F-theory and AdS$_3$/CFT$_2$

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ABSTRACT: We construct supersymmetric AdS$_3$ solutions in F-theory, that is Type IIB supergravity with varying axio-dilaton, which are holographically dual to 2d $\mathcal{N} = (0, 4)$ superconformal field theories with small superconformal algebra. In F-theory these arise from D3-branes wrapped on curves in the base of an elliptically fibered Calabi-Yau threefold Y_3 and correspond to self-dual strings in the 6d $\mathcal{N} = (1, 0)$ theory obtained from F-theory on Y_3. The non-trivial fibration over the wrapped curves implies a varying coupling of the $\mathcal{N} = 4$ Super-Yang-Mills theory on the D3-branes. We compute the holographic central charges and show that these agree with the field theory and with the anomalies of self-dual strings in 6d. We complement our analysis with a discussion of the dual M-theory solutions and a comparison of the central charges.

Contents

1 Introduction

F-theory [1–3] has a firm standing as a framework for constructing Type IIB Minkowski vacua in even dimensions, which preserve minimal supersymmetry. The main focus thus far in utilising F-theory has been on the construction and classification of Type IIB vacua with varying axio-dilaton τ, as well as (p,q) 7-branes, which are naturally encoded in the singularities of τ. The canonical setup to construct such vacua is the compactification on elliptic Calabi-Yau varieties Y_d of complex dimension d with base B_{d-1}, where the complex structure of the elliptic fiber models the axio-dilaton.

In the current paper we will deviate from this and consider supergravity solutions with AdS vacua of Type IIB supergravity, where we allow the axio-dilaton τ to vary consistently with the $\mathrm{SL}(2,\mathbb{Z})$ duality transformations. In this sense we are constructing F-theory solutions. Our main focus will be supersymmetric solutions with AdS factors, which allow for a holographic interpretation. The motivation to study these backgrounds arises from various points of view. It is in general an interesting question to characterise systematically such Type IIB solutions, as a method for exploring superconformal field theories (SCFTs). Moreover, recently[1] various D3-brane configurations in F-theory have been shown to give rise to 2d SCFTs [5–9]. These constructions are based on D3-branes wrapped on a complex curve C in the base B_{d-1} of the elliptic fibration. Our goal here is to construct holographic duals to such 2d SCFTs. In this paper we are interested in the

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[1]D3-branes wrapped on curves in compact Calabi-Yau threefolds were studied much earlier [4], however in those setups the coupling of the D3-brane remains constant.

case where the curve C is deformable, which is in contrast to the case of the strings in 6d non-Higgsable clusters (NHC) [10], where the curve is rigid.

The main novelty in our configuration is that we consider a background, where the complexified coupling τ of the $\mathcal{N} = 4$ Super-Yang Mills (SYM) theory on the D3-brane varies over the internal compact dimensions. This requires a particular topological twist of the $\mathcal{N} = 4$ SYM theory, which is known as the topological duality twist which accounts for the additional varying coupling constant, and was introduced for U(1) gauge groups in [11] and generalised to non-abelian groups in [12]. For the case of a single D3-brane wrapping a curve above which the coupling varies, as we consider here, the 2d SCFTs were studied in [8, 9] and the central charges for these abelian theories were obtained therein.

Concretely, we will consider F-theory on elliptically fibered Calabi-Yau threefolds Y_3 with base B, where the D3-branes wrap a curve $C \subset B$, and an $\mathbb{R}^{1,1}$ factor, giving rise to strings with 2d $(0, 4)$ supersymmetric theories on their worldvolume. We will see that the holographic dual AdS$_3$ solutions have a non-trivial axio-dilaton profile and the geometry of the solution is of the form

$$\text{AdS}_3 \times S^3 \times B \,, \tag{1.1}$$

with τ varying holomorphically over B, which is a Kähler surface, and a specific form of five-form flux.[2] Constraints on B follow from the existence of a minimal elliptic Calabi-Yau fibration above it, which imply that B is either $\mathbb{P}^2$, a Hirzebruch surface $\mathbb{F}_m$, blow-ups thereof, or an Enriques surface [14, 15]. As an F-theoretic solution this could formally be said to correspond to a background of the form AdS$_3 \times S^3 \times Y_3$. The fibration will necessarily have fibers which degenerate and are thus singular, i.e. there are codimension one loci in the base, above which τ will have log-singularities. Physically these correspond to the presence of (p, q) 7-branes, which are sourced by the singularity in τ. The specific type of 7-brane is canonically determined by the singularity type, using the Kodaira-Néron classification of singular fibers [16, 17]. The base is a smooth Kähler surface, however the metric induced from the Calabi-Yau fibration above it has singularities due to the presence of 7-branes. This makes the direct Type IIB analysis somewhat more delicate. We will show that these solutions exist, preserve the required supersymmetry, and we will be able to determine some properties of the dual SCFTs, despite the singular nature of the Type IIB solutions.

In view of the subtleties with regards to the singularities of the metric on B, we will corroborate these results by considering the M/F-dual solutions in 11d supergravity, which can be obtained by T-duality and uplift to M-theory, where they become solutions of the type

$$\text{AdS}_3 \times S^2 \times Y_3 \qquad \text{with} \qquad \mathbb{E}_\tau \hookrightarrow Y_3 \to B \,. \tag{1.2}$$

These fall into the classification of supersymmetric AdS$_3$ solutions presented in [18, 19]. Once lifted to 11d supergravity, the singularities of the six-dimensional metric can be resolved, yielding a smooth Ricci-flat Kähler metric on the compact Calabi-Yau variety,

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[2]Geometries of this type were discussed in [8] from the point of view of effective six-dimensional supergravity [13]. However, to the best of our knowledge the problem of lifting these to supersymmetric solutions of ten dimensional supergravity was not addressed. Our findings demonstrate that only a restricted class of 6d vacua discussed in [8] can be embedded consistently into F-theory.

which is the resolution of Y_3. This allows us to compute reliably the holographic central charges, including certain sub-leading corrections.

More generally, we will show that in the Type IIB solution one can perform a quotient of S^3 by a discrete subgroup $\Gamma = \Gamma_{ADE} \subset \mathrm{SU}(2)_L \subset \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ retaining the full $(0,4)$ supersymmetry. The resulting

$$\mathrm{AdS}_3 \times S^3/\Gamma \times B, \tag{1.3}$$

solution is the most general F-theory geometry (with vanishing three-form fluxes) preserving $(0,4)$ supersymmetry. We will concentrate on the A series,[3] where $\Gamma = \mathbb{Z}_M$, with the quotient acting on the Hopf fiber of S^3. In this case, the solution can be interpreted as the near-horizon limit of D3-branes wrapped on a curve $C \subset B$ transverse to an M-centered[4] Taub-NUT space, corresponding to the presence of M Kaluza-Klein (KK) monopoles. Of course, for $M = 1$ the near-horizon limit of the solutions corresponding to one KK or no KK-monopoles are indistinguishable. However, we will see that the explicit dual 11d supergravity solution can be argued to correspond to the configuration with *one KK-monopole* in Type IIB.

A priori the M/F-dual setup to the D3-branes wrapped on $C \times \mathbb{R}^{1,1}$ are M5-branes that in addition wrap the elliptic fiber above the curve C, i.e. the M5-branes wrap an elliptic surface, which we will denote by $\widehat{C}$, times $\mathbb{R}^{1,1}$. This is closely related to the MSW-string [20–22], although here $\widehat{C}$ is *not* an ample divisor of Y_3, and as a result the supergravity solution for this configuration does not exist. However, upon explicit T-duality and lift to 11d, we will find that our F-theory solution can be interpreted in terms of M5-branes wrapped on an ample divisor that is a linear combination of N times $\widehat{C}$ and M times the section of the fibration, B. The latter arise precisely from dualizing M KK-monopoles in Type IIB.[5] The spectrum of the wrapped M5-brane on $\widehat{C} \times \mathbb{R}^{1,1}$ and its relation to a single D3-brane wrapped on $C \times \mathbb{R}^{1,1}$ with the topological duality twist, was analysed in [9].

The entire class of F-theory solutions $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_M \times B$, and their 11d supergravity duals $\mathrm{AdS}_3 \times S^2 \times Y_3$, is expected to be holographically dual to 2d $(0,4)$ SCFTs. The details of the SCFTs will depend on a choice of Kähler manifold B and the elliptic fibration above it, which specifies the 7-brane configuration, as well as the choice of holomorphic curve $C \subset B$ that the D3-branes wrap. Therefore, in this paper we will not attempt to construct explicit 2d field theories that should flow to the SCFTs in the IR, beyond the discussion in [9]. Nevertheless, some generic features of these SCFTs can be inferred from the supergravity solutions and may be checked by using different methods, mainly based on anomaly inflow on branes as well as anomalies of strings in 6d SCFTs. For $M > 1$ the supergravity solutions include an $\mathrm{SO}(2,2) \times \mathrm{SU}(2)_R$ isometry, with the Killing spinors transforming each in the representation **2** of $\mathrm{SU}(2)_R$. This can therefore be identified as the R-symmetry of a *small* superconformal algebra [24]. For $M = 1$ the F-theory solution (but not the 11d supergravity dual) has an enhanced $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry.

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[3]The D and E series result in three-dimensional spaces with non-abelian fundamental groups, which do not posses continuous isometries.

[4]In the geometry, the M centers coincide, which results in a $\mathbb{Z}_M$ singularity.

[5]Aspects of these backgrounds were studied in [23] from a different point of view.

Since the Killing spinors do not transform under $SU(2)_L$ this corresponds to an emergent flavour symmetry in the theory arising from D3-branes, with no KK-monopoles. In the abelian case, this symmetry was shown to be realised in terms of a current algebra [8].

Information on the field theories arising from a stack of D3-branes only and those involving in addition M KK-monopoles can be obtained using complementary approaches. The former give rise to self-dual strings in 6d and we can determine the central charges c_L, c_R from their anomaly polynomial. We can also reproduce these central charges in the M-theory dual setup involving a stack of M5-branes wrapped on the (non-ample) divisor $\widehat{C}$, by evaluating the relevant anomaly polynomial [25]. However, as remarked above, this setup does not admit a supergravity dual, and there will be additional subtleties that we shall discuss in detail. Conversely, for the theories arising in the presence of M KK-monopoles, the anomaly polynomial for the 6d strings is not known, so in this paper we will not be able to compute the central charges microscopically on the F-theory side in these cases. For the M-theory configurations however, we can determine them from the M5-brane anomaly polynomial, and successfully compare the results with the gravity duals.

We arrive at the F-theory solutions described above by performing a systematic analysis of the supersymmetry equations in Type IIB supergravity. We adopt the method of G-structures, that has been employed to investigate supersymmetric solutions with AdS factors in various settings — see [26–32] for a representative list of references. In this paper we study warped AdS_3 solutions of Type IIB supergravity, allowing for a general five-form flux and arbitrary axio-dilaton, while setting the three-form fluxes to zero. Here we will concentrate on solutions preserving $(0,4)$ supersymmetry, where our analysis shows that the class of solutions is essentially unique. Generalisations of this setup will be discussed elsewhere [33].

Examples of supersymmetric AdS_3 solutions that can be thought of as the near-horizon limit of brane intersections have been known for a long time [34], although in many cases the dual SCFTs have remained somewhat elusive [35]. The conditions for AdS_3 solutions preserving at least $(0,2)$ supersymmetry, arising purely from wrapped D3-branes with constant coupling, were spelled out in [36]. Following this work, a number of explicit solutions have been found in [37–39], which also determined some properties of the putative dual 2d $(0,2)$ SCFTs. The authors of [40] presented generalisations to include a particular form of three-form fluxes. Finally, new interesting examples of solutions to the equations in [36] were found in [41–43], where the precise dual $(0,2)$ SCFTs were identified and non-trivial checks of the central charges were performed. In particular, the central charges of $(0,2)$ SCFTs can be determined by the method of c-extremisation [44]. In all these examples, however, the axio-dilaton is constant. In the present paper we will initiate a line of investigation in which this can vary non-trivially over the internal manifold, setting the stage for studying holography in the context of F-theory.

We should remark on other supergravity solutions with holographic duals, where non-trivial profiles of the axio-dilaton have appeared — however none of which include a varying axio-dilaton with the full $SL(2,\mathbb{Z})$ monodromy, which we incorporate in this paper. AdS-duals with particular constant, but not necessarily perturbative, values of the axio-dilaton, which correspond to F-theory at constant coupling were studied in [45–47].

There are also holographic setups with D3- and D7-branes, where the latter play the role of flavour symmetries in the field theory dual, see e.g. [48] for a review. Typically these correspond to configurations of D3- and D7-branes sharing four flat space-time directions, corresponding to non-conformal four-dimensional field theories. When the backreaction of the D7-branes is included, the supergravity solutions do not have an AdS factor. Another closely related setup involving D3- and D7-branes was discussed in [49–51]. These are configurations where, as in the present paper, the branes share two dimensions. Here the D3-branes are placed into the supergravity background sourced by the D7-branes, however the Type IIB solution does not possess $SO(2,2)$ isometry, and therefore it is not holographically dual to a 2d SCFT. This is distinct from the setup that we consider, in that it corresponds to a 4d gauge theory in the presence of 2d defects.

Recently, AdS_6 solutions dual to 5d SCFTs were constructed in Type IIB supergravity, which have a non-trivial τ profile that allows for poles in τ, but does not include any $SL(2,\mathbb{Z})$ monodromy [52, 53]. Furthermore there is the class of holographic duals to Janus configurations, [54–57] where the gauge coupling varies along a real line, which was later generalised to the θ-angle varying along the 1d line [58]. In contrast, in our configurations, the complexified coupling τ varies holomorphically along the base of the fibration, which is a complex surface in the present case, giving rise to an elliptic fibration with general $SL(2,\mathbb{Z})$ monodromy.

The rest of the paper is organised as follows. Section 2 contains a short overview of F-theory and elliptic Calabi-Yau manifolds. In section 3 we derive the Type IIB supergravity solutions using G-structure methods. In section 4 we present the computation of the holographic central charges for the F-theory solutions. In section 5 we discuss the duality to 11d supergravity making contact with [20]. In section 6 we present the computation of the holographic central charges for the M-theory solutions. Section 7 contains the microscopic computations of the central charges of the dual SCFTs via anomaly inflow for 6d self-dual strings and MSW-strings. We discuss our results in section 8. We include several appendices containing supergravity computations as well as summarising mathematical properties of the geometries used in the main part of the paper. Moreover, in appendix F we present a new AdS_3 solution of Type IIB supergravity, including three-form fluxes, which is thereby outside the analysis in the rest of the paper.

2 F-theory and wrapped D3-branes

2.1 F-theory

We begin with some remarks regarding F-theory and solutions of Type IIB supergravity. Consider a Calabi-Yau manifold, Y, which is elliptically fibered over a compact Kähler base B, which we abbreviate as $\mathbb{E}_\tau \hookrightarrow Y \to B$. If there are no singular fibers, the fibration is in fact topologically trivial, i.e. $Y = \mathbb{E}_\tau \times B$, implying that the base B is itself a Calabi-Yau variety, and also that τ does not vary over B. Here we will be interested in non-trivial fibrations, which necessarily include so-called Kodaira singular fibers [16, 17].[6] We consider

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[6] As a cautious remark, despite the name, these are the resolved fibers above singular loci of the fibration.

an elliptic fibration with a section, i.e. a map $B \to \mathbb{E}_\tau$, which implies the existence of a Weierstrass form of the elliptic fibration

$$y^2 = x^3 + fxw^4 + gw^6\,, \tag{2.1}$$

where f, g are sections of K_B^{-4} and K_B^{-6}, respectively, and thereby depend on the base B. Here K_B denotes the canonical class of the base. The hypersurface equation (2.1) is written in an ambient space, which is the fourfold obtained as the total space of the projectivization of the sum of line bundles over B

$$X_4 = \mathbb{P}(\mathcal{O}_B \oplus (\mathcal{O}_B \oplus K_B^{-1})^2 \oplus (\mathcal{O}_B \oplus K_B^{-1})^3)\,, \tag{2.2}$$

where $\mathcal{O}_B$ is the structure sheaf of B. The coordinates $[w, x, y]$ satisfy standard projective relations in $\mathbb{P}^{1,2,3}$ and their class is given by $[w] = \sigma$, $[x] = 2(\sigma + c_1(B))$ and $[y] = 3(\sigma + c_1(B))$. In particular, the class of the section of the elliptic fibration is denoted here by σ. For a more in depth review of elliptic fibrations and their geometry and more specifically the intersection theory used in the following, we refer the reader to e.g. [59–61]. For each point in the base, this equation defines an elliptic curve, whose complex structure can be determined via the j-function, which in turn depends on f and g. Singularities in the elliptic fibration are characterised by the vanishing of the discriminant Δ of the Weierstrass equation, i.e.

$$\Delta = 4f^3 + 27g^2 = 0\,, \tag{2.3}$$

which define complex codimension one loci in B. The type of singular fibers that can occur were classified by Kodaira-Néron, and are characterised in terms of the order of vanishing of (f, g, Δ) along the discriminant locus. More concretely, the different singularities are characterised in terms of the fibers that are obtained upon resolution.[7] The resulting Kodaira singular fibers are collections of rational curves ($\mathbb{P}^1$s) with precise intersection patterns and multiplicities, given e.g. in terms of affine Dynkin diagrams of ADE Lie algebras. The simplest Kodaira fiber is I_1, which has

$$I_1: \qquad \mathrm{ord}(f, g, \Delta) = (0, 0, 1)\,. \tag{2.4}$$

The fiber is a node, i.e. a $\mathbb{P}^1$ with a self-intersection. This fiber does not induce a singularity in the total space of the fibration and thus no resolution is necessary, as one can check directly. In F-theory, singularities of the axio-dilaton τ, i.e. vanishing of the discriminant Δ, determine the loci of the 7-branes, and the type of singularity characterises the (p, q) charges of the 7-branes under the $\mathrm{SL}(2, \mathbb{Z})$ self-duality of Type IIB. The I_1 singular fiber corresponds to a single D7-brane. The worldvolume of the 7-branes is $\Delta \times \mathbb{R}^{1,d}$. The effective theory of F-theory on $Y \times \mathbb{R}^{1,d}$ is given in terms of the gauge theories on the 7-branes coupled to $\mathcal{N} = 1$ supergravity in $d + 1$ dimensions.

For a given Weierstrass model, the complex structure τ of the elliiptic curve can be extracted from

$$j(\tau) = -1728\frac{(4f)^3}{\Delta}\,, \tag{2.5}$$

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[7]More precisely, these are Kähler deformations, which retain the Calabi-Yau property, so-called crepant resolutions.

where $j(\tau)$ is the Jacobi j-function. By expanding

$$j(\tau) = \frac{1}{q} + 744 + \cdots , \qquad (2.6)$$

where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. Using the asymptotic expansion along the loci where $\Delta = 0$ one can extract the local behaviour of τ. For example, the axio-dilaton close to a 7-brane wrapping the local divisor $z = 0$ in the base B has the profile

$$\tau = \frac{1}{2\pi\mathrm{i}} \log z + \cdots , \qquad (2.7)$$

which has a singularity at $z = 0$, and undergoes a monodromy $\tau \to \tau + 1$ around this locus. More general (p, q) 7-branes will have $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ monodromy. For elliptic K3s, this was derived in [62]. Specifically this singular behaviour of the axio-dilaton implies that the metric on the base will have singularities.

In the present context we are interested in solutions to the effective theory. Naively we would define the F-theory supergravity solutions on Y in terms of Type IIB supergravity, on B including τ which varies over B. However when the elliptic fibration has singularities as in (2.7), the metric that is induced on the base B is expected to be singular. In the case of K3 surfaces, this can be made explicit for non-compact [62] and for compact K3s [63, 64], who also give a precise measure for the divergence of the curvature scalar close to the singular fibers. Thus a supergravity approach seems at first sight to be somewhat questionable.

In the absence of a first principle formulation of F-theory, the effective action of a compactification on an elliptically fibered manifold Y is defined in terms of M/F-duality: F-theory on $Y \times S^1$ defines the same effective theory as M-theory on Y. This will be the approach that we will use to define the F-theory solution: there is a dual M-theory solution on an elliptically fibered manifold Y, which is smooth and Calabi-Yau, with a smooth Ricci-flat Kähler metric. Reducing to Type IIA, and performing a T-duality results in a Type IIB solution with varying τ. The effective theory is obtained solely by considerations in the 11d supergravity compactification. Whenever we allow for not only I_1 but enhanced singular fibers, the computations are done in the resolved geometry, where the singularities are blown up retaining the Calabi-Yau property of Y_3.

2.2 Elliptic fibrations

In the following it will be useful to have some geometric basics about elliptic fibrations in place for studying F-theory solutions. Here our main interest is in elliptic threefolds, but much can be generalised to other dimensions. We consider Calabi-Yau threefolds Y_3, which are elliptically fibered over a base B, which is a complex surface. Denote the projection map $\pi : Y_3 \to B$. Furthermore we assume there is a section, which as explained earlier implies the existence of a Weierstrass model.

It will be very important in the following to determine the possible divisors (4d submanifolds) in such a geometry, which is the content of the Shioda-Tate-Wazir theorem [65], which implies that the divisors of an elliptic Calabi-Yau threefold Y_3 with a section, fall into the following three classes:

1. <u>Section</u>: this is the divisor obtained by the image of the base B in Y_3. We denote it simply by B. The dual $(1,1)$-form will be denoted by ω_0.[8]

2. <u>Pull-back of curves in the base B</u>: for every effective curve $C_\alpha \in H_2(B)$ we have a divisor in Y_3 given by $\pi^*(C_\alpha)$. We will refer to these as $\widehat{C}_\alpha \equiv \pi^*(C_\alpha)$, and denote the dual $(1,1)$-forms by ω_α.

3. <u>Resolution/Cartan divisors</u>: these divisors occur whenever there is a singularity in the Weierstrass model of the elliptic fibration, and they are given in terms of rational curves, that are obtained from the resolution of the singularities, fibered over a curve in the base (which are components of the discriminant). In the literature these are often referred to as Cartan divisors, as they are (in many cases) labeled by the simple roots of the Lie algebra associated to the Kodaira singular fiber. The Cartan divisors will be denoted by D_i, and the dual $(1,1)$-forms by ω_i.

For most part of the paper we will consider smooth Weierstrass fibrations, i.e. there are no Cartan divisors. However this can be easily generalised and we will comment on this throughout the paper. Divisors are dual to $(1,1)$-forms, and the Shioda-Tate-Wazir theorem thus implies that the Kähler form of the Calabi-Yau can be expanded as

$$ J_{Y_3} = k_0 \omega_0 + \sum_\alpha k_\alpha \omega_\alpha + \sum_i k_i \omega_i \,. \tag{2.8} $$

We will require that the Kähler class of the base

$$ J_B = \sum_\alpha k_\alpha \omega_\alpha \,, \tag{2.9} $$

is dual inside B to a curve C, implying that $k_\alpha \in \mathbb{Z}^+$. This means that J_B is in fact the Kähler class associated to the Hodge metric on B [66]. However, we do not require any such integrality for k_0.

The non-trivial triple intersections of the basis $\omega_I = \{\omega_0, \omega_\alpha, \omega_i\}$ in the Calabi-Yau

$$ C_{IJK} = D_I \cdot_{Y_3} D_J \cdot_{Y_3} D_K = \int_{Y_3} \omega_I \wedge \omega_J \wedge \omega_K \,, \tag{2.10} $$

can be evaluated in terms of data of the base B as follows[9]

$$
\begin{aligned}
C_{000} &= \int_B c_1(B)^2 = 10 - h^{1,1}(B) \\
C_{00\alpha} &= -c_1(B) \cdot C_\alpha \\
C_{0\alpha\beta} &= Q_{\alpha\beta} \\
C_{\alpha ij} &= -\mathcal{C}_{ij} Q_{\alpha\beta} C^\beta \,,
\end{aligned}
\tag{2.11}
$$

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[8]An elliptic fibration can have more rational sections, in which case there are additional divisors and $(1,1)$ forms, which generate the Mordell-Weil group of the fibration. As this will not play any role here, we refrain from discussing these further.

[9]Whenever we write $\cdot$ without any subscript in the following, this will denote the intersection in B, unless otherwise stated.

where $Q_{\alpha\beta}$ is the intersection form on B and $\mathcal{C}_{ij}$ the Cartan matrix of the gauge algebra $\mathfrak{g}$ associated to the singularity. The triple intersection C_{ijk} were determined in [67–69] and depend on codimension two singularities, which are labeled by representations of $\mathfrak{g}$.

In deriving these intersection numbers we have made use of the intersection relation in Y_3[10]

$$\sigma \cdot_{Y_3} (\sigma + c_1(B)) = 0\,. \tag{2.12}$$

We will also need to compute intersections with $c_2(Y_3)$. The total Chern class for the Calabi-Yau can be written as

$$c(Y_3) = (1 + [w])(1 + [x])(1 + [y])\frac{c(B)}{(1 + [y^2])}\,, \tag{2.13}$$

where $c(B)$ is the total Chern class of the base and the denominator corresponds to the class of the hypersurface equation (2.1). Expanding this to second order we obtain[11]

$$c_2(Y_3) = c_2(B) + 11c_1(B)^2 + 12\omega_0 \wedge c_1(B)\,, \tag{2.14}$$

which allows the computation of integrals over $c_2(Y_3)$ using the intersection numbers in (2.11).

Finally, we will often consider curves $C \subset B$, and it will be useful to recall the adjunction formula

$$K_C = (K_B + C)|_C\,, \tag{2.15}$$

where K_C and K_B are the canonical classes of C and B, respectively. For a genus g curve this implies

$$2(g - 1) = C \cdot C - c_1(B) \cdot C\,. \tag{2.16}$$

Throughout our considerations we will assume the base B to be smooth as a variety, albeit the induced metric on B will have singularities that we will discuss in some detail later on.

2.3   D3-branes in F-theory and 2d (0, 4) SCFTs

The supergravity solutions that will be studied in this paper are dual to 2d $\mathcal{N} = (0, 4)$ SCFTs arising from D3-branes wrapped on a curve, C, in the base of an elliptic Calabi-Yau threefold, in an F-theory compactification to 6d. The field content for these 2d SCFTs was worked out in [9], where in particular the abelian zero mode spectrum and the left and right central charges were computed. The zero mode spectrum in terms of $(0, 4)$ multiplets was found to be

$(0, 4)$ multiplet	Multiplicity	(c_R, c_L)
Hyper	$\frac{1}{2}C \cdot C + \frac{1}{2}c_1(B) \cdot C$	$(6, 4)$
Twisted Hyper	1	$(6, 4)$
Fermi	$\frac{1}{2}C \cdot C - \frac{1}{2}c_1(B) \cdot C + 1$	$(0, 2)$

$$\tag{2.17}$$

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[10]This follows from the fact that w, x, y cannot vanish at the same time, and thus $[w] \cdot [x] \cdot [y] = 0$ as intersections in X_4, or noting that the class of the hypersurface (2.1) is $[y^2]$, this becomes $\sigma \cdot (\sigma + c_1(B)) \cdot [y^2] = 0$.

[11]Here we used the relation (2.12), which holds on Y_3.

In addition, one has half-Fermi multiplets arising from 3-7 strings, which contribute $c_{37} = 8c_1(B) \cdot C$ to the left-moving central charge. The left and right central charges are computed by summing the contributions from each multiplet with their appropriate multiplicity, and are given by [8, 9]

$$c_R = 3C \cdot C + 3c_1(B) \cdot C + 6 \,,$$
$$c_L = 3C \cdot C + 9c_1(B) \cdot C + 6 \,. \tag{2.18}$$

Notice that upon using the adjunction formula (2.16), the right central charge may be rewritten as

$$c_R = 6 \left(g + c_1(B) \cdot C \right) \,, \tag{2.19}$$

which is manifestly a multiple of 6, as expected generically for $(0,4)$ SCFTs with small superconformal algebra [70]. Under M/F-duality, this is equivalent to M5-branes wrapped on the elliptic surface $\widehat{C} = \pi^*(C)$ in the Calabi-Yau threefold. The 2d spectrum obtained from a single M5-brane wrapped on an elliptic surface was also determined in [9] as

$(0,4)$ multiplet	Multiplicity	(c_R, c_L)
Hyper	$\frac{1}{2}C \cdot C + \frac{1}{2}c_1(B) \cdot C + 1$	$(6,4)$
Fermi	$\frac{1}{2}C \cdot C - \frac{1}{2}c_1(B) \cdot C + 1$	$(0,2)$
Half-Fermi	$8c_1(B) \cdot C$	$(0,1)$

$$\tag{2.20}$$

Here, the half-Fermi multiplets arise directly from the reduction of the 6d $\mathcal{N} = (2,0)$ tensor multiplet. This spectrum matches that of the D3-brane wrapped on C and therefore the left and right central charges are also given by (2.18). These central charges are computed for a single D3-brane, i.e. $N = 1$. In the following we will compute these holographically for general N.

