EXTREME VALUE THEORY
FOR CONTINUOUS PARAMETER STATIONARY PROCESSES

by

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Abstract

In this paper the central distributional results of classical extreme value theory are obtained, under appropriate dependence restrictions, for maxima of continuous parameter stochastic processes. In particular we prove the basic result (here called Gnedenko's Theorem) concerning the existence of just three types of non-degenerate limiting distributions in such cases, and give necessary and sufficient conditions for each to apply. The development relies, in part, on the corresponding known theory for stationary sequences.

The general theory given does not require finiteness of the number of upcrossings of any level x. However when the number per unit time is a.s. finite and has a finite mean $\mu(x)$, it is found that the classical criteria for domains of attraction apply when $\mu(x)$ is used in lieu of the tail of the marginal distribution function. The theory is specialized to this case and applied to give the general known results for stationary normal processes (for which $\mu(x)$ may or may not be finite).

A general Poisson convergence theorem is given for high level upcrossings, together with its implications for the asymptotic distributions of r^{th} largest local maxima.

Key Words and Phrases: extreme value theory, extreme values, stationary processes, stochastic extremes

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1. Introduction.

In this paper we shall be concerned primarily with asymptotic distributional properties of the maximum

$$M(T) = \sup\{\xi(t):\ 0 \le t \le T\}$$

of a continuous parameter stationary process $\{\xi(t):\ t \ge 0\}$. A great deal is known about such properties in the important special case when the process is normal (cf. [2], [16]). Our purpose here is to delineate the types of limiting behavior which are possible when the process is not necessarily normal, obtaining, in particular, versions of the central results of classical extreme value theory which apply in this context.

The classical theory is concerned with properties of the maximum $M_n = \max(\xi_1, \xi_2, \ldots \xi_n)$ of n i.i.d. random variables as n becomes large. Central to the theory is the result which asserts that if M_n has a non-degenerate limiting distribution (under linear normalizations), i.e. if $P\{a_n(M_n - b_n) \le x\} \to G(x)$ for sequences $\{a_n > 0\}, \{b_n\}$, then G must be one of only three general types:

Type I	$G(x) = \exp(-e^{-x})$	$-\infty < x < \infty$	
Type II	$G(x) = \exp(-x^{-\alpha})$	$x > 0$	$\alpha > 0$
Type III	$G(x) = \exp -(-x)^{\alpha}$	$x < 0$	

(linear transformations of the variable x being permitted). This result, which arose from work of Frechet [5] and Fisher and Tippett [4], was later given a complete form by Gnedenko [6] and is here referred to as "Gnedenko's Theorem."

Gnedenko also obtained necessary and sufficient conditions for the domains of attraction for each of the three limiting types. These and other versions obtained subsequently (cf. [7]) concern the rate of decay of the tail $1-F(x)$ of the distribution F of each ξ_n as x increases.

A further result--trivially proved in the classical case--is that for any sequence $\{u_n\}$, $\tau > 0$, $P\{M_n \leq u_n\} \to e^{-\tau}$ if and only if $1 - F(u_n) \sim \tau/n$. This is sometimes useful in calculation of the constants a_n, b_n in Gnedenko's Theorem (when $u_n = x/a_n + b_n$).

In more recent years there has been considerable interest in extending these and other results of the classical theory to apply to stationary sequences which exhibit a "decay of dependence" which is not too slow. In particular the early work of Watson [17] concerning convergence of $P\{M_n \leq u_n\}$ applied under m-dependence, Loynes [14] proved Gnedenko's Theorem under strong mixing assumptions, and Berman [1] obtained detailed results for normal sequences under a mild condition involving correlation decay. More recently we have obtained a theory (cf. [9]) involving weak "distributional mixing" conditions, which unifies these results and provides a rather satisfying extension of the classical distributional theory to include stationary sequences.

It is not too surprising that such an extension is possible for stationary sequences, at least under suitable dependence restrictions. What may seem surprising at first sight is that a corresponding theory is possible for continuous parameter stationary processes. However this becomes intuitively clear by recognizing that the maximum up to time n, say, is just the maximum of n random variables--the "submaxima" in the fixed intervals $(i-1, i)$, $1 \leq i \leq n$. Our procedure will be, in fact, to use the existing theory for stationary sequences by means of (a slightly modified version of) this precise approach. The sequence results which will be needed are stated in Section 2.

In Section 3 we will obtain Gnedenko's Theorem for continuous parameter stationary processes, showing under appropriate conditions that if

$$P\{a_T(M(T)-b_T) \leq x\} \to G(x) \text{ as } T \to \infty$$

for some constants $a_T>0, b_T$, then G must be one of the extreme value forms.

In Section 4 we obtain a related result--again extending a classical theorem--to give necessary and sufficient conditions for the convergence of $P\{M(T) \leq u_T\}$ for sequences not necessarily of the form $u_T = x/a_T + b_T$ implicit in Gnedenko's Theorem.

As a corollary of this result we obtain necessary and sufficient criteria for the domains of attraction occurring in Gnedenko's Theorem. In the classical i.i.d. sequence case, the criteria for domains of attraction involve the rate of decay of the marginal distribution $1-F(x)$ as x increases. For the present case the very same criteria apply, provided $1-F(x)$ is replaced by another function $\psi(x)$. For processes whose mean number $\mu(x)$ of upcrossings of any level x is finite, the function $\psi(x)$ is precisely $\mu(x)$, a readily calculated quantity.

The general theory will not require that the mean number of upcrossings of a level per unit time be finite, and accordingly will include the class of stationary Gaussian processes with covariances of the form $r(\tau) = 1 - C|\tau|^{\alpha} + o(|\tau|^{\alpha})$ as $\tau \to 0$ for $0 < \alpha < 2$. In Section 5 we consider such processes, as well as (possibly non-Gaussian) cases for which the mean number of upcrossings per unit time *is* finite. Finally in Section 6 we note the general Poisson limit for the point processes of upcrossings of increasingly high levels and its implications regarding limit theorems for the distribution of the r^{th} largest local maximum of $\xi(t)$ in $0 \le t \le T$.

2. Two results for stationary sequences.

As noted, our development of extremal theory for stationary processes will rely in part on the existing sequence theory. Specifically we shall require the following definitions and results (which may be found e.g. in [10]).

Let $\{\xi_n\}$ be a stationary sequence and write $F_{i_1 \ldots i_n}(x_1 \ldots x_n)$ for the joint distribution function of $\xi_{i_1} \ldots \xi_{i_n}$. For brevity write also $F_{i_1 \ldots i_n}(u)$ to denote $F_{i_1 \ldots i_n}(u, u \ldots u) = P\{\xi_{i_1} \le u \ldots \xi_{i_n} \le u\}$. If $\{u_n\}$ is a sequence of real constants, we say that *the sequence* $\{\xi_n\}$ *satisfies the (dependence) condition* $D(u_n)$ if for each n, $1 \le i_1 < i_2 \ldots < i_p < j_1 \ldots < j_{p'} \le n$, $j_1 - i_p \ge \ell$,

(2.1) $$|F_{i_1 \ldots i_p j_1 \ldots j_{p'}}(u_n) - F_{i_1 \ldots i_p}(u_n) F_{j_1 \ldots j_{p'}}(u_n)| \le \alpha_{n,\ell}$$

where

(2.2) $$\alpha_{n,\ell_n} \to 0 \text{ for some sequence } \ell_n = 0 \text{ as } n \to \infty .$$

Note that $\alpha_{n,\ell}$ can (and will) be taken to be decreasing in ℓ for each n by simply replacing it by the smallest value it can take to make (2.1) hold (i.e. the maximum value of the left-hand side of (2.1) over all allowable sets of integers $i_1 \ldots i_p$, $j_1 \ldots j_{p'}$. Note also that (2.2) may then be shown equivalent to the condition (cf. [12] for proof)

(2.3) $$\alpha_{n,n\lambda} \to 0 \text{ as } n \to \infty \text{ for each } \lambda > 0 .$$

The condition $D(u_n)$ indicates a degree of "approximate independence" of members of the sequence separated by increasing distances. However this condition, which we refer to as "distributional mixing," is clearly potentially far less restrictive than, for example, "strong mixing." In the case of normal sequences, it is in fact satisfied when the covariance sequence $\{r_n\}$ tends to zero even just fast enough so that $r_n \log n \to 0$.

The following result is basic to the sequence theory and will be required in later sections.

