Extremes
2008, Vol. 11, No. 4, pp. 339-351.

Extremes of Shepp statistics for the Wiener process

Dmitrii Zholud*

Abstract

Define $Y(t) = \max_{0 \le s \le 1} W(t+s) - W(t)$, where $W(\cdot)$ is the standard Wiener process. We study the maximum of Y up to time T: $M_T = \max_{0 \le t \le T} Y(t)$ and determine an asymptotic expression for $\mathbf{P}(M_T > u)$ when $u \to \infty$. Further we establish the limiting Gumbel distribution of M_T when $T \to \infty$ and present the corresponding normalization sequence.

Keywords: Wiener process, increments, maximum, extreme values, high level excursions, large deviations, asymptotic behavior, Shepp statistics, distribution tail, Gumbel law, limit theorems, weak theorems, Gaussian random walk.

AMS 2000 Subject Classifications: $\frac{\text{Primary-60G70;}}{\text{Secondary-60G15, 60F05;}}$.

1 Introduction

First, we introduce two different techniques used in the asymptotic theory of Gaussian processes and fields. For a Gaussian process $Z(t)$, consider asymptotic behavior of the probability

$$\mathbf{P}\left(\max_{[0,T]} Z(t) > u\right),\ u \to \infty. \tag{1}$$

In the case when $Z(t)$ is a stationary Gaussian process with a covariance function $r(t)$ such that $r(t) - r(0)$ is a regularly varying function of index α for $t \to 0$, the exact asymptotic forms of (1) were given by Pickands, see Pickands (1969a,b).

In the non-stationary case there are a number of results for Gaussian processes with a unique point of maximum variance, see e.g. Berman (1985), Hüsler (1990) and related papers. When $Z(t)$ is a Gaussian process with continuous paths, zero mean and nonconstant variance, and there is a unique fixed point of maximum variance t_0 in the interval $[0, T]$, the asymptotic behavior of probability in (1) is known. The theory sketched out above is described in detail in Piterbarg (1996).

Next, define $X(t, s) = W(t+s) - W(t)$ and $Y(t) = \max_{0 \le s \le 1} X(t, s)$, for $W(\cdot)$ the standard Wiener process. Let $M_T = \max_{[0,T]} Y(t)$ be the maximum up to time T of $Y(t)$. The aim of this paper is to find the asymptotic behavior of $\mathbf{P}(M_T > u)$, the probability of high level excursions of $Y(t)$ as $u \to \infty$ and to obtain the limiting distribution of M_T when $T \to \infty$.

* *Department of Mathematical Statistics*
Chalmers University of Technology and University of Göteborg, Sweden.
E-mail: dmitrii@chalmers.se

For the first task it is crucial to use a representation of M_T as a maximum of the Gaussian field $X(t,s)$ over rectangle $[0,T]\times[0,1]$:

$$M_T = \max_{[0,T]\times[0,1]} X(t,s).$$

Since for fixed s, $X(\cdot,s)$ is a stationary process, and for fixed t, $X(t,\cdot)$ is a process with a unique point of maximum variance, the asymptotic behavior is obtained by combining standard techniques for the corresponding cases. Let $\psi(u)$ be the tail of the standard normal distribution function. The following result and its proof, as well as the expression for the constant H are given in Section 2.

Theorem 1.1. *If* $Tu^2\to\infty$ *and* $Tu^2\psi(u)\to 0$ *when* $u\to\infty$, *then*

