Kim, H.S., Kim, I.-B. and Takagi, R.
Osaka J. Math.
41 (2004), 853–863

EXTRINSICALLY HOMOGENEOUS REAL HYPERSURFACES WITH THREE DISTINCT PRINCIPAL CURVATURES IN $H_n(\mathbb{C})$

Dedicated to Professor Koichi Ogiue on his 60th birthday

HYANG SOOK KIM, IN-BAE KIM and RYOICHI TAKAGI

(Received March 12, 2003)

Introduction

Let $H_n(\mathbb{C})$ be a complex hyperbolic space of complex dimension n ($\geqq 2$) endowed with the metric of constant holomorphic sectional curvature $4c$, and G be the identity component of the group of all isometries of $H_n(\mathbb{C})$. A submanifold M in $H_n(\mathbb{C})$ is said to be *extrinsically homogeneous* if M is an orbit under a closed subgroup of G.

As proposed also in R. Niebergall and P.J. Ryan ([7]), the following is an open problem: *Classify all extrinsically homogeneous real hypersurfaces in* $H_n(\mathbb{C})$. As a partial answer of this problem, J. Berndt ([1]) classified all extrinsically homogeneous real hypersurfaces in $H_n(\mathbb{C})$ whose structure vector fields are principal, where an eigenvector of the shape operator is called *principal*.

Recently he constructed in [2] a subgroup B_n of G for each n ($\geqq 2$) such that a certain orbit M under B_n in $H_n(\mathbb{C})$ has three distinct principal curvatures 1, -1 and 0 with multiplicities 1, 1 and $2n-3$ respectively and the structure vector field on M is not principal. We shall call this group the *Berndt subgroup* of G. The following is due to J. Berndt and H. Tamaru.

Theorem A ([4]). *Let* $\mathfrak{F}$ *be a homogeneous foliation of codimension one on connected irreducible Riemannian symmetric space of noncompact type. Then* $\mathfrak{F}$ *is isometrically congruent to one of the model foliations* $\mathfrak{F}_l$ *or* $\mathfrak{F}_i$.

Remark that, in the above Theorem, the model foliation $\mathfrak{F}_l$ consists of leaves each of which is a real hypersurface of so called A_0 type (so with two distinct principal curvatures), and the model foliation $\mathfrak{F}_i$ consists of leaves each of which is an orbit under the Berndt subgroup (so with three distinct principal curvatures). As for the detailed, see [4].

In this paper, at first we shall establish general properties of extrinsically homogeneous real hypersurfaces in $H_n(\mathbb{C})$. Next, as its applications, we shall prove the fol-

2000 *Mathematics Subject Classification* : 53C40, 53C15.

lowing.

Theorem 1. *Let L be a connected closed subgroup of G. Assume that every real hypersurface given as an orbit under L has three distinct principal curvatures and the structure vector field is not principal. Then any of such orbits is isometrically congruent to an orbit under the Berndt subgroup.*

1. Extrinsically homogeneous real hypersurfaces

Let $H_n(\mathbb{C})$ be the complex hyperbolic space of complex dimension n ($\geqq 2$) with a Riemannian metric $\langle\ ,\ \rangle$ of constant holomorphic sectional curvature $4c$, and G be the identity component of the group of all isometries of $H_n(\mathbb{C})$. The associated Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is a subalgebra and $\mathfrak{p}$ is a vector subspace of $\mathfrak{g}$. We can identify $\mathfrak{p}$ with the tangent space $T_o(H_n(\mathbb{C}))$ of $H_n(\mathbb{C})$ at the origin o.

Let $\mathfrak{l}$ be a Lie subalgebra of $\mathfrak{g}$, and r be a positive integer. Throughout this section, we assume that *there exists a non-empty open set U of $H_n(\mathbb{C})$ such that every orbit under L through U is a real hypersurface in $H_n(\mathbb{C})$ and has r distinct principal curvatures*. We may assume that U contains the origin o. Then there exist $2n-1$ elements $Z_1, \ldots, Z_{2n-1} \in \mathfrak{l}$ such that $\{(Z_1)_\mathfrak{p}, \ldots, (Z_{2n-1})_\mathfrak{p}\}$ is an orthonormal basis for the tangent space $T_o(L(o))$ of $L(o)$. We choose a unit normal vector $Z_0 \in \mathfrak{p}$ of $L(o)$ at o. We put

$$\sigma_t = \exp tZ_0.$$

Then an orbit $\sigma_t(o)$ ($t \in \mathbb{R}$) is a geodesic in $H_n(\mathbb{C})$.

