J. DIFFERENTIAL GEOMETRY
9 (1974) 531-535

f-STRUCTURES WITH PARALLELIZABLE KERNEL ON MANIFOLDS

RICHARD S. MILLMAN

1. A structure on an n-dimensional differentiable manifold given by a non-zero tensor field f of type (1,1) and constant rank r, which satisfies $f^3 + f = 0$, is called an *f-structure*. This notion has been studied by Yano and Ishihara (among others) [5]. An f-structure is *integrable* if about each point there is a coordinate system in which f has the constant components

$$f = \begin{bmatrix} 0 & -I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

where I_p is the $p \times p$ identity matrix ($p = \frac{1}{2}r$). In [2] it is shown that the integrability of f is equivalent to the vanishing of the Nijenhuis tensor of f given by $N(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]$ where X and Y are vector fields on M. We shall write $\chi(M)$ for the set of all vector fields on $M, T_m(M)$ for the tangent space of M at $m \in M$, and $T(M)$ for the tangent bundle of M. For $m \in M$, let $(\ker f)_m = \{X \in T_mM \,|\, f_m(X) = 0\}$ and $(\operatorname{im} f)_m = \{X \in T_mM \,|\, X = f_mY \text{ for some } Y \in T_mM\}$. The *kernel* $\ker f$ *of* f is $\bigcup_m(\ker f)_m$ and the *image* $\operatorname{im} f$ *of* f is $\bigcup_m(\operatorname{im} f)_m$. An f-manifold is *k-framed* if there are $\xi_1, \cdots, \xi_{n-r} \in \chi(M)$ such that $\{\xi_1(m), \cdots, \xi_{n-r}(m)\}$ forms a basis for $(\ker f)_m$ for all $m \in M$. We write $n_0 = n - r$. If M_1 and M_2 are k-framed f-manifolds, then we define an almost complex structure J on $M_1 \times M_2$. We shall denote the k-framing on M_i by $\{\xi^i_1, \cdots, \xi^i_{n_0}\}$, and the f-structure on M_i by f_i. If in addition $[\xi^i_k, \xi^i_l] = 0$ for all $1 \leq k,\ l \leq n_0$, then M_i is called an *f-contact manifold*. The concept of f-contact manifold generalizes the basic features of almost contact structure to f-manifold of higher nullity (i.e., lower rank).

Theorem A. *Let M_1 and M_2 be two k-framed f-manifolds of the same rank with f_1- and f_2-structures respectively, and suppose that f_1 and f_2 are integrable. Then the almost complex structure J on $M_1 \times M_2$ is integrable if and only if both M_1 and M_2 are f-contact manifolds.*

If $\varphi\colon M_1 \to M_2$ and $f_2\varphi_*(X) = \varphi_*f_1(X)$ for all $X \in T_mM_1$, $m \in M_1$, then φ is an *f-map*. Here φ_* denotes, as usual, the differential of φ. If $M_1 = M_2$, then

Communicated by K. Yano, May 24, 1973.

φ is an *f-automorphism*; if φ is a diffeomorphism, then both φ and φ^{-1} are f-maps and $\varphi_*\xi_i = \xi_i$ for all $1 \leq i \leq n_0$.

Theorem B. *If M is a compact integrable f-contact manifold, then the set of all f-automorphisms of M is a Lie group in the compact-open topology.*

Theorem A generalizes a result of Morimoto [3] which states that the product of any two normal (integrable) almost contact manifolds is a complex manifold. (This includes the Calabi-Eckmann manifolds $S^{2p+1} \times S^{2q+1}$ as a special case.) Morimoto [3] also proved Theorem B for integrable almost contact manifolds. Theorem B is also valid without the assumption of integrability if M is an almost contact manifold [4].

2. We shall construct the almost complex structure J.

Lemma 1. *If f is an f-structure on an f-manifold, then* $\ker f \cap \operatorname{im} f = (0)$.

Proof. If $Y = f(X) \in \ker f$, then $0 = f(Y) = f^2(X)$, so from $f^3(X) + f(X) = 0$ we have $Y = f(X) = 0$. q.e.d.

Since $\dim T_mM = \dim (\ker f)_m + \dim (\operatorname{im} f)_m$, Lemma 1 allows us to write $T_mM = (\ker f)_m \oplus (\operatorname{im} f)_m$. Let $\pi_m : T_mM \to (\ker f)_m$ be the projection associated to this direct sum decomposition. We define the differential 1-forms η_i $(i = 1, \cdots, n_0)$ on M by $(\eta_i)_m(X) = a_i(m)$ where $\pi_m X = \sum a_i(m)\xi_i(m)$ and $X \in T_mM$.

