Tohoku Math. J.
60 (2008), 597–605

EXTRINSIC GEODESICS AND HYPERSURFACES OF TYPE (A) IN A COMPLEX PROJECTIVE SPACE

Dedicated to Professor Kouei Sekigawa on the occasion of his retirement from Niigata University

SADAHIRO MAEDA AND TOSHIAKI ADACHI

(Received March 21, 2008)

Abstract. In a complex projective space, we distinguish hypersurfaces of type (A_1) from hypersurfaces of type (A_2) in terms of the cardinality of congruence classes of their extrinsic geodesics.

1. Introduction. When we study shapes of Riemannian submanifolds, it is natural to investigate their geodesics (see [4] for example). Recall that an isometric immersion f of a Riemannian manifold M into another $\tilde{M}$ is totally geodesic if and only if every geodesic on M is also a geodesic in $\tilde{M}$. We say a geodesic on a submanifold M to be an *extrinsic* geodesic in the ambient space $\tilde{M}$ if it is also a geodesic considered as a curve in $\tilde{M}$. Note that there is no totally geodesic real hypersurface, which is a submanifold of real dimension $2n - 1$, in a complex n-dimensional complex projective space $\boldsymbol{C}P^n$. This leads us to study real hypersurfaces by the cardinality of extrinsic geodesics in $\boldsymbol{C}P^n$.

A real hypersurface in a complex projective space $\boldsymbol{C}P^n$ is said to be of type (A) if it is a tube around some totally geodesic Kähler submanifold $\boldsymbol{C}P^k$, where $0 \leqq k \leqq n-1$. If, in particular, $k = 0$ or $k = n-1$, it is called a real hypersurface of type (A_1), and otherwise it is called a real hypersurface of type (A_2). These real hypersurfaces are remarkable examples of submanifolds in $\boldsymbol{C}P^n$. Recall that hypersurfaces of type (A_1) (resp. of type (A_2)) have two (resp. three) distinct constant principal curvatures in $\boldsymbol{C}P^n$. They have many nice *common* properties, which enrich the theory of real hypersurfaces (cf. [7]). For example, they are Hopf hypersurfaces in $\boldsymbol{C}P^n$ and each geodesic on them has constant structure torsion (for details, see Section 2). In this regard, we give a characterization of all real hypersurfaces of type (A) in the class of Hopf hypersurfaces (Theorem 1). Investigating the quantity of extrinsic geodesics on real hypersurfaces and applying Theorem 1, we obtain our main results which distinguish hypersurfaces of type (A_1) from hypersurfaces of type (A_2) in $\boldsymbol{C}P^n$.

2000 *Mathematics Subject Classification.* Primary 53B25, Secondary 53C40.

Key words and phrases. Hypersurfaces of type (A), geodesic spheres, hypersurfaces of type (A_2), ruled real hypersurfaces, complex projective spaces, normal section, integral curves of the characteristic vector field, geodesics, extrinsic geodesics, structure torsion, normal curvature.

The first author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 19540084), Japan Society for the Promotion of Sciences.

The second author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 17540072, 20540071), Japan Society for the Promotion of Sciences.

2. Extrinsic geodesics on hypersurfaces of type (A). Let M be a real hypersurface in a Kähler manifold $(\tilde{M}, \langle\ ,\ \rangle, J)$ with Riemannian metric $\langle\ ,\ \rangle$ and complex structure J. The Riemannian connections $\tilde{\nabla}$ of $\tilde{M}$ and ∇ of M are related by the following Gauss and Weingarten formulas with a unit normal local vector field $\mathcal{N}$ on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N} \quad \text{and} \quad \tilde{\nabla}_X \mathcal{N} = -AX$$

for vector fields X, Y on M, where A is the shape operator of M in $\tilde{M}$. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of the ambient Kähler manifold $\tilde{M}$. That is, we have a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η on M defined by

$$\langle \phi u, v\rangle = \langle Ju, v\rangle \quad \text{and} \quad \langle \xi, u\rangle = \eta(u) = \langle Ju, \mathcal{N}\rangle$$

for all tangent vectors $u, v \in TM$. These satisfy

$$\phi^2 v = -v + \eta(v)\xi\,, \quad \xi = -J\mathcal{N}\,, \quad \text{and} \quad \phi\xi = 0\,.$$

