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Faber Polynomial Coefficient Estimates for Janowski Type bi-Close-to-Convex and bi-Quasi-Convex Functions

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Abstract: Motivated by the recent work on symmetric analytic functions by using the concept of Faber polynomials, this article introduces and studies two new subclasses of bi-close-to-convex and quasi-close-to-convex functions associated with Janowski functions. By using the Faber polynomial expansion method, it determines the general coefficient bounds for the functions belonging to these classes. It also finds initial coefficients of bi-close-to-convex and bi-quasi-convex functions by using Janowski functions. Some known consequences of the main results are also highlighted.

Keywords: analytic functions; univalent functions; bi-univalent functions; Janowski functions; Faber polynomials expansions.

MSC: Primary: 05A30; 30C45; Secondary: 11B65; 47B38

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the set of all analytic functions f in the open unit disk $E = \{z : |z| < 1\}$. The functions of $\mathcal{A}$ are normalized by

$$f(0) = 0 \text{ and } f'(0) = 1.$$

Thus, every function $f \in \mathcal{A}$ can be expressed in the series form provided as:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \tag{1}$$

Furthermore, $\mathcal{S}$ is the subclass of $\mathcal{A}$ whose members are univalent in E. For f_1, $f_2 \in \mathcal{A}$, the function f_1 is said to subordinate the function f_2 in E, denoted symbolically as $f_1(z) \prec f_2(z)$, if there exists a function $u_0 \in \mathcal{A}$ with $|u_0(z)| < 1$, $u_0(0) = 0$, such that

$$f_1(z) = f_2(u_0(z)),\ z \in E.$$

Some well-known subclasses of univalent functions class $\mathcal{S}$ are provided as:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{S} : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\},\ \ 0 \leq \alpha < 1.$$

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{S} : Re\left(\frac{\left(zf'(z)\right)'}{f'(z)} \right) > \alpha \right\}, \ 0 \leq \alpha < 1.$$

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{S}, g \in \mathcal{S}^* : Re\left(\frac{zf'(z)}{g(z)} \right) > \alpha \right\}, \ 0 \leq \alpha < 1.$$

$$\mathcal{C}^*(\alpha) = \left\{ f \in \mathcal{S}, g \in \mathcal{K} : Re\left(\frac{\left(zf'(z)\right)'}{g'(z)} \right) > \alpha \right\}, \ 0 \leq \alpha < 1.$$

These classes are starlike functions of order α, convex functions of order α, close-to-convex functions of order α, and qausi convex functions of order α, respectively, see [1–5].

For each function $f \in \mathcal{S}$ has an inverse function $f^{-1} = F$, defined as:

$$F(f(z)) = z, \ \ z \in E,$$

and

$$f(F(w)) = w, \ |w| < r_0(f), \ r_0(f) \geq \frac{1}{4}.$$

The series of the inverse function is provided by

$$F(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{2}$$

An analytic function f is called bi-univalent in E if f and $f^{-1}$are both univalent in E, and the class of all bi-univalent functions is denoted by Σ. For $f \in \Sigma$, Levin [6] proved that $|a_2| < 1.51$ and after that Branan and Clunie [7] improved this bound and proved that $|a_2| \leq \sqrt{2}$. Furthermore, for $f \in \Sigma$, Netanyahu [8] proved that $\max|a_2| = \frac{4}{3}$ (see for details [9–12]). In these recent papers, only non-sharp estimates on the initial coefficients were obtained.