$$\mathbf{P}\left(M_T > u\right) = HTu^2\psi(u)(1+o(1)).$$

When the asymptotic behavior of the tail of distribution of M_T is known, we find a limiting distribution of M_T when $T\to\infty$. In this case it is essential to use the representation of M_T as a maximum up to time T of stationary process $Y(t)$. When $|t_1-t_2|>1$, the random variables $Y(t_1)$ and $Y(t_2)$ are independent . The method of establishing the limit theorem is common. Introduce a partition of $[0,T]$ into long blocks $A_i=[i(S+1), i(S+1)+S)$ of length S, and short blocks $B_i=[i(S+1)+S,(i+1)(S+1))$ of length 1 such that $[0,T]=\bigcup_{i=0}^{n}(A_i\cup B_i)$. Then define a sequence of independent identically distributed random variables (i.i.d. r.v.) $Y_i=\max_{A_i} Y(t),\ i=1,2,..$ Letting S to infinity and following the proof of J. Pickands theorem for $\max\{Y_1,Y_2,...\}$, see Leadbetter et al. (1983), the only thing left is to show that random variables $\bar{Y}_i=\max Y(t)$ over B_i give negligible contributions to the limiting distribution of the maximum $M_T=\max\{Y_1,\bar{Y}_1,Y_2,\bar{Y}_2,Y_3,\bar{Y}_3...\}$. However, this idea is extended to obtain a more general result, see Lemma 3.1. It will be used when building limit theorems for the Shepp statistic for a Gaussian random walk, see Zholud (2009). As a corollary of the lemma stated in Section 3 we obtain the limiting Gumbel distribution for M_T, when $T\to\infty$.

Theorem 1.2. *For any fixed* x *and* $T\to\infty$, *the following relation holds:*

$$\mathbf{P}\left(\max_{(t,s)\in[0,T]\times[0,1]} a_T(W(t+s)-W(t)-b_T)\le x\right) = e^{-e^{-x}}+o(1),$$

where

$$a_T=\sqrt{2\ln T},\quad b_T=\sqrt{2\ln T}+\frac{\ln H+\frac{1}{2}\left(\ln\ln T-\ln\pi\right)}{\sqrt{2\ln T}}.$$

A similar result for the standardized Wiener process increments is obtained in Kabluchko (2007). There are also a number of works about strong laws for the increments of the Wiener process, see e.g. Csörgő and Révész (1979), Frolov (2005).

One of the applications of the result derived in this paper is given in Zholud (2009). Let $(\xi_i, i\ge 1)$ be standard normal random variables, and S_k be the corresponding random walk, $S_k=\sum_{i=1}^{k}\xi_i$, $S_0=0$. Define a random variable $\zeta_L^{(N)}(k)=\frac{1}{\sqrt{N}}\left(S_{k+L-1}-S_{k-1}\right)$. Asymptotic behavior of the probability

$$\mathbf{P}\left(\max_{\substack{0<k\le TN\\0<L\le N}}\zeta_L^{(N)}(k)>u\right),$$

when $u\to\infty$, $N\to\infty$ in some synchronized way is then examined. For fixed u, owing to the weak convergence of a random walk to the Wiener process,

$$\mathbf{P}\left(\max_{\substack{0<k\le TN\\0<L\le N}}\zeta_L^{(N)}(k)>u\right)=\mathbf{P}\left(M_T>u\right)\left(1+o(1)\right),\ N\to\infty.$$

Paper Zholud (2009) shows that this equation also holds when $u\to\infty$ and $u/\sqrt{N}\to 0$.

2 Asymptotic behavior of the distribution tail of M_T

In this section we find the asymptotic behavior of the probability

$$\mathbf{P}\left(M_T > u\right) = \mathbf{P}\left(\max_{\substack{0\le t\le T\\0\le s\le 1}} W(t+s) - W(t) > u\right), \tag{2}$$

when $u \to \infty$ and $T \to \infty$ in an appropriate way. As before, we denote $X(t,s) = W(t+s) - W(t)$. The proof is divided into two steps.

First, for any positive constant B we focus on the asymptotic behavior of maximum of X over a thin layer $[0,T] \times [1 - Bu^{-2}, 1]$. It will be shown that within this area and assuming that u is large, $X(t,s)$ and $X(t,1)$ behave in a similar way, and it is possible to determine the asymptotic behavior using the standard technique for stationary processes.

Second, knowing the asymptotic behavior of the maximum of X over the area of its maximum variance, we will show that the maximum over the complementary set $[0,T] \times [0, 1 - Bu^{-2}]$ gives neglible contribution to the probability in (2).

The proof of the first part is based on the Double Sum Method: the lemma below is the analog of Lemma 6.1, Piterbarg (1996). To proceed, let A and B be any positive constants and denote $p = Au^{-2}$, $q = Bu^{-2}$ and $A_0(u) = [0,p] \times [1-q, 1]$. Although it is possible to obtain a representation similar to what we get in Lemma 2.1 by repeating the proof of Lemma 6.1, Piterbarg (1996), our proof does not follow the standard procedure. Instead of passing on to the family of conditional distributions as in Piterbarg (1996), we extract the common part of the increment $X(t,s)$ for all $(t,s) \in A_0(u)$ and use independence of the Wiener process increments.