Let I be an open interval containing 0 such that the geodesic segment $g = \sigma_t(o)$ ($t \in I$) is contained in U, and $\mathcal{U}$ be an open neighborhood of 0 in the vector subspace

$$\mathrm{span}_\mathbb{R}\{Z_1, \ldots, Z_{2n-1}\}$$

of $\mathfrak{p}$ such that the exponential map exp of $\mathcal{U}$ onto $(\exp\mathcal{U})(o) \subset L(o)$ is a diffeomorphism. We choose a local orthonormal frame field

$$\{e'_0, e'_1, \ldots, e'_{2n-1}\}$$

along g such that

$$(e'_A)_o = (Z_A)_\mathfrak{p}. \tag{1.1}$$

The subset defined by

$$V = \{(\exp Z)(\sigma_t(o)) \mid Z \in \mathcal{U},\ \ t \in I\}$$

is a neighborhood of o in $H_n(\mathbb{C})$. We define an orthonormal frame field $\{e_A\}$ on V by

$$(e_A)_{(\exp Z)(\sigma_t(o))} = (\exp Z)_*(e'_A)_{\sigma_t(o)}. \tag{1.2}$$

Then it is clear that $\{e_A\}$ is an extension of $\{e'_A\}$.

We denote by θ^A the dual 1-forms of e_A. Let θ^A_B be the connection forms of $H_n(\mathbb{C})$ with respect to the dual 1-forms θ^A. Then the structure equations of $H_n(\mathbb{C})$ are given by

$$\begin{aligned} &d\theta^A + \sum_B \theta^A_B \wedge \theta^B = 0, \quad \theta^A_B + \theta^B_A = 0, \\ &d\theta^A_B + \sum_C \theta^A_C \wedge \theta^C_B = c \sum_{C,D} \left(\delta^A_C \delta^B_D + J^A_C J^B_D + J^A_B J^C_D\right) \theta^C \wedge \theta^D, \end{aligned} \tag{1.3}$$

where J^A_B are components of the complex structure J of $H_n(\mathbb{C})$.

If we put

$$\xi_i = J^0_i, \tag{1.4}$$

then (J^i_j, ξ_i) forms an almost contact structure on each orbit $L(\sigma_t(o))$, that is,

$$\sum_k J^i_k J^k_j = -\delta^i_j + \xi_i \xi_j, \quad \sum_j J^i_j \xi_j = 0, \quad \sum_i \xi_i \xi_i = 1, \tag{1.5}$$

where $\xi = \sum \xi_i e_i$ is said to be the *structure vector field* on $L(\sigma_t(o))$. For convenience sake, we put $M_t = L(\sigma_t(o))$.

Since, for any $t \in I$ and any $\sigma \in L$, the distance between the orbit M_t and the point $\sigma(0)$ is equal to t, we can consider the parameter t as a function around g. It is clear that

$$\theta^0 = dt. \tag{1.6}$$

Since it follows from (1.3) and the exterior derivative of (1.6) that

$$\sum \theta^0_i \wedge \theta^i = 0,$$

we can put

$$\theta^0_i = \sum_j h_{ji} \theta^j, \quad h_{ij} = h_{ji}.$$

For each t of I, the symmetric matrix $(h_{ij}(t))$ is the shape operator of the real hypersurface M_t, and the eigenvalues $\lambda_i(t)$ of $(h_{ij}(t))$ are called the *principal curvatures* of M_t.

Hereafter we retake the orthonormal frame field $\{e_A\}$ in such a way that each e_i is principal, that is,

$$\theta_i^0 = \lambda_i \theta^i. \tag{1.7}$$

It follows from (1.3), (1.4), (1.7) and the exterior derivative of (1.7) that

$$\begin{aligned}\sum_j \Big\{(\lambda_i - \lambda_j)\theta_j^i - c\sum_k (\xi_i J_k^j + \xi_j J_k^i)\theta^k \\ + (\lambda_i^2 - \lambda_i' + c)\delta_j^i \theta^0 + 3c\xi_i\xi_j\theta^0\Big\} \wedge \theta^j = 0,\end{aligned} \tag{1.8}$$

where we have put $\lambda_i' = d\lambda_i/dt$. We put

$$\begin{aligned}\sum_k A_{ijk}\theta^k = (\lambda_i - \lambda_j)\theta_j^i - c\sum_k (\xi_i J_k^j + \xi_j J_k^i)\theta^k \\ + (\lambda_i^2 - \lambda_i' + c)\delta_j^i \theta^0 + 3c\xi_i\xi_j\theta^0.\end{aligned} \tag{1.9}$$