Lemma 2. *If $X \in T_mM$, then*

(a) $\eta_i(fX) = 0$ *for* $i = 1, \cdots, n_0$,

(b) $f^2(X) - \sum_i \eta_i(X)\xi_i = -X$.

Proof. (a) If $fX = Z + \pi(fX)$ where $Z \in \operatorname{Im} f$, then $\pi(fX) = fX - Z \in (\ker f) \cap (\operatorname{im} f) = (0)$ so $\pi(fX) = 0$.

(b) Let $Y = X + f^2(X)$. Then $f(Y) = 0$ so $Y = \sum a_i\xi_i$. Thus $a_i = \eta_i(Y) = \eta_i(X) + \eta_i(f^2(X)) = \eta_i(X)$ where the last equality follows from (a). q.e.d.

Assume M_1 (resp. M_2) has f-structure f_1 (resp. f_2) with k-framing $\{\xi_1^1, \cdots, \xi_{n_0}^1\}$ (resp. $\{\xi_1^2, \cdots, \xi_{n_0}^2\}$). Note that we have assumed that the rank of f_1 is equal to the rank of f_2. If $X_1 \in T_pM_1$, $X_2 \in T_qM_2$ where $p \in M_1$, $q \in M_2$, then we define a tensor J of type (1,1) on $M_1 \times M_2$ by

$$(2) \quad J_{p,q}(X_1, X_2) = (f_1(X_1) - \sum_i \eta_i^2(X_2)\xi_i^1(p), f_2(X_2) + \sum_i \eta_i^1(X_1)\xi_i^2(q)) .$$

Proposition 3. *J is an almost complex structure on $M_1 \times M_2$.*

Proof. Clearly

$$J_{p,q}^2(X_1, X_2) = (f_1^2(X_1) - \sum \eta_i^1(X_1)\xi_i^1(p), f_2^2(X_2) - \sum \eta_i^2(X_2)\xi_i^2(q));$$

hence $J_{p,q}^2 = -I$ by Lemma 2.

3. Before proving Theorem A we need the following:

Lemma 3. *If M is an integrable k-framed f-manifold, then*

(a) $\eta_i([fX, Y] + [X, fY]) = f(X)\eta_i(Y) - (fY)\eta_i(X)$ *for all* $1 \leq i \leq n_0$, $X, Y \in \chi(M)$,

(b) $f[X, \xi_j] = [f(X), \xi_j]$ *for* $1 \leq j \leq n_0, X \in TM$.

Proof. (a) Since f is integrable, there is a coordinate system (with $s = \frac{1}{2}r$) $(x_1, \cdots, x_s, y_1, \cdots, y_s, w_1, \cdots, w_{n_0})$ such that $\{\partial/\partial x_i, \partial/\partial y_i \mid i = 1, \cdots, s\}$ forms a local basis for $\operatorname{im} f$ and $\{\partial/\partial w_i \mid i = 1, \cdots, n_0\}$ forms a basis for $\ker f$. It suffices to show (a) when $X, Y \in \ker f$, $X \in \ker f$, $Y \in \operatorname{im} f$ and $X, Y \in \operatorname{im} f$ since both sides are skew-symmetric. If $X, Y \in \ker f$, then both sides are zero. If $Y = g\xi_i$ and $X = h\partial/\partial x_j$ where $h \in C^\infty(M)$, then $fX = h\partial/\partial y_j$ and both sides are $\partial g/\partial y_j$. If $Y = g\xi_i$ and $X = h\partial/\partial y_j$, then both sides are $-h\partial g/\partial x_j$. Now assume $X, Y \in \operatorname{im} f$, and suppose $X = h\partial/\partial x_j$ and $Y = g\partial/\partial y_k$. Then $[fX, Y] + [X, fY] = [h\partial/\partial y_j, g\partial/\partial y_k] - [h\partial/\partial x_j, g\partial/\partial x_k]$ which is in $\operatorname{im} f$; hence $\eta_i([fX, Y] + [X, fY]) = 0$ for all i. On the other hand $\eta_i(Y) = \eta_i(X) = 0$, so both sides are zero. The other three cases of this part are the same.