Faber [13] introduced the Faber polynomials expansion method and used this technique to investigate the coefficient bounds $|a_m|$ for $m \geq 3$. These polynomials play an important role in mathematical sciences, particularly in Geometric Function Theory. Hamidi and Jahangiri [14,15] defined new subclasses of bi-univalent functions by using the Faber polynomials expansion technique and found some interesting and useful properties. In 1948, Schiffer [16] studied applications of the Faber polynomials in the theory of univalent functions. After that, Pommerenke [17–19] provided the substantial contributions to the available information about the structure of the Faber polynomials. Further, in 1971, Curtiss [20] studied the Faber polynomial and the Faber series, while, in 2006, Airault [21] used the Faber polynomials in the coordinate system to study the geometry of the manifold of coefficients of univalent functions. Then, in 2007, Airault [22] found symmetric sums associated with the factorizations of the Grunsky coefficients. Hamidi et al. [23] started to apply the Faber polynomial methods for meromorphic bi-starlike functions and discussed the unpredictable behaviors of the initial coefficients. In [24,25], Altinkaya and Yalcin also applied the Faber polynomial methods and investigated general coefficient bounds and different behaviors of initial coefficient bounds. Bulut [26] considered a new class of meromorphic bi-univalent functions and used the Faber polynomial technique and produced some useful results. Recently, Jia et al. [27] studied symmetric analytic functions by using Faber polynomials. Several different subclasses of the analytic and bi-univalent functions were introduced and analogously studied by the many authors (see, for example, [21,24,25,28–33]).

Now, we provide the definitions of two new subclasses of bi-close-to-convex and bi-quasi-convex functions related with Janowski functions.

Definition 1. *A function $f \in \mathcal{A}$ is said to be bi-close-to-convex in E if both f and $f^{-1} = F$ are close-to-convex in E. Furthermore, $f \in \mathcal{C}_\Sigma(A,B)$, the class of bi-close-to-convex functions associated with Janowski functions, if there is a function $g \in \mathcal{S}^*$ satisfying*

$$\frac{zf^{'}(z)}{g(z)} \prec \frac{1+Az}{1+Bz}$$

and

$$\frac{wF^{'}(w)}{G(w)} \prec \frac{1+Aw}{1+Bw},$$

where, $-1 \leq B < A \leq 1, z, w \in E$.

Definition 2. *Let f be an analytic function and be of the form (1). Then, $f \in \mathcal{C}^*_\Sigma(A,B)$, the class of bi-Quasi-convex functions associated with Janowski functions, if there is a function $g \in \mathcal{C}$ satisfying*

$$\frac{\left(zf^{'}(z)\right)^{'}}{g^{'}(z)} \prec \frac{1+Az}{1+Bz}$$

and

$$\frac{\left(wF^{'}(w)\right)^{'}}{G^{'}(w)} \prec \frac{1+Aw}{1+Bw},$$

where $-1 \leq B < A \leq 1, z, w \in E$.

Throughout, in this article, we assume $-1 \leq B < A \leq 1$.

2. The Faber Polynomial Expansion Method and Its Applications

Using the Faber polynomial technique for the analytic function f, the coefficients of its inverse map F can be written as follows (see [21,22]):

$$F(w) = f^{-1}(w) = w + \sum_{m=2}^{\infty} \frac{1}{m} \mathfrak{S}^m_{m-1}(a_2, a_3, \ldots, a_m) w^m,$$

where

$$\begin{aligned}
\mathfrak{S}^{-m}_{m-1} \quad = \quad & \frac{(-m)!}{(-2m+1)!(m-1)!} a_2^{m-1} + \frac{(-m)!}{[2(-m+1)]!(m-3)!} a_2^{m-3} a_3 \\
& + \frac{(-m)!}{(-2m+3)!(m-4)!} a_2^{m-4} a_4 \\
& + \frac{(-m)!}{[2(-m+2)]!(m-5)!} a_2^{m-5} \left[a_5 + (-m+2) a_3^2\right] \\
& + \frac{(-m)!}{(-2m+5)!(m-6)!} a_2^{m-6} [a_6 + (-2m+5) a_3 a_4] \\
& + \sum_{\mathrm{j} \geq 7} a_2^{\mathrm{j}-\mathrm{m}} \mathcal{Q}_{\mathrm{m}},
\end{aligned}$$

and $\mathcal{Q}_{\mathrm{m}}$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_m$, for $7 \leq \mathrm{j} \leq m$. Particularly, the first three terms of $\mathfrak{S}^{-m}_{m-1}$ are

$$\begin{aligned}
\frac{1}{2}\mathfrak{S}^{-2}_1 \quad &= \quad -a_2, \ \frac{1}{3}\mathfrak{S}^{-3}_2 = 2a_2^2 - a_3, \\
\frac{1}{4}\mathfrak{S}^{-4}_3 \quad &= \quad -(5a_2^3 - 5a_2 a_3 + a_4).
\end{aligned}$$

