Filomat 30:6 (2016), 1567–1575
DOI 10.2298/FIL1606567B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Faber Polynomial Coefficient Estimates for a Subclass of Analytic Bi-univalent Functions

Serap Bulut[a]

[a]*Kocaeli University, Civil Aviation College, Arslanbey Campus, TR-41285 İzmit-Kocaeli, Turkey*

Abstract. In this work, considering a general subclass of analytic bi-univalent functions, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in this class. For this purpose, we use the Faber polynomial expansions. In certain cases, our estimates improve some of those existing coeffcient bounds.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &= : w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50
Keywords. Analytic functions; Univalent functions; Bi-univalent functions; Taylor-Maclaurin series expansion; Coefficient bounds and coefficient estimates; Taylor-Maclaurin coefficients; Faber polynomials.
Received: 7 May 2014; Accepted: 14 July 2014
Communicated by Hari M. Srivastava
Email address: serap.bulut@kocaeli.edu.tr (Serap Bulut)

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [22], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [24] proved that $|a_2| \leq 4/3$. Brannan and Taha [5] and Taha [31] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [29] (see also [5]). In fact, the aforecited work of Srivastava *et al.* [29] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [14], Xu *et al.* [33, 34], Hayami and Owa [19], and others (see, for example, [2, 6–9, 11, 15, 23, 25, 26, 28]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n > 3$. This is because the bi-univalency requirement makes the behavior of the coefficients of the function f and f^{-1} unpredictable. Here, in this paper, we use the Faber polynomial expansions for a general subclass of analytic bi-univalent functions to determine estimates for the general coefficient bounds $|a_n|$.

The Faber polynomials introduced by Faber [13] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [16] and [18] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions.

In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions given by (1) using Faber polynomial expansions, [10, 17, 20, 21, 30]. Hamidi and Jahangiri [17] considered the class of analytic bi-close-to-convex functions. Jahangiri and Hamidi [20] considered the class defined by Frasin and Aouf [14]. Bulut [10] generalized the results obtained in [20]. Jahangiri *et al.* [21] considered the class of analytic bi-univalent functions with positive real-part derivatives. In this work, we generalize the results obtained by Srivastava *et al.* [30].

2. The Class $\mathcal{N}_\Sigma(\alpha, \lambda, \delta)$

Firstly, we introduce a general class of analytic bi-univalent functions as follows.

Definition 1. *For $\lambda \geq 1$ and $\delta \geq 0$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ if the following conditions are satisfied:*

$$Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z)\right) > \alpha \tag{3}$$

and

$$Re\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w)\right) > \alpha \tag{4}$$

where $0 \leq \alpha < 1$ and $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by (2).

Remark 1. In the following special cases of Definition 1, we show how the class of analytic bi-univalent functions $\mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ for suitable choices of λ and δ lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.
(**i**) For $\delta = 0$, we obtain the bi-univalent function class

$$\mathcal{N}_\Sigma(\alpha, \lambda, 0) = \mathcal{B}_\Sigma(\alpha, \lambda)$$

introduced by Frasin and Aouf [14]. This class consists of functions $f \in \Sigma$ satisfying

$$Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \alpha$$

and

$$Re\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)\right)>\alpha$$

where $0\leq\alpha<1$ and $z,w\in\mathbb{U}$ and $g=f^{-1}$ is defined by (2).
(**ii**) For $\delta=0$ and $\lambda=1$, we have the bi-univalent function class

$$\mathcal{N}_{\Sigma}(\alpha,1,0)=\mathcal{H}_{\Sigma}(\alpha)$$

introduced by Srivastava *et al.* [29]. This class consists of functions $f\in\Sigma$ satisfying

$$Re(f'(z))>\alpha$$

and

$$Re(g'(w))>\alpha$$

where $0\leq\alpha<1$ and $z,w\in\mathbb{U}$ and $g=f^{-1}$ is defined by (2).
(**iii**) For $\lambda=1$, we get the bi-univalent function class

$$\mathcal{N}_{\Sigma}(\alpha,1,\delta)=\mathcal{N}_{\Sigma}^{(\alpha,\delta)}$$

introduced by Srivastava *et al.* [30]. This class consists of functions $f\in\Sigma$ satisfying

$$Re(f'(z)+\delta zf''(z))>\alpha$$

and

$$Re(g'(w)+\delta wg''(w))>\alpha$$

where $0\leq\alpha<1$ and $z,w\in\mathbb{U}$ and $g=f^{-1}$ is defined by (2).