Lemma 2.1. *Let $u \to \infty$. Then*

$$\mathbf{P}\left(\max_{A_0(u)} W(t+s) - W(t) > u\right) = H_A^B \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}(1 + o(1)),$$

where

$$H_A^B = e^{-\frac{A+B}{2}} \mathbf{E} \exp\left(\max_{\substack{0\le t\le A\\0\le s\le B}} W(t+s+A) - W(t)\right).$$

Proof: We have that since $1 - q > p$ for large u,

$$\mathbf{P}\left(\max_{A_0(u)} W(t+s) - W(t) > u\right)$$
$$= \mathbf{P}\left(W(1-q) - W(p) + \max_{A_0(u)} W(t+s) - W(1-q) + W(p) - W(t) > u\right),$$

and by stationarity and independence of the Wiener process increments $W(t+s) - W(1-q)$ and $W(p) - W(t)$, the probability above is equal to

$$\mathbf{P}\left(\xi + \max_{A_0(u)} W(t+s-(1-q)+p) - W(t) > u\right)$$
$$= \mathbf{P}\left(\xi + \max_{\substack{0\le t\le p\\0\le s\le q}} W(t+s+p) - W(t) > u\right),$$

where random variable ξ is normally distributed with zero mean, variance $\sigma^2 = 1 - p - q$ and is independent of the expression inside the maximum sign.

Thus,

$$\mathbf{P}\left(\max_{A_0(u)} W(t+s)-W(t)>u\right)$$
$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{v^2}{2\sigma^2}}\mathbf{P}\left(\max_{\substack{0\le t\le p\\0\le s\le q}} W(t+s+p)-W(t)>u-v\right)dv.$$

After the change of variables $v=u-\frac{w}{u}$, the last expression equals

$$\frac{\sigma^{-1}}{\sqrt{2\pi}u}\int_{-\infty}^{\infty} e^{-\frac{(u-\frac{w}{u})^2}{2\sigma^2}}\mathbf{P}\left(\max_{\substack{0\le t\le p\\0\le s\le q}} u(W(t+s+p)-W(t))>w\right)dw$$
$$=\frac{e^{-\frac{u^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma u}\int_{-\infty}^{\infty} e^{-\frac{w^2/u^2}{2\sigma^2}}e^{\frac{w}{\sigma^2}}\mathbf{P}\left(\max_{\substack{0\le t\le A\\0\le s\le B}} W(t+s+A)-W(t)>w\right)dw.$$

Next, by the dominated convergence theorem, which follows from the upper estimate of the probability under the integral sign (see Borel's theorem, Piterbarg (1996), p.13), and relations $\sigma^2\to 1$ and

$$e^{-\frac{u^2}{2\sigma^2}}=e^{-\frac{u^2}{2}(1+p+q+o(u^{-2}))}(1+o(1))=e^{-\frac{u^2}{2}}e^{-\frac{A+B}{2}}(1+o(1)),$$

when $u\to\infty$, we obtain the desired representation. □

Corollary 2.1.1.
1) H_A^B is nondecreasing with respect to the parameters A and B.
2) $H_{A_1+A_2}^B\le H_{A_1}^B+H_{A_2}^B$.
3) $H_A^B\le AH_1^B$, for any integer A.

Our next aim is to move on from the rectangle $[0,Au^{-2}]\times[1-Bu^{-2},1]$ to the layer $[0,T]\times[1-Bu^{-2},1]$. We use Lemma 2.1 and the Bonferroni inequality to obtain estimates of the probability of high level excursions of the maximum of X. Then we show that estimates from below and from above are asymptotically equivalent.