3 AdS$_3$ solutions in F-theory dual to 2d $(0,4)$ theories

In this section we switch gears and turn to an explicit analysis of the supersymmetry equations of the ten-dimensional Type IIB supergravity. We will derive the AdS$_3$ solutions of interest starting from a very general ansatz and requiring the existence of $(0,4)$ supersymmetry in the boundary SCFTs. The methods we use are by now completely standard, but we nevertheless include some details here, for the benefit of readers that may not be familiar with these.

3.1 Type IIB Killing spinor equations

To find supersymmetric solutions we first determine the constraints implied by the existence of certain Killing spinors in Type IIB. We follow the Type IIB supergravity conventions presented in [28], which we briefly summarise here. The equations of motion and supersymmetry transformations for bosonic configurations were originally found in [71, 72]. The Neveu-Schwarz Neveu-Schwarz (NS-NS) sector of Type IIB supergravity consists of the metric g, a real scalar ϕ, called the dilaton, and a real two-form potential, $B^{(2)}$. The Ramond Ramond (RR) sector includes a real scalar potential $C^{(0)}$, a real two-form potential

$C^{(2)}$, and a real four-form potential, $C^{(4)}$, with self dual-field strength. It is convenient to combine these fields into complex ones: we combine the scalars as

$$\tau = \tau_1 + \mathrm{i}\tau_2 \equiv C^{(0)} + \mathrm{i}e^{-\phi}\,, \tag{3.1}$$

which we refer to as the axio-dilaton, and the three-form field strengths as

$$G \equiv \mathrm{i}e^{\phi/2}(\tau \mathrm{d}B^{(2)} - \mathrm{d}C^{(2)})\,. \tag{3.2}$$

We are interested in bosonic solutions and therefore we set the fermionic content of the theory to zero. We still wish to preserve supersymmetry, therefore the supersymmetry variations must vanish identically. The bosonic variations, being proportional to fermionic fields vanish trivially, whilst for the fermionic variations

$$\delta\psi_M = \mathcal{D}_M \epsilon - \frac{1}{96}\left(\Gamma_M{}^{P_1\dots P_3} G_{P_1\dots P_3} - 9\Gamma^{P_1 P_2} G_{M P_1 P_2}\right)\epsilon^c + \frac{\mathrm{i}}{192}\Gamma^{P_1\dots P_4} F_{M P_1 \dots P_4}\epsilon\,, \tag{3.3}$$

$$\delta\lambda = \mathrm{i}\Gamma^M P_M \epsilon^c + \frac{\mathrm{i}}{24}\Gamma^{P_1\dots P_3} G_{P_1\dots P_3}\epsilon\,, \tag{3.4}$$

we must choose the bosonic fields such that they vanish. These are the Killing spinor equations of Type IIB supergravity. The supersymmetry parameter ϵ is a Weyl spinor satisfying the projection condition $\Gamma_{11}\epsilon = -\epsilon$.

The covariant derivative $\mathcal{D}$ is with respect to both Lorentz transformations and local U(1) transformations,

$$\mathcal{D}_\mu = \nabla_\mu - \mathrm{i}q Q_\mu\,, \tag{3.5}$$

where Q is the gauge field for the U(1) transformations. It takes the form

$$Q = -\frac{1}{2\tau_2}\mathrm{d}\tau_1\,, \tag{3.6}$$

where $\tau = \tau_1 + \mathrm{i}\tau_2$ is the axio-dilaton and q is the charge under the U(1). The Killing spinors have U(1) charge $1/2$, P has charge 2, and G has charge 1. The field P, appearing in the dilatino equation, is constructed from the axio-dilaton as

$$P = \frac{\mathrm{i}}{2\tau_2}\mathrm{d}\tau\,. \tag{3.7}$$

The equations of motion consist of the Einstein equation

$$R_{MN} = 2P_{(M}P^*_{N)} + \frac{1}{96}F_{M P_1\dots P_4}F_N{}^{P_1\dots P_4} + \frac{1}{8}\left(2G_{(M}{}^{P_1 P_2}G^*_{N)P_1 P_2} - \frac{1}{6}g_{MN}G^{P^1\dots P_3}G^*_{P_1\dots P_3}\right)\,, \tag{3.8}$$

and the equations of motion for the fluxes

$$\mathcal{D}*G = P \wedge *G^* - \mathrm{i}F \wedge G\,, \tag{3.9}$$

$$\mathcal{D}*P = -\frac{1}{4}G \wedge *G\,. \tag{3.10}$$

These are supplemented by the Bianchi identities

$$\mathcal{D}P = 0\,, \tag{3.11}$$

$$\mathcal{D}G = -P \wedge G^*\,, \tag{3.12}$$

$$\mathrm{d}F = \frac{\mathrm{i}}{2}G \wedge G^*\,, \tag{3.13}$$

and the self-duality constraint

$$F = *F\,. \tag{3.14}$$

3.2 AdS$_3$ ansatz

In this paper we consider the most general class of bosonic Type IIB supergravity solutions with $\mathrm{SO}(2,2)$ symmetry and vanishing three-form fluxes. We take the 10d metric in Einstein frame to be a warped product of the form

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + \mathrm{d}s^2(\mathcal{M}_7)\right)\,, \tag{3.15}$$

where $\mathrm{d}s^2(AdS_3)$ is the metric on AdS$_3$, with Ricci tensor $R_{ab} = -2m^2 g_{ab}$, and $\mathrm{d}s^2(\mathcal{M}_7)$ is the metric on an arbitrary internal 7d manifold $\mathcal{M}_7$. To preserve the $\mathrm{SO}(2,2)$ symmetry of AdS$_3$ we impose $A \in \Omega^{(0)}(\mathcal{M}_7, \mathbb{R})$, $P \in \Omega^{(1)}(\mathcal{M}_7, \mathbb{C})$ and $\tau \in \Omega^{(0)}(\mathcal{M}_7, \mathbb{C})$. In this paper we will not consider solutions with non-trivial three-form fluxes, thus our fluxes take the form

$$F = (1 + *)\mathrm{dvol}(\mathrm{AdS}_3) \wedge F^{(2)}\,, \qquad G = 0\,, \tag{3.16}$$

with $F^{(2)} \in \Omega^{(2)}(\mathcal{M}_7, \mathbb{R})$. The Bianchi identity for the five-form flux gives two equations for $F^{(2)}$, that read

$$\mathrm{d}F^{(2)} = 0\,, \qquad \mathrm{d}\hat{*}_7 F^{(2)} = 0\,, \tag{3.17}$$

where $\hat{*}_7$ is the Hodge star on the unwarped metric $\mathrm{d}s^2(\mathcal{M}_7)$.

We now use the spinor ansatz developed in appendix A and apply the results of appendix B for Killing spinors of AdS$_3$. We consider the Killing spinor ansatz

$$\epsilon = \psi_1 \otimes \mathrm{e}^{A/2}\xi_1 \otimes \theta + \psi_2 \otimes \mathrm{e}^{A/2}\xi_2 \otimes \theta\,, \tag{3.18}$$

where ψ_i are Majorana Killing spinors on AdS$_3$ and satisfy

$$\nabla_a \psi_i = \frac{\alpha_i m}{2}\rho_a \psi_i\,. \tag{3.19}$$

The constants $\alpha_i = \pm 1$ are the eigenvalues of the matrix W discussed in appendix B. We assume that the ψ_i are independent and that the ξ_i are Dirac spinors. With this ansatz we can preserve $\mathcal{N} = \{(4,0),\ (2,2),\ (0,4)\}$, depending on the signs in (3.19), in the dual SCFT, as discussed in appendix B. Each Majorana AdS$_3$ Killing spinor will contribute a single superconformal supercharge and a single Poincaré supercharge. The two independent Dirac spinors will then imply that we preserve four superconformal supercharges and four Poincaré supercharges and hence $\mathcal{N} = \{(4,0),\ (2,2),\ (0,4)\}$ depending on the signs in (3.19).

Decomposing the 10d supersymmetry equations, (3.3) and (3.4), by using the above ansatz (3.18) we obtain the equations

$$\gamma^\mu P_\mu \xi_j^c = 0 \,, \tag{3.20}$$

$$\left(\frac{1}{2} \partial_\mu A \gamma^\mu - \frac{\mathrm{i}\alpha_j m}{2} + \frac{\mathrm{e}^{-4A}}{8} \slashed{F}^{(2)} \right) \xi_j = 0 \,, \tag{3.21}$$

$$\left(\mathcal{D}_\mu + \frac{\mathrm{i}\alpha_j m}{2} \gamma_\mu - \frac{\mathrm{e}^{-4A}}{8} F^{(2)}_{\nu_1 \nu_2} \gamma_\mu{}^{\nu_1 \nu_2} \right) \xi_j = 0 \,, \tag{3.22}$$

for each of the two spinors ξ_i. Note that we can derive some immediate consequences of the algebraic condition (3.20), which implies

$$P^2 \bar{\xi}_j \xi_i = 0 \,. \tag{3.23}$$

In particular for $i = j$ we have that $\bar{\xi}_i \xi_i \neq 0$ and therefore we see that necessarily

$$P^2 = 0 \,. \tag{3.24}$$

Together with the equation of motion $\mathrm{d} * P = 0$, this implies that τ is harmonic

$$\Box \tau = 0 \,. \tag{3.25}$$

We are specifically interested in solutions where τ varies over the compact part of the space. In a compact space without boundary the only harmonic functions are constant [73]. This was noted in [28] in the context of supersymmetric AdS$_5$ solutions and it implies that we must allow for singularities in τ, as anticipated from general F-theory considerations; in particular we will allow for log singularities as in (2.7).

3.3 Constraints for 2d (0, 4) supersymmetry

In the rest of the paper we specialise to the case where the dual 2d SCFTs have chiral supersymmetry. In particular, we choose the convention where the supercharges are right-moving i.e. $\mathcal{N} = (0, 4)$ supersymmetry, which implies we take $\alpha_1 = \alpha_2 = 1$.

To classify the solutions we shall analyse the G-structure defined by the Killing spinors of the solution. This is a standard technique for finding the necessary and sufficient conditions imposed by supersymmetry [74]. Let the Killing spinors of the solution have isotropy group G, this then defines a canonical G-structure. One may construct tensors from these Killing spinors as spinor bilinears, it follows that these tensors are then G-invariant. These G-invariant tensors will satisfy a number of algebraic relations depending on the particular G-structure defined. One then analyses the information obtained from the Killing spinor equations by computing the differential and algebraic conditions they impose on the G-invariant tensors, these define the so called "intrinsic torsion" of the G-structure. Finally one computes the integrability conditions of the Killing spinor equations and the torsion conditions of the G-invariant tensors, and determines which equations of motion and Bianchi identities are automatically satisfied, imposing the remaining conditions. This classification procedure provides the most general local form of the solution given in terms of the information contained in the G-structure.

We find that the most general solutions in this class admit an SU(2) structure. In seven-dimensions an SU(2) structure implies the existence of three independent one-forms orthogonal to a four-dimensional foliation with SU(2) structure. This is specified by a real two-form of maximal rank and a complex two-form satisfying the SU(2) structure relations which we give later. We define the G-invariant tensors obtained from the Killing spinors in appendix C. To compute the algebraic relations imposed by the SU(2) structure we shall introduce an orthonormal frame using the gamma matrices defined in appendix A. One may recover these results by making use of Fierz identities.

In the following we summarise the results, with more detailed provided in appendix C, where in particular the torsion conditions for general α_i are written down. Here we specialise to the relevant case $\alpha_1 = \alpha_2 = 1$.

From (C.18) and (C.22) we obtain the following conditions on the scalar bilinears

$$S_{11} = S_{22} = 1 \,, \tag{3.26}$$
$$A_{11} = A_{22} = A_{12} = S_{12} = 0 \,. \tag{3.27}$$

From (C.28) we see that there are three independent Killing vectors. Imposing that the Killing vectors lie along a subspace defined by the vielbeins e^5, e^6, e^7, consistent with an SU(2) structure, is equivalent to imposing the projection condition

$$\gamma_{1234}\xi_i = -\xi_i \,. \tag{3.28}$$

In addition we have the freedom to choose K_{11} to be parallel to e^7. In this frame the independent one-forms and two-forms are given by[12]

$$K_{11} = -K_{22} = e^7 \,, \tag{3.29}$$
$$K_{12} = e^5 - \mathrm{i}e^6 \,, \tag{3.30}$$
$$B_{12} = 0 \,, \tag{3.31}$$
$$U_{11} = -\mathrm{i}(e^{12} + e^{34} - e^{56}) \,, \tag{3.32}$$
$$U_{22} = -\mathrm{i}(e^{12} + e^{34} + e^{56}) \,, \tag{3.33}$$
$$V_{11} = V_{22} = 0 \,, \tag{3.34}$$
$$V_{12} = -(e^1 - \mathrm{i}e^2) \wedge (e^3 - \mathrm{i}e^4) \,. \tag{3.35}$$

The remaining forms may be expressed in terms of the forms defined above as

$$
\begin{aligned}
U_{12} &= K_{11} \wedge K_{12} \,, \\
X_{11} &= U_{11} \wedge K_{11} \,, \\
X_{22} &= U_{22} \wedge K_{22} \,, \\
X_{12} &= U_{11} \wedge K_{12} = U_{22} \wedge K_{12} \,, \\
Y_{11} &= V_{12} \wedge K_{12}^* \,, \\
Y_{22} &= -V_{12} \wedge K_{12} \,, \\
Y_{12} &= -V_{12} \wedge K_{11} = V_{12} \wedge K_{22} \,.
\end{aligned}
\tag{3.36}
$$

---

[12]For notational simplification, we shall make use of the following shorthand notation for the wedge of multiple vielbein $e^{a_1} \wedge \ldots \wedge e^{a_n} = e^{a_1 \cdots a_n}$.

3.4 F-theory AdS$_3$ solution

After introducing the frame, it is now possible to reduce the differential conditions and determine the final Type IIB supergravity solution. The remaining conditions are

$$e^{-4A}d\left(e^{4A}K_{jj}\right) = -2imU_{jj} - e^{-4A}F^{(2)}, \qquad (3.37)$$

$$e^{-4A}d\left(e^{4A}K_{12}\right) = -2imU_{12}, \qquad (3.38)$$

$$d\left(e^{4A}U_{ij}\right) = 0, \qquad (3.39)$$

$$\mathcal{D}\left(e^{6A}V_{12}\right) = 0. \qquad (3.40)$$

First we determine $F^{(2)}$. The frame computation implies $K_{11} = -K_{22} = e^7$, and inserting this into (3.37) we find

$$F^{(2)} = -ime^{4A}(U_{11} + U_{22}) = -2me^{4A}\left(e^{12} + e^{34}\right). \qquad (3.41)$$

Notice that (3.39) implies that (3.41) satisfies the Bianchi identity for $F^{(2)}$. From this explicit expression we may also compute $*F^{(2)}$ and show that it satisfies its equation of motion. Observe from the algebraic equations (C.14)–(C.17) we have the relation

$$\partial_\mu A = -\frac{e^{-4A}}{4}F_{\mu\nu}\bar{\xi}_1\gamma^\nu\xi_1, \qquad (3.42)$$

which together with (3.41) implies that the warp factor is constant

$$dA = 0. \qquad (3.43)$$

Next we can determine the Killing vectors. From (C.28) we see that there are three independent Killing vectors of the *full* solution whose dual one-forms are

$$\begin{aligned}
K_{11} &= -K_{22} = e^7, \\
\mathrm{Re}\,[K_{12}] &= e^5, \\
\mathrm{Im}\,[K_{12}] &= -e^6.
\end{aligned} \qquad (3.44)$$

From the torsion conditions (3.37) the dual one-forms to these Killing vectors satisfy the differential conditions

$$de^5 = 2me^{67}, \qquad (3.45)$$

$$de^6 = 2me^{75}, \qquad (3.46)$$

$$de^7 = 2me^{56}, \qquad (3.47)$$

which is a warped form of the equations obeyed by the SU(2) invariant one-forms.[13]

---

[13]The Maurer-Cartan left-invariant one-forms in the coordinates we are using are

$$\sigma_{1,L} = -\sin\psi d\theta + \cos\psi\sin\theta d\varphi, \quad \sigma_{2,L} = \cos\psi d\theta + \sin\psi\sin\theta d\varphi, \quad \sigma_{3,L} = d\psi + \cos\theta d\varphi.$$

The coordinates have periods $\psi \in [0, 4\pi]$, $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$. These satisfy $d\sigma_{i,L} = \frac{1}{2}\epsilon_{ijk}\sigma_{j,L}\wedge\sigma_{k,L}$. The right-invariant one-forms we shall take are

$$\sigma_{1,R} = \sin\varphi d\theta - \cos\varphi\sin\theta d\psi, \quad \sigma_{2,R} = \cos\varphi d\theta + \sin\varphi\sin\theta d\psi, \quad \sigma_{3,R} = d\varphi + \cos\theta d\psi.$$

These satisfy $d\sigma_{i,R} = -\frac{1}{2}\epsilon_{ijk}\sigma_{j,R}\sigma_{k,R}$. Of course we could take the right-invariant one-forms to solve the same equation as the left-invariant one-forms however in the present case we have the desirable property $\sigma_{1,L}\wedge\sigma_{2,L}\wedge\sigma_{3,L} = \sigma_{1,R}\wedge\sigma_{2,R}\wedge\sigma_{3,R}$.

On the 4d subspace B we have an SU(2) structure with the Kähler-form given by

$$J_B = \frac{i}{2} \left(U_{11} + U_{22} \right) = e^{12} + e^{34} \,, \tag{3.48}$$

with corresponding holomorphic two-form satisfying $\frac{1}{2} \Omega_B \wedge \bar{\Omega}_B = J_B \wedge J_B = 2\mathrm{dvol}(B)$, which takes the form

$$\Omega_B = -V_{12}^* = \left(e^1 + i e^2 \right) \wedge \left(e^3 + i e^4 \right) \,. \tag{3.49}$$

With these definitions the remaining torsion conditions become

$$\mathrm{d}J_B = 0 \,, \tag{3.50}$$

$$\bar{D}\Omega_B = 0 \,. \tag{3.51}$$

Furthermore, from (C.20) and (C.21) it follows that P may only have components along B and using (3.20) we find

$$J_{Bm}{}^n P_n = i P_m \,, \tag{3.52}$$

and hence P is holomorphic. The form of P then implies that τ is holomorphic and therefore τ satisfies

$$\mathrm{d}\tau \wedge \Omega_B = 0 \,. \tag{3.53}$$

Notice that (3.52) implies the necessary conditions (3.24) and (3.25). From (3.51) we may identify $-Q$ as the canonical Ricci-form potential on the Kähler manifold B and hence we have

$$\mathfrak{R} + \mathrm{d}Q = 0 \,, \tag{3.54}$$

where $\mathfrak{R}$ is the Ricci-form on B. Making use of the identity,

$$\mathfrak{R}^{m_1}{}_{m_2} = J^{m_1}{}_n R^n{}_{m_2} \,, \tag{3.55}$$

the condition in (3.54) may be expressed as

$$R_{mn}^{Y_3} \equiv R_{mn} - \frac{1}{2\tau_2^2} \left(\partial_m \tau_1 \partial_n \tau_1 + \partial_m \tau_2 \partial_n \tau_2 \right) = 0 \,. \tag{3.56}$$

This equation relates the Ricci tensor of the base to the variation of τ over B. In particular, this is the Ricci flatness condition for the metric of an elliptically fibered Calabi-Yau threefold Y_3 valid away from the singularities in the fiber.

Since τ is holomorphic, away from loci where the fiber degenerates, the metric for the elliptically fibered Calabi-Yau can be written as

$$\mathrm{d}s^2(Y_3) = \frac{1}{\tau_2} \left((\mathrm{d}x + \tau_1 \mathrm{d}y)^2 + \tau_2^2 \mathrm{d}y^2 \right) + \mathrm{d}s^2(B) \,. \tag{3.57}$$

Indeed, imposing that this metric is Ricci flat implies that the Ricci tensor on B satisfies (3.56). As was noted in [62] this local metric is singular over the discriminant locus of the elliptic fibration.

To exhibit the Calabi-Yau condition, we construct the Kähler form and holomorphic three-form of the Calabi-Yau threefold from the corresponding quantities of the base, which define an SU(3) structure. Let the vielbein on the fibration be

$$e^1 = \frac{1}{\sqrt{\tau_2}}(\mathrm{d}x + \tau_1 \mathrm{d}y)\,, \quad e^2 = \sqrt{\tau_2}\mathrm{d}y\,, \tag{3.58}$$

then

$$J_{Y_3} = e^{12} + J_B\,, \qquad \Omega_{Y_3} = (e^1 + \mathrm{i}e^2) \wedge \Omega_B\,. \tag{3.59}$$

With this frame on the elliptic fibration and giving indices $3, 4, 5, 6$ to the base, some relevant components of the spin connection which are useful later on are

$$\omega^1{}_2 = Q\,, \qquad \omega^3{}_4 + \omega^5{}_6 = -Q\,, \qquad \omega^4{}_5 + \omega^3{}_6 = 0\,, \qquad \omega^3{}_5 - \omega^4{}_6 = 0\,. \tag{3.60}$$

The Calabi-Yau condition in terms of this SU(3) structure is equivalent to

$$\mathrm{d}J_{Y_3} = 0\,, \qquad \mathrm{d}\Omega_{Y_3} = 0\,. \tag{3.61}$$

Upon using $\mathrm{d}J_B = 0$ it follows trivially that $\mathrm{d}J_{Y_3} = 0$. Consider instead $\mathrm{d}\Omega_{Y_3} = 0$; we have

$$\begin{aligned}
\mathrm{d}\Omega_{Y_3} &= \left(-\frac{1}{2\tau_2}\mathrm{d}\tau_2 - \mathrm{i}Q\right) \wedge \Omega_{Y_3} + \frac{1}{\tau_2}\mathrm{d}\tau \wedge \mathrm{d}y \wedge \Omega_B \\
&= \frac{\mathrm{i}}{2\tau_2}\mathrm{d}\tau \wedge \Omega_{Y_3} + \frac{1}{\tau_2}\mathrm{d}\tau \wedge \mathrm{d}y \wedge \Omega_B\,,
\end{aligned} \tag{3.62}$$

where we have used (3.51) in the first line. Upon using the holomorphicity of τ and that it depends only on the base coordinates this is identically zero. This shows that (3.51) and therefore also (3.54) and (3.56) are equivalent to B being the base of an elliptically fibered Calabi-Yau threefold.

The Type IIB equations of motion follow immediately from supersymmetry. The equation of motion for F, $\mathrm{d}F = 0$, follows immediately from (3.50). Moreover one may rewrite (3.56) as

$$R_{mn} = 2P_{(m}P_{n)}^*\,, \tag{3.63}$$

which is precisely the form in which τ appears in the Einstein equation (3.8). With the explicit form of F it is easy to show that indeed the Einstein equation is also satisfied.

In summary, the solution

$$\begin{array}{c}
\mathbb{E}_\tau \\
\downarrow \\
\mathrm{AdS}_3 \times S^3 \times B
\end{array} \tag{3.64}$$

where the elliptic fibration over B gives rise to a Calabi-Yau threefold, is thus given by[14]

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{4m^2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) + \frac{1}{m_B^2}\mathrm{d}s^2(B)\right)\,, \tag{3.65}$$

$$F = -(1 + *)\frac{2m\mathrm{e}^{4A}}{m_B^2}J_B \wedge \mathrm{dvol}(\mathrm{AdS}_3)\,, \tag{3.66}$$

$$P = \frac{\mathrm{i}}{2\tau_2}\mathrm{d}\tau\,, \tag{3.67}$$

---

[14]Of course, the one-forms σ_i can be taken to be either $\sigma_{i,L}$ or $\sigma_{i,R}$.

where J_B is the Kähler form on the base of the elliptically fibered Calabi-Yau threefold, and τ varies holomorphically over B. Here, $1/m_B$ is the length scale associated to the base B.

The possible base manifolds of an elliptically fibered Calabi-Yau threefold were determined in [14, 15] and found to be one of the following: $\mathbb{P}^2$, Hirzebruch surfaces $\mathbb{F}_m$, blow-ups thereof, and Enriques surfaces. In the case where the elliptic fibration is trivial then the base itself must be a Calabi-Yau two-fold, which is either a K3 surface or T^4. This is precisely the solution obtained in [36] which results in $(4,4)$ supersymmetry, and is the dual to the classic D1-D5 system [75].

At this point it is perhaps timely to recall that our description is valid away from the singular loci of τ.[15] As explained earlier, we will allow for singularities in τ, given by for instance by (2.7), which have a characterisation in terms of Kodaira singular fibers. The Ricci-flatness condition then takes the form

$$K_B = -\sum_i a_i D_i\,, \tag{3.68}$$

where D_i are the Cartan divisors of the resolution of the singularity and a_i depend on the Kodaira type of the singular fiber [2, 3].

For the case of an elliptically fibered K3 surface with 24 I_1 singularities, a semi-Ricci-flat metric was constructed in [63]. The metric in the neighborhood of each I_1 fiber is given by the Ooguri-Vafa metric [76]. The semi-flat metric was constructed by gluing the Ooguri-Vafa metric to the metric constructed in [62] around the 24 points where the fiber becomes singular. It was shown in [63] that in the limit $\mathrm{vol}(\mathbb{E}_\tau) \to 0$ the semi-flat metric reduced to a singular metric on $\mathbb{P}^1$, the base of the elliptic K3, where the singularities are exactly at the points where the fiber is singular. In [45, 46] the metric in [62] was used to give some estimate of the curvature singularity, and it was argued that in the large N limit, the gravity approximation can still be trusted. One expects in higher dimensions that the metric on the base is also singular in the F-theory limit. However, as we shall discuss in section 4, one is still able to compute quantities of the dual CFT using this solution. It would be interesting to estimate the curvature singularities in these higher-dimensional cases, to support these findings.

In the next subsection we shall describe a supersymmetry preserving $\mathbb{Z}_M$ quotient of these solutions. This will be important for identifying the superconformal R-symmetry of the dual $(0,4)$ SCFT in the IR, and furthermore will be a key ingredient in performing the duality to 11d supergravity in section 5.

3.5 Lens space solution

Manifest in the solution is an S^3 which has isometry group $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, a subgroup of which realises the R-symmetry of the dual SCFT geometrically. The Killing

---

[15]This manifests itself e.g. by noting that since $c_1(B) = 2\pi\mathfrak{R}$ we would have $c_1(B) \wedge c_1(B) = 0$, contradicting the global property that

$$\int_B c_1(B) \wedge c_1(B) = 10 - h^{1,1}(B)\,.$$

spinors transform non-trivially under the R-symmetry but are singlets under flavour symmetries. We shall find that the Killing spinors of this solution are only charged under one of the SU(2)s, which identifies the small $\mathcal{N} = (0,4)$ superconformal R-symmetry. Furthermore, by inspection of the Killing spinors it is apparent that one can extend the solution found above by quotienting the S^3 by a discrete group $\Gamma \subset \mathrm{SU}(2)_L$ and still preserve the same amount of supersymmetry. This generalises the solution described in section 3.4 to the class

$$\mathrm{AdS}_3 \times S^3/\Gamma \times B \, . \tag{3.69}$$

We will focus on the case that $\Gamma = \mathbb{Z}_M$, where the quotient has the effect of changing the period of ψ, the coordinate of the Hopf fiber, so that $\psi \sim \psi + 4\pi/M$ rather than being 4π periodic. We shall show that the Killing spinors we obtain are $\mathrm{SU}(2)_L$ singlets, and in particular independent of ψ, therefore quotienting by $\mathbb{Z}_M$ does not break any supersymmetry.