It follows from these equalities and the Weingarten formula that

$$\nabla_X \xi = \phi AX\,. \tag{2.1}$$

A real hypersurface in a complex projective space $\boldsymbol{C}P^n$ is said to be of type (A) if it is a tube around some totally geodesic Kähler submanifold $\boldsymbol{C}P^k$, $0 \leqq k \leqq n-1$. Real hypersurfaces of type (A) are classified into two classes. Real hypersurfaces of type (A) corresponding to the case $k = 0$ or $k = n-1$ are said to be real hypersurfaces of type (A_1), and others are said to be real hypersurfaces of type (A_2). We should note that in a complex projective space $\boldsymbol{C}P^n(c)$ of constant holomorphic sectional curvature c, a tube of radius r with $0 < r < \pi/\sqrt{c}$ around $\boldsymbol{C}P^k(c)$ is congruent to a tube of radius $(\pi/\sqrt{c}\,) - r$ around $\boldsymbol{C}P^{n-k-1}(c)$. Hence we can say that real hypersurfaces of type (A_1) are geodesic spheres. In this section, we study the quantity of extrinsic geodesics on real hypersurfaces of type (A). We call two geodesics γ_1, γ_2 on a Riemannian manifold M *congruent* to each other if there exist an isometry φ of M and a constant s_0 with $\gamma_2(s + s_0) = \varphi \circ \gamma_1(s)$ for all s. We can see the quantity of congruence classes of extrinsic geodesics on real hypersurfaces of type (A) as follows.

PROPOSITION 1. *Let M be a geodesic sphere of radius r with $0 < r < \pi/\sqrt{c}$ in $\boldsymbol{C}P^n(c)$.*

(1) *When $0 < r < \pi/(2\sqrt{c}\,)$, there exist no extrinsic geodesics on M.*

(2) *When $\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}$, there exists just one congruence class of extrinsic geodesics on M.*

PROPOSITION 2. *For a real hypersurface M of type* (A_2), *at each point $x \in M$ there exist infinitely many congruence classes of extrinsic geodesics on M passing through x.*

In order to show these propositions we here recall invariants of geodesics on real hypersurfaces of type (A) in $\boldsymbol{C}P^n$. Such hypersurfaces are characterized by properties of their shape operators as follows.

LEMMA 1 ([7]). *For a real hypersurface M in $\boldsymbol{C}P^n(c)$, the following conditions are mutually equivalent to each other*:

(1) *M is of type* (A).

(2) *$\phi A = A\phi$ holds on M.*

(3) *The covariant derivative of the shape operator A of M satisfies*

$$(\nabla_u A)v = (-c/4)\{\langle \phi u, v\rangle \xi + \eta(v)\phi u\}$$

for all tangent vectors $u, v \in TM$.

For a geodesic γ on a real hypersurface M of type (A), we define its structure torsion ρ_γ by $\rho_\gamma = \langle \dot\gamma, \xi_\gamma\rangle$. Clearly, it satisfies $-1 \leqq \rho_\gamma \leqq 1$. By making use of (2.1) and Lemma 1(2), we can see that, for each geodesic γ on M, this structure torsion ρ_γ is constant along γ in the following manner:

$$\begin{aligned}\nabla_{\dot\gamma}\rho_\gamma(s) &= \langle \dot\gamma(s), \phi A\dot\gamma(s)\rangle = \frac{1}{2}\{\langle \dot\gamma(s), \phi A\dot\gamma(s)\rangle + \langle \phi A\dot\gamma(s), \dot\gamma(s)\rangle\}\\ &= \frac{1}{2}\langle \dot\gamma(s), (\phi A - A\phi)\dot\gamma(s)\rangle = 0\,.\end{aligned} \tag{2.2}$$

For geodesics on a hypersurface of type (A_1), we can classify them by means of their structure torsions (see Proposition 2.3 in [3]):

LEMMA 2. *On a geodesic sphere M in $\boldsymbol{C}P^n$, two geodesics γ_1, γ_2 are congruent to each other with respect to the isometry group $I(M)$ of M if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.*