Then, from (1.8) and (1.9), we can easily find

$$A_{ijk} = A_{jik} = A_{ikj}. \tag{1.10}$$

Let $\lambda_i = \lambda_j$ in (1.9). Then we have

$$A_{ijk} = -c\,\xi_i J_k^j - c\xi_j J_k^i \quad \text{if } \lambda_i = \lambda_j, \tag{1.11}$$

$$\lambda_i' = \lambda_i^2 + c + 3c\xi_i^2. \tag{1.12}$$

Moreover, it follows from (1.9) that

$$(\lambda_i - \lambda_j)\theta_j^i = \sum_k (A_{ijk} + c\xi_i J_k^j + c\xi_j J_k^i)\theta^k - 3c\xi_i\xi_j\theta^0 \quad \text{for } i \neq j. \tag{1.13}$$

Using (1.4) and (1.7), the parallelism of the complex structure J of $H_n(\mathbb{C})$ implies

$$dJ_j^i = \sum_k (J_k^i\theta_j^k - J_k^j\theta_i^k) - \xi_i\lambda_j\theta^j + \xi_j\lambda_i\theta^i \tag{1.14}$$

$$d\xi_i = \sum_j (\xi_j\theta_i^j - \lambda_j J_i^j \theta^j). \tag{1.15}$$

Since $\sigma_* \circ J = J \circ \sigma_*$ for any $\sigma \in G$, the components J_j^i and ξ_i depend only on t. Therefore it follows from (1.13) and (1.15) that

$$\xi_i' = -3c\,\xi_i \sum_j^{\lambda_j \neq \lambda_i} \frac{\xi_j^2}{\lambda_j - \lambda_i}, \tag{1.16}$$

$$\sum_{j}^{\lambda_j \neq \lambda_i} \frac{\xi_j}{\lambda_j - \lambda_i}(A_{ijk} + c\xi_i J_k^j + c\xi_j J_k^i) - \lambda_k J_i^k = 0. \tag{1.17}$$

We denote by V_λ the eigenspace corresponding to the principal curvature λ. Then, under the above notation, we have the following general properties about extrinsically homogeneous real hypersurfaces in $H_n(\mathbb{C})$.

Theorem 1.1. *Let L be a connected closed subgroup of G. Assume that there exists a non-empty open interval I such that, for every t of I, the orbit $M_t = L(\sigma_t(o))$ under L is a real hypersurface in $H_n(\mathbb{C})$, and the number of distinct principal curvatures of M_t does not depend on t. Then we have the following*:
(1) *If M_t has a principal curvature λ with multiplicity $\geqq 2$, then the V_λ-component of the structure vector field ξ vanish identically on M_t,*
(2) *M_t has at least one principal curvature with multiplicity* 1,
(3) *If there exists a principal curvature λ such that the V_λ-component of the structure vector field ξ vanishes for some M_{t_0}, then λ is given by*

$$\lambda = -\sqrt{-c}\tanh\sqrt{-c}(t - t_0), \quad \lambda = \pm\sqrt{-c} \ \textit{or}\ \lambda = -\sqrt{-c}\coth\sqrt{-c}(t - t_0),$$

where t_0 is constant and $t - t_0 \in I$,
(4) *If M_t has $2n - 1$ distinct principal curvatures, then the isotropy subgroup of the group of all isometries of M_t is 0-dimensional.*

Proof. (1) If $\lambda_i = \lambda_j$ for $i \neq j$, then it follows from (1.9) that $\xi_i \xi_j = 0$. Since we see from (1.12) that $\xi_i^2 = \xi_j^2$, we have $\xi_i = \xi_j = 0$.
(2) Assume that all principal curvatures of M_t have multiplicities $\geqq 2$. Then we see from (1) that ξ vanishes identically on M_t and a contradiction.
(3) It is immediate from (1.12) and (1.16).
(4) It follows from (1.2) that the map σ_* preserves the principal directions of M_t. Therefore, if the dimension of the isotropy subgroup is not less than 1, then we see that there exists a principal curvature λ_i with multiplicity $\geqq 2$ and a contradiction. □