(b) If $N(X, Y)$ is the Nijenhuis torsion of f (which is zero since f is integrable), then

$$0 = f(N(X, Y)) = f[fX, fY] - f^2[fX, Y] - f^2[X, fY] - f[X, Y] .$$

Applying Lemma 2(b) we see

$$\begin{aligned} 0 &= f[fX, fY] + [fX, Y] + [X, fY] - f[X, Y] \\ &\quad - \sum_{i=1}^{n_0} \{\eta_i([fX, Y] + [X, fY])\xi_i\} . \end{aligned} \tag{3}$$

If we let $Y = \xi_j$ and apply part (a), (3) becomes

$$f([X, \xi_j]) = [fX, \xi_j] - \sum_{i=1}^{n_0} (f(X)\delta_{ij})\xi_i ,$$

where δ_{ij} is the Kronecker δ, so that each term in the summation is zero. q.e.d.

We shall now prove Theorem A using the notation introduced there. Let $X_i, Y_i \in \chi(M_i)$, $i = 1, 2$, and $A = (X_1, X_2)$, $B = (Y_1, Y_2)$. J is integrable if and only if

$$N(A, B) = [JA, JB] - J[JA, B] - J[A, JB] - [A, B] = 0 . \tag{4}$$

We prove this at the point $(m_1, m_2) \in M_1 \times M_2$. Let $(x_1^i, \cdots, x_s^i, y_1^i, \cdots, y_s^i, w_1^i, \cdots, w_{n_0}^i)$ be local coordinates about m_i as in the proof of Lemma 3. It suffices to prove (4) when X_1, Y_1 are one of $\partial/\partial x_i^1, \partial/\partial y_i^1, \xi_i^1$, and X_2, Y_2 are one of $\partial/\partial x_i^2, \partial/\partial y_i^2, \xi_i^2$ since N is a tensor.

We shall consider two cases—the others are similar. Suppose $A = (\partial/\partial x^1, \partial/\partial y^2)$ and $B = (\xi_i^1, \xi_j^2)$. Then $JA = (\partial/\partial y^1, -\partial/\partial x^2)$, $JB = (-\xi_j^2, \xi_i^1)$ so that

$$\begin{aligned} J[JA, B] &= J([\partial/\partial y^2, \xi_i^1], -[\partial/\partial x^2, \xi_j^2]) \\ &= (f_1([\partial/\partial y^2, \xi_i^1]) + \sum_l \eta_l^2([\partial/\partial x^2, \xi_j^2])\xi_l^1, \\ &\qquad - f_2([\partial/\partial x^2, \xi_j^2]) + \sum_l \eta_l^1([\partial/\partial y^2, \xi_i^1])\xi_l^2) . \end{aligned}$$

Using Lemma 3(b) and the fact that

$$\eta_l^2([\partial/\partial x^2, \xi_j^2]) = -\eta_l^2([f\partial/\partial y^2, \xi_j^2]) = -\eta_l^2(f[\partial/\partial y^2, \xi_j^2]) = 0 ,$$

from Lemma 2(a) we have

$$J[JA, B] = (-[\partial/\partial x^2, \xi_i^1], -([\partial/\partial y^2, \xi_j^2])) . \tag{5}$$

Similarly

$$\begin{aligned} J[A, JB] &= ([-\partial/\partial y^1, \xi_j^2], -[\partial/\partial x^2, \xi_i^2]) , \\ &\qquad ([A, B] = ([\partial/\partial x^2, \xi_i^1], [\partial/\partial y^2, \xi_j^2])) , \end{aligned} \tag{6}$$

$$[JA, JB] = (-[\partial/\partial y^1, \xi_j^2], -[\partial/\partial x^2, \xi_i^1]) . \tag{7}$$

From (5), (6) and (7) it follows that $N(A, B) = 0$ in this case.

The other case we shall study in detail is when $A = (c^1\xi_l^1, c^2\xi_m^2)$ and $B = (d^1\xi_p^1, d^2\xi_q^2)$ where $c^i, d^i \in R$ for $i = 1, 2$. Note that $JA = (-c^2\xi_m^1, c^1\xi_l^2)$ and $JB = (-d^2\xi_q^1, d^1\xi_p^2)$. Clearly

$$\begin{aligned} J[JA, B] &= -\sum_{k=0}^{n_0} (\eta_k^2([c^1\xi_l^2, d^2\xi_q^2])\xi_k^1, \eta_k^1([c^2\xi_m^1, d^1\xi_p^1])\xi_k^2) , \\ J[A, JB] &= -\sum_{k=1}^{n_0} (\eta_k^2([c^2\xi_m^2, d^1\xi_p^2])\xi_k^1, \eta_k^1([c^1\xi_l^1, d^2\xi_q^1])\xi_k^2) , \\ [JA, JB] &= ([c^2\xi_m^1, d^2\xi_q^1], [c^1\xi_l^2, d^1\xi_p^2]) . \end{aligned}$$