In order to classify geodesics on real hypersurfaces of type (A_2) we need another invariant. For a geodesic γ on a real hypersurface of type (A) in $\boldsymbol{C}P^n$ we define its *normal curvature* κ_γ by $\kappa_\gamma = \langle A\dot\gamma, \dot\gamma\rangle$. By Lemma 1(3) we have

$$\nabla_{\dot\gamma}\kappa_\gamma(s) = \langle (\nabla_{\dot\gamma(s)}A)\dot\gamma(s), \dot\gamma(s)\rangle = 0\,,$$

which shows that κ_γ is constant along γ. We can interpret this invariant in another way as follows. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. When M is a geodesic sphere of radius r with $0 < r < \pi/\sqrt{c}$ in $\boldsymbol{C}P^n(c)$, it has two principal curvatures $\sqrt{c}\cot(\sqrt{c}\,r)$ and $(\sqrt{c}\,r/2)\cot(\sqrt{c}\,r/2)$. When M is a real hypersurface of type (A_2) which is a tube of radius r ($0 < r < \pi/\sqrt{c}$) around $\boldsymbol{C}P^k(c)$ in $\boldsymbol{C}P^n(c)$, it has three principal curvatures $\sqrt{c}\cot(\sqrt{c}\,r)$, $(\sqrt{c}\,r/2)\cot(\sqrt{c}\,r/2)$ and $-(\sqrt{c}\,r/2)\tan(\sqrt{c}\,r/2)$. For a principal curvature λ of a real hypersurface of type (A), we denote by V_λ the subbundle of principal curvature vectors associated with λ. For a tube M of radius r around totally geodesic $\boldsymbol{C}P^k(c)$ with $0 \leqq k \leqq n-2$ in $\boldsymbol{C}P^n(c)$, we consider a projection $\mathrm{Proj} : TM \to V_\lambda$ of the tangent bundle onto the subbundle of principal vectors with $\lambda = (\sqrt{c}\,r/2)\cot(\sqrt{c}\,r/2)$. For a geodesic on M we define its *principal torsion* τ_γ by $\tau_\gamma = \|\mathrm{Proj}(\dot\gamma)\|$. Clearly, it satisfies $0 \leqq \tau_\gamma \leqq \sqrt{1-\rho_\gamma^2}$

and

$$\kappa_\gamma = \rho_\gamma^2\sqrt{c}\ \cot(\sqrt{c}\,r) + \tau_\gamma^2(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) - (1-\rho_\gamma^2-\tau_\gamma^2)(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)\,, \tag{2.3}$$

since the characteristic vector field ξ is principal with principal curvature $\sqrt{c}\cot(\sqrt{c}\,r)$. Hence τ_γ is also constant along γ.

Geodesics on a hypersurface of type (A_2) are classified by means of their structure torsions and normal curvatures (see Theorem 2 in [2] and Proposition 1 in [1]):

LEMMA 3. *On a hypersurface M of type* (A_2), *two geodesics γ_1, γ_2 are congruent to each other with respect to the isometry group $I(M)$ of M if and only if one of the following conditions holds*:

(1) *Their structure torsions and normal curvatures satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ and $\kappa_{\gamma_1} = \kappa_{\gamma_2}$.*
(2) *Their structure torsions and principal torsions satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ and $\tau_{\gamma_1} = \tau_{\gamma_2}$.*

We now prove Propositions 1 and 2.

PROOF OF PROPOSITION 1. By the Gauss formula, we see that a geodesic γ on M is an extrinsic geodesic in $\boldsymbol{C}P^n$ if and only if it has null normal curvature. Since its principal torsion is $\sqrt{1-\rho_\gamma^2}$, we find

$$\kappa_\gamma = \rho_\gamma^2\sqrt{c}\ \cot(\sqrt{c}\,r) + (1-\rho_\gamma^2)(\sqrt{c}/2)\cot(\sqrt{c}r/2)\,.$$