Lemma 2.1. *Let* $\{\xi_n\}$ *be a stationary sequence satisfying* $D\{u_n\}$ *for a given sequence* $\{u_n\}$ *of constants and write* $M_n = \max(\xi_1, \xi_2 \ldots \xi_n)$. *Then for any integer* $k \geq 1$ *(writing* [ ] *to denote integer part),*

$$P\{M_n \leq u_n\} - P^k\{M_{[n/k]} \leq u_n\} \to 0 \text{ as } n \to \infty .$$

This lemma indicates a degree of independence between the $[n/k]$ maxima when the first n integers are divided into k groups. We shall also need the sequence form of Gnedenko's Theorem, which is given (e.g. in [10]) as follows:

Theorem 2.2. *Let* $\{\xi_n\}$ *be a stationary sequence such that* $M_n = \max(\xi_1, \xi_2 \ldots \xi_n)$ *satisfies* $P\{a_n(M_n - b_n) \leq x\} \to G(x)$ *as* $n \to \infty$ *for some non-degenerate d.f.* G *and constants* $\{a_n > 0\}, \{b_n\}$. *Suppose that* $D(u_n)$ *holds for all* u_n *of the form* $x/a_n + b_n$, $-\infty < x < \infty$. *Then* G *is one of the three extreme value distributional types.*

The other classical result quoted--concerning convergence of $P\{M_n \leq u_n\}$ for arbitrary sequences $\{u_n\}$--is also important and holds under appropriate conditions for stationary sequences $\{\xi_n\}$. This will not be discussed here since the corresponding continuous parameter result will be independently derived.

3. Gnedenko's Theorem for stationary processes.

As indicated above, it will be convenient to relate the maximum $M(T)$ of the continuous parameter stationary process $\xi(t)$ to the maximum of n terms of a sequence of "submaxima." Specifically if $h > 0$ we write

$$\zeta_i = \sup\{\xi(t): (i-1)h \le t \le ih\} \tag{3.1}$$

so that for $n = 1,2,3\ldots$,

$$M(nh) = \max(\zeta_1, \zeta_2, \ldots \zeta_n) \ . \tag{3.2}$$

The following preliminary form of Gnedenko's Theorem (involving conditions on the ζ-sequence) is immediate.

Theorem 3.1. *Suppose that for some families of constants* $\{a_T > 0\}, \{b_T\}$ *we have*

$$P\{a_T(M(T)-b_T) \le x\} \to G(x) \text{ as } T \to \infty \tag{3.3}$$

for some non-degenerate G, *and that the* $\{\zeta_i\}$ *sequence defined by* (3.1) *satisfies* $D(u_n)$ *whenever* $u_n = x/a_{nh} + b_{nh}$ *for some fixed* $h > 0$ *and all real* x. *Then* G *is one of the three extreme value types.*

Proof. Since (3.3) holds in particular as $T \to \infty$ through values nh and the ζ_n-sequence is clearly stationary, the result follows by replacing ξ_n by ζ_n in Theorem 2.2 and using (3.2). □

Corollary 3.2. *The result holds in particular if the* $D(u_n)$ *conditions are replaced by the assumption that* $\{\xi(t)\}$ *is strongly mixing. For then the sequence* $\{\zeta_n\}$ *is strongly mixing and satisfies* $D(u_n)$. □

We now introduce the continuous analog of the condition $D(u_n)$, stated in terms of the finite dimensional distribution functions $F_{t_1 \ldots t_n}$ of $\xi(t)$ (again writing $F_{t_1 \ldots t_n}(u)$ for $F_{t_1 \ldots t_n}(u \ldots u)$.

The condition $D_c(u_T)$ *will be said to hold for the process* $\xi(t)$ *and the family of constants* $\{u_T: T > 0\}$, *with respect to a family* $\{q_T \to 0\}$, if for any points $s_1 < s_2 \ldots < s_p < t_1 \ldots < t_{p'}$ belonging to $(kq_T: 0 \le kq_T \le T)$ and satisfying $t_1 - s_p \ge \tau$, we have

$$(3.4) \qquad |F_{s_1 \ldots s_p t_1 \ldots t_{p'}}(u_T) - F_{s_1 \ldots s_p}(u_T) F_{t_1 \ldots t_{p'}}(u_T)| \le \alpha_{T,\gamma}$$

where $\alpha_{T,\gamma_T} \to 0$ for some sequence $\gamma_T = o(T)$ or, equivalently, where

$$(3.5) \qquad \alpha_{T,\lambda T} \to 0 \text{ as } T \to \infty$$

for each $\lambda > 0$.

The $D(u_n)$ condition for $\{\zeta_n\}$ required in Theorem 3.1 will now be related to $D_c(u_T)$ by approximating crossings and extremes of the continuous parameter process, by corresponding quantities for a sampled version. To achieve the approximation we require two conditions involving the maximum of $\xi(t)$ in fixed and in very small time intervals. These conditions are given here in a form which applies very generally--readily verifiable sufficient conditions for important cases are given in Section 5.

Specifically we suppose that there is a function $\psi(u)$ such that, for $h > 0$,

$$(3.6) \qquad \limsup_{u \to \infty} \frac{P\{M(h) > u\}}{h\psi(u)} \le 1 ,$$

and that for each $a > 0$, there is a family of constants $q = q_a(u) \to 0$ as $u \to \infty$ such that

$$\text{(3.7)} \qquad \limsup_{u\to\infty} \frac{P\{\xi(0)<u\,,\ \xi(q)<u\,,\ M(q)>u\}}{q\psi(u)} \to 0 \text{ as } a \to 0\ .$$

Note that Equation (3.6) specifies an asymptotic upper bound for the tail distribution of the maximum in a fixed interval, whereas (3.7) limits the probability that the maximum in a short interval exceeds u, but the process itself is less than u at both endpoints. The following result now enables us to approximate the maximum in an interval of length h by the maximum at discrete points in that interval.

Lemma 3.3.

(i) If (3.6) *holds, then* $P\{M(q)>u\} = o\psi(u)$ *as* $u \to \infty$ *for any* $q = q(u) \to 0$. *Also* $P\{\xi(0)>u\} = o\psi(u)$.

(ii) If (3.6) *and* (3.7) *both hold, and* I *is an interval of length* h, *then there are constants* λ_a *such that*

$$\text{(3.8)} \qquad 0 \le \limsup_{u\to\infty} [P\{\xi(jq)\le u\,,\ jq\in I\} - P\{M(I)\le u\}]/\psi(u) \le \lambda_a \to 0 \text{ as } a \to 0\ ,$$

where $q = q_a(u)$ *is as in* (3.7), *the convergence being uniform in all intervals of this fixed length* h.

Proof. If (3.6) holds and $q \to 0$ as $u \to \infty$, then for any fixed $h > 0$, q is eventually smaller than h and $P\{M(q)>u\} \le P\{M(h)>u\}$, so that

$$\limsup_{u\to\infty} P\{M(q)>u\}/\psi(u) \le \limsup_{u\to\infty} P\{M(h)>u\}/\psi(u) \le h \text{ by (3.6)}\ ,$$

from which it follows that $P\{M(q)>u\}/\psi(u) \to 0$, as stated. The remaining statement of *(i)* also follows since $P\{\xi(0)>u\} \le P\{M(q)>u\}$.

Suppose now that (3.6) and (3.7) both hold and that I is an interval of fixed length h. The interval I consists of no more than h/q subintervals of the form ((j-1)q , jq), together with (possibly) a shorter interval at each end. The difference in probabilities in (3.8) is clearly non-negative and (using stationarity) dominated by

$$\lambda_{a,u} = \frac{h}{q} P\{\xi(0)<u , \xi(q)<u , M(q)>u\} + 2P\{M(q)>u\} .$$

The desired result *(ii)* now follows from (3.7) and *(i)* by writing $\lambda_a = \limsup_{u\to\infty} \lambda_{a,u}$. □

It is now relatively straightforward to relate $D(u_n)$ for the sequence $\{\zeta_n\}$ to the condition $D_c(u_T)$ for the process $\xi(t)$, as the following lemma shows. (In this we use the (potentially ambiguous) notation $D(u_{nh})$ to mean $D(v_n)$ with $v_n = u_{nh}$.)

Lemma 3.4. *Suppose that* (3.6) *holds with some function* $\psi(u)$ *and let* $\{q_a(u)\}$ *be a family of constants for each* $a > 0$ *with* $q_a(u)>0$, $q_a(u)\to 0$ *as* $u \to \infty$, *and such that* (3.7) *holds. If* $D_c(u_T)$ *is satisfied with respect to the family* $q_T = q_a(u_T)$ *for each* $a > 0$, *and* $T\psi(u_T)$ *is bounded, then the sequence* $\{\zeta_n\}$ *defined by* (3.2) *satisfies* $D(u_{nh})$ *for* $h > 0$.