On the other hand, putting $i = k$ in (1.17) and making use of (1.11), we obtain

$$\xi_i \sum_{j}^{\lambda_j \neq \lambda_i} \frac{J_i^j}{\lambda_j - \lambda_i}\xi_j = 0. \tag{1.18}$$

Since it is clear that $\sum_j^{\lambda_j \neq \lambda_i} (\lambda_j - \lambda_i)/(\lambda_j - \lambda_i)\xi_j J_i^j = 0$ by (1.5), this equation and (1.18) imply that

$$\xi_i \sum_j^{\lambda_j \neq \lambda_i} \frac{\lambda_j J_i^j}{\lambda_j - \lambda_i} \xi_j = 0. \tag{1.19}$$

Thus we can state

Lemma 1.2. *If the components of the structure vector field ξ on M_t satisfy $\xi_1 \neq 0, \dots, \xi_r \neq 0$ and $\xi_i = 0$ $(i \geqq r+1)$, then the rank of the matrix $B = (b_{\alpha\beta})_{1 \leqq \alpha, \beta \leqq r}$ is not greater than $r-2$, where the entries $b_{\alpha\beta}$ of B are defined by*

$$b_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha = \beta \\ \dfrac{J_\beta^\alpha}{\lambda_\alpha - \lambda_\beta} & \text{if } \alpha \neq \beta. \end{cases}$$

Proof. Since $\xi_\alpha \neq 0$ for $\alpha = 1, 2, \dots, r$, we see from Theorem 1.1 that the multiplicity of λ_α is equal to 1 for any α, that is, $\lambda_\alpha \neq \lambda_\beta$ when $\alpha \neq \beta$ $(1 \leqq \alpha, \beta \leqq r)$. From the construction of B, we see that the matrix B is symmetric. Moreover it follows from (1.18) and (1.19) that

$$\sum_\beta^{\lambda_\beta \neq \lambda_\alpha} \frac{J_\alpha^\beta}{\lambda_\beta - \lambda_\alpha} \xi_\beta = 0 \quad \text{and} \quad \sum_\beta^{\lambda_\beta \neq \lambda_\alpha} \frac{J_\alpha^\beta}{\lambda_\beta - \lambda_\alpha} \lambda_\beta \xi_\beta = 0. \tag{1.20}$$

Define two vectors X and Y in $\mathbb{R}^r$ by

$$X = (\xi_1, \dots, \xi_r) \text{ and } Y = (\lambda_1 \xi_1, \dots, \lambda_r \xi_r).$$

Then X and Y are linearly independent because of the fact that $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$. Therefore (1.20) shows that $X, Y \in \operatorname{Ker} B$ and hence $\operatorname{rank} B \leqq r-2$. □

Now we shall quote the following formulas from [8, (2.6) in p.510].

$$\begin{aligned} &2 \sum_k^{\lambda_k \neq \lambda_i} \frac{(A_{ijk} + c\xi_k J_j^i + c\xi_i J_j^k)^2}{\lambda_k - \lambda_i} \\ &- 2 \sum_k^{\lambda_k \neq \lambda_j} \frac{(A_{ijk} + c\xi_k J_i^j + c\xi_j J_i^k)^2}{\lambda_k - \lambda_j} \\ &- 6c(\lambda_i - \lambda_j){J_j^i}^2 + 3c(\xi_j^2 \lambda_i - \xi_i^2 \lambda_j) - (\lambda_i - \lambda_j)(c + \lambda_i \lambda_j) \\ &= 0 \end{aligned} \tag{1.21}$$

if $\lambda_i \neq \lambda_j$.

The following is used later.

Proposition 1.3. *Let I' be an open interval defined by*

$$I' = \{s \in \mathbb{R} \mid L(\sigma_s(o)) \text{ is a real hypersurface in } H_n(\mathbb{C})\}.$$

If there is a finite real number $s_0 \in \partial I'$, then there exists a principal curvature $l(t)$ of $L(\sigma_t(o))$ $(t \in I')$ such that

$$\lim_{t \to s_0} \lambda(t) = \infty.$$

Proof. By changing the parameter s, it suffices to prove that if $L(\sigma_t(o))$ is a real hypersurface for $0 < t < \epsilon$ and $L(o)$ is not so, then there is a principal curvature $\lambda(t)$ of $L(\sigma_t(o))$ such that $\lim_{t \to 0} \lambda(t) = \infty$.