Since $2\cot(\sqrt{c}\,r) = \cot(\sqrt{c}\,r/2) - \tan(\sqrt{c}\,r/2)$, we conclude that $\kappa_\gamma = 0$ if and only if $\rho_\gamma = \pm\cot(\sqrt{c}\,r/2)$. This implies that $r \geqq \pi/(2\sqrt{c}\,)$ because $-1 \leqq \rho_\gamma \leqq 1$. By virtue of Lemma 2 we obtain the result. □

PROOF OF PROPOSITION 2. By Relation (2.3), a geodesic γ on M is an extrinsic geodesic (i.e., $\kappa_\gamma = 0$) if and only if

$$\rho_\gamma^2 = (1-\tau_\gamma^2)\tan^2(\sqrt{c}\,r/2) - \tau_\gamma^2$$

holds. Since $0 \leqq \rho_\gamma^2 \leqq 1$, the above equality holds if and only if

$$\sin^2(\sqrt{c}\,r/2) - \cos^2(\sqrt{c}\,r/2) \leqq \tau_\gamma^2 \leqq \sin^2(\sqrt{c}\,r/2)\,.$$

Hence we obtain the conclusion with the aid of Lemma 3. □

REMARK. (1) On a geodesic sphere of radius r in $\boldsymbol{C}P^n(c)$ with $\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}$, a geodesic is an extrinsic geodesic in $\boldsymbol{C}P^n(c)$ if and only if its structure torsion is $\pm\cot(\sqrt{c}\,r/2)$.

(2) On a real hypersurface of type (A_2) which is a tube of radius r around $\boldsymbol{C}P^k(c)$ in $\boldsymbol{C}P^n(c)$, a geodesic is an extrinsic geodesic in $\boldsymbol{C}P^n(c)$ if and only if its structure torsion and principal torsion satisfy

(i) $\sin^2(\sqrt{c}\,r/2) - \cos^2(\sqrt{c}\,r/2) \leqq \tau_\gamma^2 \leqq \sin^2(\sqrt{c}\,r/2)$,
(ii) $\rho_\gamma^2 = (1-\tau_\gamma^2)\tan^2(\sqrt{c}\,r/2) - \tau_\gamma^2$.

We here make a remark on the congruency of geodesics on real hypersurfaces M of type (A) in $\boldsymbol{C}P^n$. Let $\iota : M \to \boldsymbol{C}P^n$ be a canonical isometric embedding. If two geodesics γ_1 and γ_2 on M are congruent to each other with respect to the isometry group $\mathrm{I}(M)$ of M, then the curves $\iota \circ \gamma_1$ and $\iota \circ \gamma_2$ are also congruent to each other as curves in $\boldsymbol{C}P^n$ with respect to the isometry group $SU(n+1)$ of $\boldsymbol{C}P^n$, because ι is an equivariant isometric embedding. However, in general, the converse does not hold. Our discussion guarantees that we can take infinitely many geodesics on a real hypersurface M of type (A_2) which are not congruent to one another on M and are extrinsic geodesics on $\boldsymbol{C}P^n$. As $\boldsymbol{C}P^n$ is a Riemannian symmetric space of rank one, these extrinsic geodesics are congruent to one another with respect to $SU(n+1)$.

3. Characterizations of real hypersurfaces of type (A). In this section we give some characterizations of real hypersurfaces of type (A) in $\boldsymbol{C}P^n$ by properties of extrinsic geodesics. A real hypersurface M of a Kähler manifold $\tilde{M}$ is said to be a *Hopf hypersurface* if the shape operator A of M satisfies $A\xi = \alpha\xi$ for some function α on M, namely, the characteristic vector field ξ is principal. If we restrict ourselves on Hopf hypersurfaces in $\boldsymbol{C}P^n$, it is well-known that the function α is automatically locally constant on them. Furthermore, roughly speaking, every tube of sufficiently small constant radius around an arbitrary Kähler submanifold of $\boldsymbol{C}P^n$ is a Hopf hypersurface (see [7]). This means that the class of Hopf hypersurfaces is an abundant class in the theory of real hypersurfaces in $\boldsymbol{C}P^n$. The following proposition characterizes geometrically all Hopf hypersurfaces of $\boldsymbol{C}P^n$.