It suffices to compute the Killing spinors in Einstein frame as this will not affect the above analysis. Moreover, as we have taken the Killing spinors to be a direct product as in (3.18) we need only consider solving the seven-dimensional Killing spinor equations (3.20)–(3.22). The Killing spinor equation obtained by restricting (3.22) to the base of the elliptically fibered Calabi-Yau is

$$\nabla_m \xi - \frac{\mathrm{i}}{2} Q_m \xi = 0 \, . \tag{3.70}$$

This follows by restricting the covariantly constant Killing spinor equation of the elliptically fibered Calabi-Yau to the base by using the results for the spin connection in (3.60). Equivalently, one can notice that this is precisely the canonical spin^c Killing spinor equation on a Kähler manifold where $-Q$ is the Ricci one-form potential, as shown in the previous subsection.[16] One may take the Killing spinor on the base of an elliptically fibered Calabi-Yau manifold to be constant if one imposes suitable projection conditions. Using the relations for the spin-connection of an elliptically fibered Calabi-Yau, as computed in (3.60), one finds that the projection conditions are

$$\gamma^{34}\xi = \gamma^{56}\xi = -\mathrm{i}\xi \, , \tag{3.71}$$

where the indices are flat. In conclusion, to solve the Killing spinor equation on the base we need only to consider a constant spinor satisfying the projection conditions (3.71). Note that (3.21) is automatically satisfied thanks to (3.71) and (3.41). Moreover, holomorphicity of τ and (3.71) imply that (3.20) is also satisfied. One therefore needs only solve (3.22) for the S^3 indices.

One may use the explicit form of the flux (3.41) to reduce (3.22) on the S^3 to

$$\nabla_{\hat{a}} \xi = \frac{\mathrm{i}m}{2} \gamma_{\hat{a}} \xi \, . \tag{3.72}$$

With the vielbein

$$e^1_{(S^3)} = -\frac{1}{2m}\sigma_{1,R} \, , \quad e^2_{(S^3)} = -\frac{1}{2m}\sigma_{2,R} \, , \quad e^3_{(S^3)} = -\frac{1}{2m}\sigma_{3,R} \, , \tag{3.73}$$

---

[16]Recall that this is a local equation as Q and the metric on B are singular.

where $\sigma_{i,R}$ are right-invariant one-forms, one finds that the constant spinor solves this final set of conditions. The Killing spinor is therefore a constant spinor subject to the projection conditions (3.71), and therefore has four real components consistent with preserving $(0,4)$ supersymmetry. As the solution is constant in ψ, there is no ambiguity in the definition of the spinor if we quotient the S^3 by $\mathbb{Z}_M$. We may therefore replace the S^3 factor in the solution by the Lens spaces $S^3/\mathbb{Z}_M$ without breaking supersymmetry and still satisfying all equations of motion and Bianchi identities. We shall give a physical interpretation of this quotient in section 3.7.

Having computed the Killing spinors we may now determine the R-symmetry. On the S^3 there are six Killing vectors corresponding to the six generators of $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. These are the three dual to the left-invariant one-forms

$$k_{(1)} = \partial_\psi \,, \qquad k_{(2)} = -\cos\psi \cot\theta \partial_\psi - \sin\psi \partial_\theta + \frac{\cos\psi}{\sin\theta} \partial_\varphi \,,$$
$$k_{(3)} = -\sin\psi \cot\theta \partial_\psi + \cos\psi \partial_\theta + \frac{\sin\psi}{\sin\theta} \partial_\varphi \,, \tag{3.74}$$

and the three dual to the right-invariant one-forms

$$k_{(4)} = \frac{\sin\varphi}{\sin\theta} \partial_\psi + \cos\varphi \partial_\theta - \cot\theta \sin\varphi \partial_\varphi, \quad k_{(5)} = -\frac{\cos\varphi}{\sin\theta} \partial_\psi + \sin\varphi \partial_\theta + \cot\theta \cos\varphi \partial_\varphi, \quad k_{(6)} = \partial_\varphi, \tag{3.75}$$

with each set satisfying the $\mathrm{SU}(2)$ Lie algebra. The spinorial Lie-derivative along a Killing direction, K, is defined to be

$$\mathcal{L}_K \epsilon = \left(K^\mu \nabla_\mu + \frac{1}{8} (\mathrm{d}K)_{\nu_1 \nu_2} \gamma^{\nu_1 \nu_2} \right) \epsilon \,. \tag{3.76}$$

In order to ascertain along which directions the Killing spinor is charged one computes the spinorial Lie derivative along these directions. We find that the Killing spinor is invariant under the left-invariant Killing vectors and charged under the right-invariant Killing vectors. This implies that we can take the quotient by $\Gamma \subset \mathrm{SU}(2)_L$, preserving the same amount of supersymmetry. Moreover, as discussed above this means that we can identify $\mathrm{SU}(2)_R$ with the $\mathrm{SU}(2)_r$ R-symmetry of the dual SCFT. We note that the spinorial Lie derivative is frame independent (subject to preserving the same orientation, which is correlated with the choice of $\mathrm{SU}(2)$ under which the Killing spinors are charged) and therefore this result is non-ambiguous.

It is a well known fact in the literature that performing a T-duality along a Killing direction with vanishing spinorial Lie-derivative for the Killing spinor along the Killing vector leads to a Killing spinor in the dual solution. It is clear from the results above that one may dualize along the Hopf fiber without breaking supersymmetry, which will be used later on to determine the dual M-theory solution.

3.6 Flux quantisation

To complete the solution, we need to ensure that the five-form field strength, F, is properly quantized through all the integral five-cycles in the 7d manifold transverse to AdS$_3$. We

impose that[17]

$$n(M_\alpha) = \frac{1}{(2\pi l_s)^4} \int_{M_\alpha} F \in \mathbb{Z} \tag{3.77}$$

for all $M_\alpha \in H_5(\mathcal{M}_7, \mathbb{Z})$. The five cycles which contribute are of the form $S^3 \times \mathcal{C}$, where $\mathcal{C}$ is any two-cycle in the base B of the Calabi-Yau. We therefore find[18]

$$
\begin{aligned}
n(M_\alpha) &= -\frac{\mathrm{e}^{4A}\mathrm{vol}(S^3/\mathbb{Z}_M)}{(2\pi l_s)^4 4m^2 m_B^2} \int_{Y_3} \omega_0 \wedge \pi^* J_B \wedge \omega_\alpha \\
&= -\frac{\mathrm{e}^{4A}\mathrm{vol}(S^3/\mathbb{Z}_M)}{(2\pi l_s)^4 4m^2 m_B^2} \int_{C_\alpha} J_B \,,
\end{aligned}
\tag{3.78}
$$

where the C_α form a basis of cycles in $H_2(B, \mathbb{Z})$.

The possible bases B for an elliptic Calabi-Yau threefold, as listed earlier, are projective, and therefore also Hodge manifolds [66], and moreover they admit an integral Kähler form. As J_B is dual to a curve, we in fact have that B is not only a Hodge manifold, but we in fact pick the Hodge metric on it. This implies that we can take

$$\int_{C_\alpha} J_B = k_\alpha \ \in \ \mathbb{Z}^+ \,. \tag{3.79}$$

Using (3.79) we find that $n(M_\alpha)$ are integer if we impose

$$N = \frac{\mathrm{e}^{4A}\mathrm{vol}(S^3/\mathbb{Z}_M)}{4m^2 m_B^2 (2\pi l_s)^4} \in \mathbb{Z} \,. \tag{3.80}$$

3.7 Brane solutions and the interpretation of the quotient

In this subsection we shall give an interpretation of the $\mathbb{Z}_M$ quotient performed in section 3.5. To do so we shall construct smeared brane solutions whose near-horizon geometry is[19]

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{4m^2}\mathrm{d}s^2(S^3/\mathbb{Z}_M) + \frac{1}{m_B^2}\mathrm{d}s^2(B) \,. \tag{3.81}$$

We shall need to combine various D3-brane solutions, employing the harmonic function rule (see [77] for a review).

We shall use this strategy to obtain a UV completion of the AdS_3 solution that we have in Type IIB in the near-horizon limit, which we refer to as the "pre near-horizon limit". In fact, as we will show below, we can construct *two distinct* such solutions, both flowing to the same near-horizon geometry. We wish to consider N D3-branes wrapping $\mathbb{R}^{1,1} \times C$ where C is the curve in the base of Y_3, Poincaré dual to the Kähler form of the base. We shall first consider a solution in the background of M KK-monopoles and later in the background $\mathbb{R}^4$. To realise the D3-branes extended along the curve Poincaré dual to $J = e^{12} + e^{34}$, with $\mathrm{d}s^2(B) = e_1^2 + \cdots + e_4^2$ we shall formally view this as two stacks of

---

[17]In the following we will set $g_s = 1$.

[18]We have defined $\mathrm{dvol}(S^3/\mathbb{Z}_M) = \sigma_{1,R} \wedge \sigma_{2,R} \wedge \sigma_{3,R}$ which gives $\mathrm{vol}(S^3/\mathbb{Z}_M) = \frac{16\pi^2}{M}$. Notice that this is not the volume form of the unit radius Lens space $S^3/\mathbb{Z}_M$.

[19]For simplicity we set the warp factor, A, to 0 in this section.

D3-branes [23]. The first stack will extend along $\mathbb{R}^{1,2} \times C_{12}$, where C_{12} is the curve dual to e^{12}, and the second stack along $\mathbb{R}^{1,2} \times C_{34}$ each with the same number of branes, N.

We begin by briefly recalling the metric for M KK-monopoles and give a few comments that will be useful for later discussion. The metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_5^2 + \mathrm{d}s^2_{TN_M} , \tag{3.82}$$

where $\mathrm{d}s^2_{TN_M}$ is the Taub-NUT metric[20]

$$m_B^2 \mathrm{d}s^2_{TN_M} = \left(1 + \frac{M}{r}\right) (\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)) + \left(1 + \frac{M}{r}\right)^{-1} (\mathrm{d}\psi + M\cos\theta \mathrm{d}\varphi)^2 . \tag{3.83}$$

This metric is well-known to be hyper-Kähler and hence Ricci-flat.

This metric approaches the singular (for $M > 1$) metric on $\mathbb{R}^4/\mathbb{Z}_M$ as $r \to 0$, whilst asymptotically, as $r \to \infty$, it approaches the cylinder $\mathbb{R}^3 \times S^1$. One can set $M = 0$ in the metric, obtaining exactly the flat metric on $\mathbb{R}^3 \times S^1$. Moreover, choosing as harmonic function $H(r) = 1 + \frac{M}{r} \to \frac{M}{r}$, a simple change of coordinates shows that this is exactly the metric on $\mathbb{R}^4/\mathbb{Z}_M$. This can be interpreted as saying that in the "near-horizon" limit the Taub-NUT metric approaches the latter.

Let us first write the Type IIB solution corresponding to N D3-branes wrapping $\mathbb{R}^{1,2} \times C_{12}$,

$$m_B^2 \mathrm{d}s^2(D3, 12) = H^{-1/2}(r)(-\mathrm{d}t^2 + \mathrm{d}x^2 + e_1^2 + e_2^2) + H^{1/2}(r)(\mathrm{d}s^2_{TN} + e_3^2 + e_4^2),$$
$$C^{(4)} = \frac{1}{m_B^2} \left(1 - \frac{1}{H(r)}\right) \mathrm{d}t \wedge \mathrm{d}x \wedge e^{12} . \tag{3.84}$$

To wrap $\mathbb{R}^{1,2} \times C_{34}$ we simply relabel $12 \leftrightarrow 34$. We have inserted the D3-branes into the background of M KK-monopoles. In particular, as remarked above, we shall *smear* the D3-branes completely along the 34 directions in the manifold B, this has the affect of making the function $H(r)$ harmonic on Taub-NUT and not the overall transverse space to the stack of D3s. If we now use the harmonic function rule on these two configurations we obtain the solution

$$m_B^2 \mathrm{d}s^2 = H(r)^{-1}(-\mathrm{d}t^2 + \mathrm{d}x^2) + H(r)\mathrm{d}s^2_{TN} + \mathrm{d}s^2(B)$$
$$C^{(4)} = \frac{1}{m_B^2} \left(1 - \frac{1}{H(r)}\right) \mathrm{d}t \wedge \mathrm{d}x \wedge J_B \tag{3.85}$$

As commented above H must be harmonic on Taub-NUT, as such we may take

$$H(r) = 1 + \frac{q_N}{r} , \qquad \text{with} \qquad q_N \equiv \frac{(2\pi\ell_s)^4 N m_B^4}{16\pi^2} , \tag{3.86}$$

---

[20]Strictly speaking, the Taub-NUT metric has $M = 1$ and this is non singular near to $r \to 0$. The metric with $M > 1$ has an $\mathbb{R}^4/\mathbb{Z}_M$ singularity in the interior, and this can be resolved by replacing the single center metric with a Gibbons-Hawking multi-center metric, where near to each center the metric looks like $\mathbb{R}^4$. This metric develops $M - 1$ two-cycles, that collapse to zero size in the single center singular metric.

and N the number of D3-branes. The metric takes the form

$$m_B^2 \mathrm{d}s^2 = \frac{r}{r+q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \frac{r+q_N}{r}\mathrm{d}s_{TN}^2 + \mathrm{d}s^2(B)\,. \tag{3.87}$$

We recall that B is the base of an elliptically fibered Calabi-Yau threefold and as such this necessarily requires τ to vary in the solution. This is an Einstein-frame solution to Type IIB supergravity with D3-branes and varying τ.

Let us now take the near-horizon limit, $r \to 0$. We have

$$m_B^2 \mathrm{d}s_{r\to 0}^2 = \frac{r}{q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \mathrm{d}s^2(B) + \frac{q_N M}{r^2}\left(\mathrm{d}r^2 + r^2 \mathrm{d}s^2(S^2)\right) + \frac{q_N}{M}(\mathrm{d}\psi + M\cos\theta \mathrm{d}\phi)^2$$

$$= \frac{r}{q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + q_N M \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}s^2(B) + q_N M(\mathrm{d}s^2(S^3/\mathbb{Z}_M))\,. \tag{3.88}$$

If we make the redefinition $Mq_N = m_B^2(4m^2)^{-1}$ and the change of coordinate $r = 4q_N^2 M\rho^2$ we obtain

$$\mathrm{d}s^2 = \frac{1}{m^2}\left(\rho^2(-\mathrm{d}t^2 + \mathrm{d}x^2) + \frac{\mathrm{d}\rho^2}{\rho^2}\right) + \frac{1}{4m^2}\mathrm{d}s^2(S^3/\mathbb{Z}_M) + \frac{1}{m_B^2}\mathrm{d}s^2(B)\,, \tag{3.89}$$

whilst the five-form becomes

$$F = (1+*)\frac{2m}{m_B^2}\mathrm{dvol}(\mathrm{AdS}_3) \wedge J_B\,, \tag{3.90}$$

which recovers exactly the AdS$_3$ solution. We have done this by inserting M KK-monopoles into the background of N D3-branes wrapping a curve, C dual to J_B, on the base of an elliptically fibered Calabi-Yau threefold.

Let us now consider a different pre near-horizon limit of the AdS$_3$ solution. This will be obtained by replacing the Taub-NUT metric in the Type IIB solution by the flat space quotient $\mathbb{R}^4/\mathbb{Z}_M$. We shall see that the near-horizon solution agrees with the Taub-NUT solution. We may use the previous results to immediately write down the metric[21]

$$m_B^2 \mathrm{d}s^2 = H(R)^{-1}(-\mathrm{d}t^2 + \mathrm{d}x^2) + H(R)\left(\mathrm{d}R^2 + \frac{R^2}{4}((\mathrm{d}\psi + \cos\theta \mathrm{d}\varphi)^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\right)$$

$$+ \mathrm{d}s^2(B) \tag{3.91}$$

where now $H(R)$ is a harmonic function on $\mathbb{R}^4$ and we take

$$H(R) = 1 + \frac{\widetilde{q_N}}{R^2} \qquad \text{with} \qquad \widetilde{q_N} \equiv \frac{(2\pi\ell_s)^4 MN m_B^4}{4\pi^2}\,. \tag{3.92}$$

This harmonic function should be contrasted with (3.86) in the Taub-NUT case. The self-dual five-form flux takes the form

$$F = (1+*)\frac{1}{m_B^4}\mathrm{d}H(R)^{-1} \wedge \mathrm{d}t \wedge \mathrm{d}x \wedge J_B\,. \tag{3.93}$$

---

[21]Of course here we can simply take ψ to have period $4\pi/M$.

Taking the near-horizon limit, $R \to 0$ we obtain

$$m_B^2 \mathrm{d}s^2 = \frac{R^2}{\widetilde{q_N}}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \frac{\widetilde{q_N}}{R^2}\mathrm{d}R^2 + \frac{\widetilde{q_N}}{4}\mathrm{d}s^2(S^3/\mathbb{Z}_M) + \mathrm{d}s^2(B) \,. \tag{3.94}$$

After rescaling R as $R \to \widetilde{q_N}R$ and identifying the inverse radius of AdS$_3$ to be $\widetilde{q_N} = \frac{m_B^2}{m^2}$ one recovers precisely the AdS$_3 \times S^3/\mathbb{Z}_M$ solution

$$m_B^2 \mathrm{d}s^2 = \frac{R^2}{m^2}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \frac{1}{m^2 R^2}\mathrm{d}R^2 + \frac{1}{4m^2}\mathrm{d}s^2(S^3/\mathbb{Z}_M) + \mathrm{d}s^2(B) \tag{3.95}$$

in perfect agreement with (3.65). The flux becomes

$$F = (1 + *)\frac{2m}{m_B^2}\mathrm{dvol}(\mathrm{AdS}_3) \wedge J_B \,, \tag{3.96}$$

in agreement with (3.66).

We have constructed two different UV completions of the Type IIB AdS$_3$ solution that is our main interest. To do so, we needed to make some technical simplifications, regarding smearing of the branes and the application of the harmonic sum rule. The resulting solutions are therefore not the fully localized brane solutions, before taking the near-horizon limit, which are typically very difficult to construct. However these solutions will still be useful in our discussion. Moreover, we should also keep in mind that the metric on B and τ were singular in the near-horizon limit and this feature will remain.

Notice that for any N and any M, asymptotically the metric (3.87) goes to $\mathbb{R}^{1,1} \times \mathbb{R}^3 \times S^1 \times B$. This is the metric far away from the N D3-branes. On the other hand, the metric (3.91) asymptotically goes to $\mathbb{R}^{1,1} \times \mathbb{R}^4/\mathbb{Z}_M \times B$. So these are clearly two different UV completions of the near-horizon geometry. This becomes particularly instructive in the case of $M = 1$: in this case both asymptotic spaces are smooth, however the solution in the presence of 1 KK-monopole comprises an asymptotic $\mathbb{R}^3 \times S^1$ geometry, whilst the solution with *no* KK-monopoles comprises an asymptotic $\mathbb{R}^4$ geometry. However, they flow to exactly the same AdS$_3 \times S^3$ solution in the IR.

The interpretation of this fact is that in the IR the field theories constructed from the two different UV setups, flow to the "same" SCFT in the large N limit. This means that in this limit for example the two theories must have the same central charges, in the large N limit. However, sub-leading corrections to the central charges may be possible.

Notice that one may set $M = 0$ without any immediate problem in (3.87), obtaining the metric

$$m_B^2 \mathrm{d}s^2 = \frac{r}{r + q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \mathrm{d}s^2(B) + \frac{r + q_N}{r}\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) + \mathrm{d}\psi^2\right) \,. \tag{3.97}$$

Notice that the Calabi-Yau base is a direct product with the remaining six-dimensional metric. Computing the curvature invariants of the six-dimensional metric, we find

$$\mathrm{R} = 0 \,,$$
$$\mathrm{R}_{AB}\mathrm{R}^{AB} = \frac{3q_N^4}{2r^2(q_N + r)^6} \tag{3.98}$$
$$\mathrm{R}_{ABCD}\mathrm{R}^{ABCD} = \frac{q_N^2(11q_N^2 + 32q_N r + 48r^2)}{2r^2(q_N + r)^6}$$

and therefore the metric is singular at $r = 0$. In fact, upon taking the near-horizon limit there is no longer an AdS$_3$ factor. In other words, putting the D3-branes transverse to the space $\mathbb{R}^3 \times S^1$ gives rise to a solution that does not contain an AdS$_3$ factor in the IR, and in fact has a curvature singularity as $r \to 0$.

4 F-theory holographic central charges

In this section we compute the central charges for the solution derived in section 3.4. As was noted previously, the metric on the base B, which is induced from the Calabi-Yau metric is singular. We shall circumvent potential problems arising with singular metrics, by carrying out our computations in the smooth Calabi-Yau threefold.

4.1 Leading order central charges

The leading order term for the central charges is given by the Brown-Henneaux formula [78] as summarised in appendix D.1. Evaluating (D.6) for the solution we find the leading order central charges to be

$$
\begin{aligned}
(c_L^{\mathrm{IIB}})^{(2)} = (c_R^{\mathrm{IIB}})^{(2)} = c_{\mathrm{sugra}}^{\mathrm{IIB}} &= \frac{3\mathrm{e}^A \mathrm{vol}(M_7)}{2m G_N^{(10)}} = N^2 \frac{3\mathrm{vol}(S^3/\mathbb{Z}_M)\mathrm{vol}(B)32\pi^2}{\mathrm{vol}(S^3/\mathbb{Z}_M)^2} \\
&= 6N^2 M \mathrm{vol}(B) \,.
\end{aligned}
\tag{4.1}
$$

We denote by $c^{(a)}$ the $\mathcal{O}(N^a)$ contribution to the central charge.

In a smooth geometry we would compute the volume of the base B using the metric. However, as we emphasised repeatedly, the metric of this space is singular. There is a smooth Ricci-flat metric on the putative elliptically fibered Calabi-Yau Y_3. The way we will work around the absence of a smooth metric on B is to compute the volume in the elliptic Calabi-Yau as follows. The $(1,1)$-form dual to B is ω_0, and the volume of the divisor can be computed by

$$
\mathrm{vol}(B) = \frac{1}{2} \int_{Y_3} \omega_0 \wedge \pi^* J_B \wedge \pi^* J_B = \frac{1}{2} \int_B J_B \wedge J_B \,.
\tag{4.2}
$$

Furthermore the latter integral can be evaluated by first using the fact that the curve wrapped by the D3-branes, C, is Poincaré dual to the Kähler form J_B and then using intersection theory to write

$$
\mathrm{vol}(B) = \frac{1}{2} \int_C J = \frac{1}{2} C \cdot C \,.
\tag{4.3}
$$

Using this identification we can rewrite the central charge in terms of the self-intersection of the curve C in B as

$$
(c_L^{\mathrm{IIB}})^{(2)} = (c_R^{\mathrm{IIB}})^{(2)} = c_{\mathrm{sugra}}^{\mathrm{IIB}} = 3N^2 M \, C \cdot C \,.
\tag{4.4}
$$

Since $\mathrm{vol}(B) > 0$ the curve wrapped by the D3-branes must have positive self-intersection in B. Using the adjunction formula (2.16) one can express the constraint $C \cdot C > 0$ as

$$
C \cdot C = 2(g-1) + c_1(B) \cdot C > 0 \,.
\tag{4.5}
$$

At this point we should comment about the relation of our setup to the strings in minimal 6d SCFTs, also known as non-Higgsable clusters (NHCs) [10], whose central charges were computed in [7]. The geometric condition for the NHCs is that the base of the Calabi-Yau threefold is locally $\mathcal{O}(-n) \to \mathbb{P}^1$. The curve that is wrapped by the D3-brane is the base $C^{\text{NHC}} = \mathbb{P}^1$, which has self-intersection

$$C^{\text{NHC}} \cdot C^{\text{NHC}} = -n < 0\,, \qquad n = 3, 4, 6, 8, 12\,, \tag{4.6}$$

and can be collapsed. This singular limit corresponds to the conformal point. In appendix E.2 the geometry of these NHCs is briefly discussed. The negative self-intersection implies that C^{NHC} is not ample, and consequently that these 2d NHC strings do not directly fit into the framework discussed in this paper.

4.2 $c_L^{\text{IIB}} - c_R^{\text{IIB}}$ at sub-leading order from anomaly inflow

The sub-leading contribution is obtained using anomaly inflow [47]. The difference of the left and right central charges appears as the coefficient in front of the gravitational Chern-Simons term in the bulk action [79]

$$S_{CS}(\Gamma_{\text{AdS}_3}) = \frac{c_L^{\text{IIB}} - c_R^{\text{IIB}}}{96\pi} \int_{\text{AdS}_3} \omega_{CS}(\Gamma_{\text{AdS}_3})\,. \tag{4.7}$$

To determine this coefficient we consider the three dimensional terms which arise from the dimensional reduction of the Chern-Simons terms in the worldvolume action of 7-branes. The Chern-Simons terms for a D7-brane were computed in [80] and are given in terms of the curvature two-forms of the tangent and normal bundles of the brane worldvolume, $\mathcal{R}_T$ and $\mathcal{R}_N$, respectively,

$$\mu_7 \int_{\mathcal{W}_8} C^{(4)} \wedge \sqrt{\frac{\hat{A}(4\pi^2 \ell_{\text{s}}{}^2 \mathcal{R}_T)}{\hat{A}(4\pi^2 \ell_{\text{s}}{}^2 \mathcal{R}_N)}} \operatorname{Tr}\left(e^{2\pi \ell_{\text{s}}{}^2 \mathcal{F}}\right) \subset S_{\text{D7}}\,, \tag{4.8}$$

where

$$\mu_7 = \frac{1}{(2\pi)^7 \ell_{\text{s}}{}^8}\,, \tag{4.9}$$

is the charge of a single D7-brane, C_4 is the potential of the five-form flux and $\mathcal{F}$ is the gauge invariant field strength of the gauge fields on the D7-brane. The trace is performed in the fundamental representation of the gauge group. For the computation of the $\mathcal{O}(N)$ corrections to the central charges we will only be interested in the terms coming from the tangent bundle of the D7-brane. Thus below we simply write $\mathcal{R} \equiv \mathcal{R}_T$. Up to the required order the A-roof genus $\hat{A}$ is given by

$$\hat{A}(\mathcal{R}) = 1 + \frac{1}{12(4\pi)^2} \operatorname{Tr}(\mathcal{R} \wedge \mathcal{R})\,. \tag{4.10}$$

As we consider only I_1 singularities our set-up consists of *single* 7-branes wrapped on curves C_x in the base.[22] Note that not all of these 7-branes can be transformed into D7-branes under an $\text{SL}(2, \mathbb{Z})$ transformation. Imposing that the elliptic fibration is Calabi-Yau

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[22]This can be easily generalised to other 7-brane singularities, by including suitable normalisations to the trace appearing in (4.8).

results in the constraint

$$[\Delta] = 12c_1(B) = \sum_x \omega_x \,, \tag{4.11}$$

where ω_x are the two-forms dual to the curves C_x wrapped by the 7-branes.

Consider a single D7-brane whose world-volume extends along $\mathcal{W}_8 = \mathrm{AdS}_3 \times S^3/\mathbb{Z}_M \times C_x$. From the D7-brane Wess-Zumino term we obtain the 3d Chern-Simons term

$$\begin{aligned}
S_{CS}(\Gamma_{\mathrm{AdS}_3}) &= \frac{\mu_7 \pi^2 \ell_{\mathrm{s}}{}^4}{24} \int_{\mathcal{W}_8} C^{(4)} \wedge \mathrm{Tr}(\mathcal{R} \wedge \mathcal{R}) \\
&= -\frac{\mu_7 \pi^2 \ell_{\mathrm{s}}{}^4}{24} \int_{\mathcal{W}_8} F \wedge \omega_{CS} \\
&= \frac{\mu_7 e^{4A} \pi^2 \ell_{\mathrm{s}}{}^4 \mathrm{vol}(S^3/\mathbb{Z}_M)}{24(2m)^2 m_B^2} \int_{C_x} J_B \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \\
&= \frac{N}{192\pi} \int_B J_B \wedge \omega_x \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \,,
\end{aligned} \tag{4.12}$$

where we have used the fact the trace over the fundamental representation of the gauge group is 1 as only one D7-brane is wrapped on C_x.

