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EXTREME VALUES IN SAMPLES FROM m-DEPENDENT STATIONARY STOCHASTIC PROCESSES

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Summary. The limiting distributions for the order statistics of n successive observations in a sequence of independent and identically distributed random variables are shown to hold also when the sequence is generated by a stationary stochastic process of a certain moving average type.

A sequence of random variables $\{x_i\}$ has been called m-dependent [3] if $|i - j| > m$ implies that x_i and x_j are independent. If the variables in a strictly stationary sequence are m-dependent and have a finite upper bound to their range of variation, the largest in a sample of n successive members tends with probability one to this upper bound. This is a simple extension of Dodd's results [1] for the case of independence.

The following theorem shows that when this upper bound is infinite, the asymptotic distribution of the largest in such a sample is the same as in the case of independence.

Theorem. *Let $\{x_i\}$ be a sequence of random variables, unbounded above and generated by an m-dependent strictly stationary stochastic process with the property that*

$$(1) \qquad \lim_{c\to\infty} \frac{1}{P(x_i > c)} \max_{|i-j|\leq m} P[(x_i > c), (x_j > c)] = 0.$$

Then, if $\xi = n\,P[x_i > c_n(\xi)]$, for ξ fixed,

$$\lim_{n\to\infty} P[x_i \leq c_n(\xi);\, i = 1, \cdots, n] = e^{-\xi}$$

Proof. Using the formula for the probabilities of the joint occurrence of a set of events in terms of probabilities of occurrence of their contraries (Feller [2],

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p. 61), we have, for any even integer $l \leqq n$ and for $i = 1, \cdots, n$, that $P[x_i \leqq c_n(\xi)]$ is bounded below and above, respectively, by

$$1 - \sum P(x_i > c) + \cdots + (-1)^{l-1} \sum P[(x_{i_1} > c), \cdots, (x_{i_{l-1}} > c)],$$
$$1 - \sum P(x_i > c) + \cdots + (-1)^{l} \sum P[(x_{i_1} > c), \cdots, (x_{i_l} > c)],$$

where, for brevity, $c = c_n(\xi)$. Clearly, $\sum P(x_i > c) = nP(x_i > c) = \xi$.

Now

$$\sum P[(x_i > c), (x_j > c)]$$
$$= \sum_{i=1}^{m} (n - i) P[(x_1 > c), (x_{i+1} > c)] + P(x_i > c)^2 \tfrac{1}{2}(n - m - 1)(n - m).$$

But

$$\sum_{i=1}^{m} (n - i) P[(x_1 > c), (x_{i+1} > c)]$$
$$\leqq mn \left[1 - \frac{(m+1)}{2n}\right] \max_{|i-j| \leqq m} P[(x_i > c), (x_j > c)]$$
$$= m\xi \left[1 - \frac{m+1}{2n}\right] \max_{|i-j| \leqq m} \frac{P[(x_i > c), (x_j > c)]}{P(x_i > c)}.$$

Since, as $n \to \infty$, with ξ fixed, $c = c_n(\xi) \to \infty$, condition (1) shows that the last expression tends to zero. Hence

$$\lim_{n \to \infty} \sum P[(x_i > c), (x_j > c) = \tfrac{1}{2}\xi^2.$$

The general sum $\sum P[(x_{i_1} > c), \cdots, (x_{i_q} > c)]$ contains $\binom{n}{q}$ terms. Of these, there are order n terms in which none of the x_i appearing ever differs in its subscript by more than m from its nearest neighbours, order n^2 terms in which only one x_i differs in its subscript by more than m from its nearest neighbours, and so on, provided that n is large enough for all the cases to occur. There are $\sim n^q/q!$ terms in which each x_i is separated in its subscript by more than m from its neighbours. These terms may be said to belong to the first, second, $\cdots$, qth class. The sum of terms of the first class will be less than a constant times $n \max_{|i-j| \leqq m} P[(x_i > c), (x_j > c)]$, the sum of terms of the second class will be less than a constant times $n^2 P(x_i > c) \max_{|i-j| \leqq m} P[(x_i > c), (x_j > c)]$ and so on until we reach the qth class, where the sum is $[n^q/q! + O(n^{q-1})]\, P(x_i > c)^q$. Thus, by (1), the only terms contributing to the sum as $n \to \infty$ are those of the qth class; these yield $\xi^q/q!$ asymptotically.

Thus for any even integer l we have shown that

$$\sum_{q=0}^{l-1} \frac{(-\xi)^q}{q!} \leqq \lim_{n \to \infty} P(x_i \leqq c_n(\xi) \leqq \sum_{q=0}^{l} \frac{(-\xi)^q}{q!}, \qquad i = 1, \cdots, n$$

which proves the theorem.

To show that this theorem covers a class of stochastic processes of practical interest, it is shown next that the condition (1) of the theorem is true in strictly stationary processes which are normal. For this, it suffices to show that

$$\frac{P[(x > c), (y > c)]}{P(x > c)} \to 0, \qquad (c \to \infty), \tag{2}$$

where x and y have a bivariate normal distribution with means zero, variances unity and covariance ρ, with $|\rho| < 1$. Now

$$P[(x > c), (y > c)] = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_c^\infty \int_c^\infty \exp\left[\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right] dx\, dy.$$

The substitution $x = r/c + c$ and $y = t/c + c$ leads to

$$P[(x > c), (y > c)] = \frac{\exp[(-c^2/(1+\rho))]}{2\pi c^2\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \exp\left[-\frac{r^2 - 2\rho rt + t^2}{2c^2(1-\rho^2)}\right] \exp\left[-\frac{r+t}{1+\rho}\right] dr\, dt$$

$$\sim \frac{1}{2\pi} \exp\left(\frac{-c^2}{1+\rho}\right)\left[\frac{(1+\rho)^{3/2}}{\sqrt{1-\rho}}\frac{1}{c^2} - 0\left(\frac{1}{c^4}\right)\right], \qquad c \text{ large.}$$

Since $P(x > c) \sim (1/\sqrt{2\pi}) \exp(-\frac{1}{2}c^2)$, statement (2) follows.

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EXPRESSION OF THE k-STATISTICS k_9 AND k_{10} IN TERMS OF POWER SUMS AND SAMPLE MOMENTS

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The k statistics are of interest to workers in the theory of sampling distributions and moment statistics. They are related also to certain aspects of the theory of numbers and combinatory analysis, as indicated by Dressel [1].

The k statistics were introduced by Fisher in 1928 [2] to estimate the cumulants

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