LEMMA 4. *Let M be a real hypersurface with a unit normal local vector field $\mathcal{N}$ in $\boldsymbol{C}P^n$, $n \geqq 2$, endowed with the Kähler structure J. Then M is a Hopf hypersurface if and only if the following condition* (S) *holds*:

(S) *At each point $x \in M$, let L_x $\left(\cong \boldsymbol{C}P^1\right)$ be a totally geodesic holomorphic line in $\boldsymbol{C}P^n$ through x whose tangent space $T_x L_x$ is the complex one dimensional linear subspace of $T_x\boldsymbol{C}P^n$ spanned by ξ_x. Then the normal section $N_x = M \cap L_x$ given by L_x is the integral curve through the point x for the characteristic vector field ξ on M.*

PROOF. It follows from the Gauss formula and (2.1) that $\tilde{\nabla}_\xi \xi = \phi A\xi + \langle A\xi, \xi\rangle\mathcal{N}$. Thus M is a Hopf hypersurface if and only if

$$\tilde{\nabla}_\xi \xi = \langle A\xi, \xi\rangle\mathcal{N} = \langle A\xi, \xi\rangle J\xi\,,$$

which is nothing but Condition (S). □

In view of the constancy of structure torsions of geodesics on real hypersurfaces of type (A) (see (2.2)) and Lemma 4, we first establish a characterization of *all* hypersurfaces of type (A) (cf. [6]).

THEOREM 1. *A connected real hypersurface M of $\boldsymbol{C}P^n$, $n \geqq 2$, is of type* (A) *if and only if it satisfies Condition* (S) *and the following condition* (G) :

(G) *At each point $x \in M$, there exists an orthonormal basis $\{v_1, v_2, \dots, v_{2n-2}\}$ of the linear subspace $T_x^0 M = \{v \in T_x M \mid \langle v, \xi_x\rangle = 0\}$ of $T_x M$ satisfying the following two conditions*:

(1) *Every geodesic* γ_i *of* M *through* x *with initial vector* $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ *has constant structure torsion* ρ_{γ_i}.

(2) *Every geodesic* γ_{ij} *of* M *through* x *with initial vector* $\dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2}$ $(1 \leqq i < j \leqq 2n-2)$ *has constant structure torsion* $\rho_{\gamma_{ij}}$.

PROOF. It suffices to prove the "if" part. By Condition (1) and (2.1) we have

$$0 = \dot{\gamma}_i(s)\langle\dot{\gamma}_i(s), \xi_{\gamma_i(s)}\rangle = \langle\dot{\gamma}_i(s), \nabla_{\dot{\gamma}_i}\xi(s)\rangle = \langle\dot{\gamma}_i(s), \phi A\dot{\gamma}_i(s)\rangle\,,$$

so that at the point $x \in M$ we obtain

$$\langle(\phi A - A\phi)v_i, v_i\rangle = 0 \quad \text{for each } i \in \{1, 2, \ldots, 2n-2\} \tag{3.1}$$

as well as

$$\langle\phi A v_i, v_i\rangle = 0 \quad \text{for each } i \in \{1, 2, \ldots, 2n-2\}\,. \tag{3.2}$$

It follows from (3.2) and Condition (2) that

$$\langle\phi A(v_i + v_j), v_i + v_j\rangle = 0 \quad \text{for } 1 \leqq i < j \leqq 2n-2\,.$$

This, together with (3.2), shows that

$$\langle(\phi A - A\phi)v_i, v_j\rangle = 0 \quad \text{for } 1 \leqq i < j \leqq 2n-2\,. \tag{3.3}$$

Hence, from (3.1) and (3.3) we see that $\phi A = A\phi$ holds on the linear subspace $T_x^0 M$. Moreover, it follows from Lemma 4 that the characteristic vector ξ is principal, so in particular $\phi A\xi = 0 = A\phi\xi$. Hence we can see that $\phi A = A\phi$ holds on $T_x M$ at each point x of our real hypersurface M. Thus M is of type (A) by Lemma 1. □

In the hypothesis of Theorem 1 we do not need to take the vectors $\{v_1, \ldots, v_{2n-2}\}$ as a local smooth field of orthonormal frames in M. However, for each hypersurface M of type (A) in $\boldsymbol{C}P^n$, we can take a global smooth field of orthonormal frames $\{v_1, \ldots, v_{2n-2}\}$ in M satisfying Condition (G).