As $C^{(4)}$ is invariant under $\mathrm{SL}(2,\mathbb{Z})$ transformations, each 7-brane gives rise to the same contribution to the 3d Chern-Simons term [47]. To obtain the total contribution we therefore sum the terms arising from each 7-brane

$$\begin{aligned}
S_{CS}(\Gamma_{\mathrm{AdS}_3}) &= \frac{N}{192\pi} \sum_x \int_B J_B \wedge \omega_x \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \\
&= \frac{N}{16\pi} \int_B J_B \wedge c_1(B) \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \,.
\end{aligned} \tag{4.13}$$

We evaluate the integral over the base by pulling back to the smooth Calabi-Yau

$$\int_{Y_3} \omega_0 \wedge \pi^* J_B \wedge \pi^* c_1(B) = \int_B J_B \wedge c_1(B) = c_1(B) \cdot C \,. \tag{4.14}$$

Using this relation we determine from the coefficient of (4.13) the difference of the left and right central charges to be

$$(c_L^{\mathrm{IIB}})^{(1)} - (c_R^{\mathrm{IIB}})^{(1)} = 6N c_1(B) \cdot C \,. \tag{4.15}$$

4.3 Level of the superconformal R-symmetry

In this section we compute the level k_r of the superconformal R-symmetry. The relation $c_R = 6k_r$ and (4.4) imply that the leading order contribution to the level is given by

$$k_r^{(2)} = \frac{1}{2} N^2 M C \cdot C \,. \tag{4.16}$$

To compute the sub-leading order term we restrict to the case of $M = 1$ and proceed by gauging the $\mathrm{SO}(4)_T$ isometry of the S^3 in the supergravity solution. The procedure for

computing the level follows [81, 82], where one first deforms the metric on the S^3 to contain connections, which depend on AdS$_3$

$$ds_{S^3}^2 \to (dx^p - A^{pq}x^q)(dx^p - A^{pr}x^r)\,, \tag{4.17}$$

where $\sum_{p=1}^4 (x^p)^2 = 1$. These connections $A^{pq} = -A^{qp}$ are one-forms on AdS$_3$ and are identified with the $\mathrm{SO}(4)_T \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ gauge fields for the superconformal R-symmetry $\mathrm{SU}(2)_R$ and the flavour symmetry $\mathrm{SU}(2)_L$. The deformed five-form flux is [81]

$$F_5'|_{M=1} = -\frac{4\pi^2 e^{4A}}{m^2 m_B^2}(1 + *)\left((e_3 - \chi_3) \wedge J_B\right)\,, \tag{4.18}$$

where e_3 is the volume form on the sphere bundle satisfying $\int_{S^3} e_3 = 1$ and $de_3 = \chi_4$, χ_4 being the Euler class of the sphere bundle. The additional term χ_3, a three-form on AdS$_3$ satisfying $d\chi_3 = \chi_4$, is required for $dF_5'|_{M=1} = 0$.

The reduction of the Chern-Simons term for D7-branes wrapped on this deformed metric gives rise to Chern-Simons terms for the $\mathrm{SO}(4)_T$ gauge fields. Upon inserting the deformed flux (4.18) into the D7-brane Chern-Simons term and summing over all 7-branes as above one finds, in addition to the gravitational Chern-Simons term,

$$S_{CS}(A_T)|_{M=1} = \frac{N}{8\pi}c_1(B) \cdot C \int_{\mathrm{AdS}_3}\left(\omega_{CS}(A_R) + \omega_{CS}(A_L)\right)\,, \tag{4.19}$$

where the additional factor of 2 arises from expressing the trace over the fundamental representation of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$ instead of the vector representation of $\mathrm{SO}(4)_T$. The level of the superconformal $\mathrm{SU}(2)_r$ R-symmetry can be extracted from the coefficient of Chern-Simons term after multiplication by 4π, namely

$$S_{CS}(A) = \frac{k_r}{4\pi} \int_{\mathrm{AdS}_3} \omega_{CS}(A)\,, \tag{4.20}$$

where $A = iA^a\sigma_a/2$. From the coefficient of $\omega_{CS}(A_R)$ the sub-leading order term in the level of the superconformal R-symmetry can be extracted and found to be

$$k_r^{(1)}|_{M=1} = \frac{1}{2}Nc_1(B) \cdot C\,. \tag{4.21}$$

For the cases with $M > 1$, the isometry group of the solution is broken to $\mathrm{SU}(2)_R \times \mathrm{U}(1)_L$. Naively, to compute the level of the superconformal R-symmetry one should still gauge the $\mathrm{SU}(2)_R$ by introducing gauge fields for this isometry, analogous to the $M = 1$ case. Formally, this gives exactly the same result as (4.21); however this is not the complete contribution, as one would have to take into account the effects of the M KK-monopoles. As we shall see in section 6.3, on the 11d supergravity side this will be captured by gauging the $\mathrm{SU}(2)_{11d}$ isometry of an S^2, which arises from the base of the $S^3/\mathbb{Z}_M$ Hopf fibration. However, it should be noted that $\mathrm{SU}(2)_R$ is different from $\mathrm{SU}(2)_{11d}$, and one can check explicitly that in fact the latter is *not* an isometry of the Type IIB solution.

4.4 Summary: central charges from F-theory

From the computations carried out in this section the central charges in Type IIB supergravity for $M = 1$ are given by

$$c_R^{\text{IIB}}|_{M=1} = 3N^2 C \cdot C + 3N c_1(B) \cdot C \,,$$
$$c_L^{\text{IIB}}|_{M=1} = 3N^2 C \cdot C + 9N c_1(B) \cdot C \,.$$
(4.22)

In this section we have only computed these central charges to sub-leading order in N. We expect $\mathcal{O}(1)$ corrections to arise from one loop computations and will comment on these in section 7.4, where we compare the central charges computed via anomaly inflow and supergravity solutions. We further point out that the superconformal algebra mandates that the right-moving central charge belongs to $6\mathbb{Z}$. To see this explicitly we make use of the adjunction formula (2.16) and rewrite it as

$$c_R^{\text{IIB}}|_{M=1} = 6N^2(g-1) + 3N(N+1)c_1(B) \cdot C \,,$$
(4.23)

which exhibits manifestly that the expression is a multiple of six, generalising to any N the property of the $N = 1$ right central charge, observed in (2.19).

For $M > 1$ we obtain

$$c_R^{\text{IIB}}|_{M>1} = 3MN^2 C \cdot C + \delta c_R^{IIB}$$
$$c_L^{\text{IIB}}|_{M>1} = 3MN^2 C \cdot C + \delta c_L^{IIB} + 6N c_1(B) \cdot C \,.$$
(4.24)

As explained in the previous section, the computation of the level of the superconformal R-symmetry for $M > 1$ is troublesome. Instead, we uplift our Type IIB solution to 11d supergravity in the next section. In doing so we will be able to compute the $\mathcal{O}(N)$ contributions to the $M > 1$ central charges, as well as $\mathcal{O}(1)$ corrections.

5 M/F-duality and AdS$_3$ solutions in M-theory

The solution found above in Type IIB supergravity is singular at the loci above which τ degenerates. We circumnavigated this problem by computing the central charges of the solutions in terms of the volume of the base B in the smooth Calabi-Yau, where it is well-defined. To substantiate this we can utilize M/F-duality: by T-dualizing and uplifting to M-theory, the elliptically fibered Calabi-Yau threefold becomes manifest in the geometry. Assuming that there are only I_1 fibers, the elliptic Calabi-Yau threefold is smooth, as can be seen by direct computation. There exists a smooth Ricci-flat metric on this space by Yau's theorem [83] and we may use this metric to compute the central charge.

5.1 Dual 11d supergravity solution

In this subsection we shall perform a T-duality along the Hopf fiber of the $S^3/\mathbb{Z}_M$ to Type IIA and then perform the uplift to 11d supergravity. As noted in section 3.5 this will preserve all supersymmetries of the original solution.

Recall that the Type IIB solution in string frame takes the form

$$\mathrm{d}s^2(\mathcal{M}_{IIB}) = \frac{\mathrm{e}^{2A}}{\sqrt{\tau_2}}\left[\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{m_B^2}\mathrm{d}s^2(B) + \frac{1}{4m^2}(\sigma_{1,L}^2 + \sigma_{2,L}^2 + \sigma_{3,L}^2)\right], \quad (5.1)$$

$$F = -\frac{2m}{m_B^2}\mathrm{e}^{4A}J_B \wedge \mathrm{dvol}(\mathrm{AdS}_3) - \frac{\mathrm{e}^{4A}}{4m^2m_B^2}J_B \wedge \sigma_{1,L} \wedge \sigma_{2,L} \wedge \sigma_{3,L}, \quad (5.2)$$

$$\tau = \tau_1 + \mathrm{i}\tau_2 = C^{(0)} + \mathrm{i}\mathrm{e}^{-\phi}. \quad (5.3)$$

The metric defined by $\frac{1}{4}(\sigma_{1,L}^2 + \sigma_{2,L}^2 + \sigma_{3,L}^2)$ is that of the round, unit radius Lens space, $S^3/\mathbb{Z}_M$. This is obtained by quotienting the Hopf fiber, $\sigma_{3,L}$ in our conventions, by the discrete group $\mathbb{Z}_M$ which has the effect of reducing the period of ψ from 4π to $4\pi/M$. Recall that M corresponds to the number of KK-monopoles in the solution before going to the near-horizon limit, as was discussed in section 3.7.

Before performing the T-duality along the Killing vector ∂_ψ we shall absorb the factor of $4m^2$ into the definition of ψ by making the change of coordinates

$$y = \frac{m_B}{2m}\psi, \quad (5.4)$$

where y now has period $\frac{2\pi m_B}{Mm}$. If we now perform the T-duality along ∂_y we obtain the Type IIA solution

$$\mathrm{d}s^2(\mathcal{M}_{IIA}) = \frac{\mathrm{e}^{2A}}{\sqrt{\tau_2}}\left[\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{1}{m_B^2}\mathrm{d}s^2(B) + \frac{1}{4m^2}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right] + \frac{\mathrm{e}^{-2A}\sqrt{\tau_2}}{m_B^2}\mathrm{d}y^2,$$

$$C^{(1)} = \tau_1\mathrm{d}y,$$

$$\mathrm{e}^{-\hat{\phi}} = \tau_2^{3/4}\mathrm{e}^A, \quad (5.5)$$

$$B^{IIA} = -\frac{\cos\theta}{2mm_B}\mathrm{d}y \wedge \mathrm{d}\varphi,$$

$$F_4^{IIA} = \frac{\mathrm{e}^{4A}}{2mm_B^2}\mathrm{dvol}(S^2) \wedge J_B.$$

Uplifting to 11d supergravity and performing a redefinition of the torus coordinates we have

$$\mathrm{d}s^2(\mathcal{M}_{11}) = \mathrm{e}^{\frac{8A}{3}}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + \frac{\mathrm{d}s^2(S^2)}{4m^2} + \frac{1}{m_B^2}\left[\mathrm{d}s^2(B) + \frac{1}{\tau_2}(\mathrm{d}\tilde{x} + \tau_1\mathrm{d}\tilde{y})^2 + \tau_2\mathrm{d}\tilde{y}^2\right]\right) \quad (5.6)$$

$$G_4 = \frac{\mathrm{e}^{4A}}{2mm_B^2}\mathrm{dvol}(S^2) \wedge (J_B + \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}), \quad (5.7)$$

where J_B is the Kähler form on the base. We have redefined the torus coordinates to be

$$\tilde{x} = \mathrm{e}^{-2A}x, \quad \tilde{y} = \mathrm{e}^{-2A}y. \quad (5.8)$$

The periods of the two coordinates are given by

$$R_{\tilde{y}} = \frac{\mathrm{e}^{-2A}l_s^2}{R_{IIB}}, \quad R_{\tilde{x}} = \frac{\mathrm{e}^{-2A}l_s^2}{R_{IIB}}, \quad (5.9)$$

where $R_{IIB} = \frac{1}{Mm}$ is the radius of the S^1 in Type IIB which we have T-dualised along, whose coordinate has been normalised to give the canonical 2π period.

As remarked earlier, the Type IIB solution is singular over the discriminant locus where the fiber degenerates. As such the 11d supergravity metric we obtain from the explicit T-duality and uplift is only valid away from the singular loci. To make progress, we exploit the fact that the algebraic variety $\mathbb{E}_\tau \hookrightarrow Y_3 \to B$, with only I_1 singular fibers,[23] is smooth and compact, and has $c_1(Y_3) = 0$, thus, by Yau's theorem, there exists a global non-singular Ricci-flat metric, of which (3.57) is an approximation valid only away from the singularities. The 11d supergravity solution is therefore given by

$$ds^2(\mathcal{M}_{11}) = e^{\frac{8A}{3}} \left(ds^2(\text{AdS}_3) + \frac{1}{4m^2}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{m_B^2}ds^2(Y_3) \right) \quad (5.10)$$

$$G_4 = \frac{e^{4A}}{2mm_B^2}\text{dvol}(S^2) \wedge J_{Y_3}. \quad (5.11)$$

This solution falls within the classification of [18], specialised to elliptically fibered Calabi-Yau threefolds. Despite the fact that we do not know this metric explicitly, we will be able to compute the central charges for this solution as we discuss in section 6.

As commented in [18], this solution agrees locally with the geometry discussed in [20]. The M5-branes therefore wrap the 4-cycle Poincaré dual to the Kähler form J_{Y_3}, which is an ample divisor in the Calabi-Yau. Using the expansion (2.8) we see that this divisor is a linear combination of B and $\widehat{C}_\alpha$, which are divisors arising from pullbacks of curves in the base. As we only consider I_1 singularities in the fiber there are no Cartan divisors D_i. The presence of M5s wrapping the base of the Calabi-Yau is consistent with the M KK-monopoles in the Type IIB supergravity solution described in section 3.7. The sequence of dualities relating these two supergravity solutions is described in detail in [23]. The T-duality of M KK-monopoles in Type IIB gives rise to M NS5-branes along $\text{AdS}_3 \times B$, which uplift to M M5-branes wrapped on the base. The D3-branes wrapped on the curve C in the base are uplifted to M5-branes wrapped on the elliptic surface $\widehat{C}$ as described in section 2.3. As noted in [23], these two stacks of M5-branes can be deformed into one stack wrapped on a linear combination of B and $\widehat{C}$ provided the curve C is sufficiently ample in the base.

5.2 M5-brane solutions

Analogous to the discussion conducted in subsection 3.7 we shall construct the explicit smeared brane solution which gives (5.6) in the near-horizon limit. To construct this solution one may either T-dualize the "pre near-horizon" solution obtained in (3.87) along the Hopf fiber and then uplift or use the brane smearing techniques employed previously to combine N M5-branes wrapping $\mathbb{R}^{1,1} \times \widehat{C}$ and M M5-branes wrapping $\mathbb{R}^{1,1} \times B$, in the background $\mathbb{R}^{1,1} \times \mathbb{R}^3 \times Y_3$ with $M > 0$. Both methods result in the same solution given

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[23]As has been previously stated if Y_3 contains singularities then one can construct the smooth and compact resolution of the singularities of Y_3 and the following analysis generalises.

by[24]

$$m_B^2 \mathrm{d}s^2(\mathcal{M}_{11}) = \left(\frac{r+q_N}{r+q_M}\right)^{-\frac{2}{3}} \left(\frac{1}{\tau_2}(\mathrm{d}y + \tau_1 \mathrm{d}\psi)^2 + \tau_2 \mathrm{d}\psi^2\right) \tag{5.12}$$

$$+ \left(\frac{r+q_N}{r+q_M}\right)^{\frac{1}{3}} \left(\frac{(r+q_N)(r+q_M)}{r^2}(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2))\right)$$

$$+ \left(\frac{r+q_N}{r+q_M}\right)^{\frac{1}{3}} \left(\frac{r}{r+q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \mathrm{d}s^2(B)\right)$$

$$G_4 = \frac{1}{m_B^3}\mathrm{dvol}(S^2) \wedge (q_N J_B + q_M \mathrm{dvol}(\mathbb{E}_\tau)), \tag{5.13}$$

where

$$q_N = 2\pi^2 \ell_{\mathrm{p}}^3 m_B^3 N \quad \text{and} \quad q_M = \frac{2\pi^2 \ell_{\mathrm{p}}^3 m_B^3 M}{\mathrm{vol}(\mathbb{E}_\tau)}. \tag{5.14}$$

Of course, as already mentioned, this solution has singularities arising from B and also τ. Notice that the Calabi-Yau metric is now warped and we are unable to resolve these singularities as in the previous subsection. However here we are interested in understanding the behaviour in the radial direction r and so we shall not discuss this issue further.

Taking the near-horizon limit one obtains the metric

$$m_B^2 \mathrm{d}s^2(\mathcal{M}_{11})_{r\to 0} = \left(\frac{q_N}{q_M}\right)^{\frac{1}{3}} \left[\frac{r}{q_N}(-\mathrm{d}t^2 + \mathrm{d}x^2) + \frac{q_N q_M}{r^2}\mathrm{d}r^2 + q_N q_M(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right.$$

$$\left. + \mathrm{d}s^2(B) + \frac{q_M}{q_N}\left(\frac{1}{\tau_2}(\mathrm{d}y + \tau_1 \mathrm{d}\psi)^2 + \tau_2 \mathrm{d}\psi^2\right)\right]. \tag{5.15}$$

Upon identifying the warp factor to be $\mathrm{e}^{8A} = \frac{q_N}{q_M}$, the inverse radius squared of AdS$_3$ to be $\frac{m_B^2}{m^2} = 4q_N q_M$ and performing the change of coordinates $r = 4q_N q_M \rho^2$, $y = \sqrt{\frac{q_N}{q_M}}\tilde{y}$, and $\psi = \sqrt{\frac{q_N}{q_M}}\tilde{\psi}$ one recovers (5.6) exactly and therefore an unwarped Calabi-Yau metric which may now be resolved. One also finds that the flux matches exactly with (5.7). Asymptotically, that is $r \to \infty$, the metric approaches the space $\mathbb{R}^{1,1} \times \mathbb{R}^3 \times Y_3$, this is the space far away from the M5-branes. We emphasise that this geometry arises from N M5-branes wrapped on $\mathbb{R}^{1,1} \times \widehat{C}$ plus M M5-branes wrapped on $\mathbb{R}^{1,1} \times B$, with B the base of Y_3, the latter M5-branes can be seen to arise from the initial M KK-monopoles in the Type IIB solution.

One may also consider the case of N M5-branes wrapping only $\mathbb{R}^{1,1} \times \widehat{C}$ in the background $\mathbb{R}^{1,1} \times \mathbb{R}^3 \times Y_3$. This is the formal definition of $M = 0$. The solution of this setup

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[24]Note that q_N is the same as the constant appearing in (3.86) upon using the relation $\ell_{\mathrm{p}}^3 = \frac{l_s^4}{R_{IIB}}$ and the fact that $R_{IIB} = \frac{2}{m_B}$ for this T-duality and uplift. Recall that R_{IIB} is the radius of the S^1 in Type IIB along which we have T-dualized with the S^1 coordinate having the canonical 2π period.

obtained from brane smearing is

$$
m_B^2 \mathrm{d}s^2(\mathcal{M}_{11}) = \left(\frac{r+q_N}{r}\right)^{-\frac{2}{3}}\left(\frac{1}{\tau_2}(\mathrm{d}y+\tau_1\mathrm{d}\psi)^2+\tau_2\mathrm{d}\psi^2\right)+\left(\frac{r+q_N}{r}\right)^{\frac{1}{3}}\mathrm{d}s^2(B)
$$
$$
+\left(\frac{r+q_N}{r}\right)^{\frac{1}{3}}\left(\frac{r}{r+q_N}(\mathrm{d}x^2-\mathrm{d}t^2)+\frac{(r+q_N)}{r}(\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2))\right),
$$
$$
G_4 = \mathrm{dvol}(S^2)\wedge q_N J_B,
$$

with q_N as before. Notice that this agrees with taking the limit $M \to 0$ in (5.12). Recall that $\widehat{C}$ is not an ample divisor and therefore the M5-branes do *not* wrap an ample divisor as in the $M \neq 0$ case. Asymptotically the metric approaches $\mathbb{R}^{1,1}\times\mathbb{R}^3\times Y_3$ as before, however the metric is singular at $r = 0$ now. To see this one computes the Ricci scalar to be[25]

$$
\mathrm{R} = \frac{q_N^2}{3r^{2/3}(r+q_N)^{10/3}}\,, \tag{5.16}
$$

which clearly diverges at $r = 0$. Upon taking the near-horizon limit one does not obtain an AdS factor, this of course matches with our previous analysis that we can only get an AdS$_3$ solution if the divisor wrapped by the branes is ample. Note that this does not imply that when N M5-branes wrap $\mathbb{R}^{1,1}\times\widehat{C}$, the dual 2d field theory does *not* flow to a SCFT in the IR. It just means that the IR SCFT does not have an AdS$_3$ gravity dual in 11d supergravity.

Recall, as discussed in section 3.7, that in Type IIB the $M = 1$ case has two UV completions. One may consider either N D3-branes wrapping $\mathbb{R}^{1,1}\times C$ in the presence of a single KK-monopole or replacing the Taub-NUT space by flat space $\mathbb{R}^4/\mathbb{Z}_M$. Applying T-duality along the Hopf fiber of (3.91) and uplifting we obtain the 11d supergravity solution

$$
\mathrm{d}s^2 = (R^2+q_N)^{1/3}\left(\frac{R^2}{R^2+q_N}(-\mathrm{d}t^2+\mathrm{d}x^2)+\frac{R^2+q_N}{R^2}(\mathrm{d}R^2+R^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2))+\mathrm{d}s^2(B)\right.
$$
$$
\left.+\frac{1}{R^2+q_N}\left(\frac{1}{\tau_2}(\mathrm{d}y+\tau_1\mathrm{d}\psi)^2+\tau_2\mathrm{d}\psi^2\right)\right), \tag{5.17}
$$
$$
G_4 = \mathrm{dvol}(S^2)\wedge\left(\frac{q_N}{4}J_B+q_M\mathrm{dvol}(\mathbb{E}_\tau)\right), \tag{5.18}
$$

with q_N and q_M as before. Of course in the near-horizon limit we obtain (5.6), however asymptotically the metric is now degenerate. This should be contrasted with the M KK-monopoles solution which has a good UV completion.[26]

To summarise, in this section we have found the "pre near-horizon" solution to the 11d supergravity AdS$_3$ solution (5.6). One may obtain such a near-horizon solution from two 11d supergravity solutions, both can be seen as the solution arising from a T-duality along a Hopf fiber and uplift of a Type IIB solution, (3.87) and (3.91) respectively. The solution arising from M KK-monopoles has a good UV completion whilst the solution arising from no KK-monopoles has a degenerate UV completion.

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[25]For ease of reading we present the result of replacing Y_3 with T^6 though the singularity persists if one reinstates the Y_3.

[26]One may, as before, consider $\mathbb{R}^4/\mathbb{Z}_M$ in place of $\mathbb{R}^4$, however similarly one obtains a degenerate UV completion.

5.3 Flux quantisation

For an 11d supergravity solution to be well-defined one must quantize the fluxes through all integral cycles in the geometry. Following [84], the correct quantization condition to impose is that for all $\Sigma_4 \in H_4(\mathcal{M}_{11}, \mathbb{Z})$,

$$n(\Sigma_4) = \int_{\Sigma_4} \left[\frac{1}{(2\pi\ell_\mathrm{p})^3} G_4 - \frac{p_1}{4} \right] \in \mathbb{Z} \,, \qquad (5.19)$$

where ℓ_p is the eleven-dimensional Planck length and p_1 is the first Pontryagin class of $\mathcal{M}_{11}$ defined as

$$p_1 = -\frac{1}{8\pi^2} \, \mathrm{Tr}[\mathcal{R}^2] \,. \qquad (5.20)$$

There are two types of integral four-cycles in $\mathcal{M}_{11}$ to consider: the divisors D in the Calabi-Yau threefold Y_3 as summarised in section 2.2, and the four-cycles $S^2 \times \mathbb{E}_\tau$ and $S^2 \times C_\alpha$ with C_α, as before, forming a basis of $H_2(B, \mathbb{Z})$.

We shall first consider the contributions from the $p_1/4$ term and show that they are all integral. As the metric is a product space we have

$$\mathcal{R}_{\mathcal{M}_{11}} = \mathcal{R}_{\mathrm{AdS}_3} + \mathcal{R}_{S^2} + \mathcal{R}_{Y_3} \,. \qquad (5.21)$$

where $\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. In particular, p_1 is non-trivial only on the Calabi-Yau, thus $p_1(\mathcal{M}_{11}) = -2c_2(Y_3)$, which is given in (2.14). This implies that the $p_1/4$ term integrated over the four-cycles $S^2 \times \mathbb{E}_\tau$ and $S^2 \times C_\alpha$ vanishes. On the other hand, the integral of $c_2(Y_3)$ over every divisor D is always divisible by two, as shown in (E.8),

$$\int_D c_2(Y_3) = 2(h^{1,1}(D) - 4h^{0,2}(D) + 2h^{0,1}(D) - 4) \,, \qquad (5.22)$$

therefore the flux quantization condition reduces simply to

$$n(\Sigma_4) = \frac{1}{(2\pi\ell_\mathrm{p})^3} \int_{\Sigma_4} G_4 \in \mathbb{Z} \,. \qquad (5.23)$$

The form of the G_4-flux implies that the quantization over the divisors of Y_3 is trivial, $n(D) = 0$, and therefore the relevant four-cycles to perform the quantization over are $S^2 \times \mathbb{E}_\tau$ and $S^2 \times C_\alpha$. Then we have

$$n(S^2 \times C_\alpha) = \frac{2\pi \mathrm{e}^{4A}}{m m_B^2 (2\pi\ell_\mathrm{p})^3} \int_{C_\alpha} J_{Y_3} \,, \qquad (5.24)$$

where J_{Y_3} is given in (2.8). Recalling that $\int_{C_\alpha} J_{Y_3} = k_\alpha \in \mathbb{Z}^+$, we see that imposing the condition

$$\mathbb{Z}^+ \ni \widetilde{N} = \frac{2\pi \mathrm{e}^{4A}}{m m_B^2 (2\pi\ell_\mathrm{p})^3} \,, \qquad (5.25)$$

guarantees that $n(S^2 \times C_\alpha)$ is correctly quantized. For later, we shall also need the volume of the elliptic fibration. This is constant over the base. We define the integer $\widetilde{M}$ as

$$\widetilde{M} = \widetilde{N} \int_{\mathbb{E}_\tau} J_{Y_3} = \widetilde{N} k^0 \,, \qquad (5.26)$$

that is $\mathrm{vol}(\mathbb{E}_\tau) = \frac{\widetilde{M}}{\widetilde{N}}$. We shall show that $\widetilde{M} = M$ where the latter M is that arising in Type IIB from the Lens space quotient. To see this we must use the periods of the elliptic fiber coordinates arising from the Type IIB solution, (5.9). As the volume is constant over the base we may compute it away from any singularity. We find

$$\mathrm{vol}(\mathbb{E}_\tau) = (2\pi)^2 R_{\tilde{x}} R_{\tilde{y}} = (2\pi)^2 m m_B^2 M \mathrm{e}^{-4A} \ell_\mathrm{p}{}^3 = \frac{M}{\widetilde{N}}\,, \tag{5.27}$$

where we have used the relation

$$\ell_\mathrm{p}{}^3 = \frac{l_s^4}{R_{IIB}}\,, \tag{5.28}$$

which follows from the T-duality and uplift. Using this relation we may also show that the N in Type IIB is the same as the $\widetilde{N}$ in 11d supergravity. Observe that

$$N = \frac{16\pi^2 \mathrm{e}^{4A}}{4m^2 m_B^2 M (2\pi\ell_\mathrm{s})^4} = \frac{2\pi \mathrm{e}^{4A}}{m m_B^2 (2\pi\ell_\mathrm{p})^3} = \widetilde{N}\,. \tag{5.29}$$

We conclude that the two integers appearing in Type IIB and 11d supergravity solutions can be identified, namely $N = \widetilde{N}$ and $M = \widetilde{M}$. For notational clarity we shall drop the tildes from now on as there is no confusion. We remark that in Type IIB M corresponds to the number of KK-monopoles in the geometry whilst in 11d supergravity it is proportional to the volume of the elliptic fibration.

6 Holographic central charges from M-theory

6.1 Leading order central charges

The gravitational central charge for the 11d supergravity solution $\mathrm{AdS}_3 \times S^2 \times Y_3$ was computed in [85]. We reproduce it here for completeness using (D.6)

$$
\begin{aligned}
(c_L^{11})^{(3)} = (c_R^{11})^{(3)} = c_{\mathrm{sugra}}^{11} &= \frac{3\mathrm{e}^{12A}}{2m m_B^2} \frac{2^5 \pi^2}{(2\pi\ell_\mathrm{p})^9} \int_{M_8} \frac{1}{4m^2 m_B^4} \mathrm{dvol}(S^2) \wedge \mathrm{dvol}(Y_3) \\
&= \frac{3\pi^3 2^4 \mathrm{e}^{12A}}{((2\pi\ell_\mathrm{p})^3 m m_B^2)^3} \int_{Y_3} \frac{1}{6} J_{Y_3} \wedge J_{Y_3} \wedge J_{Y_3} \\
&= N^3 C_{IJK} k^I k^J k^K\,,
\end{aligned} \tag{6.1}
$$

where we have expanded the Kähler form in a basis of $(1,1)$-forms on the Calabi-Yau threefold as in (2.8) and C_{IJK} are the triple intersection numbers as given in section 2.2, with $I = 0$ included in this expansion. This result, as noted in [85], matches the original field theory computation in [20] and [25].

