the type I is the most commonly referred in discussions of extreme values. Indeed, the Gumbel distribution $\{\Lambda(x;\lambda,\delta) = \Lambda((x-\lambda)/\delta); \lambda \in \mathbb{R}, \delta > 0\}$, is often coined "*the*" extreme value distribution.

Proposition 1 (Moments and Mode of EVD).
The mean, variance and mode of the EVD as in definition 1 are, respectively:

(i) Gumbel – Λ*:* $E[X] = \gamma = 0.5772\ldots = Euler's\ constant$*;* $Var[X] = \pi^2/6$*;* $Mode = 0$*;*

(ii) Fréchet – Φ_α*:* $E[X] = \Gamma(1-1/\alpha)$*, for* $\alpha > 1$*;* $Var[X] = \Gamma(1-2/\alpha) - \Gamma^2(1-1/\alpha)$*, for* $\alpha > 2$*;* $Mode = (1+1/\alpha)^{-1/\alpha}$*;*

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(iii) Weibull – Ψ_α*:* $E[X] = -\Gamma(1+1/\alpha)$*;* $Var[X] = \Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)$*;* $Mode = -(1-1/\alpha)^{-1/\alpha}$*, for* $\alpha > 1$*, and* $Mode = 0$*, for* $0 < \alpha \leq 1$*;*

here Γ *denotes the gamma function* $\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx$*,* $s > 0$*.*

Definition 2 (Extreme Value Distributions *for minima*)**.**
The standard converse EVD's for minima are defined as: $\Lambda^*(x) = 1 - \Lambda(-x)$*,* $\Phi^*_\alpha(x) = 1 - \Phi_\alpha(-x)$ *and* $\Psi^*_\alpha(x) = 1 - \Psi_\alpha(-x)$*.*

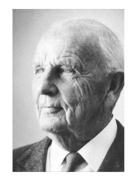
Emil Gumbel

The Gumbel distribution, named after one of the pioneer scientists in practical applications of the Extreme Value Theory (EVT), the German mathematician Emil Gumbel (1891-1966), has been extensively used in various fields including hydrology for modeling extreme events. Gumbel applied EVT on real world problems in engineering and in meteorological phenomena such as annual flood flows(Gumbel, 1958):

"It seems that the rivers know the theory. It only remains to convince the engineers of the validity of this analysis."

Maurice Fréchet

The EVD of type II was named after Maurice Fréchet (1878-1973), a French mathematician who devised one possible limiting distribution for a sequence of maxima, provided convenient scale normalization (Fréchet, 1927). In applications to finance, the Fréchet distribution has been of great use apropos to the adequate modeling of market-returns which are often heavy-tailed.



Waloddi Weibull

The EVD of type III was named after Waloddi Weibull (1887-1979), a Swedish engineer and scientist well-known for his work on strength of materials and fatigue analysis (Weibull, 1939). Even though the Weibull distribution was originally developed to address the problems for minima arising in material sciences, it is widely used in many other areas thanks to its flexibility. If $\alpha = 1$, the Weibull distribution function for *minima*, Ψ^*_α, reduces to the Exponential model, whereas for $\alpha = 2$ it mimics the Rayleigh distribution which is mainly used in the telecommunications field. Furthermore, Ψ^*_α resembles the Normal distribution when $\alpha = 3.5$.

Owing to the equality

$$\min(X_1, \ldots, X_n) = -\max(-X_1, \ldots, -X_n)$$

it suffices to consider henceforth only the EVD's for maxima featuring in Definition 1. In probability theory and statistics, the Generalized Extreme Value (GEV) distribution is a family of continuous probability distributions developed under the extreme value theory in order to combine the Gumbel, Fréchet and Weibull families. The GEV distribution arises from the extreme value theorem (Fisher-Tippett, 1928 and Gnedenko, 1943) as the limiting distribution of properly normalized maxima of a sequence of independent and identically distributed (i.i.d.) random variables. Because of this, the GEV distribution is fairly used as an approximation to model the maxima of long (finite) sequences of random variables. In some fields of application the GEV distribution is in fact known as the Fisher-Tippett distribution, named after Sir Ronald Aylmer Fisher (1890-1962) and Leonard Henry Caleb Tippett (1902-1985) who recognized the only three possible limiting functions outlined above in Definition 1.

2 Extreme Value Theory and Max-Stability

Richard von Mises (1883-1953) studied the EVT in 1936, giving in particular the von Mises sufficient conditions on the hazard rate (assuming the density exists) in order to give a situation in which EVT behaviour occurs, leading to one of the above three types of limit law that is, giving an extremal domain of attraction $\mathcal{D}(G)$ for the extreme-value distribution G. Later on, and motivated by a storm surge in the North Sea (31 January-1 February 1953) which caused extensive flooding and many deaths, the Netherlands Government gave top priority to understanding the causes of such tragedies with a view to risk mitigation. Since it is the maximum sea level which is the danger, EVT became a Netherlands scientific priority. A relevant work in the field is the doctoral thesis of Laurens de Haan in 1970.