In general, for $r \in \mathbb{Z}$ ($\mathbb{Z} := 0, \pm 1, \pm 2, \ldots$) and $m \geq 2$, an expansion of $\mathfrak{S}_m^r$ is of the form:

$$\mathfrak{S}_m^r = r a_m + \frac{r(r-1)}{2}\mathcal{V}_m^2 + \frac{r!}{(r-3)!3!}\mathcal{V}_m^3 + \ldots + \frac{r!}{(r-m)!(m)!}\mathcal{V}_m^m,$$

where

$$\mathcal{V}_m^r = \mathcal{V}_m^r(a_2, a_3, \ldots),$$

and, by [22], we have

$$\mathcal{V}_m^v(a_2, \ldots, a_m) = \sum_{m=1}^{\infty} \frac{v!(a_2)^{\mu_1} \ldots (a_m)^{\mu_m}}{\mu_1!, \ldots, \mu_m!}, \text{ for } a_1 = 1 \text{ and } v \leq m.$$

The sum is adopted over all non-negative integers $\mu_1, \ldots, \mu_m$ which satisfy

$$\mu_1 + \mu_2 + \ldots + \mu_m = v$$

and

$$\mu_1 + 2\mu_2 + \ldots + m\mu_m = m.$$

For more details, see [34]. Clearly,

$$\mathcal{V}_m^m(a_1, \ldots, a_m) = \mathcal{V}_1^m,$$

and the first and last polynomials are

$$\mathcal{V}_m^m = a_1^m, \text{ and } \mathcal{V}_m^1 = a_m.$$

To prove our main results, we shall need the following well-known lemmas (see Jahangiri [35], Duren [1]).

Lemma 1. *Let $\Phi(z) = 1 + \sum\limits_{m=1}^{\infty} c_m z^m$ be a positive real part function so that*

$$Re\Phi(z) > 0$$

for $|z| < 1$. If $\alpha \geq -\frac{1}{2}$, then

$$\left|c_2 + \alpha c_1^2\right| \leq 2 + \alpha|c_1|^2.$$

Lemma 2. *Let $\varphi(z) = \sum\limits_{m=1}^{\infty} \varphi_m z^m$ be a Schwarz function so that*

$$|\varphi(z)| < 1$$

for $|z| < 1$. If $\gamma \geq 0$, then

$$\left|\varphi_2 + \gamma\varphi_1^2\right| \leq 1 + (\gamma - 1)|\varphi_1|^2.$$

Now, by using the Faber Polynomial technique, we obtain general coefficients $|a_m|$, for the classes $\mathcal{C}_\Sigma(A, B)$ and $\mathcal{C}_\Sigma^*(A, B)$. We also show the unpredictable behavior of the initial coefficients for these classes.

3. Main Results

Theorem 1. *Let $f \in \mathcal{C}_\Sigma(A, B)$ be an analytic function and if $a_k = 0, 2 \leq k \leq m-1$, then*

$$|a_m| \leq \frac{(A-B)}{m} + 1, \quad for \quad m \geq 3.$$

Proof. For $f \in \mathcal{C}_{\Sigma}(A,B)$, there exists a function $g \in \mathcal{S}^*$, then the Faber polynomial expansion for $\frac{zf'(z)}{g(z)}$ is provided by

$$\frac{zf'(z)}{g(z)} = 1 + \sum_{m=2}^{\infty}\left[(ma_m - b_m)\sum_{l=1}^{m-2}\mathfrak{S}_l^{-1}(b_2, b_3, \ldots b_{l+1})((m-l)a_{m-l} - b_{m-l})\right]z^{m-1}. \quad (3)$$

For the inverse mappings $F = f^{-1}$ and $G = g^{-1}$, the Faber polynomial expansion for $\frac{wF'(w)}{G(w)}$ is

$$\frac{wF'(w)}{G(w)} = 1 + \sum_{m=2}^{\infty}\left[\begin{array}{c}(mA_m - B_m)\sum_{l=1}^{m-2}\mathfrak{S}_l^{-1}(B_2, B_3, \ldots B_{l+1}) \\ ((m-l)A_{m-l} - B_{m-l})\end{array}\right]w^{m-1}. \quad (4)$$