3. Coefficient Estimates

Using the Faber polynomial expansion of functions $f\in\mathcal{A}$ of the form (1), the coefficients of its inverse map $g=f^{-1}$ may be expressed as, [1]:

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots)w^n, \tag{5}$$

where

$$\begin{aligned}K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!\,(n-1)!}a_2^{n-1}\\ &\quad+\frac{(-n)!}{(2(-n+1))!\,(n-3)!}a_2^{n-3}a_3\\ &\quad+\frac{(-n)!}{(-2n+3)!\,(n-4)!}a_2^{n-4}a_4\\ &\quad+\frac{(-n)!}{(2(-n+2))!\,(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]\\ &\quad+\frac{(-n)!}{(-2n+5)!\,(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]\\ &\quad+\sum_{j\geq7}a_2^{n-j}V_j,\end{aligned} \tag{6}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$, [3]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} K_1^{-2} &= -2a_2, \\ K_2^{-3} &= 3\left(2a_2^2 - a_3\right), \\ K_3^{-4} &= -4\left(5a_2^3 - 5a_2a_3 + a_4\right). \end{aligned} \tag{7}$$

In general, for any $p \in \mathbb{N} := \{1, 2, 3, \ldots\}$, an expansion of K_n^p is as, [1],

$$\begin{aligned} K_n^p &= pa_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!\,3!} D_n^3 \\ &\quad + \cdots + \frac{p!}{(p-n)!\,n!} D_n^n, \end{aligned} \tag{8}$$

where

$$D_n^p = D_n^p(a_2, a_3, \ldots),$$

and by [32],

$$D_n^m(a_1, a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{m!}{i_1! \ldots i_n!} a_1^{i_1} \ldots a_n^{i_n}$$

while $a_1 = 1$, and the sum is taken over all non-negative integers $i_1, \ldots, i_n$ satisfying

$$\begin{aligned} i_1 + i_2 + \cdots + i_n &= m \\ i_1 + 2i_2 + \cdots + ni_n &= n. \end{aligned}$$

It is clear that

$$D_n^n(a_1, a_2, \ldots, a_n) = a_1^n.$$

Consequently, for functions $f \in \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ of the form (1), we can write:

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) = 1 + \sum_{n=2}^{\infty} [1 + (n-1)\lambda + n(n-1)\delta] a_n z^{n-1}. \tag{9}$$

Our first theorem introduces an upper bound for the coefficients $|a_n|$ of analytic bi-univalent functions in the class $\mathcal{N}_\Sigma(\alpha, \lambda, \delta)$.

Theorem 2. *For $\lambda \geq 1$, $\delta \geq 0$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ be given by (1). If $a_k = 0$ $(2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{1 + (n-1)\lambda + n(n-1)\delta} \qquad (n \geq 4).$$

Proof. For the function $f \in \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ of the form (1), we have the expansion (9) and for the inverse map $g = f^{-1}$, considering (2), we obtain

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) = 1 + \sum_{n=2}^{\infty} [1 + (n-1)\lambda + n(n-1)\delta] A_n w^{n-1}, \tag{10}$$

with

$$A_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n). \tag{11}$$

On the other hand, since $f \in \mathcal{N}_{\Sigma}(\alpha,\lambda,\delta)$ and $g = f^{-1} \in \mathcal{N}_{\Sigma}(\alpha,\lambda,\delta)$, by definition, there exist two positive real-part functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}$$

and

$$q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{A},$$

where

$$Re(p(z)) > 0 \quad \text{and} \quad Re(q(w)) > 0$$

in $\mathbb{U}$ so that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) = \alpha + (1-\alpha)p(z) = 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(c_1,c_2,\ldots,c_n)z^n \tag{12}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) = \alpha + (1-\alpha)q(w) = 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(d_1,d_2,\ldots,d_n)w^n. \tag{13}$$