The Kähler form is expanded as in (2.8), where the coefficient in front of the zero-section ω_0 is

$$k_0 = \mathrm{vol}(\mathbb{E}_\tau) = \frac{M}{N}\,, \tag{6.2}$$

the volume of the elliptic fiber. The central charge (6.1) can then be expanded into three terms

$$
\begin{aligned}
(c_L^{11})^{(3)} = (c_R^{11})^{(3)} &= N^3 \left(3k_0 k_\alpha k_\beta \int_{Y_3} \omega_0 \wedge \omega_\alpha \wedge \omega_\beta + 3k_0^2 k_\alpha \int_{Y_3} \omega_0 \wedge \omega_0 \wedge \omega_\alpha + k_0^3 \int_{Y_3} \omega_0^3 \right) \\
&= N^3 \left(3k_0 k_\alpha k_\beta \int_B \omega_\alpha \wedge \omega_\beta - 3k_0^2 k_\alpha \int_B c_1(B) \wedge \omega_\alpha + k_0^3 \int_B c_1(B)^2 \right) \\
&= 3N^2 M C \cdot C - 3N M^2 c_1(B) \cdot C + M^3 (10 - h^{1,1}(B)),
\end{aligned}
\tag{6.3}
$$

where we have made use of (2.12).

6.2 Chern-Simons terms and $c_L^{11} - c_R^{11}$

We now calculate $c_L^{11} - c_R^{11}$ by using the eight derivative corrections as presented in [86]. The term that will be relevant for us is the Chern-Simons term [87]

$$
S_{CS} = -\frac{(4\pi\kappa_{11})^{2/3}}{2\kappa_{11}^2} \int_{M_{11}} C_3 \wedge X_8,
\tag{6.4}
$$

where

$$
X_8 = \frac{1}{(2\pi)^4 2^6 \cdot 3} \left(\mathrm{Tr}[\mathcal{R}^4] - \frac{1}{4}(\mathrm{Tr}[\mathcal{R}^2])^2 \right).
\tag{6.5}
$$

We wish to dimensionally reduce this to obtain Chern-Simons terms in the 3d action. From the coefficient in front of the 3d Chern-Simons term one can extract $c_L^{11} - c_R^{11}$ by using (D.7). Using (5.21) one can see that $\mathrm{Tr}[\mathcal{R}^4] = 0$. We wish to find the term proportional to (D.7) and so we shall drop terms that do not contribute to this if necessary

$$
\begin{aligned}
S_{CS} &= \frac{(4\pi\kappa_{11})^{2/3}}{2^3 \kappa_{11}^2} \frac{1}{(2\pi)^4 2^6 \cdot 3} \int_{M_{11}} C_3 \wedge \mathrm{Tr}[\mathcal{R}^2] \wedge \mathrm{Tr}[\mathcal{R}^2] \\
&= \frac{(4\pi\kappa_{11})^{2/3}}{2^3 \kappa_{11}^2} \frac{1}{(2\pi)^4 2^5 \cdot 3} \int_{M_{11}} G_4 \wedge \mathrm{Tr}[\mathcal{R}^2] \wedge \omega_{CS} \\
&= \frac{\pi}{(2\pi\ell_\mathrm{p})^3 m m_B^2} \frac{\mathrm{e}^{4A}}{(2\pi)^4 2^7 \cdot 3} \int_{M_{11}} \mathrm{dvol}(S^2) \wedge J_{Y_3} \wedge \mathrm{Tr}[\mathcal{R}^2] \wedge \omega_{CS} \\
&= \frac{N\pi}{2^4 \cdot 3(2\pi)^2} \int_{Y_3} J_{Y_3} \wedge c_2(Y_3) \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}),
\end{aligned}
\tag{6.6}
$$

from which we obtain

$$
c_L^{11} - c_R^{11} = \frac{N}{2} \int_{Y_3} J_{Y_3} \wedge c_2(Y_3),
\tag{6.7}
$$

which is in agreement with [85].[27]

To evaluate (6.7) we use the expansion of the Kähler form in (2.8) and the form of $c_2(Y_3)$ as in (2.14). With this information we reduce the integrals in (6.7) to integrals over the base of the fibration, namely[28]

$$
\int_{Y_3} c_2(Y_3) \wedge \omega_0 = \int_B c_2(B) - c_1(B)^2 = 2h^{1,1}(B) - 8.
\tag{6.8}
$$

---

[27]Note that $\mathrm{Tr}[\mathcal{R}^2] = 16\pi^2 c_2(Y_3)$ which is valid for a Calabi-Yau threefold whilst working in real coordinates with the normalisation as in [13].

[28]Note the integral $\int_{Y_3}$ can be translated into one over B by using the intersection ring relations, and extracting the coefficient of σ^2.

For the remaining term, the Poincaré dual to ω_α are divisors $D_\alpha = \widehat{C}_\alpha$ which are pull-backs of curves in the base. Thus the integral over Y_3 is only non-vanishing for those terms in $c_2(Y_3)$, which have fiber components, i.e. the $12\omega_0 \wedge c_1(B)$ term, which leads to

$$\int_{Y_3} c_2(Y_3) \wedge \omega_\alpha = 12c_1(B) \cdot C_\alpha. \tag{6.9}$$

Combining these terms we find

$$c_L^{11} - c_R^{11} = 6Nc_1(B) \cdot C + M(h^{1,1}(B) - 4). \tag{6.10}$$

6.3 Chern-Simons couplings from 11d supergravity

The 11d supergravity solution $\text{AdS}_3 \times S^2 \times Y_3$ has dual SCFTs with small $\mathcal{N} = (0,4)$ superconformal symmetry. In order to determine the left and right central charges one must also calculate the level k_r of the superconformal $\text{SU}(2)_r$ R-symmetry at sub-leading order. The leading and sub-leading corrections to the level k_r were computed in [81, 85] to be

$$k_r = \frac{N^3}{6} C_{IJK} k^I k^J k^K + \frac{N}{12} \int_{Y_3} J_{Y_3} \wedge c_2(Y_3). \tag{6.11}$$

These terms are computed by deforming the metric on the two-sphere to contain connections which depend on AdS_3 only

$$\text{d}s_{S^2}^2 \rightarrow (\text{d}x^a - A^{ab}x^b)(\text{d}x^a - A^{ac}x^c), \tag{6.12}$$

where $\sum_{a=1}^3 (x^a)^2 = 1$. These connections are identified with the SO(3) gauge fields for the R-symmetry.

The leading order term is computed from the 11d term

$$S_{AFF} = -\frac{1}{12\kappa_{11}^2} \int A_3' \wedge G_4' \wedge G_4', \tag{6.13}$$

where we have used the conventions of [86]. For the deformed metric the fluxes are corrected by terms involving the R-symmetry gauge fields and are given by

$$\begin{aligned}
A_3' &= \frac{2\pi \text{e}^{4A}}{m m_B^2} e_1^{(0)} \wedge J_{Y_3} \\
G_4' &= \frac{2\pi \text{e}^{4A}}{m m_B^2} e_2 \wedge J_{Y_3},
\end{aligned} \tag{6.14}$$

where e_2 is the unique two-form for the S^2 bundle satisfying $\int_{S^2} e_2 = 1$ and $\text{d}e_2 = 0$. The one-form $e_1^{(0)}$ is defined by $\text{d}e_1^{(0)} = e_2$. The overall factors in G_4' have been fixed by requiring that the quantization pre-deformation is the same as that post-deformation. Inserting these expressions into (6.13) we obtain

$$S_{AFF} = -\frac{(2\pi)^3 \text{e}^{12A}}{12\kappa_{11}^2 m^3 m_B^6} \int_{Y_3} J_{Y_3} \wedge J_{Y_3} \wedge J_{Y_3} \int_{\text{AdS}_3 \times S^2} e_1^{(0)} \wedge e_2 \wedge e_2. \tag{6.15}$$

To simplify this expression we make use of the formula derived in [81]

$$\int_{\text{AdS}_3 \times S^2} e_1^{(0)} \wedge e_2 \wedge e_2 = -\frac{1}{2(2\pi)^2} \int_{\text{AdS}_3} \omega_{CS}(A)\,, \tag{6.16}$$

which originates from [88]. Recalling the expression $N = \frac{2\pi}{(2\pi l_p)^3 m m_B^2}$ we obtain

$$\begin{aligned}
S_{AFF} &= \frac{\pi \text{e}^{12A}}{12\kappa_{11}^2 m^3 m_B^6} \int_{Y_3} J_{Y_3} \wedge J_{Y_3} \wedge J_{Y_3} \int_{\text{AdS}_3} \omega_{CS}(A) \\
&= \frac{N^3}{24\pi} C_{IJK} k^I k^J k^K \int_{\text{AdS}_3} \omega_{CS}(A)\,,
\end{aligned} \tag{6.17}$$

The level k_r is extracted from the coefficient of the Chern-Simons term from the definition in (4.20). From this we obtain the leading order term in (6.11).

The sub-leading order term is found by computing S_{CS} for the deformed metric, which now contains a contribution from the R-symmetry gauge fields

$$S_{CS} = \frac{N}{192\pi} \int_{CY} J_{Y_3} \wedge c_2(Y_3) \left(\int_{\text{AdS}_3} \omega_{CS}(\Gamma_{\text{AdS}_3}) + 4 \int_{\text{AdS}_3} \omega_{CS}(A) \right)\,, \tag{6.18}$$

where the trace in $\omega_{CS}(A)$ is taken over the fundamental representation of SU(2). The factor of 4 appearing in the gauge Chern-Simons term arises from changing the trace from over the vector representation of SO(3) to SU(2) fundamental. Comparing (6.18) to (4.20) the sub-leading term indeed matches that in (6.11).

Using the results from section 6.2 the level can be expressed as

$$\begin{aligned}
k_r &= \frac{N^3}{6} C_{IJK} k^I k^J k^K + \frac{N}{12} \int_{Y_3} c_2(Y_3) \wedge J_{Y_3} \\
&= \frac{1}{2} N^2 M C \cdot C + \frac{N}{2}(2 - M^2) c_1(B) \cdot C + \frac{M^3}{6}(10 - h^{1,1}(B)) + \frac{M}{6}(h^{1,1}(B) - 4)\,.
\end{aligned} \tag{6.19}$$

The left and right central charges can now be deduced by using the relation $c_R^{11} = 6k_r$ [89]. We obtain the central charges

$$\begin{aligned}
c_R^{11} &= 3N^2 M C \cdot C + 3N(2 - M^2) c_1(B) \cdot C + M^3(10 - h^{1,1}(B)) + M(h^{1,1}(B) - 4), \\
c_L^{11} &= 3N^2 M C \cdot C + 3N(4 - M^2) c_1(B) \cdot C + M^3(10 - h^{1,1}(B)) + 2M(h^{1,1}(B) - 4).
\end{aligned} \tag{6.20}$$

Interestingly, we note that the right-moving central charge c_R^{11} can be shown to be an integer multiple of 6 as expected [70]. To see this we rewrite c_R^{11} as

$$\begin{aligned}
c_R^{11} = {}&6N^2 M(g - 1) + (6N + 3NM(N - M))\, c_1(B) \cdot C \\
&+ 6M^3 + (M - 1)M(M + 1)(4 - h^{1,1}(B))\,.
\end{aligned} \tag{6.21}$$

It is an elementary exercise to show that each term in the expression above is indeed a multiple of 6, for arbitrary values of $N, M \in \mathbb{Z}$. We regard this as a non-trivial check on the interpretation of c_R^{11} as the right-moving central charge of a $(0, 4)$ SCFT with small superconformal algebra.

In section 4 for $M > 1$ we were only able to determine the leading order central charge. To the contrary here, we have the all order expression. It would be very interesting to extend the analysis in section 4.3 to include $M > 1$ and to compare with the above expression.

7   Central charges from anomalies and comparisons

In this section we shall determine the central charges of the 2d SCFTs microscopically, using a UV description in terms of world-volume theories on wrapped branes. To determine these we will essentially need to compute only the anomaly polynomials of the corresponding branes, although we will discuss some subtleties involved in these computations. This complements and extends the central charge computation in section 2.3 from the dimensional reduction of the abelian $\mathcal{N} = 4$ SYM theory. Below we will invert the order of presentation with respect to the previous sections as we find it more convenient to begin with the M5-branes in the M-theory picture and address the D3-branes in the F-theory picture after. We also include a section summarising the results of the computations in the different setups and their comparison.

7.1   Anomalies from M5-branes

In this section we wish to determine the anomaly polynomial associated to the $(0, 4)$ theory on the worldvolume of the string in 5d arising from a stack of M5-branes wrapping a compact 4-cycle in a Calabi-Yau threefold.

A single M5-brane has an anomaly [90] from the chiral modes living on the 6d worldvolume of the brane; this anomaly must be cancelled by anomaly inflow from the M-theory bulk. In [91] a certain deformation of the cubic Chern-Simons term in M-theory was found to cancel the anomaly from a single M5-brane, and this was generalised in [25] to compute the total anomaly polynomial of the 6d worldvolume theory on a stack of N M5-branes. The anomaly polynomial is

$$I_8[N] = NI_8[1] + \frac{1}{24}(N^3 - N)p_2(\mathcal{N}) \,, \tag{7.1}$$

where

$$I_8[1] = \frac{1}{48}\left[p_2(\mathcal{N}) - p_2(W) + \frac{1}{4}(p_1(W) - p_1(\mathcal{N}))^2\right] \,, \tag{7.2}$$

is the anomaly polynomial for the free abelian tensor multiplet that lives on the worldvolume of a single M5-brane and W, $\mathcal{N}$ are respectively the 6d submanifold the M5-brane wraps, and the normal, or SO(5) R-symmetry, bundle associated to the transverse directions of the M5-brane worldvolume in the 11d spacetime.

The theory living on the worldvolume of N M5-branes in flat space is the interacting $(2, 0)$ superconformal field theory of type A_{N-1} coupled to the free abelian tensor multiplet. We can determine the anomaly polynomial of the A_{N-1} theory by subtracting off the contribution from the latter,

$$I_8^{\text{int}}[N] = (N - 1)I_8[1] + \frac{1}{24}(N^3 - N)p_2(\mathcal{N}) \,. \tag{7.3}$$

This agrees with [92] where the anomaly polynomial of the 6d $(2, 0)$ theories associated to ADE Lie algebras was conjectured to be

$$I_8(G) = r(G)I_8[1] + \frac{1}{24}d(G)h^{\vee}(G)p_2(\mathcal{N}) \,, \tag{7.4}$$

where r, d, and $h^\vee$ are the rank, dimension, and the dual Coxeter number of the ADE group G, respectively.

Following [25] the anomaly polynomial I_4 for the string arising from the M5-brane wrapping a compact surface P inside a Calabi-Yau threefold,[29] Y_3, can be determined by integrating the 6d anomaly polynomial over P. For such an M-theory setup the tangent and normal bundles decompose as

$$
\begin{aligned}
TW &= TP \oplus TW_2\,, \\
\mathcal{N} &= \mathcal{N}_{P/Y_3} \oplus \mathcal{N}_3\,,
\end{aligned}
\tag{7.5}
$$

where W_2 is the worldvolume of the string, $\mathcal{N}_{P/Y_3}$ is the normal bundle of P inside of the Calabi-Yau, and $\mathcal{N}_3$ is the bundle associated to the $SO(3)_T$ global symmetry from the rotations of the 3 transverse directions to the string in 5d. Under these bundle decompositions the Pontryagin classes decompose, via the splitting principle, to

$$
\begin{aligned}
p_1(\mathcal{N}_{P/Y_3} \oplus \mathcal{N}_3) &= p_1(\mathcal{N}_{P/Y_3}) + p_1(\mathcal{N}_3) \\
p_2(\mathcal{N}_{P/Y_3} \oplus \mathcal{N}_3) &= p_2(\mathcal{N}_{P/Y_3}) + p_2(\mathcal{N}_3) + p_1(\mathcal{N}_{P/Y_3})p_1(\mathcal{N}_3)\,,
\end{aligned}
\tag{7.6}
$$

and similarly for $p_i(W)$.

First, let us consider the integration of the anomaly polynomial of a single M5-brane:

$$
48 \int_P I_8[1] = 2p_1(\mathcal{N}_3) \int_P p_1(\mathcal{N}_{P/Y_3}) - \frac{1}{2}(p_1(W_2) + p_1(\mathcal{N}_3)) \int_P (p_1(P) + p_1(\mathcal{N}_{P/Y_3}))\,.
\tag{7.7}
$$

We can use the adjunction formula

$$
TY_3 = TP \oplus \mathcal{N}_{P/Y_3}\,,
\tag{7.8}
$$

to rewrite the last integrand as $p_1(Y_3)$. Finally we can use the representation of the Pontryagin classes in terms of the Chern classes,

$$
p_1(Y_3) = -2c_2(Y_3) + c_1(Y_3)^2\,,
\tag{7.9}
$$

and the Calabi-Yau property of Y_3, $c_1(Y_3) = 0$, to rewrite the two integrands in terms of the Chern classes of P and Y_3. In conclusion, the integral over the total anomaly polynomial I_8, combining both the free and interacting theories living on the M5-brane, is [25]

$$
I_4[N] = \int_P I_8[N] = N I_4[1] + \frac{1}{24}(N^3 - N)P^3 p_1(\mathcal{N}_3)\,,
\tag{7.10}
$$

where we have rewritten the integrals over P as integrals over Y_3 using intersection notation, and

$$
I_4[1] = \int_P I_8[1] = \frac{1}{48}\left[2P^3 p_1(\mathcal{N}_3) + c_2(Y_3) \cdot_{Y_3} P(p_1(W_2) + p_1(\mathcal{N}_3))\right]\,.
\tag{7.11}
$$

---

[29]Note that the Calabi-Yau threefold does not, at this point, need to be elliptically fibered. Moreover, P does not have to be a (very) ample divisor.

The gravitational anomaly determines the difference between the left- and right-moving central charges of the $(0,4)$ SCFT on the string, and can be read off from the anomaly

$$I_4 \supset \frac{c_L - c_R}{24} p_1(W_2) \, . \tag{7.12}$$

Thus we immediately determine that

$$c_L - c_R = \frac{1}{2} N c_2(Y_3) \cdot_{Y_3} P \, . \tag{7.13}$$

From the anomaly polynomial it is also possible to read off the level associated to the $SO(3)_T$ global symmetry by studying the $k_3 p_1(\mathcal{N}_3)/4$ term. We find

$$k_3 = \frac{1}{6} N^3 P^3 + \frac{1}{12} N c_2(Y_3) \cdot_{Y_3} P \, . \tag{7.14}$$

For future reference we also note that k_3 can be expressed directly in terms of the Hodge numbers of P. Using the expansion of the Chern numbers in terms of the Hodge numbers we have

$$P^3 = 10 h^{0,2}(P) - 8 h^{0,1}(P) - h^{1,1}(P) + 10$$
$$c_2(Y_3) \cdot_{Y_3} P = 2 h^{1,1}(P) - 8 h^{0,2}(P) + 4 h^{0,1}(P) - 8 \, . \tag{7.15}$$

At this point we shall specialise to considering that Y_3 is an elliptic fibration. From the Shioda-Tate-Wazir theorem as described in section 2.2 we know the divisors in Y_3 that generate the Neron-Severi lattice, and we would like to compute these quantities, $c_L - c_R$ and k_3, for representatives of certain linear systems of these divisors on Y_3. Recall that we are interested in elliptically fibered Calabi-Yau threefolds $\pi : Y_3 \to B$, with section, and that the two types of basis divisors of principle interest are the base, B, and the pullbacks of curves in the base, $\widehat{C}_\alpha = \pi^* C_\alpha$, such that the curve is not contained inside the discriminant locus of the elliptic fibration.

Let us consider an M5-brane wrapping a smooth irreducible divisor in the linear system

$$D \in |MB + N\widehat{C}| \, , \tag{7.16}$$

where $\widehat{C}$ is a linear combination of the $\widehat{C}_\alpha$, and compute the above quantities for $P = D$. The cohomology class of D can be written as

$$[D] = M[B] + N[\widehat{C}] \, , \tag{7.17}$$

and thus the first intersection number that must be computed is

$$[D]^3 = M^3 [B]^3 + N^2 [\widehat{C}]^3 + 3 M^2 N [B] \cdot_{Y_3} [B] \cdot_{Y_3} [\widehat{C}] + 3 M N^2 [B] \cdot_{Y_3} [\widehat{C}] \cdot_{Y_3} [\widehat{C}]$$
$$= M^3 (10 - h^{1,1}(B)) + 3 M^2 N (-c_1(B) \cdot_B C) + 3 M N^2 C \cdot_B C \, , \tag{7.18}$$

where for the final two intersections we have used the triple intersection numbers for elliptic Calabi-Yau varieties of section 2.2. Furthermore

$$\frac{1}{2} c_2(Y_3) \cdot_{Y_3} [D] = \frac{1}{2} M c_2(Y_3) \cdot_{Y_3} [B] + \frac{1}{2} N c_2(Y_3) \cdot_{Y_3} [\widehat{C}]$$
$$= M(h^{1,1}(B) - 4) + 6 N c_1(B) \cdot_B C \, . \tag{7.19}$$

Therefore we have determined that for an M5-brane wrapping an arbitrary divisor D belonging to such a linear system

$$c_L - c_R = 6N c_1(B) \cdot C + M(h^{1,1}(B) - 4), \tag{7.20}$$

and

$$k_3 = \frac{1}{2} M N^2 C \cdot C + \frac{1}{2} N(2 - M^2) c_1(B) \cdot C \\ + \frac{1}{6} \left(M^3 (10 - h^{1,1}(B)) + M(h^{1,1}(B) - 4) \right). \tag{7.21}$$

Note that to compute these coefficients we had to use the anomaly polynomial for a single M5-brane, $I_4[1]$, as M and N may be coprime, however when either M or N vanishes we see the correct result for multiple M5-branes wrapping a single divisor as in (7.10).[30]

At this point we have determined the difference in left- and right-moving central charges and the anomaly coefficient for the $SO(3)_T$ normal bundle anomaly for an the 2d $(0,4)$ theory on the worldvolume of the string from an M5-brane wrapping an arbitrary divisor D in Y_3. From [20] it is known that if D is a very ample divisor in Y_3 then the computation of k_3 is a suitable substitute for the computation of k_r, the level of the superconformal $SU(2)_r$ R-symmetry in the IR, and thus one can compute the right-moving central charge through the superconformal algebra relation

$$c_R = 6 k_r. \tag{7.22}$$

In fact, when D is ample the existence of an 11d supergravity dual of the type $AdS_3 \times S^2 \times Y_3$ guarantees that $SO(3)_T$ can be identified exactly[31] with the $SU(2)_r$ R-symmetry rotating the S^2. Thus $c_R = 6 k_r = 6 k_3$ is valid more generally for an ample divisor D.

From the information just described it is possible to compute the left- and right-moving central charges for the $(0,4)$ SCFT living on the string from a stack of M5-branes wrapping a compact complex surface inside a Calabi-Yau threefold, assuming that the surfaces satisfy sufficient topological properties that the level associated to the superconformal R-symmetry, k_r, is the same as k_3. For a divisor D inside the linear system that we are interested in, $|MB + N\widehat{C}|$, a discussion of exactly when this divisor may be ample in Y_3 is contained in appendix E.2. A necessary condition for D to be an ample divisor is that

$$D \cdot C = (N - M) C \cdot C + M(2g - 2) > 0, \tag{7.23}$$

as pointed out in (E.16). It is clear that such an inequality cannot be satisfied for arbitrary values of M, N, and g, however in the large N limit, where $N \gg M$, and when C is ample

---

[30] For arbitrary values of M and N one can consider the anomaly of a single M5-brane wrapping either the divisor D as in (7.11), or one can factor D as $D = \gcd(M, N) D'$, and consider $\gcd(M, N)$ M5-branes wrapping the divisor D' as in (7.10), by computing $I_4[\gcd(M, N)]$ for the divisor D'. It is straightforward to verify that both approaches produce the same result.

[31] More specifically the $SO(3)_T$ acting on the fields of the interacting SCFT, is then exactly the $SU(2)_r$ superconformal R-symmetry of that interacting SCFT. One can see directly from the spectrum that only after the universal centre-of-mass hypermultiplet is separated out is the $SO(3)_T$ consistent with the superconformal algebra.

in the base, this is always satisfied. For any ample D, which then satisfies this inequality, we can use (7.20) and (7.21), to compute the right- and left-moving central charges on the M5-brane wrapping D and we find

$$c_R = 3N^2 M C \cdot C + 3N(2-M^2)c_1(B) \cdot C + M^3(10-h^{1,1}(B)) + M(h^{1,1}(B)-4),$$
$$c_L = 3N^2 M C \cdot C + 3N(4-M^2)c_1(B) \cdot C + M^3(10-h^{1,1}(B)) + 2M(h^{1,1}(B)-4). \tag{7.24}$$

To determine these central charges we have used that the level k_3 of the $SO(3)_T$ normal bundle anomaly is the same as the level of the superconformal R-symmetry anomaly, however this only holds if D is ample in Y_3, which is exactly the requirement for when a supergravity dual of this 2d theory exists. From the field theory side we are justified in considering a setup where $M = 0$ and we just have a stack of N M5-branes wrapping the elliptic surface $\widehat{C}$. In appendix E.2 we show that $\widehat{C}$ is never itself an ample divisor, but in such a situation we would like to be able to determine a prescription for computing the central charge of the $(0,4)$ theory for such a stack of M5-branes, applicable even when the divisor wrapped by the M5-branes is not ample. This will correspond to the Type IIB/D3-brane setup where there are no KK-monopoles. We postpone this discussion for M5-branes until section 7.3, while we now turn to the F-theory picture for this setup.

7.2 Anomalies of 6d self-dual strings

A stack of N M5-branes wrapping an elliptic surface $\widehat{C}$ inside an elliptic Calabi-Yau three-fold is T-dual to a stack of N D3-branes wrapping a curve in the base of the elliptic Calabi-Yau. Such D3-brane stacks give rise to self-dual strings in 6d, and the anomaly polynomial for such strings was determined via inflow from the 6d theory in [93, 94] and extended to include arbitrary genus curves in [9]. We will assume that the curve, C, on which the D3-branes wrap has only transversal intersections with the discriminant locus of the elliptic fibration. The $(0,4)$ worldvolume theory on the string has the global symmetry group

$$SU(2)_R \times SU(2)_L \times SU(2)_I, \tag{7.25}$$

where $SO(4)_T \cong SU(2)_R \times SU(2)_L$ is the rotation group to the non-compact directions transverse to the string and $SU(2)_I$ is the R-symmetry group of the 6d theory. The $SO(4)_R$ UV R-symmetry group for the $(0,4)$ theory on the worldvolume of the string is $SU(2)_R \times SU(2)_I$.