Let G_t be a geodesic hypersphere in $H_n(\mathbb{C})$ centered at o with radius t $(0 < t < \epsilon)$. Then the unit vector field $N_t = (d/dt)\sigma_t(o)$ is normal to both $L(\sigma_t(o))$ and G_t. By the hypothesis, there is a vector $X \in \mathfrak{l}$ such that $X(\sigma_t(o)) \neq 0$ for $0 < t < \epsilon$ and $X(o) = 0$. We consider the curve $\tau(s) = (\exp sX)(\sigma_t(o))$ on $L(\sigma_t(o))$. Then we see that $\tau(s)$ is also on G_t since $(\exp sX)(o) = o$. The unit vector field defined by

$$\widetilde{N}_t = (\exp sX)_* N_t$$

along $\tau(s)$ is normal to $L(\sigma_t(o))$ and G_t in common.

It is known ([1]) that the principal curvatures of G_t are given by $\lambda = 2\coth 2t$ and $\mu = \coth t$ with multiplicities 1 and $2n - 2$ respectively. For a unit vector field e_1 belonging to the eigenspace V_λ along $\sigma_t(o)$, a vector field $X(\sigma_t(o))$ is expressed by

$$X(\sigma_t(o)) = |X|(\cos\theta e_1 + \sin\theta e_2),$$

where e_2 is a unit vector field belonging to V_μ and $|X|$ indicates the length of $X(\sigma_t(o))$. We can choose an orthonormal frame field $\{e_2, \ldots, e_{2n-1}\}$ in V_μ. Then it is easy to see that $S_t(e_1) = \lambda e_1$ and $S_t(e_i) = \mu e_i$ $(2 \leqq i \leqq 2n-1)$, where S_t is the shape operator of G_t. With respect to this local orthonormal frame field $\{e_1, e_2, \ldots, e_{2n-1}\}$ along $\sigma_t(o)$, we shall denote the components of the shape operator T_t of $L(\sigma_t(o))$ by h_{ij}.

Let ∇ be the Riemannian connection of $H_n(\mathbb{C})$. The tangent space $T_{\sigma_t(o)}(G_t)$ of G_t at $\sigma_t(o)$ is the just vector space $\mathfrak{l}(\sigma_t(o))$, which is denoted by M_{t_1}. Since $\widetilde{N}_t$ is the unit normal vector field of $L(\sigma_t(o))$ and G_t in common, we have

$$\left(\nabla_{X(\sigma_t(o))}\widetilde{N}\right)|_{M_{t_1}} = -T_t(X(\sigma_t(o))) = -S_t(X(\sigma_t(o))).$$

Since it is easy to see that

$$S_t(X(\sigma_t(o))) = |X|(\lambda\cos\theta e_1 + \mu\sin\theta e_2),$$
$$T_t(X(\sigma_t(o))) = |X|\left(\cos\theta\sum h_{i1}e_i + \sin\theta\sum h_{i2}e_i\right),$$

we obtain the equations

$$(h_{11} - l)\cos\theta + h_{12}\sin\theta = 0,$$
$$h_{12}\cos\theta + (h_{22} - \mu)\sin\theta = 0,$$

which implies that

$$\mu - \left(\frac{\mu}{\lambda}h_{11} + h_{22}\right) + \frac{1}{\lambda}\left(h_{11}h_{22} - {h_{12}}^2\right) = 0.$$

If all principal curvatures of $L(\sigma_t(o))$ are bounded for $0 < t < \epsilon$, then we have $\lim_{t\to 0} h_{ij} < \infty$ for each h_{ij} and this shows that the last equation gives a contradiction. □

2. Proof of Theorem 1

In this section we shall prove Theorem 1. We can use the notation and the results in the previous section. For conveniences sake, we assume that the constant holomorphic sectional curvature of $H_n(\mathbb{C})$ is equal to -4, that is, $c = -1$.

We take the interval I' defined in Proposition 1.3, which is the maximal interval satisfying the assumption of Theorem 1.1. From Theorem 1.1 and Lemma 1.2, we see that there is an orthonormal frame field $\{e_1, \dots, e_{2n-1}\}$ on $M_t = L(\sigma_t(o))$, $t \in I'$, such that λ_1, λ_2 and λ_3 are distinct principal curvatures with multiplicities 1, 1 and $2n-3$ respectively, and the components of the structure vector field ξ are given by $\xi_1 \neq 0$, $\xi_2 \neq 0$ and $\xi_i = 0$ ($i \geqq 3$) with respect to the frame field. Moreover we see that $J^1_2 = 0$ by Lemma 1.2.