Note that, by the Caratheodory lemma (e.g., [12]),

$$|c_n| \le 2 \qquad \text{and} \qquad |d_n| \le 2 \quad (n \in \mathbb{N}).$$

Comparing the corresponding coefficients of (9) and (12), for any $n \ge 2$, yields

$$[1 + (n-1)\lambda + n(n-1)\delta]\, a_n = (1-\alpha) K_{n-1}^1(c_1,c_2,\ldots,c_{n-1}), \tag{14}$$

and similarly, from (10) and (13) we find

$$[1 + (n-1)\lambda + n(n-1)\delta]\, A_n = (1-\alpha) K_{n-1}^1(d_1,d_2,\ldots,d_{n-1}). \tag{15}$$

Note that for $a_k = 0 \ (2 \le k \le n-1)$, we have

$$A_n = -a_n$$

and so

$$\begin{aligned} [1 + (n-1)\lambda + n(n-1)\delta]\, a_n &= (1-\alpha)c_{n-1}, \\ -[1 + (n-1)\lambda + n(n-1)\delta]\, a_n &= (1-\alpha)d_{n-1}. \end{aligned}$$

Taking the absolute values of the above equalities, we obtain

$$|a_n| = \frac{(1-\alpha)|c_{n-1}|}{1 + (n-1)\lambda + n(n-1)\delta} = \frac{(1-\alpha)|d_{n-1}|}{1 + (n-1)\lambda + n(n-1)\delta} \le \frac{2(1-\alpha)}{1 + (n-1)\lambda + n(n-1)\delta},$$

which completes the proof of the Theorem 2. □

The following corollaries are immediate consequences of the above theorem.

Corollary 3. *[20, Theorem 1] For $\lambda \ge 1$ and $0 \le \alpha < 1$, let the function $f \in \mathcal{B}_{\Sigma}(\alpha,\lambda)$ be given by (1). If $a_k = 0 \ (2 \le k \le n-1)$, then*

$$|a_n| \le \frac{2(1-\alpha)}{1 + (n-1)\lambda} \qquad (n \ge 4).$$

Corollary 4. *[30, Theorem 1] For $\delta \geq 0$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_{\Sigma}^{(\alpha,\delta)}$ be given by (1). If $a_k = 0$ $(2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{n[1+(n-1)\delta]} \qquad (n \geq 4).$$

Theorem 5. *For $\lambda \geq 1$, $\delta \geq 0$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_{\Sigma}(\alpha,\lambda,\delta)$ be given by (1). Then one has the following*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{1+2\lambda+6\delta}} &, \quad 0 \leq \alpha < 1 - \frac{(1+\lambda+2\delta)^2}{2(1+2\lambda+6\delta)} \\ \frac{2(1-\alpha)}{1+\lambda+2\delta} &, \quad 1 - \frac{(1+\lambda+2\delta)^2}{2(1+2\lambda+6\delta)} \leq \alpha < 1 \end{cases}, \tag{16}$$

$$|a_3| \leq \frac{2(1-\alpha)}{1+2\lambda+6\delta}, \tag{17}$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{1+2\lambda+6\delta}.$$

Proof. If we set $n = 2$ and $n = 3$ in (14) and (15), respectively, we get

$$(1+\lambda+2\delta)a_2 = (1-\alpha)c_1, \tag{18}$$

$$(1+2\lambda+6\delta)a_3 = (1-\alpha)c_2, \tag{19}$$

$$-(1+\lambda+2\delta)a_2 = (1-\alpha)d_1, \tag{20}$$

$$(1+2\lambda+6\delta)\left(2a_2^2 - a_3\right) = (1-\alpha)d_2. \tag{21}$$

From (18) and (20), we find (by the Caratheodory lemma)

$$|a_2| = \frac{(1-\alpha)|c_1|}{1+\lambda+2\delta} = \frac{(1-\alpha)|d_1|}{1+\lambda+2\delta} \leq \frac{2(1-\alpha)}{1+\lambda+2\delta}. \tag{22}$$