As immediate consequences of Theorem 1 and Propositions 1 and 2 we obtain the following three theorems which distinguish real hypersurfaces of type (A_1) from those of type (A_2) in $\boldsymbol{C}P^n$. We assume always $n \geqq 2$.

THEOREM 2. *A connected real hypersurface* M *of* $\boldsymbol{C}P^n(c)$ *is a geodesic sphere of radius* r *with* $0 < r < \pi/(2\sqrt{c}\,)$ *if and only if* M *satisfies Conditions* (S), (G) *and has a point* $x_0 \in M$ *such that there exist no extrinsic geodesics passing through this point.*

THEOREM 3. *A connected real hypersurface* M *of* $\boldsymbol{C}P^n(c)$ *is a geodesic sphere of radius* r *with* $\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}$ *if and only if it satisfies Conditions* (S), (G) *and has a point* $x_0 \in M$ *such that there exists just one congruence class of extrinsic geodesics passing through this point.*

COROLLARY. *A connected real hypersurface M of $\boldsymbol{C}P^n(c)$ is a hypersurface of type* (A_1) *if and only if it satisfies Conditions* (S), (G) *and has a point $x_0 \in M$ such that there exist at most finite congruence classes of extrinsic geodesics passing through this point.*

THEOREM 4. *A connected real hypersurface M of $\boldsymbol{C}P^n(c)$ is a hypersurface of type* (A_2) *if and only if it satisfies Conditions* (S), (G) *and has infinitely many congruence classes of extrinsic geodesics.*

Finally, we characterize hypersurfaces of type (A) in $\boldsymbol{C}P^n(c)$ whose radius is just $\pi/(2\sqrt{c}\,)$.

THEOREM 5. *A connected real hypersurface M in $\boldsymbol{C}P^n(c)$ is of type* (A) *of radius $\pi/(2\sqrt{c}\,)$ if and only if it satisfies Conditions* (S), (G) *and has a point $x_0 \in M$ such that the normal section N_{x_0} at this point is an extrinsic geodesic.*

PROOF. By virtue of Lemma 4 and Theorem 1 we see that M is of type (A). If its radius is r, we can set $A\xi = \sqrt{c}\cot(\sqrt{c}\,r)\xi$ (see [7]). On the other hand, as the normal section N_{x_0} can be seen as an extrinsic geodesic, we have

$$0 = \tilde{\nabla}_\xi \xi(x_0) = \nabla_\xi \xi(x_0) + \langle A\xi_{x_0}, \xi_{x_0}\rangle \mathcal{N}_{x_0} = \langle A\xi_{x_0}, \xi_{x_0}\rangle \mathcal{N}_{x_0}\,.$$

Therefore $r = \pi/(2\sqrt{c}\,)$. □

4. Two comments on our theorems. We first remark that Theorem 1 does not hold without Condition (S), that is, the assumption that M is a Hopf hypersurface. To see this, we take ruled real hypersurfaces. A real hypersurface M is called a *ruled real hypersurface* in $\boldsymbol{C}P^n$ if the holomorphic distribution T^0M is integrable and each of its maximal integral manifolds is a totally geodesic hyperplane $\boldsymbol{C}P^{n-1}$ of $\boldsymbol{C}P^n$. The following proposition gives a fundamental property of geodesics on ruled real hypersurfaces.