For simplicity, we put $\xi_1 = \alpha$ and $\xi_2 = \beta$. Then we have $\alpha^2 + \beta^2 = 1$ by (1.5). Since we obtain $J^i_1\alpha + J^i_2\beta = 0$ ($i \geqq 3$) by (1.5), we may put $J^3_1 = \beta$ and $J^3_2 = -\alpha$. Using $J^1_2 = 0$, $\alpha^2 + \beta^2 = 1$ and (1.5), we see that

$$\xi = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad J = \left(\begin{array}{ccc|c} 0 & 0 & -\beta & \\ 0 & 0 & \alpha & 0 \\ \beta & -\alpha & 0 & \\ \hline & 0 & & * \end{array}\right), \quad \det(*) \neq 0. \tag{2.1}$$

In the following we shall prove

$$\lambda_1 + \lambda_2 = 3\lambda_3 \text{ and } \lambda_1\lambda_2 = 3\lambda_3^2 - 1. \tag{2.2}$$

Putting $i = 2$ and $i = 3$ in (1.15) and making use of (2.1), we obtain

$$\theta_2^1 = -\lambda_3\theta^3, \tag{2.3}$$

$$\alpha\theta_3^1 + \beta\theta_3^2 = -\beta\lambda_1\theta^1 + \alpha\lambda_2\theta^2 \tag{2.4}$$

respectively. Since we have $\lambda_i = \lambda_3$ $(i \geqq 3)$, it follows from (1.9), (1.10), (1.11) and (2.1) that

$$\begin{aligned} &A_{113} = -A_{223} = -2\alpha\beta, \\ &A_{ijk} = 0 \text{ otherwise except } i = 1,\ j = 2 \text{ and } k = 3. \end{aligned} \tag{2.5}$$

If we put $i = 1$ and $j = 2$ in (1.9) and take account of (2.1), (2.3) and (2.5), then we have

$$A_{123} = -\lambda_3(\lambda_1 - \lambda_2) + \alpha^2 - \beta^2. \tag{2.6}$$

As a similar argument as in (2.6), putting $i = 1$, $j = 3$ and $i = 2$, $j = 3$ in (1.9) respectively and using (2.5) yield

$$\theta_3^1 = -\frac{3\alpha\beta}{\lambda_1 - \lambda_3}\theta^1 + \frac{A_{123} + \alpha^2}{\lambda_1 - \lambda_3}\theta^2, \tag{2.7}$$

$$\theta_3^2 = \frac{A_{123} - \beta^2}{\lambda_2 - \lambda_3}\theta^1 + \frac{3\alpha\beta}{\lambda_2 - \lambda_3}\theta^2. \tag{2.8}$$

If we compare (2.4) with (2.7) and (2.8), then we have

$$A_{123} = -\lambda_1(\lambda_2 - \lambda_3) + 3\alpha^2\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3} + \beta^2, \tag{2.9}$$

$$A_{123} = \lambda_2(\lambda_1 - \lambda_3) - 3\beta^2\frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} - \alpha^2 \tag{2.10}$$

respectively. Eliminating A_{123} from (2.6) and (2.9), and from (2.6) and (2.10) respectively, we can find

$$3\alpha^2(\lambda_1 - \lambda_2) = -(\lambda_1 - \lambda_3)\{\lambda_1(\lambda_2 - \lambda_3) - \lambda_3(\lambda_1 - \lambda_2) - 2\}, \tag{2.11}$$

$$3\beta^2(\lambda_1 - \lambda_2) = (\lambda_2 - \lambda_3)\{\lambda_2(\lambda_1 - \lambda_3) + \lambda_3(\lambda_1 - \lambda_2) - 2\}. \tag{2.12}$$

Thus the sum of (2.11) and (2.12) gives the equation

$$3\lambda_3^2 - 2(\lambda_1 + \lambda_2)\lambda_3 + \lambda_1\lambda_2 + 1 = 0 \tag{2.13}$$

because $\alpha^2 + \beta^2 = 1$.