Proof. For a given n, let $i_1 < i_2 \ldots < i_p < j_1 \ldots < j_{p'} < n$, $j_1 - i_p \geq \ell$. Write $I_r = [(i_r-1)h , i_r h]$, $J_s = [(j_s-1)h , j_s h]$. For brevity let q denote one of the families $\{q_a(\cdot)\}$ and

$$A_q = \bigcap_{r=1}^{p} \{\xi(jq) \leq u_{nh} , jq \in I_r\} , \qquad A = \bigcap_{r=1}^{p} \{\zeta_{i_r} \leq u_{nh}\}$$

$$B_q = \bigcap_{s=1}^{p'} \{\xi(jq) \leq u_{nh} , jq \in J_s\} , \qquad B = \bigcap_{s=1}^{p'} \{\zeta_{j_s} \leq u_{nh}\} .$$

It follows in an obvious way from Lemma 3.3 that

$$0 \le \limsup_{n\to\infty}\{P(A_q \cap B_q) - P(A\cap B)\} \le \limsup_{n\to\infty}(p+p')\psi(u_{nh})\lambda_a$$

$$\le n\psi(u_{nh})\lambda_a \le K\lambda_a$$

for some constant K (since $nh\psi(u_{nh})$ is bounded) and where $\lambda_a \to 0$ as $a \to 0$. Similarly

$$\limsup|P(A_q)-P(A)| \le K\lambda_a \ , \ \ \limsup|P(B_q)-P(B)| \le K\lambda_a \ .$$

Now

$$|P(A\cap B) - P(A)P(B)| \le |P(A\cap B) - P(A_q\cap B_q)| + |P(A_q\cap B_q) - P(A_q)P(B_q)|$$

(3.9) $$+ P(A_q)|P(B_q) - P(B)| + P(B)|P(A_q) - P(A)|$$

$$= R_{n,a} + |P(A_q\cap B_q) - P(A_q)P(B_q)|$$

where $\limsup_{n\to\infty} R_{n,a} \le 3K\lambda_a$.

Since the largest jq in any I_r is at most $i_p h$, and the smallest in any J_s is at least $(j_1-1)h$, their difference is at least $(\ell-1)h$. Also the largest jq in $J_{p'}$ does not exceed $j_{p'}h \le nh$ so that from (3.4) and (3.9)

(3.10) $$|P(A\cap B) - P(A)P(B)| \le R_{n,a} + \alpha^{(a)}_{nh,(\ell-1)h}$$

(in which the dependence of $\alpha_{T,\ell}$ on a is explicitly indicated). Write now $\alpha^*_{n,\ell} = \inf_{a>0}\{R_{n,a} + \alpha^{(a)}_{nh,(\ell-1)h}\}$. Since the left-hand side of (3.9) does not depend on a we have

$$|P(A\cap B) - P(A)P(B)| \le \alpha^*_{n,\ell} \ ,$$

which is precisely the desired conclusion of the lemma, provided we can show that $\lim_{n\to\infty} \alpha^*_{n,\lambda n} = 0$ for any $\lambda > 0$ (cf. (2.3)). But for any a > 0

$$\alpha^*_{n,\lambda n} \leq R_{n,a} + \alpha^{(a)}_{nh,(\lambda n-1)h} \leq R_{n,a} + \alpha^{(a)}_{nh,\frac{1}{2}\lambda nh}$$

when n is sufficiently large (since $\alpha^{(a)}_{T,\ell}$ decreases in ℓ), and hence by (3.5)

$$\limsup_{n\to\infty} \alpha^*_{n,\lambda n} \leq 3K\lambda_a ,$$

and since a is arbitrary and $\lambda_a \to 0$ as $a \to 0$, it follows that $\alpha^*_{n,\lambda n} \to 0$ as desired. □

The general continuous version of Gnedenko's Theorem is now readily restated in terms of conditions on $\xi(t)$ itself.

Theorem 3.5. *With the above notation for the stationary process* $\xi(t)$ *satisfying* (3.6) *for some function* ψ, *suppose that, for some families of constants* $\{a_T>0\},\{b_T\}$,

$$P\{a_T(M(T)-b_T) \leq x\} \to G(x)$$

for a non-degenerate G. *Suppose that* $T\psi(u_T)$ *is bounded and* $D_c(u_T)$ *holds for* $u_T = x/a_T + b_T$ *for each real* x, *with respect to families of constants* $\{q_a(u)\}$ *satisfying* (3.7). *Then* G *is one of the three extreme value distributional types.*

Proof. This follows at once from Theorem 3.1 and Lemma 3.4. □

As noted the conditions of this theorem are of a general kind, and more specific sufficient conditions will be given in the applications in Section 5.

4. Convergence of $P\{M(T) \leq u_T\}$.

Gnedenko's Theorem involved consideration of $P\{a_T\{M(T)-b_T\} \leq x\}$, which may be rewritten as $P\{M(T) \leq u_T\}$ with $u_T = a_T^{-1} x + b_T$. We turn now to the question of convergence of $P\{M(T) \leq u_T\}$ as $T \to \infty$ for families u_T which are not necessarily linear functions of a parameter x. (This is analogous to the convergence of $P(M_n \leq u_n)$ for sequences, of course.) These results are of interest in their own right, but also since they make it possible to simply modify the classical criteria for domains of attraction to the three limiting distributions, to apply in this continuous parameter context.

The discussion will be carried out in terms of so-called "ε-upcrossings" of a level by the stationary process--a concept originally introduced by Pickands [15] to deal with extremes of processes whose sample functions were so irregular that the "ordinary" upcrossings could be infinite in number in a finite interval. (Here we make essential use of this concept whether the process is irregular or not.)

Briefly, if $\varepsilon > 0$, $\xi(t)$ is said to have an ε-upcrossing of u at a point t_0 if $\xi(t) \leq u$ for all t in the interval $(t_0-\varepsilon, t_0)$, but $\xi(t) > u$ for some point $t \in (t_0, t_0+\eta)$ for each $\eta > 0$. Since the interval $(t_0-\varepsilon, t_0)$ contains no upcrossings, the number of ε-upcrossings in a unit interval does not exceed $1/\varepsilon$. We write $N_{\varepsilon,u}(t)$, $N_{\varepsilon,u}(I)$ for the number of ε-upcrossings in the intervals $(0,t)$, I respectively and $\mu_{\varepsilon,u} = \mathcal{E}N_{\varepsilon,u}(1)$ so that $\mathcal{E}N_{\varepsilon,u}(t) = t\mu_{\varepsilon,u}$. The following small result indicates some connections between ε-upcrossings and maxima.

<u>Lemma 4.1.</u>

(i) For $h > 0$, $P\{M(h) > u\} \geq h\mu_{h,u}$.

(ii) $h\mu_{h,u} \geq P\{M(2h) > u\} - P\{M(h) > u\}$.

(iii) If (3.6) *holds (i.e.* $\limsup_{u\to\infty} P\{M(h)>u\}/(h\psi(u)) \leq 1$ *for some* $\psi(u)$, $h > 0$) *then* $\limsup_{u\to\infty} \mu_{h,u}/\psi(u) \leq 1$.

(iv) If

(4.1) $$P\{M(h)>u\} \sim h\psi(u) \text{ as } u \to \infty \text{ for } 0 < h < h_0$$

then $\mu_{\varepsilon,u} \sim \psi(u)$ *for all (sufficiently small)* $\varepsilon > 0$.

<u>Proof</u>. Since clearly $N_{h,u}(h)$ is either zero or one, we have

$$h\mu_{h,u} = EN_{h,u}(h) = P\{N_{h,u}(h)=1\} \leq P\{M(h)>u\}$$

so that *(i)* follows at once. To prove *(ii)* we note that

$$\begin{aligned} P\{M(2h)>u\} &\leq P\{M(h)>u\} + P\{N_{h,u}(h,2h) \geq 1\} \\ &= P\{M(h)>u\} + h\mu_{h,u} . \end{aligned}$$

If (3.6) holds *(iii)* follows at once from *(i)*.

Finally, if (4.1) holds, the conclusion of *(iv)* follows from *(iii)* and the inequality *(ii)*, which gives

$$\liminf_{u\to\infty} \mu_{\varepsilon,u}/\psi(u) \geq \lim_{u\to\infty}\left[\frac{P\{M(2\varepsilon)>u\}}{\varepsilon\psi(u)} - \frac{P\{M(\varepsilon)>u\}}{\varepsilon\psi(u)}\right] = 1 . \qquad \square$$

Our main purpose is to demonstrate the equivalence of the relations $P\{M(h)>u_T\} \sim \tau/T$ and $P\{M(T)\leq u_T\} \to e^{-\tau}$ under appropriate conditions. The following condition will be referred to as $D_c'(u_T)$, and is analogous to D' conditions required for similar purposes for sequences (cf. [10]).