Let $A_r(u)=[rAu^{-2},(r+1)Au^{-2}]\times[1-Bu^{-2},1]$. For ease of notation we suppress dependence on u. Using stationarity of $X(t,s)$ with respect to t, we obtain that

$$(\tfrac{Tu^2}{A}+1)\mathbf{P}\left(\max_{(t,s)\in A_0} X(t,s)>u\right)\ge\mathbf{P}\left(\max_{\substack{0\le t\le T\\1-Bu^{-2}\le s\le 1}} X(t,s)>u\right)\ge$$
$$\ge(\tfrac{Tu^2}{A}-1)\mathbf{P}\left(\max_{(t,s)\in A_0} X(t,s)>u\right)-$$
$$-\sum_{\substack{0\le l,m\le\frac{Tu^2}{A}+1\\l\ne m}}\mathbf{P}\left(\max_{(t,s)\in A_l} X(t,s)>u,\ \max_{(t,s)\in A_m} X(t,s)>u\right).\quad(3)$$

Let $p_{l,m}$ denote the summands in the last sum in (3). The sum, owing to stationarity, does not exceed

$$2\left(\frac{Tu^2}{A}+1\right)\sum_{n=1}^{\frac{Tu^2}{A}+1} p_{0,n}.\quad(4)$$

Estimating the probabilities $p_{0,n}$ from above, we will show that the sum (4) is negligible, and thus the upper and lower estimates in (3) are asymptotically equivalent.

The estimates are obtained in slightly different ways, in the same manner as in Lemma 7.1, Piterbarg (1996). The next lemma is a modification of Lemma 6.3, Piterbarg (1996).

Lemma 2.2. *There exists an absolute constant C such that inequality*

$$\mathbf{P}\left(\max_{(t,s)\in A_0} X(t,s) > u, \max_{(t,s)\in A_r} X(t,s) > u\right) \le C(AB)^2\psi(u)e^{-\frac{(r-1)A}{4}}$$

holds for any A, B any $1 < r \le 1 + \frac{u^2}{A}$, and for any u, $u \ge u_0$,

$$u_0 = \inf\left\{u : e^{-4Au^{-2}} \le 1 - 2Au^{-2}\ , \quad Bu^{-2} \le \frac{1}{2}\right\}.$$

Proof: The Gaussian field $X(t,s)$ has zero mean, is stationary in t, and its covariance function is

$$K(t,s;t_1,s_1) = mes\left([t,t+s]\bigcap[t_1,t_1+s_1]\right). \tag{5}$$

Consequently, a global Hölder condition holds, that is,

$$\mathbf{E}\left(X(t,s) - X(t_1,s_1)\right)^2 \le 2(|s-s_1| + |t-t_1|). \tag{6}$$

Introducing $Y(\mathbf{v},\mathbf{w}) = X(\mathbf{v}) + X(\mathbf{w})$, where $\mathbf{v} = (t,s)$ and $\mathbf{w} = (t_1,s_1)$, we obtain

$$\mathbf{P}\left(\max_{(t,s)\in A_0} X(t,s) > u, \max_{(t,s)\in A_r} X(t,s) > u\right) \le \mathbf{P}\left(\max_{A_0\times A_r} Y(\mathbf{v},\mathbf{w}) > 2u\right).$$

Using (5), (6) and restrictions on r and u it is straightforward to derive bounds on the variance of $Y(\mathbf{v},\mathbf{w})$ and then to obtain an estimate from below of the covariance function of normalized field $Y^*(\mathbf{v},\mathbf{w})$, see Lemma 6.3, Piterbarg (1996). Further steps repeat the proof of the lemma. □

Corollary 2.2.1. *When $r > 1 + \frac{u^2}{A}$ and $u \ge u_0$ the following inequality holds*

$$\mathbf{P}\left(\max_{(t,s)\in A_0} X(t,s) > u, \max_{(t,s)\in A_r} X(t,s) > u\right) \le C(AB)^2\psi(u)^2.$$

Indeed, for $r > 1 + \frac{u^2}{A}$ the events inside the probability are independent and this finishes the proof.

Corollary 2.2.2. *When $r = 1$ and $u \ge u_0$, the following inequality holds*

$$\mathbf{P}\left(\max_{(t,s)\in A_0} X(t,s) > u, \max_{(t,s)\in A_r} X(t,s) > u\right)$$
$$\le \left(C(AB)^2e^{-\frac{1}{4}\sqrt{A}} + (\sqrt{A}+1)H_1^B\right)\psi(u).$$