Putting $i = 1$, $j = 3$ and $i = 2$, $j = 3$ in (1.21) and making use of (1.17), (2.1) and (2.5), we have

$$\begin{aligned}(2.14)\qquad &\frac{2(A_{123}-\alpha^2+\beta^2)^2}{\lambda_1-\lambda_2}+\frac{18\alpha^2\beta^2}{\lambda_1-\lambda_3}+\frac{2(A_{123}-\beta^2)^2}{\lambda_2-\lambda_3}\\ &\qquad+(\lambda_1-\lambda_3)(\lambda_1\lambda_3-1-6\beta^2)-3\alpha^2\lambda_3=0,\\ (2.15)\qquad &-\frac{2(A_{123}-\alpha^2+\beta^2)^2}{\lambda_1-\lambda_2}+\frac{2(A_{123}+\alpha^2)^2}{\lambda_1-\lambda_3}+\frac{18\alpha^2\beta^2}{\lambda_2-\lambda_3}\\ &\qquad+(\lambda_2-\lambda_3)(\lambda_2\lambda_3-1-6\alpha^2)-3\beta^2\lambda_3=0\end{aligned}$$

respectively. Using (2.9), (2.10), (2.11), (2.12) and (2.13), it is easy to see that the sum of (2.14) and (2.15) is reduced to

$$(2.16)\qquad \lambda_3(\lambda_1^2-4\lambda_1\lambda_2+\lambda_2^2+3)+(\lambda_1+\lambda_2)(\lambda_1\lambda_2-2)=0.$$

The equations (2.13) and (2.16) imply (2.2). It is easily seen from (2.2) that the principal curvatures $\lambda_1(t)$ and $\lambda_2(t)$ of the real hypersurface $M_t = L(\sigma_t(o))$ $(t \in I)$ are distinct solutions of the quadratic equation

$$x^2-3\lambda_3(t)x+3\lambda_3(t)^2-1=0,$$

and the discriminant of this equation implies that $|\lambda_3(t)| < 2/\sqrt{3}$. Therefore all of the principal curvatures $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$ of M_t $(t \in I')$ are bounded. Thus by Proposition 1.3 we see $I' = \mathbb{R}$.

Since the collection $\{M_t \mid t \in \mathbb{R}\}$ is a homogeneous foliation of codimension one on $H_n(\mathbb{C})$, it follows from Theorem A that M_t is congruent to an orbit under the Berndt subgroup B_n, as explained in Introduction. ☐

ACKNOWLEDGEMENT. The authors would like to express their hearty thanks to the referee for his useful advices.

References

[1] J. Berndt, *Real hypersurfaces with constant principal curvatures in a complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132–141.

[2] J. Berndt, *Homogeneous hypersurfaces in hyperbolic spaces*, Math. Z. **229** (1998), 589–600.

[3] J. Berndt and M. Brück, *Cohomogeneity one actions on hyperbolic spaces*, J. Reine Angew. Math. **541** (2001), 209–235.

[4] J. Berndt and H. Tamaru, *Homogeneous codimension one foliations on noncompact symmetric spaces*, preprint.

[5] S. Helgason, Differential Geometry, Lie groups, and Symmetric Spaces, Academic Press, New York-London, 1978.

[6] M. Lohnherr and H. Reckziegel, *On ruled real hypersurfaces in a complex space forms*, Geom. Dedicate, **74** (1999), 267–286.

[7] R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms*; in Tight and taut submanifolds, edited by T.E. Cecil et al., Math. Sci. Res. Inst. Publications, Cambridge, 1997, 233–305.

[8] R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures* I, II, J. Math. Soc. Japan, **27** (1975), 43–53, 507–516.

[9] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*: in Differential geometry (in honor of K. Yano), edited by S. Kobayashi et al. Kinokuniya, Tokyo, 469–481, 1972.

Hyang Sook Kim
Computational Mathematics
School of Computer Aided Science
Inje University
Kimhae 621-749, Korea
e-mail: mathkim@ijnc.inje.ac.kr

In-Bae Kim
Department of Mathematics
College of Natural Science
Hankuk University of Foreign Studies
Seoul 130-791, Korea
e-mail: ibkim@maincc.hufs.ac.kr

Ryoichi Takagi
Department of Mathematics and Informatics
Faculty of Science
Chiba University
Chiba-shi, Chiba 263-8522, Japan
e-mail: takagi@math.s.chiba-u.ac.jp