If $\{u_T\}$ is a given family of constants the condition $D_c'(u_T)$ will be said to hold (for the process $\{\xi(t)\}$ satisfying (4.1)) if

$$\limsup_{T\to\infty} T|\mu_{\varepsilon T, u_T} - \psi(u_T)| \to 0 \text{ as } \varepsilon \to 0 \ . \tag{4.2}$$

We now state and prove the first part of the desired equivalence.

Theorem 4.2. *Suppose that* (4.1) *holds for some function* ψ *and let* $\{u_T\}$ *be a family of constants such that* $D_c(u_T)$ *holds (with respect to a family* (q) *satisfying* (3.7)*), and that* $D_c'(u_T)$ *holds. Then*

$$T\psi(u_T) \to \tau > 0 \tag{4.3}$$

implies

$$P\{M(T) \le u_T\} \to e^{-\tau} \ . \tag{4.4}$$

Proof. Let $0 < h < h_0$ (cf. (4.1)), and let n,k be integers, writing $n' = [n/k]$. By Lemma 3.4, the sequence of "submaxima" $\{\zeta_n\}$ defined by (3.1) satisfies $D(u_{nh})$ and hence, from Lemma 2.1,

$$P\{M(nh) \le u_{nh}\} - P^k\{M(n'h) \le u_{nh}\} \to 0 \text{ as } n \to \infty \ . \tag{4.5}$$

Now writing $u_{nh} = u$, Lemma 4.1 (*i*) gives

$$n'h\mu_{n'h,u} \le P\{M(n'h) > u\} \le n'P\{M(h) > u\} \sim n'h\psi(u)$$

so that

(4.6) $$1 - n'h\psi(u)(1+o(1)) \le P\{M(n'h) \le u\} \le 1 - n'h\mu_{n'h,u} \ .$$

But $n'h\psi(u) = nh\psi(u_{nh})/k \to \tau/k$ by (4.3). Further

$$n'h\mu_{n'h,u} = \frac{nh}{k}[\mu_{\frac{nh}{k},u_{nh}} - \psi(u_{nh})] + \frac{nh}{k}\psi(u_{nh})$$

so that letting $n \to \infty$ in (4.6) and using $D_c'(u_T)$ we have

$$1 - \tau/k \le \liminf_{n\to\infty} P\{M(n'h) \le u\} \le \limsup_{n\to\infty} P\{M(n'h) \le u\}$$
$$\le 1 - \tau/k + o(1/k) \ .$$

By taking k^{th} powers and using (4.5) we see that

$$(1-\tau/k)^k \le \liminf P\{M(nh) \le u\} \le \limsup P\{M(nh) \le u\} \le (1-\tau/k+o(1/k))^k$$

and hence, letting $k \to \infty$, that

(4.7) $$P\{M(nh) \le u_{nh}\} \to e^{-\tau} \ .$$

Now if n is chosen so that $nh \le T < (n+1)h$, and if $u_{nh} \le u_T$,

$$P\{M(nh) \le u_T\} = P\{M(nh) \le u_{nh}\} + P\{u_{nh} < M(nh) \le u_T\}$$

where the last term does not exceed

$$nP\{u_{nh} \le M(h) < u_T\} = n[P\{M(h) > u_{nh}\} - P\{M(h) > u_T\}]$$
$$= [\frac{\tau h}{nh}(1+o(1)) - \frac{h\tau}{T}(1+o(1))]$$

by (4.1) and (4.3). Since $nh \sim T$ this clearly tends to zero. A corresponding calculation where $u_{nh} > u_T$ thus shows from (4.7) that

$$P\{M(nh) \le u_T\} \to e^{-\tau}$$

with $T = [n/h]$. Finally

$$\begin{aligned} P\{M(T) \le u_T\} &\le P\{M(nh) \le u_T\} \\ &\le P\{M(T) \le u_T\} + P\{M(nh) \le u_T < M(T)\} . \end{aligned}$$

But the event $M(nh) \le u_T < M(T)$ implies that the maximum in the interval $[nh,(n+1)h]$ exceeds u_T, which has probability $P\{M(h) > u_T\}$, giving

$$P\{M(nh) \le u_T\} - P\{M(h) > u_T\} \le P\{M(T) \le u_T\} \le P\{M(nh) \le u_T\} ,$$

from which (4.4) follows since $P\{M(h) > u_T\} \sim h\tau/T \to 0$. □

In our treatment of the converse result it will be convenient to use the innocuous further assumption

(4.8) $$\psi(u_T) \sim \psi(u_{[T/h]h}) \text{ as } n \to \infty$$

(for some given $h > 0$). This assumption is possibly dispensable but certainly commonly holds (e.g. when $\psi(u_T) \sim (2 \log T)^{\frac{1}{2}}$ for stationary normal processes) and, of course, always holds if the function u_T is replaced by the step function $u_{[T/h]h}$, constant between consecutive points nh.

The first step of the derivation exhibits approximate independence of maxima in disjoint intervals.

Lemma 4.3. *Suppose that* (4.1) *holds for some* ψ *and let* $\{u_T\}$ *be a family of constants (satisfying* (4.8) *for some* $h > 0$*) such that* $D_c(u_T)$ *holds with respect to a family* (q) *satisfying* (3.7) *and such that* $T\psi(u_T)$ *is bounded. Let*

$$P\{M(T) \le u_T\} \to e^{-\tau} \tag{4.9}$$

for some $\tau > 0$. *Then for* $k = 1,2...$

$$P\{M(T/k) \le u_T\} \to e^{-\tau/k} \quad \text{as} \quad T \to \infty \ . \tag{4.10}$$

Proof. As in the previous result the assumptions surrounding $D_c(u_T)$ imply that

$$P\{M(nh) \le u\} - P^k\{M(n'h) \le u\} \to 0 \quad \text{as} \quad n \to \infty \tag{4.11}$$

where $n' = [n/k]$ and $u = u_{nh}$. Now if $n = [T/h]$ it is readily checked that $[n/k]h \le T/k < ([n/k]+1)h$, so that

$$P\{M(T/k) \le u\} \le P\{M(n'h) \le u\} \to e^{-\tau/k}$$

by (4.5) (which holds here by the same argument as in Theorem 4.2), and (4.9) with $T = nh$. But also

$$\begin{aligned} P\{M(T/k) \le u\} &\ge P\{M((n'+1)h) \le u\} \\ &= P\{M(n'h) \le u\} - P\{M(n'h) \le u < M((n'+1)h)\} \\ &\ge P\{M(n'h) \le u\} - P\{M(h) > u\} \ . \end{aligned}$$

The first term on the right tends to $e^{-\tau/k}$ and the second is asymptotically equivalent to $h\psi(u_{nh}) \sim n^{-1}[nh\psi(u_{nh})] \to 0$ since $T\psi(u_T)$ is bounded. Hence we have

$$\text{(4.12)} \qquad P\{M(T/k)\le u_{nh}\} \to e^{-\tau/k} \ .$$

But it follows simply that u_{nh} may be replaced by u_T in (4.12) to give the desired result since if, for example, $u_{nh} \le u_T$ we have

$$\begin{aligned} &0 \le P\{M(T/k)\le u_T\} - P\{M(T/k)\le u_{nh}\} \\ \text{(4.13)} \qquad &= P\{u_{nh} < M(T/k) \le u_T\} \\ &\le (n/k)P\{u_{nh} < M(h)\le u_T\} + P\{M(h)>u_{nh}\} \end{aligned}$$

since if the maximum in $(0\,,T/k)$ lies between u_{nh} and u_T this must also occur in one of the first n' intervals $((i-1)h\,, ih)$ or in $(n'h\,, T/k)$. The first term of (4.13) is readily seen to be

$$(nh/k)\,[\psi(u_{nh})(1+o(1)) - \psi(u_T)(1+o(1))] \ ,$$

which is easily seen to tend to zero by (4.8) since $nh\psi(u_{nh})$ is bounded. Boundedness of $nh\psi(u_{nh})$ also implies that the second term of (4.13) tends to zero. A corresponding calculation applies for $u_{nh} \ge u_T$ so that $P\{M(T/k)\le u_T\} - P\{M(T/k)\le u_{nh}\} \to 0$, giving the desired result. □

<u>Lemma 4.4</u>. *Under the same assumptions as in Lemma 4.3 we have*

$$\frac{1}{\varepsilon} \limsup_{T\to\infty} |P\{M(\varepsilon T)\le u_T\} - e^{-\varepsilon\tau}| \to 0 \ \textit{as} \ \varepsilon \to 0 \ .$$