In [93, 94] the anomaly polynomial for a self-dual string, of charges Q_i with respect to the two-form potentials B_i, with dB_i self-dual, in a 6d $\mathcal{N} = (1,0)$ theory was determined by applying a similar analysis as that was introduced in [25], and which was used in section 7.3 for the anomaly polynomial on a stack of M5-branes. The translation of the charges Q_i of the strings into the curve classes from the interpretation of the strings as coming from D3-branes wrapping the curve C was included in [9]. The final result for the anomaly polynomial, I_4, of the string in terms of the characteristic classes of the bundles associated to the symmetry groups (7.25) is

$$I_4 = c_2(R)\left[\frac{1}{2}N^2 C \cdot C + \frac{1}{2}Nc_1(B) \cdot C\right] + c_2(L)\left[-\frac{1}{2}N^2 C \cdot C + \frac{1}{2}Nc_1(B) \cdot C\right]$$
$$+ c_2(I)[N] - \frac{1}{24}p_1(T)[6Nc_1(B) \cdot C], \tag{7.26}$$

where we have ignored contributions from any additional global (flavour) symmetries other than those discussed above, and where we recall that the genus of the curve is contained inside the above expressions implicitly via adjunction (2.16). First we can determine the difference between the left- and right-moving central charges from the gravitational anomaly term

$$c_L - c_R = 6Nc_1(B) \cdot C \,. \tag{7.27}$$

One can also read off from the anomaly polynomial the levels of the $SU(2)_{R,L,I}$ global symmetries

$$
\begin{aligned}
k_R &= \frac{1}{2}N^2 C \cdot C + \frac{1}{2}Nc_1(B) \cdot C \\
k_L &= -\frac{1}{2}N^2 C \cdot C + \frac{1}{2}Nc_1(B) \cdot C \\
k_I &= N \,.
\end{aligned}
\tag{7.28}
$$

Note that the $SU(2)_r$ superconformal R-symmetry can in principle be a mix [35] of the two $SU(2)$ factors in the $SO(4)_R$ UV R-symmetry. We observe from the spectrum for $N = 1$ that the IR R-symmetry for the SCFT must be $SU(2)_R$ as this is the only factor under which the bosons of all the hypermultiplets constituting the theory are uncharged. Moreover, in the next subsection we will argue (using only the information on k_L from this section) that the correct R-symmetry in the IR should be simply $SU(2)_R$ for any N. Thus there is no mixing with $SU(2)_I$ and we conclude that

$$c_R = 6k_R = 3N^2 C \cdot C + 3Nc_1(B) \cdot C \,, \tag{7.29}$$

and from (7.27) we also obtain

$$c_L = 3N^2 C \cdot C + 9Nc_1(B) \cdot C \,. \tag{7.30}$$

7.3 Anomaly from M5-branes on $\widehat{C}$

Let us now return to the M5-brane anomaly inflow, in the case that the branes wrap the elliptic surface $\widehat{C}$ in Y_3, which is not ample. We recall that in this instance there does not exist an AdS_3 dual, without three-form flux, because of the lack of ampleness of the divisor. However, in this section, one shall see that it is still possible to determine the central charges of the SCFT. We can immediately see from a study of the spectrum of a single M5-brane [9] that k_3 is not a suitable substitute computation for k_r when the wrapped divisor is not ample.[32] Let us first consider an arbitrary divisor P inside an arbitrary Calabi-Yau threefold. We can read off from the expressions in terms of Hodge numbers in (7.15) that

$$k_3 = h^{0,2}(P) - h^{0,1}(P) + 1 \,, \tag{7.31}$$

but a direct computation of the right-moving central charge from the spectrum reveals that

$$k_r = h^{0,2}(P) + 1 \,. \tag{7.32}$$

---

[32]This puzzle was raised in [95, 96].

This is consistent, as for P an ample divisor inside a Calabi-Yau threefold then $h^{0,1}(P) = 0$ by the Lefschetz hyperplane theorem.

Now, let us consider multiple (N) M5-branes wrapping the divisor $P = \widehat{C}$; hence $M = 0$ in the notation of section 7.1. Using standard mathematical results for the cohomologies of elliptic surfaces

$$h^{0,2}(\widehat{C}) = \frac{1}{2} \left(C \cdot C + c_1(B) \cdot C \right) ,$$
$$h^{0,1}(\widehat{C}) = \frac{1}{2} \left(C \cdot C - c_1(B) \cdot C \right) + 1 = g , \qquad (7.33)$$
$$h^{1,1}(\widehat{C}) = C \cdot C + 9c_1(B) \cdot C + 2 ,$$

we can see that

$$\widehat{C}^3 = 0 , \qquad c_2(Y_3) \cdot \widehat{C} = 12c_1(B) \cdot C , \qquad (7.34)$$

and thus

$$k_3 = Nc_1(B) \cdot C , \qquad (7.35)$$

for an M5-brane wrapping any elliptic surface embedded inside an elliptic Calabi-Yau as discussed. Such a result of course also follows directly from the expression (7.21) for k_3 when one sets $M = 0$.

When the divisor is not ample we follow the idea in [97] that k_3 is really a substitute for computing the anomaly associated with the diagonal of the superconformal R-symmetry, k_r, with an additional flavour symmetry that only emerges, from the M5-brane point of view, in the IR

$$k_r = k_3 - k_F , \qquad (7.36)$$

where k_F is the level of the emergent $\mathrm{SU}(2)_F$ flavour symmetry.

In order to make progress in determining this flavour symmetry, we go back to the D3 brane setup in Type IIB. The reason why this is useful is that in the Type IIB side a *flavour* (i.e. non-R) symmetry is realized geometrically,[33] simply because the normal bundle of the wrapped D3 branes is $\mathrm{SO}(4)_T$, while the normal bundle of the wrapped M5 branes is $\mathrm{SO}(3)_T$. Notice that while R-symmetries are ambiguous, because mixing an R-symmetry with a flavour symmetry is still an R-symmetry, flavour symmetries do not have this ambiguity.

From the self-dual string in 6d, as is discussed in section 7.2, we know exactly one flavour symmetry, which is the $\mathrm{SU}(2)_L$ arising from the transverse rotations to the string, and further we can observe from the spectrum that the $\mathrm{SO}(3)_T$ charges of the multiplets from the M5-brane on $\widehat{C}$ are the diagonal of the $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$ charges of the multiplets from the D3-brane on C [9]. As it is the only flavour symmetry that we know is always present, and since it combines with the superconformal R-symmetry in the correct way to form $\mathrm{SO}(3)_T$ we are justified in conjecturing that the flavour symmetry, $\mathrm{SU}(2)_F$, which we do not observe the origin of in the M-theory, has level, k_F, which we must subtract off to compute the k_r is none other than k_L.

---

[33]Although the two setups are related by a T-duality, the symmetries on the two sides do not necessarily match manifestly. This is also true at the level of isometries of supergravity solutions, although notice that presently one of the two setups does not admit a supergravity description.

From the analysis of the self-dual string we have that

$$k_L = -\frac{1}{2}N^2 C \cdot C + \frac{1}{2}N c_1(B) \cdot C \,, \tag{7.37}$$

however, as discussed in [8], this anomaly coefficient is not quite identified with the level of the $SU(2)_L$ symmetry on the combined theory. In the anomaly coefficient of the $SU(2)_L$ anomaly there is a fictitious contribution from the centre-of-mass hypermultiplet. This universal hypermultiplet is charged under the $SU(2)_L$ however there is no $SU(2)_L$ current algebra acting on these modes. The level of the $SU(2)_L$ current algebra on the combined theory is then determined by subtracting the contribution[34] of $k_L^{\mathrm{CoM}} = +1$ from k_L to find that the level is

$$k_L - 1 \,. \tag{7.38}$$

This is then the level of the flavour symmetry of the combined theory including the centre of mass which we then subtract from k_3, which is the level of the $SO(3)_T$ normal bundle anomaly of the combined theory, to determine the level of the superconformal R-symmetry of the combined theory.

As such the right-moving central charge as determined via the M5-brane anomaly inflow when $M = 0$ is

$$c_R = 6(k_3 - (k_L - 1)) = 3N^2 C \cdot C + 3N c_1(B) \cdot C + 6 \,. \tag{7.39}$$

We emphasise again that, as expected, this is the central charge for the combined theory, i.e. the interacting theory together with the centre of mass. Further, we can observe that this identifies the superconformal R-symmetry level as

$$k_r = k_3 - (k_L - 1) = k_R \,, \tag{7.40}$$

demonstrating our statement in the previous subsection that the superconformal R-symmetry is identified with $SU(2)_R$ for all N. In this analysis we are working under the assumption that generically there is only one $SU(2)$ flavour symmetry in the IR, and that that flavour symmetry is $SU(2)_L$. If there are additional flavour symmetries then these could in principle also mix with the superconformal R-symmetry to form k_3 and these would need to be subtracted in addition.

7.4 Summary and comparison

Let us finally summarise and compare the results of all the computations (from anomalies and holography) of central charges presented in this paper. The theory to which the worldvolume theory on the string flows in the IR consists of a direct sum of two SCFTs; the generically non-trivial and the centre-of-mass conformal field theories. We shall refer to the former as the SCFT part. Depending on the method used we either compute properties of the SCFT, or else of the combined theory. Generally speaking we shall be interested in comparing the central charges of the SCFT, not including the centre of mass; these are the quantities naturally computed by the AdS duals as the centre of mass decouples in the near-horizon geometry.

---

[34]We note that there is a difference of an overall minus sign between here and [8].

The spectrum. For a single D3-brane wrapping a curve C in the base of an elliptic threefold, or equivalently for a single M5-brane wrapping the elliptic surface $\widehat{C}$, the massless spectrum can be computed explicitly. The central charges as computed directly from the UV spectrum are

$$\text{Spectrum } (N = 1) : \qquad \begin{aligned} c_R &= 3C \cdot C + 3c_1(B) \cdot C + 6 \,, \\ c_L &= 3C \cdot C + 9c_1(B) \cdot C + 6 \,. \end{aligned} \tag{7.41}$$

These are the central charges for the combined theory, including the centre-of-mass modes. The scalar fields parametrising the position of the string in the transverse 5d or 6d space are contained inside of a single hypermultiplet, which is then referred to as the centre-of-mass hypermultiplet, and contributes to the central charges

$$(c_L^{\text{CoM}}, c_R^{\text{CoM}}) = (4, 6) \,. \tag{7.42}$$

Subtracting off these modes gives the central charges for the IR SCFT on the worldvolume of the string.

Anomaly polynomial of self-dual strings. In [94] the anomaly polynomial for the self-dual string in 6d was written down, as we discussed in section 7.2. This is the anomaly polynomial for the combined theory including both the centre-of-mass and SCFT sectors. The combined theory on the string has a global symmetry group

$$\text{SU}(2)_R \times \text{SU}(2)_L \times \text{SU}(2)_I \,, \tag{7.43}$$

where $\text{SU}(2)_R \times \text{SU}(2)_L$ comes from the transverse rotations to the string in 6d, and $\text{SU}(2)_R \times \text{SU}(2)_I$ is the UV R-symmetry of the worldvolume theory of the string. We are interested in computing from this anomaly polynomial the central charges of the SCFT in the IR. First one can determine the difference of the central charges of the combined theory from the gravitational anomaly

$$c_L - c_R = 6Nc_1(B) \cdot C \,. \tag{7.44}$$

To determine the right-moving central charge of the SCFT we need to know the level of the superconformal $\text{SU}(2)_r$ R-symmetry, which should be one of the $\text{SU}(2)$ factors inside the $\text{SO}(4)$ UV R-symmetry. Furthermore, identifying $\text{SU}(2)_R$ with the IR R-symmetry, as discussed in the previous subsection, we have computed that

$$c_R = 6k_R = 3N^2 C \cdot C + 3Nc_1(B) \cdot C \,. \tag{7.45}$$

This matches the right-moving central charge computed for the SCFT from the spectrum for $N = 1$, as expected. If we subtract the free hypermultiplet constituting the centre-of-mass degree of freedom from the difference of the right- and left-moving central charges then we can also determine the left-moving central charge for the SCFT as

$$c_L = 3N^2 C \cdot C + 9Nc_1(B) \cdot C + 2 \,. \tag{7.46}$$

Again this matches the spectrum when $N = 1$ as expected.

Type IIB supergravity. As discussed in section 4 we can also compute the central charges for the same setup from the Type IIB supergravity dual. As such a supergravity computation is necessarily in the near-horizon limit then the centre-of-mass modes are decoupled and we compute directly only the central charges of the SCFT. We will first consider the case *without KK-monopoles*, where in (4.22) we found that

$$c_R^{\rm IIB} = 3N^2 C \cdot C + 3N c_1(B) \cdot C \,, \tag{7.47}$$

which exactly matches the right-moving central charge of the theory from the spectrum and the anomaly analyses discussed previously. This would lead us to conclude that the Type IIB supergravity computation of c_R *is* in fact exact, meaning that there would be no quantum corrections, in this precise situation, as any sub-subleading correction would ruin the precise matching with the result in (7.45).

In (4.22) we also determined the left-moving central charge to be

$$c_L^{\rm IIB} = 3N^2 C \cdot C + 9N c_1(B) \cdot C \,, \tag{7.48}$$

where we remind the reader that this result is only expected to be accurate to order in $\mathcal{O}(N)$, and we expect from the alternate approaches to the computation of the same quantity that the full result, including quantum corrections, should have an additional $+2$.

In principle, from the Type IIB supergravity one should be able to determine the holographic central charges also for $M \geq 1$, where there are in addition M KK-monopoles in the system. However, as we discussed in section 4, in this case we can compute reliably only the leading order, $\mathcal{O}(N^2)$, coefficients. To determine the correct $\mathcal{O}(N)$ contributions to the anomalies we would need to incorporate the effect of the KK-monopoles.

11d supergravity. In order for the 11d supergravity solution to exist it is necessary that the divisor wrapped by the M5-brane is an ample divisor in the Calabi-Yau threefold, and from appendix E.2 we can see that this generally requires that $M > 0$. In this section we shall take $M = 1$ principally so as to compare with the majority of the different approaches, and we will show a matching for $M > 0$ result at the end. For $M = 1$ (the 11d supergravity setup dual to *one* KK-monopole in Type IIB) in section 6 we computed the central charges to be

$$\begin{aligned}
c_R^{11} &= 3N^2 C \cdot C + 3N c_1(B) \cdot C + 6 \,, \\
c_L^{11} &= 3N^2 C \cdot C + 9N c_1(B) \cdot C + 2 + h^{1,1}(B) \,.
\end{aligned} \tag{7.49}$$

These central charges are said to be exact in [79] as they can be determined from an anomaly analysis. Since the exactness follows from an anomaly argument these central charges should be the central charges for the full combined theory, including the centre-of-mass degrees of freedom. Given that the centre-of-mass contribution should be universal, regardless of the values of M, N, we can similarly subtract one universal hypermultiplet to determine the central charges of the IR SCFT. We notice that the leading and sub-leading terms are consistent with all other methods of computations for one or no KK-monopole. As discussed in section 1, in the near-horizon limit there is no difference between the setup with one or no KK-monopoles, and thus the leading contribution to the central charges

must be identical. We find that the central charges match also at the subleading order, and in fact the expression for c_R matches the results obtained in the case without KK-monopoles exactly, but it is not clear to us whether this is accidental or not. On the other hand, it makes sense that both c_R and c_L do not both match exactly across the configurations with one or no KK-monopole, as the difference $c_L - c_R$ is a quantity that can be computed purely in the UV, and in the UV the single KK-monopole is apparent.

The general result for all $M > 0$ was given in (6.20) and reads

$$
\begin{aligned}
c_R^{11} &= 3N^2 M C{\cdot}C + 3N(2-M^2)c_1(B){\cdot}C + M^3(10-h^{1,1}(B)) + M(h^{1,1}(B)-4),\\
c_L^{11} &= 3N^2 M C{\cdot}C + 3N(4-M^2)c_1(B){\cdot}C + M^3(10-h^{1,1}(B)) + 2M(h^{1,1}(B)-4),
\end{aligned}
\tag{7.50}
$$

but as discussed previously, we have not determined these in the Type IIB picture, beyond the leading $\mathcal{O}(N^2)$ order. At this order we indeed find perfect agreement for any N and M, see (4.24).

M5-brane anomaly inflow. Another M-theory approach that one can take to determine the central charges involves computing the anomaly polynomial to the string via M5-brane anomaly inflow as described in section 7.1. When the divisor wrapped by the M5-brane is ample in the Calabi-Yau then this approach involves effectively the same computation as was used to determine the central charges from 11d supergravity, and is also a computation of the central charges of the combined theory. The results for the central charges for $M > 0$ from the anomaly inflow are then the same as those given in (6.20) from the 11d supergravity.

The inflow computation however is valid for any divisor D even if it is not ample in the Calabi-Yau. As such, here we shall be mainly interested in the central charges for the $M = 0$ case where the M5-brane wraps simply $\widehat{C}$. As described in section 7.3 this approach does not directly compute the central charge, but instead computes the anomaly coefficient associated to the $SO(3)_T$ normal bundle anomaly, and the gravitational anomaly which fixes

$$
c_L - c_R = 6N c_1(B) \cdot C \,.
\tag{7.51}
$$

It is known that when the divisor wrapped is ample the computation of the anomaly coefficient k_3 is a suitable substitute computation for the anomaly coefficient of the superconformal R-symmetry, k_r. However when the wrapped divisor is not ample one must subtract an emergent IR flavour symmetry from k_3 to determine the superconformal R-symmetry. As discussed in section 7.3 we can determine the flavour symmetry which mixes with the superconformal R-symmetry and we can then compute

$$
c_R = 6(k_3 - (k_L - 1)) = 3N^2 C \cdot C + 3N c_1(B) \cdot C + 6 \,,
\tag{7.52}
$$

which is the central charge of the combined theory. Further one can determine the left-moving central charge of the combined theory as

$$
c_L = 3N^2 C \cdot C + 9N c_1(B) \cdot C + 6 \,.
\tag{7.53}
$$

8 Conclusions and outlook

New holographic setups which allow for a controlled computational framework for both the perturbative gauge theory as well as the dual gravitational/string theory, are difficult to come by. In this paper we studied a new class of solutions of Type IIB supergravity, which allow for a varying axio-dilaton τ, that is consistent with the $\mathrm{SL}(2,\mathbb{Z})$ duality, i.e. F-theory solutions. In particular, we classified the AdS$_3$ solutions in F-theory dual to 2d SCFTs with $(0,4)$ supersymmetry, in the absence of three-form fluxes. The field theory duals arise from D3-branes wrapped on curves in the base of elliptic Calabi-Yau threefold compactifications studied in [8, 9]. The solutions that we have found to be the most general of this kind are of the type AdS$_3 \times S_3/\Gamma \times B$, where B is the base of an elliptic Calabi-Yau threefold, and the profile of the axio-dilaton is determined in terms of the complex structure of the elliptic fiber.

Conceptually there are various points that make this duality more subtle than those involving Type IIB solutions with constant τ. First of all the profile of the axio-dilaton has to be such that τ is singular along curves in the base B. This in turn implies that the metric on the base cannot be smooth everywhere, and thus some care needs to be taken in order to reliably apply a supergravity analysis. This is in particular subtle in Type IIB as the compactification manifold does not include the elliptic fiber, but only the base. Key to corroborating the consistency of this solution is the duality to 11d supergravity, that we can perform for the solutions with $\Gamma = \mathbb{Z}_M$. We showed that in 11d supergravity these solutions are of the form AdS$_3 \times S^2 \times Y_3$, where the elliptic Calabi-Yau threefold Y_3 can be resolved and has a smooth Ricci-flat Kähler metric.

Following up on this paper, there are numerous immediate questions of interest to pursue: an obvious extension of the present results is to include three-form fluxes and to potentially classify all $(0,4)$ AdS$_3$ solutions in Type IIB supergravity. We presented an example of such solution in appendix F. Based on this, and other examples in the literature [98, 99], we expect that this class of solutions may be quite rich. Furthermore, the classification obtained here can be applied to $\mathcal{N} = (2,2)$ supersymmetry in 2d, and it will be interesting to explore this class of solutions.

Another class of $(0,4)$ strings in F-theory compactifications to 6d are the so-called non-Higgsable cluster strings. As we recalled earlier, these are obtained from D3-branes wrapped on collapsed curves in Calabi-Yau threefolds, which have singular algebraic varieties as base manifolds. In particular, these singularities can be thought of as arising from the collapse of a curve $C^{\mathrm{NHC}} \simeq \mathbb{P}^1$ in the local geometry of $\mathcal{O}(-n) \to \mathbb{P}^1$, where the curve has self-intersection $C^{\mathrm{NHC}} \cdot C^{\mathrm{NHC}} = -n < 0$. These can be embedded in a compact geometry by projectivizing, which results in the Hirzebruch surfaces $\mathbb{F}_n$. It is then tantalizing to speculate that our solutions might capture some features of the NHC strings by choosing the Kähler base to be $B = \mathbb{F}_n$, or their singular limits, i.e. the weighted projective spaces $\mathbb{P}^{(1,1,n)}$. On the other hand, since C^{NHC} is not ample, this simple setup cannot be found within the class of solutions discussed in this paper. Our attempts to reproduce features of the NHC strings in this holographic setup have not been successful, and it remains an open problem to determine what the appropriate holographic duals of these SCFTs, if these exist, are.

In [9] a class of 2d $(0,2)$ theories were obtained, from D3-branes wrapped in the base of elliptic Calabi-Yau four- and fivefolds. These are very closely related setups to the ones studied here, and naturally finding AdS$_3$ duals to these 2d SCFTs would be very interesting. In relation to the solutions found here, the case of Calabi-Yau fivefolds is closely related to our F-theory solutions with KK-monopole. The F-theory compactification space is $Y_3 \times TN_M$, which is a special Calabi-Yau fivefold. F-theory on elliptic Calabi-Yau fivefolds has only recently been investigated in [100, 101] and result in 2d (0,2) theories for generic Calabi-Yau fivefolds. In view of this, it would be interesting to study our AdS$_3$ solutions with $\Gamma = \mathbb{Z}_M$ in relation to the near horizon limits of D3-branes in Calabi-Yau fivefold compactifications of the type $Y_3 \times TN_M$ and determine the spectrum for general M as in [9].

Finally, the question of AdS$_5$ solutions in F-theory arises, which would generalise the solutions AdS$_5 \times S^5/\Gamma$ of [45, 47, 102] to F-theory solutions[35] with non-trivially varying τ. In particular, this would be interesting in relation to dimensional reductions of the recently obtained classification of 6d $(1,0)$ SCFTs in F-theory [104], which upon compactification on curves yield 4d $\mathcal{N} = 1$ SCFTs. These theories could arise also in terms of F-theory on Calabi-Yau fourfolds and may have F-theoretic AdS$_5$ duals.

We hope to return to these interesting questions in the near future [33].

Acknowledgments

A Conventions for gamma matrices and spinors

We shall use the letters $M, N, \ldots$ for the 10d indices, $a, b, \ldots$ takes values $0, 1, 2$ and are used for the AdS$_3$ indices and $\mu, \nu, \ldots \in \{1, \ldots, 7\}$ for the indices for $\mathcal{M}_7$. Following [105]

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[35]Such solutions were briefly alluded to in [103].

we decompose the 10d Gamma matrices as

$$\Gamma^a = \rho^a \otimes 1 \otimes \sigma_2 \,, \tag{A.1}$$

$$\Gamma^\mu = 1 \otimes \gamma^\mu \otimes \sigma_1 \,, \tag{A.2}$$

where ρ^a generate Cliff(1,2) and γ^μ generate Cliff(7). Explicitly we shall take

$$\rho_0 = \mathrm{i}\,\sigma^1 \,, \qquad \rho_1 = \sigma^2 \,, \qquad \rho_2 = \sigma^3 \,, \tag{A.3}$$

with $\rho_{012} = -1$. For the Cliff(7) gamma matrices we shall take

$$\gamma^1 = -\sigma_1 \otimes 1 \otimes 1 \,, \tag{A.4}$$

$$\gamma^2 = \sigma_3 \otimes \sigma_2 \otimes 1 \,, \tag{A.5}$$

$$\gamma^3 = -\sigma_2 \otimes 1 \otimes 1 \,, \tag{A.6}$$

$$\gamma^4 = \sigma_3 \otimes \sigma_1 \otimes 1 \,, \tag{A.7}$$

$$\gamma^5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \,, \tag{A.8}$$

$$\gamma^6 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_2 \,, \tag{A.9}$$

$$\gamma^7 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \,, \tag{A.10}$$

and we have $\gamma_{1\ldots7} = -\mathrm{i}1$. With these conventions we have

$$\Gamma_{11} = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3 \,. \tag{A.11}$$

We follow the definitions in [106] for the various intertwiners. For the A intertwiner we have

$$A_{10}\Gamma_M A_{10}^{-1} = \Gamma_M^\dagger \,, \tag{A.12}$$

$$A_7 \gamma_\mu A_7^{-1} = \gamma_\mu^\dagger \,, \tag{A.13}$$

$$A_{1,2}\rho_a A_{1,2}^{-1} = -\rho_a^\dagger \,, \tag{A.14}$$

$$A_{10} = A_{1,2}\ \otimes A_7 \otimes \sigma_1 \,, \tag{A.15}$$

$$A_7 = 1 \,, \tag{A.16}$$

$$A_{1,2} = \sigma_1 \,. \tag{A.17}$$

For the charge conjugation intertwiner C we take

$$C_{10}^{-1}\Gamma_M C_{10} = -\Gamma_M^{\mathrm{T}} \,, \tag{A.18}$$

$$C_7^{-1}\gamma_\mu C_7 = -\gamma_\mu^{\mathrm{T}} \,, \tag{A.19}$$

$$C_{1,2}^{-1}\rho_a C_{1,2} = -\rho_a^{\mathrm{T}} \,, \tag{A.20}$$

$$C_{10} = C_{1,2} \otimes C_7 \otimes \sigma_1 \,, \tag{A.21}$$

$$C_7 = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \,, \tag{A.22}$$

$$C_{1,2} = \sigma_2 \,. \tag{A.23}$$

We have

$$C_{10}^{\mathrm{T}} = -C_{10} \,, \qquad C_7^{\mathrm{T}} = C_7 \,, \qquad C_{1,2}^{\mathrm{T}} = -C_{1,2} \,. \tag{A.24}$$

Finally the D intertwiner satisfies

$$D_{10}^{-1}\Gamma_M D_{10} = \Gamma_M^* \,, \tag{A.25}$$

$$D_7^{-1}\gamma_\mu D_7 = -\gamma_\mu^* \,, \tag{A.26}$$

$$D_{1,2}^{-1}\rho^a D_{1,2} = \rho_a^* \,, \tag{A.27}$$

$$D_{10} = D_{1,2} \otimes D_7 \otimes \sigma_3 \,, \tag{A.28}$$

$$D_7 = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \,, \tag{A.29}$$

$$D_{1,2} = -\mathrm{i}\,\sigma_3 \tag{A.30}$$

They satisfy

$$D_{10}^* = D_{10}^{-1} \,, \qquad D_7^* = D_7^{-1} \,, \qquad D_{1,2}^* = D_{1,2}^{-1} \,. \tag{A.31}$$

We now wish to decompose a 10d Majorana-Weyl spinor consistent with these conventions. We shall decompose the spinor, ϵ as $\epsilon = \psi \otimes \chi \otimes \theta$ where ψ is a two-component spinor, χ an eight-component spinor and θ a two-component spinor. The chirality condition in 10d is

$$\Gamma_{11}\epsilon = -\epsilon \tag{A.32}$$

which is solved by

$$\sigma_3\theta = -\theta \,. \tag{A.33}$$

For the Majorana condition we impose that both χ and ψ are Majorana and also that θ is purely imaginary. Type IIB supersymmetry is parametrised by two 10d Majorana-Weyl spinors. We may complexify the two Majorana-Weyl spinors into

$$\epsilon = \psi_1 \otimes \xi_1 \otimes \theta \tag{A.34}$$

where $\xi = \chi_1 + \mathrm{i}\chi_2$ is a Dirac spinor. This will generically preserve $(0,2)$ supersymmetry however we are also interested in finding the equations for preserving $(0,4)$ explicitly and so the ansatz we use to accommodate both cases is

$$\epsilon = \psi_1 \otimes \mathrm{e}^{A/2}\xi_1 \otimes \theta + \psi_2 \otimes \mathrm{e}^{A/2}\xi_2 \otimes \theta \,. \tag{A.35}$$

The $(0,2)$ case is obtained by setting one of the ξ's to zero. The warp factor appears here for later convenience. Here the ψ_i are Killing spinors on AdS$_3$ and satisfy the most general Killing spinor equations for two Killing spinors on AdS$_3$

$$\nabla_a \psi_i = \frac{m}{2} \sum_{j=1}^{2} W_{ij}\rho_a \psi_j \,. \tag{A.36}$$

In appendix B we show that we may diagonalize W.

B Killing spinors of AdS$_3$

In general two Killing spinors on AdS$_3$ may satisfy an equation of the form

$$\nabla_a \psi_i = \frac{m}{2} \sum_{j=1}^{2} W_{ij}\rho_a \psi_j \,, \tag{B.1}$$

with W an arbitrary matrix with possible coordinate dependence. In this section we show that W is in fact constant for our purposes and moreover may be diagonalised, allowing us without loss of generality, to take the Killing spinors on AdS$_3$ to satisfy

$$\nabla_a \psi_i = \pm \frac{m}{2} \rho_a \psi_i \tag{B.2}$$

which we have done in the main text.