PROPOSITION 3. *On a ruled real hypersurface M in $\boldsymbol{C}P^n$ every geodesic γ, whose initial vector $\dot\gamma(0)$ is orthogonal to the vector $\xi_{\gamma(0)}$, has constant structure torsion, namely the tangential vector $\dot\gamma(s)$ is orthogonal to $\xi_{\gamma(s)}$ for every s, and is an extrinsic geodesic.*

PROOF. Let $\boldsymbol{C}P^{n-1}$ be the maximal integral manifold through the point $x = \gamma(0)$ for the holomorphic distribution T^0M. We take a geodesic γ_1 on this leaf $\boldsymbol{C}P^{n-1}$ satisfying the initial condition $\gamma_1(0) = x$ and $\dot\gamma_1(0) = \dot\gamma(0)$. Since the leaf $\boldsymbol{C}P^{n-1}$ is totally geodesic in $\boldsymbol{C}P^n$, the curve γ_1 is a geodesic on $\boldsymbol{C}P^n$. Hence γ_1, considered as a curve on our real hypersurface M, is also a geodesic on M. Therefore by the uniqueness of geodesics, we see that $\gamma(s) = \gamma_1(s)$ for each s with $-\varepsilon < s < \varepsilon$ for some $\varepsilon > 0$, so that γ has null structure torsion and is an extrinsic geodesic. □

Proposition 3 shows that Theorem 1 is no longer true without the assumption that M is a Hopf hypersurface.

In order to guarantee the existence of ruled real hypersurfaces, we review the construction of such hypersurfaces in $\boldsymbol{C}P^n$. For an arbitrary regular real curve γ in $\boldsymbol{C}P^n$, we attach at each point $\gamma(s)$ a totally geodesic hyperplane M_s ($\cong \boldsymbol{C}P^{n-1}$) whose tangent space is orthogonal to the complex one-dimensional linear subspace $\{\dot\gamma(s), J\dot\gamma(s)\}$ and obtain a ruled real hypersurface $M = \bigcup_s M_s$. We deal with a ruled real hypersurface locally, because generally this hypersurface has self-intersections and singularities. We recall the shape operator A of a ruled real hypersurface M. We set differentiable functions μ, ν on M by $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$. Then on the open dense subset $M_* = \{x \in M; \nu(x) > 0\}$ of M, the shape operator A of M satisfies the following equalities with a unit vector field U orthogonal to ξ for an arbitrary tangent vector X orthogonal to ξ and U:

$$A\xi = \mu\xi + \nu U\,, \quad AU = \nu\xi\,, \quad AX = 0\,.$$

In the theory of real hypersurfaces, for a ruled real hypersurface M we omit the points where ξ is principal and suppose that $M_* = M$. All ruled real hypersurfaces are typical examples of non-Hopf hypersurfaces. Furthermore, every ruled real hypersurface in $\boldsymbol{C}P^n(c)$ is *not* complete (see (18) and (19) in [5]). On the contrary, every hypersurface of type (A) in $\boldsymbol{C}P^n$ is compact.

We next consider real hypersurfaces M in a complex n-dimensional complex hyperbolic space $\boldsymbol{C}H^n(c)$ of constant holomorphic sectional curvature $c < 0$. We note that there exist no totally geodesic real hypersurfaces neither in $\boldsymbol{C}H^n(c)$. The following hypersurfaces are basic examples in the theory of real hypersurfaces in $\boldsymbol{C}H^n(c)$ (cf. [7]):

(1) A horosphere in $\boldsymbol{C}H^n(c)$.
(2) A geodesic sphere of radius r, where $0 < r < \infty$.
(3) A tube of radius r around totally geodesic $\boldsymbol{C}H^{n-1}(c)$, where $0 < r < \infty$.
(4) A tube of radius r around totally geodesic $\boldsymbol{C}H^k(c)$, where $1 \leqq k \leqq n-2$ and $0 < r < \infty$.

We shall call them real hypersurfaces of type (A), or individually call them of type (A_0), ($A_{1,0}$), ($A_{1,1}$), (A_2) in serial order. Usually both of real hypersurfaces of type ($A_{1,0}$) and of ($A_{1,1}$) are said to be of type (A_1). These real hypersurfaces are Hopf hypersurfaces in a complex hyperbolic space $\boldsymbol{C}H^n$. For real hypersurfaces in $\boldsymbol{C}H^n$, we consider the following condition (S$'$) corresponding to Condition (S) for real hypersurfaces in $\boldsymbol{C}P^n$. Since the assertion of Lemma 1 also holds for real hypersurfaces in $\boldsymbol{C}H^n$, by the same discussion as that in the proof of Theorem 1, we obtain the following theorem which characterizes all hypersurfaces of type (A) in $\boldsymbol{C}H^n(c)$.