The proof follows Lemma 7.1 on p.107 in Piterbarg (1996). We are now ready to estimate (4) from above. Since

$$\sum_{n=1}^{\frac{Tu^2}{A}+1} p_{0,n} = p_{0,1} + \sum_{n=2}^{\frac{u^2}{A}+1} p_{0,n} + \sum_{n=\frac{u^2}{A}+2}^{\frac{Tu^2}{A}+1} p_{0,n}$$

and estimating the first summand by using Corollary 2.2.2, the second using Lemma 2.2 and the last using Corollary 2.2.1, we obtain

$$(4) \le 2\left(\tfrac{Tu^2}{A}+1\right)\psi(u)\left\{\left(C(AB)^2e^{-\frac{1}{4}\sqrt{A}}+(\sqrt{A}+1)H_1^B\right)+ \right.$$
$$\left. + C(AB)^2\sum_{n=2}^{\infty} e^{-\frac{1}{4}(n-1)A}+\tfrac{Tu^2}{A}C(AB)^2\psi(u)\right\}.$$

Assuming that $Tu^2\to\infty$ and $Tu^2\psi(u)\to 0$ it follows from (3), (4), Lemma 2.1 and the estimate of (4) above that

$$\varlimsup_{u\to\infty}\mathbf{P}\left(\max_{\substack{0\le t\le T\\ 1-Bu^{-2}\le s\le 1}} X(t,s)>u\right)\Big/ Tu^2\psi(u)\le A^{-1}H_A^B$$

and (7)

$$\varliminf_{u\to\infty}\mathbf{P}\left(\max_{\substack{0\le t\le T\\ 1-Bu^{-2}\le s\le 1}} X(t,s)>u\right)\Big/ Tu^2\psi(u)\ge (A')^{-1}H_{A'}^B -$$
$$-2\tfrac{C}{A'}\left\{\left((A'B)^2e^{-\frac{\sqrt{A'}}{4}}+\tfrac{\sqrt{A'}+1}{C}H_1^B\right)+(A'B)^2\sum_{n=2}^{\infty}e^{-\frac{(n-1)A'}{4}}\right\}.$$

Thus, noticing that the expression in the last line tends to zero when $A'\to\infty$, and applying Corollary 2.1.1 3), we see that

$$\varliminf_{A\to\infty} A^{-1}H_A^B\le \varlimsup_{A'\to\infty}(A')^{-1}H_{A'}^B\le H_1^B<\infty.$$

Finally, we want to show that the limit

$$H^B=\lim_{A\to\infty}A^{-1}H_A^B,\quad 0<H^B\le H_1^B<\infty, \tag{8}$$

that exists as a consequence of the estimate above, is positive. This is done by following the proof of D.16 in Piterbarg (1996) when considering the probability of high level excursion over the subset $D=\bigcup_i A_{2i}\cap[0,T]\times[0,1]$.

Thus, assuming A and A' in (7) tend to infinity and applying (8), we obtain the asymptotic behavior of the probability of high level excursion of the maximum of $X(t,s)$ over the upper layer $[0,T]\times[1-Bu^{-2},1]$:

Lemma 2.3. *Assuming $Tu^2\to\infty$ and $Tu^2\psi(u)\to 0$, the following equality holds*

$$\mathbf{P}\left(\max_{\substack{0\le t\le T\\ 1-Bu^{-2}\le s\le 1}} X(t,s)>u\right)=H^BTu^2\psi(u)(1+o(1)).$$

Below we give the second part of the proof. It shows that the asymptotic behavior of the probability of the high level excursion of the maximum of $X(t,s)$ over the upper layer, which corresponds to the area of the maximum variance of the field, gives the main contribution to (2).

Let $B_n(u)=[0,T]\times[1-(n+1)Bu^{-2},1-nBu^{-2}]$ and assume that the conditions $Tu^2\to\infty$ and $Tu^2\psi(u)\to 0$ are satisfied. As before, we suppress the dependence of B_n on u and continue with

Lemma 2.4. *Starting from large enough values of u, if $nBu^{-2}\le\frac{1}{2}$, then*

$$\mathbf{P}\left(\max_{(t,s)\in B_n}X(t,s)>u\right)\le 4H^{2B}e^{-\frac{1}{2}nB}Tu^2\psi(u)(1+c(u)),$$

where $c(u)\to 0$, when $u\to\infty$.