We shall first show that W_{ij} is necessarily a constant matrix and finally show that it may be diagonalised. Observe that by multiplying (B.1) by $\bar{\psi}_k$ one has the bilinear

$$\bar{\psi}_k \rho^a \nabla_a \psi_i = \frac{3m}{2} W_{ij} \bar{\psi}_k \psi_j \,. \tag{B.3}$$

The Majorana condition and the antisymmetry of C_{12} imply $\bar{\psi}_i \psi_j = -\bar{\psi}_j \psi_i$ and in particular $\bar{\psi}_i \psi_i = 0$ (no sum). Observe that (B.3) gives four equations for the four components of W:

$$\bar{\psi}_1 \slashed{\nabla} \psi_1 = \frac{3m}{2} W_{12} \bar{\psi}_1 \psi_2 \,, \tag{B.4}$$

$$\bar{\psi}_1 \slashed{\nabla} \psi_2 = \frac{3m}{2} W_{22} \bar{\psi}_1 \psi_2 \,, \tag{B.5}$$

$$\bar{\psi}_2 \slashed{\nabla} \psi_1 = \frac{3m}{2} W_{11} \bar{\psi}_2 \psi_1 \,, \tag{B.6}$$

$$\bar{\psi}_2 \slashed{\nabla} \psi_2 = \frac{3m}{2} W_{21} \bar{\psi}_2 \psi_1 \,. \tag{B.7}$$

As the left-hand side of all four equations and the spinors on AdS$_3$ are independent of the internal manifold coordinates it follows that W is dependent only on the coordinates of AdS$_3$. As we wish to preserve the symmetry of AdS$_3$ in the solution this requires that the components of W must in fact be independent of the AdS$_3$ coordinates and therefore constant.

As the components of W are constant we are able to compute the integrability condition for these spinors and commute the derivatives past the components of W. One has the identity

$$[\nabla_{a_1}, \nabla_{a_2}] \psi_i = \frac{1}{4} R_{a_1 a_2}{}^{b_1 b_2} \rho_{b_1 b_2} \psi_i \,. \tag{B.8}$$

Explicitly computing this using (B.1) we find

$$[\nabla_{a_1}, \nabla_{a_2}] \psi_i = \frac{m^2}{2} W_{ij} W_{jk} \rho_{a_1 a_2} \psi_k \,. \tag{B.9}$$

Upon equating the two expressions and contracting with ρ^{a_2} whilst recalling our normalisation of AdS$_3$ we find

$$W_{ij} W_{jk} = \delta_{ik} \,. \tag{B.10}$$

It is therefore clear from this expression that the eigenvalues of W are ± 1, and furthermore that W is diagonalizable over $\mathbb{R}$. Therefore there is no ambiguity in changing basis of the Killing spinors on AdS$_3$ to make W diagonal. We shall interpret the eigenvalues of W as determining the preserved supersymmetry of the putative dual SCFT. After diagonalizing W the Killing spinor equation on AdS$_3$ is

$$\nabla_a \psi = \pm \frac{m}{2} \rho_a \psi \,. \tag{B.11}$$

It can be seen that the two equations give Killing spinors of opposite chirality on the boundary of AdS$_3$. Let us take global coordinates on AdS$_3$ in which the metric takes the form

$$\mathrm{d}s^2(\mathrm{AdS}_3) = -\frac{r^2 m^2 + 1}{m^2}\mathrm{d}t^2 + r^2\mathrm{d}\phi^2 + \frac{1}{m^2 r^2 + 1}\mathrm{d}r^2\,, \tag{B.12}$$

and satisfies $R_{ab} = -2m^2 g_{ab}$. First computing the Majorana Killing spinors on AdS$_3$ for the positive sign in (B.11), and using the gamma matrix conventions of appendix A, we find

$$\psi_+ = \mathrm{e}^{-\frac{\mathrm{i}(t+\phi)}{2}}\left(\begin{array}{c}\sqrt{r + \frac{1}{m}\sqrt{1 + m^2 r^2}}\left(a - \mathrm{i}\mathrm{e}^{\mathrm{i}(t+\phi)}a^*\right)\\ -\frac{1}{m\sqrt{r + \frac{1}{m}\sqrt{1 + m^2 r^2}}}(a + \mathrm{i}\mathrm{e}^{\mathrm{i}(t+\phi)}a^*)\end{array}\right)\,, \tag{B.13}$$

whilst for the negative sign in (B.11) we find the Killing spinor

$$\psi_- = \mathrm{e}^{-\frac{\mathrm{i}(t+\phi)}{2}}\left(\begin{array}{c}-\frac{1}{m\sqrt{r + \frac{1}{m}\sqrt{1 + m^2 r^2}}}(b\mathrm{e}^{\mathrm{i}\phi} - \mathrm{i}b^*\mathrm{e}^{\mathrm{i}t})\\ -\sqrt{r + \frac{1}{m}\sqrt{1 + m^2 r^2}}\left(b\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}b^*\mathrm{e}^{\mathrm{i}t}\right)\end{array}\right)\,. \tag{B.14}$$

Following [107] the divergent piece of each spinor in the limit as we go to the boundary of AdS$_3$ ($r \to \infty$ in the coordinates we have chosen), becomes the 2d supersymmetry parameter. In our conventions the chirality matrix in 2d is σ_3 and therefore ψ_+ gives rise to a supersymmetry parameter in 2d with positive chirality, whilst ψ_- gives rise to one with negative chirality. We conclude that to preserve $(0,2)$ supersymmetry in the boundary theory we must use a Killing spinor on AdS$_3$ which solves (B.11) with positive sign in our spinor ansatz (A.35), whilst to preserve $(2,0)$ we use one which solves (B.11) with negative sign.

C Torsion conditions on spinor bilinears

In the following subsections we compute the torsion conditions of the spinor bilinears. We shall keep the α_i's arbitrary in this appendix and specialise in the main text. Our notation for the various spinor bilinears is given by

$$S_{ij} \equiv \bar{\xi}_i \xi_j \tag{C.1}$$

$$A_{ij} \equiv \bar{\xi}_i^c \xi_j \tag{C.2}$$

$$K_{ij}^\mu \equiv \bar{\xi}_i \gamma^\mu \xi_j \tag{C.3}$$

$$B_{ij}^\mu \equiv \bar{\xi}_i^c \gamma^\mu \xi_j \tag{C.4}$$

$$U_{ij}^{\mu_1\mu_2} \equiv \bar{\xi}_i \gamma^{\mu_1\mu_2} \xi_j \tag{C.5}$$

$$V_{ij}^{\mu_1\mu_2} \equiv \bar{\xi}_i^c \gamma^{\mu_1\mu_2} \xi_j \tag{C.6}$$

$$X_{ij}^{\mu_1\mu_2\mu_3} \equiv \bar{\xi}_i \gamma^{\mu_1\mu_2\mu_3} \xi_j \tag{C.7}$$

$$Y_{ij}^{\mu_1\mu_2\mu_3} \equiv \bar{\xi}_i^c \gamma^{\mu_1\mu_2\mu_3} \xi_j\,. \tag{C.8}$$

Higher order bilinears are related to the ones presented above by Hodge duality.

Using the representation of the Clifford algebra in appendix A we find that the spinor bilinears have following symmetries

$$A_{ij} = A_{ji}, \qquad B_{ij} = -B_{ji}, \qquad V_{ij} = -V_{ji}, \qquad Y_{ij} = Y_{ji}. \qquad \text{(C.9)}$$

Furthermore, the following identities hold

$$\bar{\xi}_i^c \xi_j^c = \bar{\xi}_j \xi_i, \qquad \text{(C.10)}$$

$$\bar{\xi}_i^c \gamma^\mu \xi_j^c = -\bar{\xi}_j \gamma^\mu \xi_i, \qquad \text{(C.11)}$$

$$\bar{\xi}_i^c \gamma^{\mu\nu} \xi_j^c = -\bar{\xi}_j \gamma^{\mu\nu} \xi_i, \qquad \text{(C.12)}$$

$$\bar{\xi}_i^c \gamma^{\mu\nu\rho} \xi_j^c = \bar{\xi}_j \gamma^{\mu\nu\rho} \xi_i. \qquad \text{(C.13)}$$

C.1 Simplifying relations

Let η be a spinor (ξ_i or ξ_i^c) and let γ be an arbitrary product of antisymmetrized gamma matrices, from (3.21) we have the algebraic relations

$$\bar{\eta}\gamma \left(\frac{1}{2} \partial_\mu A \gamma^\mu - \frac{\mathrm{i}\alpha_j m}{2} + \frac{\mathrm{e}^{-4A}}{8} \slashed{F}^{(2)} \right) \xi_j = 0, \qquad \text{(C.14)}$$

$$\bar{\xi}_i \left(\frac{1}{2} \partial_\mu A \gamma^\mu + \frac{\mathrm{i}\alpha_i m}{2} - \frac{\mathrm{e}^{-4A}}{8} \slashed{F}^{(2)} \right) \gamma\eta = 0, \qquad \text{(C.15)}$$

$$\bar{\xi}_i^c \left(\frac{1}{2} \partial_\mu A \gamma^\mu + \frac{\mathrm{i}\alpha_i m}{2} + \frac{\mathrm{e}^{-4A}}{8} \slashed{F}^{(2)} \right) \gamma\eta = 0, \qquad \text{(C.16)}$$

$$\bar{\eta}\gamma \left(-\frac{1}{2} \partial_\mu A \gamma^\mu + \frac{\mathrm{i}\alpha_j m}{2} + \frac{\mathrm{e}^{-4A}}{8} \slashed{F}^{(2)} \right) \xi_j^c = 0. \qquad \text{(C.17)}$$

These relations are useful in simplifying the expressions obtained from computing the torsion conditions and have been used extensively in deriving the formulae in the following sections.

C.2 Algebraic equations

We begin by computing some algebraic equations that will be useful in our analysis. First note, by taking the difference of (C.14) and (C.16) one can derive

$$(\alpha_i + \alpha_j) A_{ij} = 0. \qquad \text{(C.18)}$$

Notice this condition implies that the scalars A_{11} and A_{22} must vanish irrespective of the value of α_i. We can use the algebraic relations of section C.1 with $\gamma = 1$ to find

$$\partial_\mu A \bar{\xi}_i \gamma^\mu \xi_j = \frac{\mathrm{i}m}{2} (\alpha_j - \alpha_i) \bar{\xi}_i \xi_j. \qquad \text{(C.19)}$$

Finally, we have two conditions involving the one-form P, which follow from (3.20)

$$P_\mu \bar{\xi}_i \gamma^\mu \xi_j = 0, \qquad \text{(C.20)}$$

$$P_\mu \bar{\xi}_i^c \gamma^\mu \xi_j = 0. \qquad \text{(C.21)}$$

C.3 Differential conditions: scalars

The torsion conditions for the scalars, S_{ij} and A_{ij}, take the form

$$\mathrm{d}S_{ij} = \frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j)K_{ij}, \tag{C.22}$$

$$\mathrm{e}^{-2A}\mathcal{D}(\mathrm{e}^{2A}A_{ij}) = -\frac{\mathrm{i}m}{2}(\alpha_i - \alpha_j)B_{ij}. \tag{C.23}$$

From (C.22) we observe that S_{ii} are constants and so we choose to normalise them to have unit norm.

Notice that our equations are invariant under $\mathrm{GL}(2,\mathbb{C})$ transformations of the spinors. Furthermore, when $\alpha_1 = \alpha_2$ we find that $S_{12} = \bar{\xi}_1\xi_2$ is also a constant. Let us consider this constraint in more detail. By the Cauchy-Schwarz inequality we have

$$|\bar{\xi}_1\xi_2|^2 \leq |\bar{\xi}_1\xi_1||\bar{\xi}_2\xi_2| = 1. \tag{C.24}$$

We separate the cases where this bound is saturated and when it is not and consider first when the bound is not saturated, namely $|\bar{\xi}_1\xi_2| = x < 1$. We may multiply ξ_1 by a phase to make $\bar{\xi}_1\xi_2$ real. Now consider the following rotation on the spinors

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -\frac{x}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-x^2}} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{C.25}$$

This transformation preserves $S_{ii} = \bar{\xi}_i'\xi_i' = 1$, where no sum is intended over the index i, and sets $|S_{12}| = |\bar{\xi}_1\xi_2| = 0$. Now consider the case when the bound is saturated, one finds

$$\xi_1 = \lambda\xi_2, \tag{C.26}$$

where λ can be shown to be just a phase. This relation, however, reduces the amount of supersymmetry preserved. Therefore, to preserve four supercharges, we set $S_{12} = 0$ when $\alpha_1 = \alpha_2$.

C.4 Differential conditions: one-forms

The covariant derivative of the one-forms K_{ij} is given by

$$\nabla_{\mu_1}\bar{\xi}_i\gamma_{\mu_2}\xi_j = \frac{\mathrm{i}m}{2}(\alpha_i+\alpha_j)\bar{\xi}_i\gamma_{\mu_1\mu_2}\xi_j - \frac{\mathrm{i}m}{2}(\alpha_i-\alpha_j)g_{\mu_1\mu_2}\bar{\xi}_i\xi_j - \frac{\mathrm{e}^{-4A}}{4}F_{\nu_1\nu_2}\bar{\xi}_i\gamma_{\mu_1\mu_2}{}^{\nu_1\nu_2}\xi_j. \tag{C.27}$$

Symmetrizing, we find

$$\nabla_{(\mu_1}\bar{\xi}_i\gamma_{\mu_2)}\xi_j = \frac{\mathrm{i}m}{2}(\alpha_j - \alpha_i)g_{\mu_1\mu_2}S_{ij}, \tag{C.28}$$

from which we observe that there are at least two Killing vectors when $i = j$. Notice that if $\alpha_1 = \alpha_2$ there is an additional Killing vector, $\bar{\xi}_1\gamma^{(1)}\xi_2$. For $\alpha_1 \neq \alpha_2$ the existence of a third Killing vector depends on whether the scalar S_{12} is vanishing or not. Note also that (C.19) vanishes whenever (C.28) vanishes, which implies that the Lie derivative of the warp factor along each Killing vector is vanishing.

The differential conditions on the one-forms read

$$\mathrm{e}^{-4A}\mathrm{d}\left(\mathrm{e}^{4A}K_{ij}\right) = -\mathrm{i}m(\alpha_i + \alpha_j)U_{ij} - S_{ij}\mathrm{e}^{-4A}F^{(2)}, \tag{C.29}$$

$$\mathcal{D}(\mathrm{e}^{2A}B_{ij}) = 0. \tag{C.30}$$

C.5 Differential conditions: higher forms

The differential conditions on higher degree forms are given by

$$e^{-4A}d(e^{4A}U_{ij}) = -\frac{im}{2}(\alpha_i - \alpha_j)X_{ij}\,, \tag{C.31}$$

$$e^{-6A}\mathcal{D}\left(e^{6A}V_{ij}\right) = -\frac{3im}{2}(\alpha_i - \alpha_j)Y_{ij} + e^{-4A}F^{(2)} \wedge B_{ij}\,, \tag{C.32}$$

$$e^{-8A}d\left(e^{8A}X_{ij}\right) = 2m(\alpha_i + \alpha_j) * X_{ij} - e^{-4A}F^{(2)} \wedge U_{ij}\,, \tag{C.33}$$

$$e^{-6A}\mathcal{D}\left(e^{6A}Y_{ij}\right) = m(\alpha_i + \alpha_j) * Y_{ij}\,, \tag{C.34}$$

$$e^{-8A}d\left(e^{8A} * X_{ij}\right) = -\frac{3}{2}im(\alpha_i - \alpha_j) * U_{ij}\,, \tag{C.35}$$

$$e^{-10A}\mathcal{D}\left(e^{10A} * Y_{ij}\right) = -\frac{5im}{2}(\alpha_i - \alpha_j) * V_{ij} - ie^{-4A}F^{(2)} \wedge Y_{ij}\,, \tag{C.36}$$

$$e^{-6A}\mathcal{D}\left(e^{6A} * Y_{ij}\right) = -\frac{im}{2}(\alpha_i - \alpha_j) * V_{ij} - e^{-4A}A_{ij} * F^{(2)}\,, \tag{C.37}$$

$$e^{-8A}d\left(e^{8A} * U_{ij}\right) = im(\alpha_i + \alpha_j) * K_{ij}\,, \tag{C.38}$$

$$e^{-10A}\mathcal{D}\left(e^{10A} * V_{ij}\right) = im(\alpha_i + \alpha_j) * B_{ij}\,, \tag{C.39}$$

$$e^{-12A}d\left(e^{12A} * K_{ij}\right) = -\frac{5im}{2}(\alpha_i - \alpha_j)S_{ij}\mathrm{dvol}(\mathcal{M}_7)\,, \tag{C.40}$$

$$e^{-10A}\mathcal{D}\left(e^{10A} * B_{ij}\right) = -\frac{3im}{2}(\alpha_i - \alpha_j)A_{ij}\mathrm{dvol}(\mathcal{M}_7)\,. \tag{C.41}$$

D Supergravity central charges

In this appendix we give details on the formulae used to compute the holographic central charges in the paper.

D.1 Holographic central charges at leading order

The leading order term in the central charge is given by the Brown-Henneaux formula [78]

$$c_{\mathrm{sugra}} = \frac{3}{2mG_N^{(3)}}\,, \tag{D.1}$$

where $G_N^{(3)}$ is the three dimensional Newton constant obtained by the reduction of the Type IIB/11d supergravity action on the internal manifold. The relevant part of the action in dimension d is

$$S_d = \frac{1}{16\pi G_N^{(d)}} \int_{M_d} *_d R^{(d)}\,. \tag{D.2}$$

We are interested in dimension $D = 10/11$ warped backgrounds of the form

$$ds^2(M_D) = e^{2A}ds^2(\mathrm{AdS}_3) + ds^2(M_{D-3})\,, \tag{D.3}$$

where A a function of the internal manifold only. In this background the action in (D.2) can be expressed as

$$S_D = \frac{1}{16\pi G_N^{(D)}} \int_{M_{D-3}} e^A(*_{D-3}1) \int_{\mathrm{AdS}_3} *_3(R^{(3)} + \ldots)\,. \tag{D.4}$$

This leading order piece is exactly the action (D.2) in three dimensions. From this we identify the $d = 3$ Newton constant to be

$$\frac{1}{G_N^{(3)}} = \frac{1}{G_N^{(D)}} \int_{M_{D-3}} \mathrm{e}^A \mathrm{dvol}(M_{D-3}) \,, \tag{D.5}$$

and hence

$$c_{\mathrm{sugra}} = \frac{3}{2m G_N^{(D)}} \int_{M_{D-3}} \mathrm{e}^A \mathrm{dvol}(M_{D-3}) \,. \tag{D.6}$$

In 10d the Newton's constant is $G_N^{(10)} = 2^3 \pi^6 \ell_s^8$, whilst in 11d it is given by $G_N^{(11)} = 2^4 \pi^7 \ell_{\mathrm{p}}{}^9$.

D.2   Holographic central charges at sub-leading order

We may compute the sub-leading order terms in 11d supergravity by making use of the X_8 anomaly inflow polynomial [86] and the relation

$$
\begin{aligned}
c_L - c_R &= 96\pi\beta \,, \\
S_{CS} &= \beta \int_{\mathrm{AdS}_3} \omega_{CS}(\Gamma) \subset S_{3d} \,,
\end{aligned}
\tag{D.7}
$$

as found in [85]. We reduce the 11d Chern-Simons term

$$S_{CS} = -\frac{(4\pi\kappa_{11})^{2/3}}{2\kappa_{11}^2} \int_{M_{11}} C_3 \wedge X_8 \tag{D.8}$$

on the internal space, with X_8 given by

$$X_8 = \frac{1}{(2\pi)^4 2^6 \cdot 3} \left(\mathrm{Tr}[\mathcal{R}^4] - \frac{1}{4}(\mathrm{Tr}[\mathcal{R}^2])^2 \right) \,, \tag{D.9}$$

and

$$
\begin{aligned}
\mathcal{R}^a{}_b &= \frac{1}{2} R^a{}_{b\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \\
2\kappa_{11}^2 &= (2\pi)^8 \ell_{\mathrm{p}}{}^9 \,.
\end{aligned}
\tag{D.10}
$$

Integrating (D.8) by parts we have

$$S_{CS} = \frac{(4\pi\kappa_{11})^{2/3}}{2\kappa_{11}^2} \int_{M_{11}} G_4 \wedge X_7 \,, \tag{D.11}$$

where $X_8 \equiv \mathrm{d}X_7$. In our solution the internal eight-dimensional space is $S^2 \times Y_3$, and given the form of the G_4 flux (5.11), we have that $X_7 = \frac{1}{3(2\pi)^4 2^7} \omega_{CS}(\Gamma_{\mathrm{AdS}_3}) \wedge \mathrm{Tr}[\mathcal{R}^2_{Y_3}]$ so that we determine

$$\beta = \frac{\mathrm{e}^{4A}}{3(2\pi\ell_{\mathrm{p}})^3 m (2\pi)^4 2^7} \int_{Y_3} J_{Y_3} \wedge \mathrm{Tr}[\mathcal{R}^2_{Y_3}] \,. \tag{D.12}$$

E   Properties of Kähler and Calabi-Yau varieties

In this appendix we collect some essential theorems related to the elliptically fibered Calabi-Yau threefolds that we consider as our compactification spaces throughout the body of this paper.

E.1 Useful relations

First let Y be a Kähler manifold with a given Kähler metric, $g_{\mu\bar{\nu}}$. Then the Kähler form associated to this metric is

$$J = \mathrm{i}g_{\mu\bar{\nu}}\mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\bar{\nu}}\,, \qquad (\text{E.1})$$

which is a closed $(1,1)$-form that is a representative of the cohomology class known as the Kähler class; where it would be otherwise unambiguous we shall abuse notation and use J to refer to both the explicit representative and the class. As J is formed from the Kähler metric then it is real and positive. This means that

$$\int_{C} J > 0\,, \quad \int_{S} J \wedge J > 0\,, \quad \cdots\,, \qquad (\text{E.2})$$

where C is any curve in Y, S any surface, and so on. One can find a summary of this standard information in, for example, [108]. Further, it is known that a compact complex manifold admits an holomorphic embedding into projective space if and only if it admits a Kähler metric whose associated Kähler form is an integral class [109]. As a corollary to Yau's theorem, any compact strict Calabi-Yau, Y_{n}, of dimension $n \geq 3$ can be embedded as a complex submanifold of a complex projective space, and thus we can conclude that any Calabi-Yau threefold permits an integral Kähler class.

After these introductory remarks we now collect several useful formulas. For this we will specialise to the case of elliptically fibered Calabi-Yau threefolds as in section 2, with base B, which is a Kähler surface. Various properties of the base B will feature in the main text, in particular relations for topological invariants such as

$$3\sigma(B) + 2\chi(B) = \int_{B} c_{1}(B)^{2}\,, \qquad (\text{E.3})$$

where $\sigma(B)$ is the signature of the manifold and $\chi(B)$ is the Euler number. In terms of the Hodge numbers of B these can be written as

$$\begin{aligned}
\chi(B) &= 2 - 4h^{0,1}(B) + 2h^{0,2}(B) + h^{1,1}(B) \\
\sigma(B) &= (2h^{0,2}(B) + 1) - (h^{1,1}(B) - 1) = b_{2}^{+} - b_{2}^{-}\,,
\end{aligned} \qquad (\text{E.4})$$

where $b_{2}^{\pm}$ are the number of self-dual and anti-self-dual two-forms of B. So far we have only assumed that B is a compact Kähler surface.

Now let us further suppose that B is the base of an elliptic fibration $\pi : Y_{3} \to B$ with section. As explained in the main text this does restrict the type of Kähler surfaces that can function as B. In particular, the existence of the section implies that

$$\pi_{1}(B) = 0 \quad \Longrightarrow \quad h^{0,1}(B) = 0\,. \qquad (\text{E.5})$$

Furthermore the elliptic fibration must be Calabi-Yau which means that

$$h^{0,2}(B) = 0\,, \qquad (\text{E.6})$$

as otherwise any $(0,2)$ forms on B would give rise to $(0,3)$ forms on the Calabi-Yau. Summarising, if B is the base of an elliptically fibered Calabi-Yau threefold then

$$\int_B c_1(B)^2 = 3\sigma(B) + 2\chi(B) = 10 - h^{1,1}(B). \tag{E.7}$$

This agrees with the results given in [13], and is a general result for any base B which may support a non-trivial Calabi-Yau elliptic fibration over it.

We will require in the main text to determine the second Chern class of the Calabi-Yau threefold when integrated over an arbitrary divisor P of Y_3. We have that

$$\int_P c_2(Y_3) = \int_P (c_2(P) - c_1(P)^2) = 2(h^{1,1}(P) - 4h^{0,2}(P) + 2h^{0,1}(P) - 4), \tag{E.8}$$

where the first equality follows via adjunction. As we can see the integral over the second Chern class over any divisor is always an even integer.

E.2 Ample divisors in elliptically fibered Calabi-Yau threefolds

We shall now collect results about the ampleness properties of divisors in an elliptically fibered Calabi-Yau threefold. An M5-brane wrapping a divisor D will only have an AdS dual when D is ample, as the divisor must be dual to a $(1,1)$-form in the Kähler cone of the Calabi-Yau, following from the 11d supergravity solution in section 5.

First we shall be general and consider Y any smooth algebraic variety, with D a divisor on Y. The Nakai-Moishezon [110, 111] criterion for ampleness (see e.g. [112] for an in depth discussion) is that

$$D^{\dim(X)} \cdot X > 0, \tag{E.9}$$

for every closed subvariety X in Y. We remark that since the Nakai-Moishezon criterion is just the intersection theory dual of the statement that

$$\int_X \omega^{\dim(X)} > 0, \tag{E.10}$$

where ω is the dual $(1,1)$-form to the divisor D; in this way we can see that every ample divisor is dual to a $(1,1)$-form inside of the Kähler cone of Y.[36]

With this in hand we shall now specifically consider a smooth elliptically fibered Calabi-Yau threefold, $\pi : Y_3 \to B$, and the ampleness of the divisors thereon. It was described in section 2.2 that an elliptic fibration, with trivial Mordell-Weil group, has three distinct classes of divisors which span the Néron-Severi lattice of divisors of Y_3. These are the zero-section, which provides a copy of B in the fiber, the pullbacks of the curves in the base, $\widehat{C}_\alpha = \pi^*(C_\alpha)$, and the Cartan divisors associated to the resolution of singularities, D_i. We will be interested in the triple intersection numbers of these divisors. The triple intersection numbers that are of interest to us were determined in [13], and were recapped in (2.11).

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[36]A subset of the ample divisors consists of the *very* ample divisors, which are those divisors which are linearly equivalent to the hyperplane class of a projective embedding of Y [113].

Let us first consider a smooth Weierstrass model Y_3, where we recall that there are no resolution divisors, we consider a divisor in the linear system

$$D \in |MB + N\widehat{C}| \,. \tag{E.11}$$

We are interested in knowing for what values of $M, N \geq 0$ is this divisor not ample. We know from the Nakai-Moishezon criterion for ampleness that

$$D \cdot \Sigma > 0 \,, \tag{E.12}$$

for every curve Σ in Y_3, which includes the curve C in B which $\widehat{C}$ is the pullback of, i.e. $\widehat{C}$ is the elliptic surface obtained by restricting the fibration to C. We can then compute

$$D \cdot C = MB \cdot C + N\widehat{C} \cdot C = MB \cdot B \cdot \widehat{C} + NB \cdot \widehat{C} \cdot \widehat{C} \,, \tag{E.13}$$

where in the final equality we have used that

$$C = B \cdot \widehat{C} \,. \tag{E.14}$$

Using the triple intersection numbers listed in (2.11), along with adjunction,

$$c_1(B) \cdot C = C \cdot C + 2 - 2g \,, \tag{E.15}$$

we can see that there is the constraint

$$D \cdot C = (N - M)C \cdot C + M(2g - 2) > 0 \,. \tag{E.16}$$

For $N \gg M$, this is equivalent to the statement that D is not ample in Y_3 if C is not ample in B. It is also clear from this formula that, for example, when $M = N$ we need we consider an elliptic surface $\widehat{C}$, where the base curve C is such that

$$g \geq 2 \,, \tag{E.17}$$

and ampleness clearly implies a non-trivial interdependence between M, N, and g. Further one would like to determine whether there are constraints on ampleness when $M = 0$. While the constraint (E.16) only requires that C must have a strictly positive self-intersection in the base we further note that the Nakai-Moishezon criterion for ampleness requires also that the triple-intersection of the divisor in Y_3 be strictly positive. For an elliptic surface we observe that

$$\widehat{C} \cdot \widehat{C} \cdot \widehat{C} = 0 \,, \tag{E.18}$$

as was evidenced directly from the Hodge numbers in (7.34), and thus we determine that when $M = 0$ the divisor cannot be ample.

For the case that is not a smooth Weierstrass model we can consider a divisor in the linear system

$$D \in |MB + N\widehat{C} + M_i D_i| \,, \tag{E.19}$$

and consider again $D \cdot C$, however we should not include in this sum the Cartan divisor associated to the affine node of the Dynkin diagram as it is not an independent divisor

inside the Neron-Severi lattice, and so the M_0 will not be a free parameter. We can see again from the triple intersection numbers (2.11) that

$$D_i \cdot B \cdot \widehat{C}\,, \tag{E.20}$$

is only non-zero when D_i is precisely the divisor associated to the affine node, and so the same conclusion on the constraints on M, N will hold as in the smooth Weierstrass case.