<u>Proof</u>. Choose the integer k depending on $\varepsilon > 0$, so that $\frac{1}{k+1} \le \varepsilon < \frac{1}{k}$. Then

$$P\{M(T/k)\le u_T\} \le P\{M(\varepsilon T)\le u_T\} \le P\{M(T/(k+1))\le u_T\} \ .$$

By subtracting $e^{-\varepsilon\tau}$ from each term, and using the facts from Lemma 4.3 that $P\{M(T/k)\le u_T\}\to e^{-\tau/k}$, $P\{M(T/(k+1))\le u_T\}\to e^{-\tau/(k+1)}$, we see that

$$\limsup_{T\to\infty}|P\{M(\varepsilon T)\le u_T\} - e^{-\varepsilon\tau}| \le \max[|e^{-\tau/(k+1)} - e^{-\varepsilon\tau}| , |e^{-\tau/k} - e^{-\varepsilon\tau}|] .$$

But

$$\frac{1}{\varepsilon}|e^{-\tau/(k+1)} - e^{-\varepsilon\tau}| \le \frac{1}{\varepsilon}|1 - e^{-\tau(\varepsilon - 1/(k+1))}|$$

$$= \frac{1}{\varepsilon}[1 - e^{-\tau(\varepsilon - \varepsilon(1+o(1)))}] ,$$

which tends to zero as $\varepsilon \to 0$. Similarly $\frac{1}{\varepsilon}|e^{-\tau/k} - e^{-\varepsilon\tau}| \to 0$ so that the desired result follows. □

The next lemma gives a conclusion which is interesting in itself and from which the main result will follow immediately.

Lemma 4.5. *Again suppose that the conditions of Lemma 4.3 hold. Then*

(4.14) $$\limsup_{T\to\infty}|T\mu_{\varepsilon T,u_T} - \tau| \to 0 \text{ as } \varepsilon \to 0 .$$

Proof. By Lemma 4.1 (*ii*),

$$P\{M(2\varepsilon T)> u_T\} \le P\{M(\varepsilon T)>u_T\} + \varepsilon T\mu_{\varepsilon T,u_T}$$

so that

$$T\mu_{\varepsilon T,u_T} - \tau \ge \frac{1}{\varepsilon}[P\{M(\varepsilon T)\le u_T\} - e^{-\varepsilon\tau}]$$

$$+ \frac{1}{\varepsilon}[e^{-2\varepsilon\tau} - P\{M(2\varepsilon T)\le u_T\}] + \frac{1}{\varepsilon}[e^{-\varepsilon\tau} - e^{-2\varepsilon\tau} - \varepsilon\tau] ,$$

giving

$$\liminf_{T\to\infty}[T\mu_{\varepsilon T,u_T} - \tau] \ge \frac{1}{\varepsilon}[e^{-\varepsilon\tau} - e^{-2\varepsilon\tau} - \varepsilon\tau]$$

$$- \frac{1}{\varepsilon} \limsup_{T\to\infty}|P\{M(\varepsilon T)\le u_T\} - e^{-\varepsilon\tau}|$$

$$- \frac{1}{\varepsilon} \limsup_{T\to\infty}|P\{M(2\varepsilon T)\le u_T\} - e^{-2\varepsilon T}| .$$

The latter two terms tend to zero as $\varepsilon \to 0$ by Lemma 4.4, so that

$$\liminf_{T\to\infty}[T\mu_{\varepsilon T,u_T} - \tau] \ge a_\varepsilon \to 0 \text{ as } \varepsilon \to 0 .$$

Similarly from the inequality $\varepsilon T\mu_{\varepsilon T,u_T} \le P\{M(\varepsilon T) > u_T\}$ (Lemma (4.1) (*i*)) it follows that

$$\limsup_{T\to\infty}[T\mu_{\varepsilon T,u_T} - \tau] \le b_\varepsilon$$

where $b_\varepsilon \to 0$ as $\varepsilon \to 0$. Since if $\liminf \beta_n \ge \lambda$ and $\limsup \beta_n \le \Lambda$ it is easily shown that $\limsup|\beta_n| \le \max(|\lambda|,|\Lambda|)$, we have

$$\limsup_{T\to\infty}|T\mu_{\varepsilon T,u_T} - \tau| \le \max(|a_\varepsilon|,|b_\varepsilon|) ,$$

which tends to zero as $\varepsilon \to 0$, giving the conclusion of the lemma. □

It will be noted that (4.14) is very similar to the condition $D_c'(u_T)$ and follows as a conclusion from the assumption $P\{M(T)\le u_T\} \to e^{-\tau}$ under appropriate conditions. For this we do not require that $D_c'(u_T)$ hold. If we now do assume that $D_c'(u_T)$ holds we immediately obtain the main converse result.

Theorem 4.6. *Suppose that* (4.1) *holds for some* ψ *and let* $\{u_T\}$ *be a family of constants (satisfying* (4.8) *for some* $h > 0$*) such that* $D_c(u_T)$ *holds with respect to a family* (q) *satisfying* (3.7). *Suppose also that* $D_c'(u_T)$ *holds. Then* (4.4) *implies* (4.3).

Proof. Since $T\mu_{\varepsilon T, u_T} \leq T/(\varepsilon T) = 1/\varepsilon$ it is implicit in the assumption $D_c'(u_T)$ that $T\psi(u_T)$ is bounded. Thus the conditions of the previous lemma are satisfied if (4.4) holds and hence

$$\limsup_{T\to\infty} |T\mu_{\varepsilon T, u_T} - \tau| \to 0 \text{ as } \varepsilon \to 0 .$$

But $D_c'(u_T)$ requires that

$$\limsup_{T\to\infty} |T\mu_{\varepsilon T, u_T} - T\psi(u_T)| \to 0 \text{ as } \varepsilon \to 0 ,$$

from which it follows simply that $T\psi(u_T) \to \tau$, as required. □

Theorems 4.2 and 4.6 may be related to the corresponding results for i.i.d. sequences in the following way.

Theorem 4.7. *Let* $\{u_T\}$ *be a family of constants such that the conditions of Theorem 4.6 hold, let* $0 < \rho < 1$, *and let* h *be chosen as in* (4.1) *and* (4.8). *Then*

(4.15) $$P\{M(T) \leq u_T\} \to \rho \text{ as } T \to \infty$$

if and only if there is a sequence $\{\zeta_n\}$ *of i.i.d. random variables with common d.f.* F *satisfying* $1 - F(u) \sim h\psi(u)$ *as* $u \to \infty$ *and such that* $\hat{M}_n = \max(\zeta_1, \zeta_2 \ldots \zeta_n)$ *satisfies*

(4.16) $$P\{\hat{M}_n \leq u_{nh}\} \to \rho \ .$$

Proof. If there is an i.i.d. sequence $\{\zeta_n\}$ with common d.f. F such that (4.16) holds then (as noted in the introduction) we have $1 - F(u_{nk}) \sim \tau/n$, where $\rho = e^{-\tau}$. Since $1 - F(u) \sim h\psi(u)$ we have $\psi(u_{nh}) \sim \tau/nh$, from which (by (4.8)) $\psi(u_T) \sim \tau/T$. Hence Theorem 4.2 gives $P\{M(T) \leq u_T\} \to e^{-\tau}$ so that (4.15) holds.

Conversely if (4.15) holds it follows from Theorem 4.6 that $T\psi(u_T) \to \tau$ and hence $nh\psi(u_{nh}) \to \tau$. Let $\{\zeta_n\}$ be i.i.d. random variables with the same d.f. F, say, as M(h), so that by (4.1)

$$1 - F(u_{nh}) \sim h\psi(u_{nh}) \sim \tau/n \ ,$$

from which it follows that $\hat{M}_n = \max(\zeta_1, \zeta_2 \ldots \zeta_n)$ satisfies $P\{M_n \leq u_n\} \to e^{-\tau} = \rho$, as required. □

These results show how the function ψ may be used in the classical criteria for domains of attraction to determine the asymptotic distribution of M(T). We write $\mathcal{D}(G)$ for the domain of attraction to the (extreme value) d.f. G, i.e. the set of all d.f.'s F such that $F^n(x/a_n + b_n) \to G(x)$ for some sequences $\{a_n > 0\}, b_n$.

Theorem 4.8. *Suppose that the conditions of Theorem 4.6 hold for all families* $u_T = x/a_T + b_T$, $-\infty < x < \infty$, *when* $\{a_T > 0\}, \{b_T\}$ *are given constants and*

(4.17) $$P\{a_T(M(T) - b_T) \leq x\} \to G(x) \ .$$

Then

(4.18) $\quad \psi(u) \sim 1 - F(u)$ *as* $u \to \infty$ *for some* $F \in \mathcal{D}(G)$.