THEOREM 6. *A connected real hypersurface M in $\boldsymbol{C}H^n(c)$ for $n \geqq 2$ is of type* (A) *if and only if it satisfies Condition* (G) *and the following condition* (S$'$):

(S$'$) *At each point $x \in M$, let L_x $\left(\cong \boldsymbol{C}H^1\right)$ be a totally geodesic holomorphic line in $\boldsymbol{C}H^n$ through x whose tangent space T_xL_x is the complex one-dimensional linear subspace of $T_x\boldsymbol{C}H^n$ spanned by ξ_x. Then the normal section $N_x = M \cap L_x$ given by L_x is the integral curve through the point x for the characteristic vector field ξ on M.*

The principal curvatures of hypersurfaces of type (A) in $\boldsymbol{C}H^n(c)$ are given as follows ([7]):

Type	λ_1	λ_2	α
(A_0)	$\frac{\sqrt{\lvert c\rvert}}{2}$	—	$\sqrt{\lvert c\rvert}$
$(\mathrm{A}_{1,0})$	$\left(\frac{\sqrt{\lvert c\rvert}}{2}\right)\coth\left(\frac{\sqrt{\lvert c\rvert}\,r}{2}\right)$	—	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$
$(\mathrm{A}_{1,1})$	$\left(\frac{\sqrt{\lvert c\rvert}}{2}\right)\tanh\left(\frac{\sqrt{\lvert c\rvert}\,r}{2}\right)$	—	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$
(A_2)	$\left(\frac{\sqrt{\lvert c\rvert}}{2}\right)\coth\left(\frac{\sqrt{\lvert c\rvert}\,r}{2}\right)$	$\left(\frac{\sqrt{\lvert c\rvert}}{2}\right)\tanh\left(\frac{\sqrt{\lvert c\rvert}\,r}{2}\right)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$

In view of this table of principal curvatures, we can easily see that there exist no extrinsic geodesics on every hypersurface of type (A) in $\boldsymbol{C}H^n$. Therefore, we emphasize that we cannot characterize individually these real hypersurfaces of type (A_0), of type (A_1) and of type (A_2) by investigating the number of congruence classes of their extrinsic geodesics.

REFERENCES

[1] T. ADACHI, Geodesics on real hypersurfaces of type A_2 in a complex space form, Monatsh. Math. 153 (2008), 283–293.

[2] T. ADACHI AND S. MAEDA, A congruence theorem of geodesics on some naturally reductive Riemannian homogeneous manifolds, C. R. Math. Rep. Acad. Sci. Canada 26 (2004), 11–17.

[3] T. ADACHI, S. MAEDA AND M. YAMAGISHI, Length spectrum of geodesic spheres in a non-flat complex space form, J. Math. Soc. Japan 54 (2002), 373–408.

[4] D. FERUS AND S. SCHIRRMACHER, Submanifolds in Euclidean space with simple geodesics, Math. Ann. 260 (1982), 57–62.

[5] M. KIMURA, Sectional curvatures of holomorphic planes on a real hypersurface in $P^n(\boldsymbol{C})$, Math. Ann. 276 (1987), 487–497.

[6] S. MAEDA AND T. ADACHI, Characterizations of hypersurfaces of type A_2 in a complex projective space, Bull. Aust. Math. Soc. 77 (2008), 1–8.

[7] R. NIEBERGALL AND P. J. RYAN, Real hypersurfaces in complex space forms, Tight and taut submanifolds (Berkeley, CA, 1994), 233–305, Math. Sci. Res. Inst. Publ. 32, T. E. Cecil and S. S. Chern, eds., Cambridge University Press, Cambridge, 1997.

DEPARTMENT OF MATHEMATICS
SAGA UNIVERSITY
1 HONZYO, SAGA 840–8502
JAPAN

E-mail address: smaeda@ms.saga-u.ac.jp

DEPARTMENT OF MATHEMATICS
NAGOYA INSTITUTE OF TECHNOLOGY
GOKISO, NAGOYA 466–8555
JAPAN

E-mail address: adachi@nitech.ac.jp