Finally one can study the case where C is a smooth rational curve of self-intersection $C \cdot C = -n$ for $n = 3, \cdots, 12$, excepting $n = 9, 10, 11$. These setups involve C not being ample in B, and correspond to the non-Higgsable clusters [10]. The self-intersection of the curve in the base is severe enough that it mandates a total space singularity above that curve in the Weierstrass model, with a specific kind of singular fiber located above the curve C depending on n. In such a setup one can again compute

$$D \cdot C = (n-2)M - nN > 0\,, \tag{E.21}$$

which, given that n is a positive integer generally requires that $M > N$, if the divisor D is to be ample.

F $\mathrm{AdS}_3 \times S^3 \times S^2 \times \mathbb{P}^1$ solution with three-form fluxes

In this section we present an AdS_3 solution of Type IIB supergravity, preserving $(0,4)$ supersymmetry, that includes non-zero three-form flux G. This is obtained by reduction and T-duality of an 11d supergravity solution constructed in [19]. Of course this does not fit into the classification of this paper as it has three-form flux, however it can be derived from the 11d supergravity of [19] exactly as the solutions in the main body of the paper. We include it here as it may be interesting to explore this class of solutions in the future.

The 11d supergravity geometry in [19] has the form $\mathrm{AdS}_3 \times S^2 \times S^2 \times Y_2$, where Y_2 is a K3 surface, with a particular four-form flux. Here we will assume that Y_2 is again an elliptically fibered Calabi-Yau and in keeping with the notation in the main body of the paper, we shall denote the base of this fibration as $B_1 \simeq \mathbb{P}^1$. We shall call the vielbeins on B_1 e^7 and e^8. The solution of [19] can be shown to take the form

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{AdS}_3) + R_1^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\varphi_1^2) + R_2^2(\mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\varphi_2^2) + \mathrm{d}s^2(Y_2)\,, \quad \text{(F.1)}$$

$$G_4 = k_1 R_1 R_2 \mathrm{vol}(S_1^2) \wedge \mathrm{vol}(S_2^2) + R_1 \mathrm{vol}(S_1^2) \wedge J_{Y_2} + R_2 \mathrm{vol}(S_2^2) \wedge \mathrm{Re}[\Omega_{Y_2}]\,, \quad \text{(F.2)}$$

where J_{Y_2} is the Kähler form of the Calabi-Yau two-fold Y_2 and Ω_{Y_2} is the holomorphic two-form. The constants k_1 and m,[37] are fixed in terms of the radii of the two S^2's to be

$$k_1 = \sqrt{R_1^2 + R_2^2}\,, \quad m = \frac{\sqrt{R_1^2 + R_2^2}}{2R_1 R_2}\,. \tag{F.3}$$

We then chose the Calabi-Yau two-fold to be elliptically fibered and we may then reduce on one of the cycles of the torus and then T-dualize along the other, again we take the metric ansatz for the fibration analogous to that given in (3.57) for the threefold. We present this duality chain below first reducing on the x direction to Type IIA supergravity and then T-dualizing along the y direction to Type IIB supergravity.

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[37]As before m is the inverse radius of AdS_3 and appears in the solution implicitly through $\mathrm{d}s^2(\mathrm{AdS}_3)$.

Type IIA supergravity solution from dimensional reduction. Reducing the 11d supergravity solution along the x direction one obtains the solution

$$\mathrm{d}s^2 = \frac{1}{\sqrt{\tau_2}}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + R_1^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\varphi_1^2) + R_2^2(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\varphi_2^2) + \mathrm{d}s^2(B_2) + \tau_2\mathrm{d}y^2\right),$$

$$\mathrm{d}C^{(3)} = k_1 R_1 R_2 \mathrm{vol}(S_1^2) \wedge \mathrm{vol}(S_2^2) + R_1 \mathrm{vol}(S_1^2) \wedge J_B - \frac{\tau_2}{R_2}\mathrm{vol}(S_2^2) \wedge e^8 \wedge \mathrm{d}y, \tag{F.4}$$

$$\mathrm{d}C^{(1)} = \mathrm{d}\tau_1 \wedge \mathrm{d}y, \tag{F.5}$$

$$B^{IIA} = R_1 \cos\theta_1 \mathrm{d}\varphi_1 \wedge \mathrm{d}y - \frac{R_2}{\sqrt{\tau_2}}\cos\theta_2 \mathrm{d}\varphi_2 \wedge e^7, \tag{F.6}$$

$$\mathrm{e}^{-2\Phi_{IIA}} = \tau_2^{\frac{3}{2}}. \tag{F.7}$$

We may now perform the T-duality along y to obtain a Type IIB supergravity solution.

Type IIB supergravity solution from T-duality. The metric and fluxes in *string frame* following from the T-duality are

$$\mathrm{d}s^2 = \frac{1}{\sqrt{\tau_2}}\left(\mathrm{d}s^2(\mathrm{AdS}_3) + R_1^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\varphi_1^2) + R_2^2(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\varphi_2^2) + \mathrm{d}s^2(B_2)\right.$$
$$\left. + (\mathrm{d}y - R_1\cos\theta_1\mathrm{d}\varphi_1)^2\right), \tag{F.8}$$

$$F = (1 + *)(k_1 R_1 R_2 \mathrm{vol}(S_1^2) \wedge \mathrm{vol}(S_2^2) + R_1 \mathrm{vol}(S_1^2) \wedge J_B) \wedge (\mathrm{d}y - R_1\cos\theta_1\mathrm{d}\varphi_1), \tag{F.9}$$

$$\mathrm{d}C^{(2)} = -\sqrt{\tau_2}R_2\mathrm{vol}(S_2^2) \wedge e^8, \tag{F.10}$$

$$H = \frac{R_2}{\sqrt{\tau_2}}\mathrm{vol}(S_2^2) \wedge e^7, \tag{F.11}$$

$$\mathrm{e}^{-\Phi} = \tau_2, \tag{F.12}$$

$$C^{(0)} = \tau_1. \tag{F.13}$$

We may now put this solution into Einstein frame and write the fluxes in the $SU(1,1)$ formalism. Inserting into the definition of G,

$$G = \frac{\mathrm{i}}{\sqrt{\tau_2}}(\tau\mathrm{d}B - \mathrm{d}C^{(2)}), \tag{F.14}$$

equations (F.10) and (F.11) one obtains for the three-form flux

$$G = -R_2\mathrm{vol}(S_2^2) \wedge \bar{\Omega}_B, \tag{F.15}$$

and the metric in Einstein frame is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{AdS}_3) + R_1^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\varphi_1^2) + R_2^2(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\varphi_2^2) + \mathrm{d}s^2(B_1)$$
$$+ (\mathrm{d}y - R_1\cos\theta_1\mathrm{d}\varphi_1)^2, \tag{F.16}$$

whilst the five-form flux remains the same. If one redefines the y coordinate as $y \to R_1\psi$ then one sees that this is just a Hopf fibration over S_1^2 and the metric is $\mathrm{AdS}_3 \times S^3 \times S^2 \times B_1$. One finds that this solution preserves $(0,4)$ supersymmetry, after explicitly computing the Killing spinors, and that this corresponds to the *large* superconformal algebra of the dual SCFT [19].

References

[1] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [`hep-th/9602022`] [ɪɴSPIRE].

[2] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 1*, *Nucl. Phys.* **B 473** (1996) 74 [`hep-th/9602114`] [ɪɴSPIRE].

[3] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2*, *Nucl. Phys.* **B 476** (1996) 437 [`hep-th/9603161`] [ɪɴSPIRE].

[4] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, *Topological reduction of 4D SYM to 2D σ-models*, *Nucl. Phys.* **B 448** (1995) 166 [`hep-th/9501096`] [ɪɴSPIRE].

[5] B. Haghighat, A. Iqbal, C. Kozçaz, G. Lockhart and C. Vafa, *M-strings*, *Commun. Math. Phys.* **334** (2015) 779 [`arXiv:1305.6322`] [ɪɴSPIRE].

[6] B. Haghighat, A. Klemm, G. Lockhart and C. Vafa, *Strings of minimal 6d SCFTs*, *Fortsch. Phys.* **63** (2015) 294 [`arXiv:1412.3152`] [ɪɴSPIRE].

[7] M. Del Zotto and G. Lockhart, *On exceptional instanton strings*, `arXiv:1609.00310` [ɪɴSPIRE].

[8] B. Haghighat, S. Murthy, C. Vafa and S. Vandoren, *F-theory, spinning black holes and multi-string branches*, *JHEP* **01** (2016) 009 [`arXiv:1509.00455`] [ɪɴSPIRE].

[9] C. Lawrie, S. Schäfer-Nameki and T. Weigand, *Chiral 2d theories from N = 4 SYM with varying coupling*, *JHEP* **04** (2017) 111 [`arXiv:1612.05640`] [ɪɴSPIRE].

[10] D.R. Morrison and W. Taylor, *Classifying bases for 6D F-theory models*, *Central Eur. J. Phys.* **10** (2012) 1072 [`arXiv:1201.1943`] [ɪɴSPIRE].

[11] L. Martucci, *Topological duality twist and brane instantons in F-theory*, *JHEP* **06** (2014) 180 [`arXiv:1403.2530`] [ɪɴSPIRE].

[12] B. Assel and S. Schäfer-Nameki, *Six-dimensional origin of N = 4 SYM with duality defects*, *JHEP* **12** (2016) 058 [`arXiv:1610.03663`] [ɪɴSPIRE].

[13] F. Bonetti and T.W. Grimm, *Six-dimensional (1, 0) effective action of F-theory via M-theory on Calabi-Yau threefolds*, *JHEP* **05** (2012) 019 [`arXiv:1112.1082`] [ɪɴSPIRE].

[14] A. Grassi, *On minimal models of elliptic threefolds*, *Math. Ann.* **290** (1991) 287.

[15] A. Grassi, *The singularities of the parameter surface of a minimal elliptic threefold*, *Internat. J. Math.* **4** (1993) 203.

[16] K. Kodaira, *On compact analytic surfaces II*, *Ann. Math.* **77** (1963) 563.

[17] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux* (in French), *Inst. Hautes Études Sci. Publ. Math.* **21** (1964) 128.

[18] E.Ó. Colgáin, J.-B. Wu and H. Yavartanoo, *Supersymmetric AdS$_3$ × S^2 M-theory geometries with fluxes*, *JHEP* **08** (2010) 114 [`arXiv:1005.4527`] [ɪɴSPIRE].

[19] Ö. Kelekci, Y. Lozano, J. Montero, E.Ó. Colgáin and M. Park, *Large superconformal near-horizons from M-theory*, *Phys. Rev.* **D 93** (2016) 086010 [`arXiv:1602.02802`] [ɪɴSPIRE].

[20] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [`hep-th/9711053`] [ɪɴSPIRE].

[21] C. Vafa, *Black holes and Calabi-Yau threefolds*, *Adv. Theor. Math. Phys.* **2** (1998) 207 [`hep-th/9711067`] [ɪɴSPIRE].

[22] R. Minasian, G.W. Moore and D. Tsimpis, *Calabi-Yau black holes and* $(0, 4)$ *σ-models*, *Commun. Math. Phys.* **209** (2000) 325 [`hep-th/9904217`] [ɪɴSPIRE].

[23] I. Bena, D.-E. Diaconescu and B. Florea, *Black string entropy and Fourier-Mukai transform*, *JHEP* **04** (2007) 045 [`hep-th/0610068`] [ɪɴSPIRE].

[24] M. Ademollo et al., *Supersymmetric strings and color confinement*, *Phys. Lett.* **B 62** (1976) 105 [ɪɴSPIRE].

[25] J.A. Harvey, R. Minasian and G.W. Moore, *Non-Abelian tensor multiplet anomalies*, *JHEP* **09** (1998) 004 [`hep-th/9808060`] [ɪɴSPIRE].

[26] D. Martelli and J. Sparks, *G structures, fluxes and calibrations in M-theory*, *Phys. Rev.* **D 68** (2003) 085014 [`hep-th/0306225`] [ɪɴSPIRE].

[27] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS$_5$ solutions of M-theory*, *Class. Quant. Grav.* **21** (2004) 4335 [`hep-th/0402153`] [ɪɴSPIRE].

[28] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS$_5$ solutions of type IIB supergravity*, *Class. Quant. Grav.* **23** (2006) 4693 [`hep-th/0510125`] [ɪɴSPIRE].

[29] M. Gabella, D. Martelli, A. Passias and J. Sparks, $N = 2$ *supersymmetric AdS$_4$ solutions of M-theory*, *Commun. Math. Phys.* **325** (2014) 487 [`arXiv:1207.3082`] [ɪɴSPIRE].

[30] F. Apruzzi, M. Fazzi, D. Rosa and A. Tomasiello, *All AdS$_7$ solutions of type-II supergravity*, *JHEP* **04** (2014) 064 [`arXiv:1309.2949`] [ɪɴSPIRE].

[31] F. Apruzzi, M. Fazzi, A. Passias, D. Rosa and A. Tomasiello, *AdS$_6$ solutions of type-II supergravity*, *JHEP* **11** (2014) 099 [*Erratum ibid.* **05** (2015) 012] [`arXiv:1406.0852`] [ɪɴSPIRE].

[32] F. Apruzzi, M. Fazzi, A. Passias and A. Tomasiello, *Supersymmetric AdS$_5$ solutions of massive IIA supergravity*, *JHEP* **06** (2015) 195 [`arXiv:1502.06620`] [ɪɴSPIRE].

[33] C. Couzens, C. Lawrie, D. Martelli and S. Schäfer-Nameki, in progress.

[34] H.J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter space-times and dual superconformal theories*, *Nucl. Phys.* **B 533** (1998) 127 [`hep-th/9803231`] [ɪɴSPIRE].

[35] D. Tong, *The holographic dual of* $AdS_3 \times S^3 \times S^3 \times S^1$, *JHEP* **04** (2014) 193 [`arXiv:1402.5135`] [ɪɴSPIRE].

[36] N. Kim, *AdS$_3$ solutions of IIB supergravity from D3-branes*, *JHEP* **01** (2006) 094 [`hep-th/0511029`] [ɪɴSPIRE].

[37] J.P. Gauntlett, O.A.P. Mac Conamhna, T. Mateos and D. Waldram, *New supersymmetric AdS$_3$ solutions*, *Phys. Rev.* **D 74** (2006) 106007 [`hep-th/0608055`] [ɪɴSPIRE].

[38] J.P. Gauntlett, N. Kim and D. Waldram, *Supersymmetric AdS$_3$, AdS$_2$ and bubble solutions*, *JHEP* **04** (2007) 005 [hep-th/0612253] [INSPIRE].

[39] J.P. Gauntlett and O.A.P. Mac Conamhna, *AdS spacetimes from wrapped D3-branes*, *Class. Quant. Grav.* **24** (2007) 6267 [arXiv:0707.3105] [INSPIRE].

[40] A. Donos, J.P. Gauntlett and N. Kim, *AdS solutions through transgression*, *JHEP* **09** (2008) 021 [arXiv:0807.4375] [INSPIRE].

[41] F. Benini and N. Bobev, *Two-dimensional SCFTs from wrapped branes and c-extremization*, *JHEP* **06** (2013) 005 [arXiv:1302.4451] [INSPIRE].

[42] N. Bobev, K. Pilch and O. Vasilakis, $(0, 2)$ *SCFTs from the Leigh-Strassler fixed point*, *JHEP* **06** (2014) 094 [arXiv:1403.7131] [INSPIRE].

[43] F. Benini, N. Bobev and P.M. Crichigno, *Two-dimensional SCFTs from D3-branes*, *JHEP* **07** (2016) 020 [arXiv:1511.09462] [INSPIRE].

[44] F. Benini and N. Bobev, *Exact two-dimensional superconformal R-symmetry and c-extremization*, *Phys. Rev. Lett.* **110** (2013) 061601 [arXiv:1211.4030] [INSPIRE].

[45] O. Aharony, A. Fayyazuddin and J.M. Maldacena, *The large-N limit of $N = 2$, $N = 1$ field theories from three-branes in F-theory*, *JHEP* **07** (1998) 013 [hep-th/9806159] [INSPIRE].

[46] C.-H. Ahn, K. Oh and R. Tatar, *The large-N limit of $N = 1$ field theories from F-theory*, *Mod. Phys. Lett.* **A 14** (1999) 369 [hep-th/9808143] [INSPIRE].

[47] O. Aharony and Y. Tachikawa, *A holographic computation of the central charges of $D = 4$, $N = 2$ SCFTs*, *JHEP* **01** (2008) 037 [arXiv:0711.4532] [INSPIRE].

[48] J. Erdmenger, N. Evans, I. Kirsch and E. Threlfall, *Mesons in gauge/gravity duals — a review*, *Eur. Phys. J.* **A 35** (2008) 81 [arXiv:0711.4467] [INSPIRE].

[49] E.I. Buchbinder, J. Gomis and F. Passerini, *Holographic gauge theories in background fields and surface operators*, *JHEP* **12** (2007) 101 [arXiv:0710.5170] [INSPIRE].

[50] J.A. Harvey and A.B. Royston, *Gauge/gravity duality with a chiral $N = (0, 8)$ string defect*, *JHEP* **08** (2008) 006 [arXiv:0804.2854] [INSPIRE].

[51] J.A. Harvey and A.B. Royston, *Localized modes at a D-brane-O-plane intersection and heterotic Alice atrings*, *JHEP* **04** (2008) 018 [arXiv:0709.1482] [INSPIRE].

[52] E. D'Hoker, M. Gutperle, A. Karch and C.F. Uhlemann, *Warped AdS$_6 \times S^2$ in type IIB supergravity I: local solutions*, *JHEP* **08** (2016) 046 [arXiv:1606.01254] [INSPIRE].

[53] E. D'Hoker, M. Gutperle and C.F. Uhlemann, *Warped AdS$_6 \times S^2$ in type IIB supergravity II: global solutions and five-brane webs*, *JHEP* **05** (2017) 131 [arXiv:1703.08186] [INSPIRE].

[54] D. Bak, M. Gutperle and S. Hirano, *A dilatonic deformation of AdS$_5$ and its field theory dual*, *JHEP* **05** (2003) 072 [hep-th/0304129] [INSPIRE].

[55] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS type IIB interface solutions I. Local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [arXiv:0705.0022] [INSPIRE].

[56] A.B. Clark, D.Z. Freedman, A. Karch and M. Schnabl, *Dual of the Janus solution: an interface conformal field theory*, *Phys. Rev.* **D 71** (2005) 066003 [hep-th/0407073] [INSPIRE].

[57] C. Bachas, E. D'Hoker, J. Estes and D. Krym, *M-theory solutions invariant under* $D(2, 1; \gamma) \oplus D(2, 1; \gamma)$, *Fortsch. Phys.* **62** (2014) 207 [arXiv:1312.5477] [INSPIRE].

[58] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings, and the θ-angle in N = 4 super Yang-Mills theory*, *JHEP* **06** (2010) 097 [arXiv:0804.2907] [INSPIRE].

[59] F. Denef, *Les Houches lectures on constructing string vacua*, in *String theory and the real world: from particle physics to astrophysics. Proceedings, Summer School in Theoretical Physics,* 87th *Session*, Les Houches France, 2–27 July 2007, pg. 483 [arXiv:0803.1194] [INSPIRE].

[60] T. Weigand, *Lectures on F-theory compactifications and model building*, *Class. Quant. Grav.* **27** (2010) 214004 [arXiv:1009.3497] [INSPIRE].

[61] J. Marsano and S. Schäfer-Nameki, *Yukawas, G-flux and spectral covers from resolved Calabi-Yau's*, *JHEP* **11** (2011) 098 [arXiv:1108.1794] [INSPIRE].

[62] B.R. Greene, A.D. Shapere, C. Vafa and S.-T. Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, *Nucl. Phys.* **B 337** (1990) 1 [INSPIRE].

[63] M. Gross and P.M.H. Wilson, *Large complex structure limits of K3 surfaces*, *J. Diff. Geom.* **55** (2000) 475.

[64] P.S. Aspinwall et al., *Dirichlet branes and mirror symmetry*, in *Clay Mathematics Monographs* **4**, American Mathematical Society, Providence RI U.S.A., (2009) [INSPIRE].

[65] R. Wazir, *Arithmetic on elliptic threefolds*, *Compos. Math.* **140** (2004) 567.

[66] K. Kodaira, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, *Ann. Math.* **60** (1954) 28.

[67] K.A. Intriligator, D.R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, *Nucl. Phys.* **B 497** (1997) 56 [hep-th/9702198] [INSPIRE].

[68] T.W. Grimm and H. Hayashi, *F-theory fluxes, chirality and Chern-Simons theories*, *JHEP* **03** (2012) 027 [arXiv:1111.1232] [INSPIRE].

[69] H. Hayashi, C. Lawrie, D.R. Morrison and S. Schäfer-Nameki, *Box graphs and singular fibers*, *JHEP* **05** (2014) 048 [arXiv:1402.2653] [INSPIRE].

[70] A. Schwimmer and N. Seiberg, *Comments on the N = 2, N = 3, N = 4 superconformal algebras in two-dimensions*, *Phys. Lett.* **B 184** (1987) 191 [INSPIRE].

[71] J.H. Schwarz, *Covariant field equations of chiral N = 2 D = 10 supergravity*, *Nucl. Phys.* **B 226** (1983) 269 [INSPIRE].

[72] P.S. Howe and P.C. West, *The complete N = 2, D = 10 supergravity*, *Nucl. Phys.* **B 238** (1984) 181 [INSPIRE].

[73] C. Voisin, *Hodge theory and complex algebraic geometry I*, in *Cambridge Studies in Advanced Mathematics* **76**, Cambridge University Press, Cambridge U.K., (2007).

[74] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G structures and wrapped NS5-branes*, *Commun. Math. Phys.* **247** (2004) 421 [hep-th/0205050] [INSPIRE].

[75] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [hep-th/9711200] [INSPIRE].

[76] H. Ooguri and C. Vafa, *Summing up D instantons*, *Phys. Rev. Lett.* **77** (1996) 3296 [hep-th/9608079] [INSPIRE].

[77] A.W. Peet, *TASI lectures on black holes in string theory*, in *Strings, branes and gravity. Proceedings, Theoretical Advanced Study Institute, TASI'99*, Boulder U.S.A., 31 May–25 June 1999, pg. 353 [hep-th/0008241] [INSPIRE].

[78] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Commun. Math. Phys.* **104** (1986) 207 [INSPIRE].

[79] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022 [hep-th/0508218] [INSPIRE].

[80] M.B. Green, J.A. Harvey and G.W. Moore, *I-brane inflow and anomalous couplings on D-branes*, *Class. Quant. Grav.* **14** (1997) 47 [hep-th/9605033] [INSPIRE].

[81] J. Hansen and P. Kraus, *Generating charge from diffeomorphisms*, *JHEP* **12** (2006) 009 [hep-th/0606230] [INSPIRE].

[82] A. Dabholkar, J. Gomes, S. Murthy and A. Sen, *Supersymmetric index from black hole entropy*, *JHEP* **04** (2011) 034 [arXiv:1009.3226] [INSPIRE].

[83] S.T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977) 1798 [INSPIRE].

[84] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122] [INSPIRE].

[85] P. Kraus and F. Larsen, *Microscopic black hole entropy in theories with higher derivatives*, *JHEP* **09** (2005) 034 [hep-th/0506176] [INSPIRE].

[86] A.A. Tseytlin, R^4 *terms in* 11 *dimensions and conformal anomaly of* $(2,0)$ *theory*, *Nucl. Phys.* **B 584** (2000) 233 [hep-th/0005072] [INSPIRE].

[87] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string-string duality: a one loop test*, *Nucl. Phys.* **B 452** (1995) 261 [hep-th/9506126] [INSPIRE].

[88] R. Bott and A.S. Cattaneo, *Integral invariants of* 3-*manifolds II*, *J. Diff. Geom.* **53** (1999) 1.

[89] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029] [INSPIRE].

[90] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234] [INSPIRE].

[91] D. Freed, J.A. Harvey, R. Minasian and G.W. Moore, *Gravitational anomaly cancellation for M-theory five-branes*, *Adv. Theor. Math. Phys.* **2** (1998) 601 [hep-th/9803205] [INSPIRE].

[92] K.A. Intriligator, *Anomaly matching and a Hopf-Wess-Zumino term in* 6d, $N = (2,0)$ *field theories*, *Nucl. Phys.* **B 581** (2000) 257 [hep-th/0001205] [INSPIRE].

[93] D.S. Berman and J.A. Harvey, *The self-dual string and anomalies in the M5-brane*, *JHEP* **11** (2004) 015 [hep-th/0408198] [INSPIRE].

[94] H. Shimizu and Y. Tachikawa, *Anomaly of strings of* 6d $N = (1,0)$ *theories*, *JHEP* **11** (2016) 165 [arXiv:1608.05894] [INSPIRE].

[95] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and nonholomorphic corrections for supersymmetric black holes*, *Nucl. Phys.* **B 567** (2000) 87 [`hep-th/9906094`] [INSPIRE].

[96] N. Lambert, *The M5-brane on $K3 \times T^2$*, *JHEP* **02** (2008) 060 [`arXiv:0712.3166`] [INSPIRE].

[97] P. Putrov, J. Song and W. Yan, *(0, 4) dualities*, *JHEP* **03** (2016) 185 [`arXiv:1505.07110`] [INSPIRE].

[98] Y. Lozano, N.T. Macpherson, J. Montero and E.Ó. Colgáin, *New $AdS_3 \times S^2$ T-duals with $N = (0, 4)$ supersymmetry*, *JHEP* **08** (2015) 121 [`arXiv:1507.02659`] [INSPIRE].

[99] L. Wulff, *All symmetric space solutions of eleven-dimensional supergravity*, *J. Phys.* **A 50** (2017) 245401 [`arXiv:1611.06139`] [INSPIRE].

[100] S. Schäfer-Nameki and T. Weigand, *F-theory and 2d (0, 2) theories*, *JHEP* **05** (2016) 059 [`arXiv:1601.02015`] [INSPIRE].

[101] F. Apruzzi, F. Hassler, J.J. Heckman and I.V. Melnikov, *UV completions for non-critical strings*, *JHEP* **07** (2016) 045 [`arXiv:1602.04221`] [INSPIRE].

[102] A. Fayyazuddin and M. Spalinski, *Large-N superconformal gauge theories and supergravity orientifolds*, *Nucl. Phys.* **B 535** (1998) 219 [`hep-th/9805096`] [INSPIRE].

[103] A. Kehagias, *New type IIB vacua and their F-theory interpretation*, *Phys. Lett.* **B 435** (1998) 337 [`hep-th/9805131`] [INSPIRE].

[104] J.J. Heckman, D.R. Morrison and C. Vafa, *On the classification of 6D SCFTs and generalized ADE orbifolds*, *JHEP* **05** (2014) 028 [*Erratum ibid.* **06** (2015) 017] [`arXiv:1312.5746`] [INSPIRE].

[105] H. Lü, C.N. Pope and J. Rahmfeld, *A construction of Killing spinors on S^n*, *J. Math. Phys.* **40** (1999) 4518 [`hep-th/9805151`] [INSPIRE].

[106] M.F. Sohnius, *Introducing supersymmetry*, *Phys. Rept.* **128** (1985) 39 [INSPIRE].

[107] V. Balasubramanian, E.G. Gimon, D. Minic and J. Rahmfeld, *Four-dimensional conformal supergravity from AdS space*, *Phys. Rev.* **D 63** (2001) 104009 [`hep-th/0007211`] [INSPIRE].

[108] T. Hubsch, *Calabi-Yau manifolds: a bestiary for physicists*, World Scientific, Singapore, (1994) [INSPIRE].

[109] C. Voisin, *Hodge theory and complex algebraic geometry II*, in *Cambridge Studies in Advanced Mathematics* **77**, Cambridge University Press, Cambridge U.K., (2007).

[110] Y. Nakai, *A criterion of an ample sheaf on a projective scheme*, *Amer. J. Math.* **85** (1963) 14.

[111] B.G. Moishezon, *A projectivity criterion of complete algebraic abstract varieties*, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964) 179.

[112] R. Lazarsfeld, *Positivity in algebraic geometry I*, in *A Series of Modern Surveys in Mathematics* **48**, Springer-Verlag, Berlin Germany, (2004).

[113] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York U.S.A. and Heidelberg Germany, (1977).