Conversely if (4.1) *holds and* $\psi(u)$ *satisfies* (4.18) *there are families of constants* $\{a_T>0\},\{b_T\}$ *such that* (4.17) *holds, provided that the conditions of Theorem 4.6 are satisfied for each* $u_T = x/a_T + b_T$, $-\infty < x < \infty$.

Proof. If (4.17) holds, together with the conditions stated, Theorem 4.7 shows that

$$P\{a_{nh}(\hat{M}_n - b_{nh}) \le x\} \to G(x)$$

where $\hat{M}_n$ is the maximum of n i.i.d. random variables with a common d.f. F_0, say, and where $h\psi(u) \sim 1 - F_0(u)$ as $u \to \infty$, and $F_0 \in \mathcal{D}(G)$. We may choose a d.f. F such that $1 - F(u) = \frac{1}{h}(1-F_0(u))$ when u is large and the classical domain of attraction criteria show that $F \in \mathcal{D}(G)$. But $\psi(u) \sim 1 - F(u)$ as desired, showing (4.18).

Conversely if (4.18) holds and $h > 0$ we may choose $F_0 \in \mathcal{D}(G)$ such that $h\psi(u) \sim 1 - F_0(u)$ and hence define an i.i.d. sequence $\{\zeta_n\}$ with common d.f. F_0, $\hat{M}_n = \max(\zeta_1,\zeta_2 \ldots \zeta_n)$, such that

$$P\{a_n'(\hat{M}_n - b_n') \le x\} \to G(x)$$

for some constants $a_n'>0$, b_n'. Define $a_T = a_n'$, $b_T = b_n'$ for $nh \le T < (n+1)h$, $n = 0,1,2 \ldots$. Then (4.16) holds with $\rho = G(x)$. If the conditions of Theorem 4.6 hold for each $u_T = x/a_T + b_T$ then (4.15) holds, which yields (4.17). □

5. Particular classes of processes.

In this section we first show how the conditions required for the previous theory may be simplified when the mean number $\mu(u)$ of upcrossings of each level u by $\xi(t)$ per unit time is finite, and then briefly indicate applications to stationary normal processes (whether or not $\mu(u) < \infty$). Throughout $N_u(I)$ $(N_u(t))$ will denote the number of upcrossings of the level u in the interval I (or in $(0,t)$ respectively).

First we write for $q > 0$

$$I_q(u) = P\{\xi(0) < u < \xi(q)\}/q \; . \tag{5.1}$$

Clearly $I_q(u) \le P\{N_u(q) \ge 1\}/q \le EN_u(q)/q = \mu$. Further, it is readily shown (by a standard dissection of the unit interval into subintervals of length q) that

$$\mu(u) = \lim_{q \to 0} I_q(u) \; , \tag{5.2}$$

which, for now, we assume finite for each u. It is apparent from (5.2) that $\mu(u)$ may, at least in principle, be readily calculated from the bivariate distributions of the process. It may also happen (as for many normal processes) that $I_q(u) \sim \mu(u)$ as $u \to \infty$ when q depends on u, $q = q(u) \to 0$. For greater flexibility we shall use the following variant of such a property. Specifically we shall assume, when needed, that for each $a > 0$ there is a family $\{q_a(u) \to 0 \text{ as } u \to \infty\}$ such that (with $q_a = q_a(u)$, $\mu = \mu(u)$)

$$\liminf_{u \to \infty} I_{q_a}(u)/\mu \ge \nu_a \tag{5.3}$$

where $\nu_a \to 1$ as $a \to 0$. As indicated below, for many normal processes we may take $q_a(u) = a/u$ and more generally as $aP\{\xi(0) > u\}/\mu(u)$.

We shall assume as needed that

$$P\{\xi(0)>u\} = o\mu(u) \quad \text{as} \quad u \to \infty \,, \tag{5.4}$$

which holds under general conditions. For example, it is readily verified if for some $q = q(u) \to 0$ as $u \to \infty$,

$$\limsup_{u\to\infty} \frac{P\{\xi(0)>u\,,\ \xi(q)>u\}}{P\{\xi(0)>u\}} < 1 \tag{5.5}$$

since (5.5) implies that $\liminf_{u\to\infty} qI_q(u)/P\{\xi(0)>u\} > 0$, from which it follows that $P\{\xi(0)>u\}/I_q(u) \to 0$, and hence (5.4) holds since $I_q(u) \le \mu(u)$.

We may now recast the conditions (3.6) and (3.7) in terms of the function $\mu(u)$.

Lemma 5.1.

(i) Suppose $\mu(u) < \infty$ *for each* u *and that* (5.4) *(or the sufficient condition* (5.5)*) holds. Then* (3.6) *holds with* $\psi(u) = \mu(u)$.

(ii) If (5.3) *holds (for some family* $\{q_a(u)\}$*) then* (3.7) *holds with* $\psi(u) = \mu(u)$.

Proof. Since clearly

$$P\{M(h)>u\} \le P\{N_u(h)\ge 1\} + P\{\xi(0)>u\} \le \mu h + P\{\xi(0)>u\} \,,$$

(3.6) follows at once from (5.4), which proves *(i)*.

Now if (5.3) holds, then with $q = q_a(u)$, $\mu=\mu(u)$,

$$\begin{aligned} P\{\xi(0)<u\,,\ \xi(q)<u\,,\ M(q)>u\} &= P\{\xi(0)<u\,,\ M(q)>u\} - P\{\xi(0)<u<\xi(q)\} \\ &\le P\{N_u(q)\ge 1\} - qI_q(u) \\ &\le \mu q - \mu q \nu_a(1+o(1)) \end{aligned}$$

so that

$$\limsup_{u\to\infty} P\{\xi(0)<u\ ,\ \xi(q)<u\ ,\ M(q)>u\}/(q\mu)\ \le\ 1-\nu_a\ ,$$

which tends to zero as $a \to 0$, giving (3.7). □

In view of this lemma, Gnedenko's Theorem now applies to processes of this kind using the more readily verifiable conditions (5.3) and (5.4), as follows.

Theorem 5.2. *Theorem 3.5 holds for a stationary process* $\xi(t)$ *with* $\psi(u) = \mu(u) < \infty$ *for each* u *if the conditions* (3.6) *and* (3.7) *are replaced by* (5.4) *and* (5.3). □

Finally, the condition $D_c'(u_T)$ may, in certain circumstances, be replaced by a sufficient condition involving the second moment of $N_u(1)$ when this is finite. This condition is not necessarily simpler to verify, but the second moment involved may usually be obtained in terms of (integrals containing) the joint densities of the process and its derivative at two general points t_1, t_2.

Lemma 5.3. *Suppose that for the stationary process* $\xi(t)$, $EN_n^2(1) < \infty$, *and for a given family* $\{u\} = \{u_T\}$,

$$\text{(5.6)} \qquad \frac{1}{\varepsilon}\limsup_{T\to\infty} EN_u(\varepsilon T)(N_u(\varepsilon T)-1) \to 0 \quad as \quad \varepsilon \to 0\ .$$

Then $\xi(t)$ *satisfies* $D_c'(u_T)$, *with* $\psi(u) = \mu(u) = EN_u(1)$.

Proof. Clearly, writing $\mu = \mu(u_T)$,

$$0 \le T(\mu-\mu_{\varepsilon T,u}) = \frac{1}{\varepsilon} E(N_u(\varepsilon T)-N_{\varepsilon T,u}(\varepsilon T)) .$$

Now if $N_u(\varepsilon T) > 1$, $N_u(\varepsilon T) - N_{\varepsilon T,u}(\varepsilon T) \le N_u(\varepsilon T)(N_u(\varepsilon T)-1)$. Also $N_u(\varepsilon T) - N_{\varepsilon T,u}(\varepsilon T)$ is zero if $N_u(\varepsilon T) = 0$ and is zero or 1 if $N_u(\varepsilon T) = 1$, the latter case requiring that $N_u(-\varepsilon T,0) \ge 1$ also. Hence we have

$$0 \le E\{N_u(\varepsilon T) - N_{\varepsilon T,u}(\varepsilon T)\} \le EN_u(\varepsilon T)(N_u(\varepsilon T)-1) + P\{N(2\varepsilon T)>1\}$$

$$\le EN_u(\varepsilon T)(N_u(\varepsilon T)-1) + EN_u(2\varepsilon T)(N_u(2\varepsilon T)-1)$$

so that $D_c(u_T)$ follows by applying (5.6) twice (once with 2ε replacing ε). □

For stationary normal processes, finiteness of $EN_u^2(1)$ requires a little more than existence of the second spectral moment used to ensure finiteness of μ (cf. [3]). We turn now to the consideration of stationary normal processes, but will not restrict attention to those for which even $\mu = EN_u(1)$ is finite. Specifically we assume that $\xi(t)$ is a (zero mean) stationary normal process with covariance function

$$r(\tau) = 1 - C|\tau|^\alpha + o|\tau|^\alpha \text{ as } \tau \to 0 \tag{5.7}$$

for some α, $0 < \alpha \le 2$. (The case $\alpha = 2$ gives $\mu < \infty$.) There is a considerable literature dealing with extremal properties of such processes, and of slightly more general cases (which could be included here) in which the term $|\tau|^\alpha$ is multiplied by a slowly varying function as $\tau \to 0$ (cf. [2], [16]). Of course a number of the same arguments (which in some cases are rather intricate) used in these papers are required to verify our general conditions here. We will not attempt to reproduce these arguments but rather to simply indicate the basic considerations used and where they may be found. However it will be convenient to summarize these results as a theorem even though formal proofs are not given.