Proof: Normalizing by the maximum standard deviation of $X(t,s)$ over B_n we get

$$\mathbf{P}\left(\max_{(t,s)\in B_n} X(t,s) > u\right) = \mathbf{P}\left(\max_{(t,s)\in B_n} \frac{X(t,s)}{\sqrt{1-nBu^{-2}}} > \frac{u}{\sqrt{1-nBu^{-2}}}\right)$$

$$= \mathbf{P}\left(\max_{\substack{0\le t\le T/(1-nBu^{-2})\\ 1-\frac{Bu^{-2}}{1-nBu^{-2}}\le s\le 1}} X(t,s) > \frac{u}{\sqrt{1-nBu^{-2}}}\right)$$

$$\le \mathbf{P}\left(\max_{\substack{0\le t\le 2T\\ 1-2Bu^{-2}\le s\le 1}} X(t,s) > \frac{u}{\sqrt{1-nBu^{-2}}}\right).$$

The expression on the right-hand side satisfies all the conditions of Lemma 2.3, and for large enough u inequality $\psi(\frac{u}{\sqrt{1-nBu^{-2}}}) \le 2\psi(u)e^{-\frac{1}{2}nB}$ holds uniformly in n. □

Lemma 2.5. *If $nBu^{-2} > \frac{1}{2}$, then*

$$\mathbf{P}\left(\max_{(t,s)\in[0,T]\times[0,1]\setminus\bigcup_{i=0}^{n} B_i} X(t,s) > u\right) \le CTu^4\psi(\sqrt{2}u).$$

Proof: Expanding the set under the maximum sign, we get

$$\mathbf{P}\left(\max_{(t,s)\in[0,T]\times[0,1]\setminus\bigcup_{i=0}^{n} B_i} X(t,s) > u\right) \le \mathbf{P}\left(\max_{\substack{0\le t\le T\\ 0\le s\le \frac{1}{2}}} X(t,s) > u\right).$$

The maximum of the variance of $X(t,s)$ over the set $[0,T]\times[0,\frac{1}{2}]$ equals $\frac{1}{2}$. Theorem 8.1, Piterbarg (1996) finishes the proof. □

Now follows the proof of Theorem 1.1: Lemmas 2.3, 2.4 and 2.5 imply that

$$\underline{\lim}_{u\to\infty} \mathbf{P}\left(\max_{[0,T]\times[0,1]} X(t,s) > u\right)\Big/ Tu^2\psi(u) \ge \underline{\lim}_{u\to\infty} \frac{\mathbf{P}\left(\max_{(t,s)\in B_0} X(t,s)>u\right)}{Tu^2\psi(u)} = H^B$$

and

$$\overline{\lim}_{u\to\infty} \mathbf{P}\left(\max_{[0,T]\times[0,1]} X(t,s) > u\right)\Big/ Tu^2\psi(u) \le \overline{\lim}_{u\to\infty} \frac{1}{Tu^2\psi(u)}\left[\mathbf{P}\left(\max_{(t,s)\in B_0} X(t,s) > u\right)\right.$$

$$\left. + \sum_{n=1}^{\frac{u^2}{2B}} \mathbf{P}\left(\max_{(t,s)\in B_n} X(t,s) > u\right) + \mathbf{P}\left(\max_{(t,s)\in \hat{B}} X(t,s) > u\right)\right]$$

$$\le H^{B'} + 4H^{2B'} \times \sum_{n=1}^{\infty} e^{-\frac{1}{2}nB'},$$

where $\hat{B}$ denotes $[0,T]\times[0,1]\setminus \bigcup_{n=0}^{\frac{u^2}{2B}+1} B_n$. Now note that the constant $H^B = \lim_{A\to\infty} A^{-1}H_A^B$ is non-decreasing with respect to the parameter B, and the last inequalities show that it is bounded from above. Thus, $\lim\limits_{B\to\infty} H^B = H$, say, exists, finite and positive, and $\lim\limits_{B'\to\infty} H^{B'} + 4H^{2B'} \times \sum\limits_{n=1}^{\infty} e^{-\frac{1}{2}nB'}$ also equals H. □

3 Limit theorem for M_T

In this section we consider the case where T goes to infinity, and we obtain the limit distribution of $(M_T - a_T)/b_T$ for the appropriate normalization functions a_T and b_T. First we prove a general lemma, which can serve as a template for obtaining limiting theorems not only for random fields, but for a family of fields as well. We assume that the specific asymptotic behavior of the tail of the distribution of the maximum of some field takes place and that this asymptotic behavior is defined by an asymptotic relation between threshold u, parameter S that defines the set over which the maximum is taken, and parameter N discussed below. The condition that defines the asymptotic behavior will be denoted by, say, $\mathcal{D}(u, N, S)$. The following lemma shows that knowing asymptotic behavior under $\mathcal{D}(u, N, S)$ we can derive a new condition involving T and N such that if it holds when T goes to infinity, M_T has limiting Gumbel distribution.