The fundamental extreme value theorem (Fisher-Tippett 1928; Gnedenko, 1943) ascertains the Generalized Extreme Value distribution in the von Mises-Jenkinson parametrization (von Mises, 1936; Jenkinson, 1955) as an unified version of all possible non-degenerate weak limits of partial maxima of sequences comprising i.i.d. random variables $X_1, X_2, \ldots$. That is:

Theorem 1 (Fisher-Tippett 1928; Gnedenko, 1943)**.**
If there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} P\left\{a_n^{-1}\left(\max(X_1, \ldots, X_n) - b_n\right) \leq x\right\} = G(x),$$

for some non-degenerate distribution function G, then it is possible to redefine the normalizing constants in such a way that

$$G(x) = G_\xi(x) := \exp(-(1+\xi x)^{-1/\xi}),$$

for all x such that $1+\xi x > 0$, with extreme value index $\xi \in \mathbb{R}$. Taking $\xi \to 0$, then $G_\xi(x)$ reduces to $\Lambda(x)$ for all $x \in \mathbb{R}$ (cf. Definition 1). Thus the distribution function F belongs to the domain of attraction of G_ξ, which is denoted by $F \in \mathcal{D}(G_\xi)$.

Remark 1. *Note that, as $n \to \infty$, the $\max(X_1, \ldots, X_n)$ detached of any normalization converges in distribution to a degenerate law assigning probability one to the right endpoint of F, $x_F := \sup\{x : F(x) < 1\}$.*

For $\xi < 0$, $\xi = 0$ and $\xi > 0$, the G_ξ distribution function reduces to Weibull, Gumbel and Fréchet distributions, respectively. More precisely,

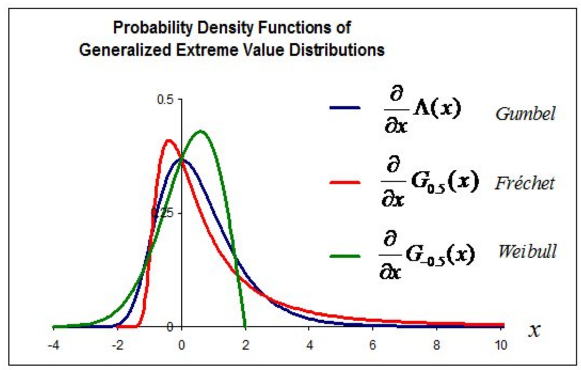
$\Lambda(x) \equiv G_0(x)$,

$\Phi_\alpha(x) \equiv G_{1/\alpha}(\alpha(x-1))$,

and

$\Psi_\alpha(x) \equiv G_{-1/\alpha}(\alpha(1+x))$.

For exhaustive details on EVD see Chapter 22 of Johnson, Balakrishnan, and Kotz (1995).



Proposition 2 (Moments and Mode of GEV)**.**
The mean, variance and mode of the GEV as in Theorem 1 are, respectively:
$E[X] = -\frac{1}{\xi}[1-\xi(1-\xi)]$, *for* $\xi < 1$; $Var[X] = \frac{1}{\xi^2}[\Gamma(1-2\xi) - \Gamma^2(1-\xi)]$, *for* $\xi < 1/2$; $Mode = \frac{1}{\xi}[(1+\xi)^{-\xi} - 1]$, *for* $\xi \neq 0$.

Proposition 3 (Skewness of GEV)**.**
The skewness coefficient of GEV distribution, defined as $skew_{G_\xi} := E[\{X - E[X]\}^3]/\{Var[X]\}^{3/2}$, is equal to zero at $\xi_0 \simeq -2.8$. Moreover, $skew_{G_\xi} > 0$, for $\xi > \xi_0$, and $skew_{G_\xi} < 0$, for $\xi < \xi_0$. Furthermore, for the Gumbel distribution, $skew_{G_0} \simeq 1.14$.

The Fréchet domain of attraction contains distributions with polynomially decay tails. All distribution functions belonging to Weibull domain of attraction are light-tailed with finite right endpoint. The intermediate case $\xi = 0$ is of particular interest in many applied sciences where extremes are relevant, not only because of the simplicity of inference within the Gumbel domain G_0 but also for the great variety of distributions possessing an exponential tail whether having finite right endpoint or not. In fact, separating statistical inference procedures according to the most suitable domain of attraction for the sampled distribution has become an usual practice. In this respect we refer to Neves and Fraga Alves (2008) and references therein.

Definition 3 (Univariate Max-Stable Distributions)**.**
A random variable X with distribution function F is max-stable if there are normalizing sequences $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the independent copies $X_1, X_2, \cdots, X_n$ satisfy the equality in distribution $\max(X_1, \cdots, X_n) \stackrel{d}{=} a_n X + b_n$. Equivalently, F is a max-stable distribution function if $[F(x)]^n = F((x - b_n)/a_n)$, all $n \in \mathbb{N}$.

The class GEV, up to location and scale parameters, $\{G_\xi(x;\lambda,\delta) = G_\xi((x-\lambda)/\delta), \lambda \in \mathbb{R}, \delta > 0\}$, represents the only possible max-stable distributions.

Additional information can be found in Kotz and Nadarajah (2000), a monograph which describes in an organized manner the central ideas and results of probabilistic extreme-value theory and related extreme-value distributions – both univariate and multivariate – and their applications, and it is aimed mainly at a novice in the field. De Haan and Ferreira (2006) constitutes an excellent introduction to EVT at the graduate level, however requiring some mathematical maturity in regular variation, point processes, empirical distribution functions, and Brownian motion. Reference Books in Extreme Value Theory and in the field of real world applications of EVD's and Extremal Domains of Attraction are: Embrechts, Klüppelberg and Mikosch (2001), Beirlant, Goegebeur, Segers and Teugels (2004), David and Nagaraja (2003), Gumbel (1958), Castillo, Hadi, Balakrishnan, and Sarabia (2005) and Reiss and Thomas (2007).

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