Also from (19) and (21), we obtain

$$2(1+2\lambda+6\delta)a_2^2 = (1-\alpha)(c_2+d_2). \tag{23}$$

Using the Caratheodory lemma, we get

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{1+2\lambda+6\delta}},$$

and combining this with inequality (22), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (16).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (21) from (19). We thus get

$$(1+2\lambda+6\delta)\left(-2a_2^2 + 2a_3\right) = (1-\alpha)(c_2-d_2)$$

or

$$a_3 = a_2^2 + \frac{(1-\alpha)(c_2-d_2)}{2(1+2\lambda+6\delta)}. \tag{24}$$

Upon substituting the value of a_2^2 from (18) into (24), it follows that

$$a_3 = \frac{(1-\alpha)^2 c_1^2}{(1+\lambda+2\delta)^2} + \frac{(1-\alpha)(c_2-d_2)}{2(1+2\lambda+6\delta)}.$$

We thus find (by the Caratheodory lemma) that

$$|a_3| \leq \frac{4(1-\alpha)^2}{(1+\lambda+2\delta)^2} + \frac{2(1-\alpha)}{1+2\lambda+6\delta}. \tag{25}$$

On the other hand, upon substituting the value of a_2^2 from (23) into (24), it follows that

$$a_3 = \frac{(1-\alpha)c_2}{1+2\lambda+6\delta}.$$

Consequently (by the Caratheodory lemma), we have

$$|a_3| \leq \frac{2(1-\alpha)}{1+2\lambda+6\delta}. \tag{26}$$

Combining (25) and (26), we get the desired estimate on the coefficient $|a_3|$ as asserted in (17).

Finally, from (21), we deduce (by the Caratheodory lemma) that

$$\left|a_3 - 2a_2^2\right| = \frac{(1-\alpha)|d_2|}{1+2\lambda+6\delta} \leq \frac{2(1-\alpha)}{1+2\lambda+6\delta}.$$

This evidently completes the proof of Theorem 5. □

Remark 2. The above estimates for $|a_2|$ and $|a_3|$ show that Theorem 5 is an improvement of the estimates obtained by Sivasubramanian *et al.* [27, Theorem 3.2].

By setting $\delta = 0$ in Theorem 5, we obtain the following consequence.

Corollary 6. *[20, Theorem 2] For $\lambda \geq 1$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{B}_\Sigma(\alpha, \lambda)$ be given by (1). Then one has the following*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{1+2\lambda}} & , \quad 0 \leq \alpha < \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \\ \frac{2(1-\alpha)}{1+\lambda} & , \quad \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \leq \alpha < 1 \end{cases},$$

$$|a_3| \leq \frac{2(1-\alpha)}{1+2\lambda},$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{1+2\lambda}.$$

Remark 3. The above estimates for $|a_2|$ and $|a_3|$ show that Corollary 6 is an improvement of the estimates obtained by Frasin and Aouf [14, Theorem 3.2].

By setting $\delta = 0$ and $\lambda = 1$ in Theorem 5, we obtain the following consequence.

Corollary 7. *For $0 \leq \alpha < 1$, let the function $f \in \mathcal{H}_\Sigma(\alpha)$ be given by (1). Then one has the following*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3}} & , \quad 0 \leq \alpha < \frac{1}{3} \\ 1-\alpha & , \quad \frac{1}{3} \leq \alpha < 1 \end{cases},$$

$$|a_3| \leq \frac{2(1-\alpha)}{3},$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{3}.$$

Remark 4. The above estimates for $|a_2|$ and $|a_3|$ show that Corollary 7 is an improvement of the estimates obtained by Srivastava *et al*. [29, Theorem 2].

By setting $\lambda = 1$ in Theorem 5, we obtain the following consequence.

Corollary 8. *[30, Theorem 2] For $\delta \geq 0$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{N}_{\Sigma}^{(\alpha,\delta)}$ be given by (1). Then one has the following*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3(1+2\delta)}} & , \quad 0 \leq \alpha < \frac{1+2\delta-2\delta^2}{3(1+2\delta)} \\ \\ \frac{1-\alpha}{1+\delta} & , \quad \frac{1+2\delta-2\delta^2}{3(1+2\delta)} \leq \alpha < 1 \end{cases},$$