Lemma 3.1. *Assume that*

1) $X^N(t, s)$ $N = 1, 2...$ is a family of fields stationary with respect to the parameter t, and defined on the set $[0, \infty) \times [0, 1]$.

2) For any N, any t, t_1 such that $|t - t_1| > 1$ and any $s, s_1 \in [0, 1]$, the random variables $X^N(t, s)$ and $X^N(t_1, s_1)$ are independent.

3) By $\mathcal{D}(u, N, S)$ we refer to some logical statement that involves variables u, N, S and such that if $\mathcal{D}(u, N, S)$ holds then the following asymptotic behavior of the tail of the distribution of a maximum of $X^N(t, s)$ over the set $D_S = [0, S] \times [0, 1]$ takes place:

$$\mathbf{P}\left(\max_{D_S} X^N(t, s) > u\right) \sim SF(u, N) \tag{9}$$

for some function $F(u, N)$. We also demand that if $\mathcal{D}(u, N, 1)$, then (9) holds for $S \equiv 1$.

4)Let $T \to \infty$ and suppose there exist appropriate normalizing functions a_T and b_T such that

$$\lim_{\substack{T\to\infty \\ (N\to\infty)}} TF(u_T, N) = e^{-x}$$

for any fixed x, where $u_T = b_T + \frac{x}{a_T}$. Functions a_T and b_T may also depend on N.

5)Let $S = S(T)$ be such a function that $S \to \infty$ and $n = \frac{T}{S+1} \to \infty$ when $T \to \infty$.

Then, if $\mathcal{D}(u_T, N, 1)$ and $\mathcal{D}(u_T, N, S(T))$ hold,

$$\mathbf{P}\left(\max_{D_T} X^N(t, s) > u_T\right) \to 1 - e^{-e^{-x}}. \tag{10}$$

Proof: Let us introduce a partition $[0, T] = \bigcup\limits_{i=0}^{n} (A_i \cup B_i)$, with $A_i = [i(S + 1), i(S + 1) + S]$ and $B_i = [i(S + 1) + S, (i + 1)(S + 1)]$ so that $|A_i| = S$ and $|B_i| = 1$ for all i.

For the expression on the left-hand side of (10) we have that

$$\mathbf{P}\left(\max_{D_T} X^N(t,s) \le u_T\right) = 1 - \mathbf{P}\left(\bigcup_{i=0}^{n}\left\{\max_{A_i\times[0,1]} X^N(t,s) > u_T \;\cup \max_{B_i\times[0,1]} X^N(t,s) > u_T\right\}\right).$$

Applying stationarity of $X^N(t,s)$ with respect to t we obtain the following estimate

$$1 - n\mathbf{P}\left(\max_{[0,1]^2} X^N(t,s) > u_T\right) - \mathbf{P}\left(\bigcup_{i=0}^{n} \max_{A_i\times[0,1]} X^N(t,s) > u_T\right) \le$$

$$\le \mathbf{P}\left(\max_{D_T} X^N(t,s) \le u_T\right) \le 1 - \mathbf{P}\left(\bigcup_{i=0}^{n} \max_{A_i\times[0,1]} X^N(t,s) > u_T\right). \tag{11}$$

Here the term $n\mathbf{P}\left(\max_{[0,1]^2} X^N(t,s) > u_T\right)$ is estimated using $\mathcal{D}(u_T, N, 1)$ and 3) and, for penultimate equality, 4)

$$n\mathbf{P}\left(\max_{[0,1]^2} X^N(t,s) > u_T\right) = nF(u_T,N)(1+o(1)) = \tfrac{TF(u_T,N)}{S+1}(1+o(1)) = \tfrac{e^{-x}(1+o(1))}{S+1} = o(1).$$