Theorem 5.4. *Let* $\xi(t)$ *be a zero mean stationary normal process with covariance function* $r(t)$ *satisfying* (5.7). *Then*

(i) (3.6), *and in fact* (4.1), *hold with* $\psi(u) = C^{1/\alpha} H_\alpha u^{2/\alpha} \phi(u)/u$, *in which* ϕ *is the standard normal density* C *is as in* (5.7), *and* H_α *is a constant depending only on* α.

(ii) (3.7) *holds with* $q_a(u) = au^{-2/\alpha}$.

(iii) $D_c(u_T)$ *holds with respect to a family* $\{q\}$ *if* $T\psi(u_T)$ *is bounded and*

$$\frac{T}{q} \sum_{\lambda T \le kq \le T} |r(kq)| e^{-u_T^2/(1+|r(kq)|)} \to 0 \quad as \quad T \to \infty \tag{5.8}$$

for each $\lambda > 0$. *This holds, in particular, if* $T\psi(u_T)$ *is bounded (with* ψ *defined as in (i)) and* $r(t)\log t \to 0$ *as* $t \to \infty$.

(iv) If $r(t)\log t \to 0$ *and* $T\psi(u_T) \to \tau > 0$, *then* $D_c'(u_T)$ *holds and* $P\{M(T) \le u_T\} \to e^{-\tau}$.

(v) If $r(t)\log t \to 0$, $M(T)$ *has the limiting distribution given by*

$$P\{a_T(M(T)-b_T) \le x\} \to e^{-e^{-x}}$$

where

$$a_T = (2\log T)^{\frac{1}{2}}$$

$$b_T = (2\log T)^{\frac{1}{2}} + (2\log T)^{-\frac{1}{2}} \{(\frac{1}{\alpha} - \frac{1}{2})\log\log T + \log(2^{\alpha^{-1}-1} \pi^{-\frac{1}{2}} C^{1/\alpha} H_\alpha)\}.$$

Indications and sources of proof.

(i) A derivation of (4.1) (from which (3.6) follows) appears in several developments of the normal theory (e.g. Theorem 2.1 of [16]). In the case $\alpha = 2$, (3.6) is incidentally simply obtained from "Rice's formula" $\mu = [(-r''(o))^{\frac{1}{2}}/2\pi]e^{-u^2/2}$.

(ii) This may be shown, for example, along the lines of Lemma 2.4 of [16], although a more direct derivation is obtainable from the normal theory given by Lindgren and Rootzen in [13].

(iii) The proof of this involves a standard calculation using "Slepian's Lemma" (cf. Lemma 3.5 of [15]), from which it follows that for two sets of standard normal random variables $\xi_1 \ldots \xi_n$, $\eta_1 \ldots \eta_n$ with covariance matrices $[\lambda_{ij}], [\nu_{ij}]$, $|\lambda_{ij}| \ge |\nu_{ij}|$

$$|P\{\bigcap_{j=1}^{n} (\xi_j \le u)\} - P\{\bigcap_{j=1}^{n} (\eta_j \le u)\}| \le K \sum_{i<j} |\lambda_{ij} - \nu_{ij}| (1-\lambda_{ij}^2)^{-\frac{1}{2}} e^{-u^2/(1+|\lambda_{ij}|)} .$$

In this application (using the notation of (3.4)), the ξ_i are identified with the r.v.'s $\xi(s_1) \ldots \xi(s_p), \xi(t_1) \ldots \xi(t_{p'})$ and the η_i with $p+p'$ standard normal r.v.'s having the same correlations except that $\text{cov}(\xi(s_i), \xi(t_j))$ is replaced by zero for $1 \le i \le p$, $1 \le j \le p'$.

The fact that boundedness of $T\psi(u_T)$ together with $r(t) \log t \to 0$ implies (5.8) follows by standard calculations (cf. [1] or Lemma 3.1 of [13]).

(iv) If $r(t) \log t \to 0$ and $T\psi(u_T) \to \tau > 0$ then $D_c'(u_T)$ may be obtained from arguments leading to Theorem 3.1 of [15], though it seems likely that a shorter route via our Lemma 3.3 may be possible. It then follows from Theorem 4.2 that $P\{M(T) \le u_T\} \to e^{-\tau}$. Of course any proof (of which there are several) that $r(t) \log t \to 0$ and $T\psi(u_T) \to \tau$ implies that

$P\{M(T) \le u_T\} \to e^{-\tau}$ must also imply that $D_c'(u_T)$ holds by virtue of our Lemma 4.5. That is $D_c'(u_T)$ may be regarded as a *necessary* condition for (4.3) to imply (4.4).

(v) This follows at once from the (relatively) straightforward verification of the fact that $T\psi(u_T) \to \tau = e^{-x}$ when $u_T = x/a_T + b_T$, using the above results. □

6. Poisson and related properties.

In this section we shall just briefly indicate the Poisson properties associated with high level upcrossings. We confine the discussion to the case where the number $N_u(I)$ of upcrossings in a bounded interval I has a finite mean, writing again $\mu = \mu(u) = EN_u(1)$. Cases where this is not so are similarly dealt with in terms of ε-upcrossings.

Our objective is to show, under D_c and D_c' conditions, that the point process of upcrossings of a high level takes on a Poisson character--as is well-known in the case when the stationary process $\xi(t)$ is normal. Since the upcrossings of increasingly high levels will tend to become rare, a normalization is required. To that end we consider a time period T and a level u_T, both increasing in such a way that $T\mu \to \tau$, $(\mu = \mu(u_T))$, and define a normalized point process of upcrossings by

$$N_T^*(I) = N_{u_T}(TI)\ ,\ (N_T^*(t) = N_{u_T}(tT))$$

for each interval (or more general Borel set) I, so that, in particular,

$$EN_T^*(1) = EN_T(T) = \mu T \to \tau\ . \tag{6.1}$$

This shows that the "intensity" (i.e. mean number of events per unit time) of the (normalized) upcrossing point process converges to τ. Our task is to show that the upcrossing point process actually converges (weakly) to a Poisson process with mean τ.

The derivation of this result is based on the following two extensions of Theorem 4.2, which are proved by similar arguments to those used in obtaining Theorem 4.2.

Theorem 6.1. *Under the conditions of Theorem 4.2, if* $\theta < 1$ *and* $\mu T \to \tau$, *then*

$$P\{M(\theta T) \le u_T\} \to e^{-\theta\tau} \text{ as } T \to \infty . \tag{6.2}$$ □

Theorem 6.2. *If* $I_1, I_2 \ldots I_k$ *are disjoint subintervals of* $[0,1]$ *and* $I_j^* = TI_j = \{t : t/T \in I\}$ *then under the conditions of Theorem 4.2, if* $\mu T \to \tau$,

$$P\{\bigcap_{j=1}^{k} M(I_j^*) \le u_T\} - \prod_{j=1}^{k} P\{M(I_j^*) \le u_T\} \to 0 , \tag{6.3}$$

so that by Theorem 6.1

$$P\{\bigcap_{j=1}^{k} (M(I_j^*) \le u_T)\} \to e^{-\tau\Sigma\theta_i} , \tag{6.4}$$

where θ_j *is the length of* I_j, $1 \le j \le k$. □

It is now a relatively straightforward matter to show that the point processes N_T^* converge (in the full sense of weak convergence) to a Poisson process N with intensity τ.