$$|a_3| \leq \frac{2(1-\alpha)}{3(1+2\delta)}.$$

References

[1] H. Airault and A. Bouali, Differential calculus on the Faber polynomials, *Bull. Sci. Math.* **130** (3) (2006), 179–222.
[2] M. K. Aouf, R. M. El-Ashwah and A. M. Abd-Eltawab, New subclasses of biunivalent functions involving Dziok-Srivastava operator, *ISRN Math. Anal.* **2013**, Art. ID 387178, 5 pp.
[3] H. Airault and J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, *Bull. Sci. Math.* **126** (5) (2002), 343–367.
[4] D. A. Brannan, J. G. Clunie (Eds.), Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1 20, 1979), Academic Press, New York and London, 1980.
[5] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in *Mathematical Analysis and Its Applications* (S. M. Mazhar, A. Hamoui and N. S. Faour, Editors) (Kuwait; February 18–21, 1985), KFAS Proceedings Series, Vol. **3**, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53–60; see also *Studia Univ. Babeş -Bolyai Math.* **31** (2) (1986), 70–77.
[6] S. Bulut, Coefficient estimates for initial Taylor-Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator, *The Scientific World Journal* **2013**, Art. ID 171039, 6 pp.
[7] S. Bulut, Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator, *J. Funct. Spaces Appl.* **2013**, Art. ID 181932, 7 pp.
[8] S. Bulut, Coefficient estimates for a new subclass of analytic and bi-univalent functions, *Annals of the Alexandru Ioan Cuza University - Mathematics*, *in press*.
[9] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, *Novi Sad J. Math.* **43** (2) (2013), 59–65.
[10] S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, *C. R. Acad. Sci. Paris, Ser. I* **352** (6) (2014), 479–484.
[11] M. Çağlar, H. Orhan and N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, *Filomat* **27** (7) (2013), 1165–1171.
[12] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983.
[13] G. Faber, Über polynomische Entwickelungen, *Math. Ann.* **57** (3) (1903), 389–408.
[14] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* **24** (2011), 1569–1573.
[15] S. P. Goyal and P. Goswami, Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, *J. Egyptian Math. Soc.* **20** (2012), 179–182.
[16] S. G. Hamidi, S. A. Halim and J. M. Jahangiri, Coefficient estimates for a class of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris, Ser. I* **351** (9-10) (2013), 349–352.
[17] S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, *C. R. Acad. Sci. Paris, Ser. I* **352** (1) (2014), 17–20.
[18] S. G. Hamidi, T. Janani, G. Murugusundaramoorthy and J. M. Jahangiri, Coefficient estimates for certain classes of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris, Ser. I* **352** (4) (2014), 277–282.
[19] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *Pan Amer. Math. J.* **22** (4) (2012), 15–26.
[20] J. M. Jahangiri and S. G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, *Int. J. Math. Math. Sci.* **2013**, Art. ID 190560, 4 pp.
[21] J. M. Jahangiri, S. G. Hamidi and S. A. Halim, Coefficients of bi-univalent functions with positive real part derivatives, *Bull. Malays. Math. Sci. Soc. (2)* **37** (3) (2014), 633–640.
[22] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* **18** (1967), 63–68.
[23] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent function, *Abstr. Appl. Anal.* **2013**, Art. ID 573017, 3 pp.
[24] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.* **32** (1969), 100–112.
[25] S. Porwal and M. Darus, On a new subclass of bi-univalent functions, *J. Egyptian Math. Soc.* **21** (3) (2013), 190–193.

[26] S. Prema and B. S. Keerthi, Coefficient bounds for certain subclasses of analytic functions, *J. Math. Anal.* **4** (1) (2013), 22–27.
[27] S. Sivasubramanian, T. N. Shanmugam and R. Sivakumar, Coefficient bound for certain subclasses of analytic and bi-univalent functions, *Far East J. Math. Sci. (FJMS)* **79** (1) (2013), 123–134.
[28] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat* **27** (5) (2013), 831–842.
[29] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23** (2010) 1188–1192.
[30] H. M. Srivastava, S. Sümer Eker and R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, *Filomat* **29** (8) (2015), 1839–1845.
[31] T. S. Taha, Topics in Univalent Function Theory, Ph.D. Thesis, University of London, 1981.
[32] P. G. Todorov, On the Faber polynomials of the univalent functions of class Σ, *J. Math. Anal. Appl.* **162** (1) (1991), 268–276.
[33] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25** (2012) 990–994.
[34] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218** (2012) 11461–11465.