Using the fact that $\max_{A_i\times[0,1]} X^N(t,s)$ and $\max_{A_j\times[0,1]} X^N(t,s)$ are independent for $i \ne j$, see 2), and, again, stationarity, we estimate the expression on the right-hand side of (11) using $\mathcal{D}(u_T, N, S(T))$ and 3) in the third step, and 4) and 5) in the fifth, we get

$$1-\mathbf{P}\left(\bigcup_{i=0}^{n} \max_{A_i\times[0,1]} X^N(t,s) > u_T\right) = \prod_{i=0}^{n} \mathbf{P}\left(\max_{A_i\times[0,1]} X^N(t,s) \le u_T\right) = \left(1 - \mathbf{P}\left(\max_{A_0\times[0,1]} X^N(t,s) > u_T\right)\right)^n$$

$$= (1 - SF(u_T,N))^n = e^{n\ln(1-SF(u_T,N))} = e^{-nSF(u_T,N)(1+o(SF(u_T,N)))} = e^{-e^{-x}}(1+o(1)).$$

It therefore follows from (11) that

$$e^{-e^{-x}}(1+o(1)) + o(1) \le \mathbf{P}\left(\max_{D_T} X^N(t,s) \le u_T\right) \le e^{-e^{-x}}(1+o(1)),$$

and this finishes the proof. □

Corollary 3.1.1 (The Wiener process)**.**

Put $X^N(t,s) \equiv W(t+s) - W(t)$. *We say that* $\mathcal{D}(u,N,S)$ *holds if and only if* $Su^2 \to \infty$ *and* $Su^2\psi(u) \to 0$, $u \to \infty$. *Thus, conditions 1), 2) and 3) of the lemma are satisfied by Theorem 1.1.*

It is easy to verify that Condition 4) is satisfied for

$$u_T = \frac{x}{\sqrt{2\ln T}} + \sqrt{2\ln T} + \frac{\ln H + \frac{1}{2}(\ln\ln T - \ln\pi)}{\sqrt{2\ln T}}.$$

In 5) we set $S(T) = \sqrt{T}$. *Condition* $\mathcal{D}(u_T, N, 1)$ *becomes equivalent to* $u_T \to \infty$ *that is equivalent to* $T \to \infty$ *owing to our choice of* u_T. *Finally, using 3) it is easy to show that*

$$S(T)u_T^2\psi(u_T) = S(T)/T \times TF(u_T,N) = e^{e^{-x}}(1+o(1))/\sqrt{T} = o(1).$$

Thus $\mathcal{D}(u_T, N, S)$ *is equivalent to* $T \to \infty$ *and Theorem 1.2 holds.*

Acknowledgements

The author is grateful to Professor Piterbarg V.I. for constant assistance and fruitful discussions on the Double Sum method, and wishes to thank Professor Holger Rootzén for careful reading and pointing out numerous typographical errors and unclear points.

References

S.M. Berman. An asymptotic formula for the distribution of the maximum of a Gaussian process with stationary increments. *Journal of Applied Probability*, 22(2):454–460, 1985. 1

M. Csörgő and P. Révész. How big are the increments of a Wiener process? *The Annals of Probability*, 7(4):731–737, 1979. 2

A.N. Frolov. Unified limit theorems for increments of processes with independent increments. *Theory of Probability and Its Applications*, 49(3):531–539, 2005. 2

J. Hüsler. Extreme values and high boundary crossings of locally stationary Gaussian processes. *Annals of Probability*, 18(3):1141–1158, 1990. 1

Z. Kabluchko. Extreme-value analysis of standardized Gaussian increments. Technical Report arXiv:0706.1849, The University of Ulm, 2007. 2

M.R. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and related properties of random sequences and processes*. Springer series in statistics. Springer-Verlag, 1983. 2

J. Pickands. Upcrossings probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.*, 145:51–73, 1969a. 1

J. Pickands. Asymptotic properties of the maximum in a stationary Gaussian process. *Trans. Amer. Math. Soc.*, 145:75–86, 1969b. 1

V.I. Piterbarg. *Asymptotic Methods in the Theory of Gaussian Processes and Fields*. Translation of mathematical monographs. AMS, 1996. 1, 3, 4, 5, 6, 7

D. Zholud. Extremes of Shepp statistics for Gaussian random walk. *Extremes*, 12(1):1–17, 2009. 2