Theorem 6.3. *Under the conditions of Theorem 4.2, if* $T\mu \to \tau$ *where* $\mu = \mu(u_T)$, *then the family* N_T^* *of (normalized) point processes of upcrossings of* u_T *on the unit interval converges in distribution to a Poisson process* N *with intensity* τ *on the unit interval as* $T \to \infty$.

Proof. By Theorem 4.7 of [8] it is sufficient to prove that

(i) $\mathcal{E}N_T^*\{(a,b]\} \to \mathcal{E}N\{(a,b]\} = \tau(b-a)$ as $T \to \infty$ for all a,b, $0 \le a \le b \le 1$.

(ii) $P\{N_T^*(B)=0\} \to P\{N(B)=0\}$ as $T \to \infty$ for all sets B of the form $\bigcup_1^n B_i$ where n is any integer and B_i are disjoint intervals $(a_i, b_i] \subset (0,1]$.

Now (i) follows trivially since

$$\mathcal{E}N_T^*\{(a,b]\} = \mu T(b-a) \to \tau(b-a) \quad .$$

To obtain (ii) we note that

$$0 \le P\{N_T^*(B)=0\} - P\{M(TB) \le u_T\}$$
$$= P\{N_u(TB)=0 \,,\, M(TB) > u_T\}$$
$$\le \sum_{i=1}^{n} P\{\xi(Ta_i) > u_T\}$$

since if the maximum in $TB = \bigcup_{i=1}^{n} (Ta_i, Tb_i]$ exceeds u_T, but there are no upcrossings of u_T in these intervals, then ξ must exceed u at the initial point of at least one such interval. But the last expression is just $nP\{\xi(0) > u_T\} \to 0$ as $T \to \infty$. Hence

$$P\{N_T^*(B)=0\} - P\{M(TB)\le u_T\} \to 0 .$$

But $P\{M(TB)\le u_T\} = P\{\bigcap_{i=1}^{n} (M(TB_i)\le u_T\} \to e^{-\tau\Sigma(b_i - a_i)}$ by Theorem 6.2 so that (ii) follows since $P\{N(B)=0\} = e^{-\tau\Sigma(b_i - a_i)}$. □

Corollary. *If* B_i *are disjoint (Borel) subsets of the unit interval and if the boundary of each* B_i *has zero Lebesgue measure then*

$$P\{N_T^*(B_i)=r_i , 1\le i\le n\} \to \prod_{i=1}^{n} e^{-\tau m(B_i)} \frac{[\tau m(B_i)]^{r_i}}{r_i!}$$

where $m(B_i)$ *denotes the Lebesgue measure of* B_i.

Proof. This is an immediate consequence of the full weak convergence proved (cf. Lemma 4.4 of [8]). □

The above results concern convergence of the point processes of upcrossings of u_T in the unit interval to a Poisson process in the unit interval. A slight modification (requiring D_c and D_c' to hold for all families $u_{\theta T}$ in place of u_T for all $\theta > 0$) enables a corresponding result to be shown for the upcrossings on the whole positive real line, but we do not pursue this here. Instead we show how Theorem 6.3 yields the asymptotic distribution of the r^{th} largest local maximum in $(0,T)$.

Suppose, then, that $\xi(t)$ has a continuous derivative a.s. and define $N_u'(T)$ to be the number of local maxima in the interval $(0,T)$ for which the process value exceeds u, i.e. the number of downcrossing points t of zero by ξ' in $(0,T)$ such that $\xi(t) > u$. Clearly $N_u'(T) \ge N_u(T) - 1$ since at least one local maximum occurs between two upcrossings. It is also reasonable to expect that if the sample function behavior is not too irregular that there will tend to be just one local maximum between most successive upcrossings of u when u is large, so that $N'_u(T)$ and $N_u(T)$ will tend to be approximately equal. The following result makes this precise.

Theorem 6.4. *With the above notation let* $\{u_T\}$ *be constants such that* $T\mu(=T\mu(u_T)) \to \tau > 0$. *Suppose that* $EN'_u(1)$ *is finite for each* u *and that* $EN'_u(1) \sim \mu(u)$ *as* $u \to \infty$. *Then, writing* $u_T = u$, $E|N'_u(T) - N_u(T)| \to 0$.

If also the conditions of Theorem 6.3 hold (so that $P\{N_u(T)=r\} \to e^{-\tau}\tau^r/r!$) it follows that $P\{N'_u(T)=r\} \to e^{-\tau}\tau^r/r!$.

Proof. As noted above, $N'_u(T) \geq N_u(T) - 1$, and it is clear, moreover, that if $N'_u(T) = N_u(T) - 1$, then $\xi(T) > u$. Hence

$$E|N'_u(T) - N_u(T)| = E\{N'_u(T) - N_u(T)\} + 2P\{N'_u(T) = N_u(T) - 1\}$$
$$\leq TEN'_u(1) - \mu T + 2P\{\xi(T) > u\},$$

which tends to zero as $T \to \infty$ since $P\{\xi(T) > u_T\} = P\{\xi(0) > u_T\} \to 0$ and $TEN'_{u_T}(1) - \mu T = \mu T[(1+o(1)) - 1] \to 0$, so that the first part of the theorem follows. The second part now follows immediately since the integer-valued r.v. $N'_u(T) - N_u(T)$ tends to zero in probability, giving $P\{N'_u(T) \neq N_u(T)\} \to 0$ and hence $P\{N'_u(T)=r\} - P\{N_u(T)=r\} \to 0$ for each r. □

Now write $M^{(r)}(T)$ for the r^{th} largest local maximum in the interval $(0,T)$. Since the events $\{M^{(r)}(T) \leq u\}, \{N'_u(T) < r\}$ are identical we obtain the following corollary:

Corollary 1. *Under the conditions of the theorem*

$$P\{M^{(r)}(T) \le u_T\} \to e^{-\tau} \sum_{s=1}^{r-1} \tau^s/s! \ . \qquad \square$$

As a further corollary we obtain the limiting distribution of $M^{(r)}(T)$ in terms of that for $M(T)$.

Corollary 2. *Suppose that* $P\{a_T(M(T)-b_T) \le x\} \to G(x)$ *and that the conditions of Theorem 4.6 hold with* $u_T = x/a_T + b_T$ *for each real* x *(and* $\psi = \mu$*). Suppose also that* $EN_u'(1) \sim EN_u(1)$ *as* $u \to \infty$. *Then*

$$(6.5) \qquad P\{a_T(M^{(r)}(T)-b_T) \le x\} \to G(x) \sum_{s=0}^{r-1} [-\log G(x)]^s/s! \ ,$$

where $G(x) > 0$ *(and zero if* $G(x) = 0$*).*

Proof. This follows from Corollary 1 by writing $G(x) = e^{-\tau}$ since Theorem 4.6 implies that $T\mu \to \tau$. $\square$

Note that for a stationary *normal* process with finite second and fourth spectral moments λ_2, λ_4 it may be shown (Section 11.6 of [3]) that

$$EN_u'(1) = \mu\Phi(u\lambda_2/\Delta) + (\lambda_4/\lambda_2)^{\frac{1}{2}} [1 - \Phi\{u(\lambda_4/\Delta)^{\frac{1}{2}}\}$$

where $\Delta = \lambda_4 - \lambda_2^2$ and Φ is the standard normal d.f., so that clearly $EN_u'(1) \sim \mu$ as $u \to \infty$.

The relation (6.5) gives the asymptotic distribution of the r^{th} largest local maximum $M^{(r)}(T)$ as a corollary of the Poisson result, Theorem 6.4. This Poisson result may itself be generalized to apply to joint convergence of upcrossings of several levels to a point process in the plane composed of successive "thinnings" of a Poisson process. From a result of this kind it is possible to obtain the joint asymptotic distribution of any number of the $M^{(r)}(T)$, and also of their time locations.

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In this paper the central distributional results of classical extreme value theory are obtained, under appropriate dependence restrictions, for maxima of continuous parameter stochastic processes. In particular we prove the basic result (here called Gnedenko's Theorem) concerning the existence of just three types of non-degenerate limiting distributions in such cases, and give necessary and sufficient conditions for each to apply. The development relies, in part, on the corresponding known theory for stationary sequences.

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20. The general theory given does not require finiteness of the number of upcrossings of any level x. However when the number per unit time is a.s. finite and has a finite mean $\mu(x)$, it is found that the classical criteria for domains of attraction apply when $\mu(x)$ is used in lieu of the tail of the marginal distribution function. The theory is specialized to this case and applied to give the general known results for stationary normal processes (for which $\mu(x)$ may or may not be finite).

A general Poisson convergence theorem is given for high level upcrossings, together with its implications for the asymptotic distributions of r-th local maxima.