Thus

$$\begin{aligned} N(A, B) = ([c^2\xi_m^1, d^2\xi_q^1] &- [c^1\xi_l, d^1\xi_p^1] \\ &+ \sum_k \eta_k^2([c^1\xi_l^2, d^2\xi_q^2] + [c^2\xi_m^2, d^1\xi_p^2])\xi_k^1, [c^1\xi_l^2, d^1\xi_p^2] - [c^2\xi_m^2, d^2\xi_q^2] \\ &+ \sum_k \eta_k^1([c^2\xi_m^1, d^1\xi_p^1] + [c^1\xi_l^1, d^2\xi_q^1])\xi_k^2) . \end{aligned} \tag{8}$$

If M is an f-contact manifold, then $[\xi_k^i, \xi_l^i] = 0$ for all $i \leq k, l \leq n_0$, $i = 1, 2$, so that $N(A, B) = 0$ in this case. If $N(A, B) = 0$, then set $c^2 = d^2 = 1$, $c^1 = d^1 = 0$ in (8) so that $0 = N(A, B) = ([\xi_m^1, \xi_q^1], [\xi_l^2, \xi_p^2])$. Since m, q, l, p are arbitrary, we conclude that both M_1 and M_2 are f-contact manifolds.

4. Let $A(M_i)$ be the set of all f-automorphisms of the k-framed f-manifold M_i, and $A(M_1 \times M_2)$ be the almost complex diffeomorphisms of $M_1 \times M_2$ with the almost complex structure J.

Proposition 4. *If $\varphi_i \in A(M_i)$ for $i = 1, 2$, then $\varphi_1 \times \varphi_2 \in A(M_1 \times M_2)$.*

Proof. This is a routine computation once we see that $\varphi^* \eta_k^i = \eta_k^i$ for $i = 1, 2$, and $k = 1, \cdots, n_0$. If $X_i \in TM_i$, then $X_i = Z_i + \sum a_j^i \xi_j^i$ for some $Z_i \in \operatorname{im} f_i$ and $a_j^i \in R$, so that $\varphi_* Z_i \in \operatorname{im} f_i$ and hence that

$$\eta_k^i(\varphi_* X_i) = \sum_j \eta_k^i(\varphi_*(a_j^i \xi_j^i)) = \sum_j a_j^i \eta_k^k(\xi_j^i) = a_k^i = \eta_k^i(X_i) \ .$$

Corollary 1. *Let M_1 and M_2 be f-contact manifolds. If $A(M_i)$ acts transitively on M_i for $i = 1, 2$, then $A(M_1 \times M_2)$ operates transitively on $M_1 \times M_2$.*

Corollary 2. *If M is an integrable f-contact manifold, and $A(M)$ operates transitively on M, then $M \times M$ is a complex homogeneous manifold.*

To prove Theorem B we define $H(\varphi) = \varphi \times \varphi$ for $\varphi \in A(M)$. By Proposition 4, $H(\varphi) \in A(M \times M)$. Using the function $H: A(M) \to A(M \times M)$ we may view $A(M)$ as a subset of $A(M \times M)$ which is a Lie group. By means of this we can show that $A(M)$ is locally compact and that any element of $A(M)$ leaving fixed a nonempty open set of M is the identity map of $A(M)$. Hence by a theorem of Bochner-Montgomery [1], $A(M)$ is a Lie transformation group. The details of the proof are quite similar to Morimoto's proof in the almost contact case [3, Theorem 5] and we refer the reader there for details.

Bibliography

[1] S. Bochner & D. Montgomery, *Locally compact groups of differentiable transformations,* Ann. of Math **47** (1946) 639–653.

[2] S. Ishihara & K. Yano, *On integrability conditions of a structure f satisfying $f^3+f=0$,* Quart. J. Math. Oxford Ser. **15** (1964) 217–222.

[3] A. Morimoto, *On normal almost contact structures,* J. Math. Soc. Japan **15** (1963) 420–436.

[4] S. Sasaki, *Almost contact manifolds. Part* II, Tôhoku University, 1967.

[5] K. Yano & S. Ishihara, *Structure defined by f satisfying $f^3+f=0$,* Proc. United States—Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, Nippon Hyoransha, Tokyo, 1966, 153–166.

Southern Illinois University, Carbondale