As opposed to that, since $\frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}$ in E, by the definition of subordination, there exist a Schwarz function

$$w(z) = \sum_{m=1}^{\infty} c_m z^m, \ \ z \in E$$

such that

$$\begin{aligned}\frac{zf'(z)}{g(z)} &= \frac{1+Aw(z)}{1+Bw(z)} = \\ &= 1 + \sum_{m=1}^{\infty}(A-B)\mathfrak{S}_m^{-1}(c_1, c_2, \ldots c_m, B)z^m. \quad (5)\end{aligned}$$

Similarly, $\frac{wF'(w)}{G(w)} \prec \frac{1+Az}{1+Bz}$ in E, there exists a Schwarz function

$$\phi(w) = \sum_{m=1}^{\infty} d_m w^m$$

such that

$$\begin{aligned}\frac{wF'(w)}{G(w)} &= \frac{1+A\phi(w))}{1+B\phi(w))} \\ &= 1 + \sum_{m=1}^{\infty}(A-B)\mathfrak{S}_m^{-1}(d_1, d_2, \ldots d_m, B)w^m. \quad (6)\end{aligned}$$

In general (e.g., see [21,28]), the coefficients $\mathfrak{S}_m^p(k_1, k_2, \ldots k_m, B)$ are provided by

$$\begin{aligned}\mathfrak{S}_m^p(k_1, k_2, \ldots k_m, B) &= \frac{p!}{(p-m)!m!}k_1^m B^{m-1} + \frac{p!}{(p-m+1)!(m-2)!}k_1^{m-2}k_2 B^{m-2} \\ &\ldots + \frac{p!}{(p-m+2)!(m-3)!}k_1^{m-3}k_3 B^{m-3} \\ &+ \frac{p!}{(p-m+3)!(m-4)!}k_1^{m-4}\left[k_4 B^{m-4} + \frac{p-m+3}{2}k_3^2 B\right] \\ &+ \frac{p!}{(p-m+4)!(m-5)!}k_1^{m-5}\left[k_5 B^{m-5} + (p-m+4)k_3 k_4 B\right] \\ &+ \sum_{j\geq 6}^{\infty} k_1^{m-j} X_j,\end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, \ldots k_m$.

Evaluating the coefficients of Equations (3) and (5), for any $m \geq 2$, yields

$$\left\{ \begin{array}{c} (ma_m - b_m) \sum_{l=1}^{m-2} \mathfrak{S}_l^{-1}(b_2, b_3, \dots b_{l+1}) \\ \times((m-l)a_{m-l} - b_{m-l}) \end{array} \right\} = (A-B)\mathfrak{S}_{m-1}^{-1}(c_1, c_2, \dots c_{m-1}, B). \tag{7}$$

Evaluating the coefficients of Equations (4) and (6) for any $m \geq 2$, yields

$$\left\{ \begin{array}{c} (mA_m - B_m) \sum_{l=1}^{m-2} \mathfrak{S}_l^{-1}(B_2, B_3, \dots B_{l+1}) \\ \\ \times((m-l)A_{m-l} - B_{m-l}) \end{array} \right\} = (A-B)\mathfrak{S}_{m-1}^{-1}(d_1, d_2, \dots d_{m-1}, B). \tag{8}$$

For special case $m = 2$, from Equations (7) and (8), we obtain

$$\begin{aligned} 2a_2 - b_2 &= (A-B)c_1 \\ 2A_2 - B_2 &= (A-B)d_1. \end{aligned}$$

Solving for a_2 and adopting the absolute values for the coefficients of the Schwarz functions p and q,, e.g., $|c_m| \leq 1$ and $|d_m| \leq 1$ (e.g, see Duren [1]), we can obtain

$$|a_2| \leq \frac{(A-B)}{2} + 1.$$

Furthermore, from the assumption $2 \leq k \leq m-1$, and $a_k = 0$, respectively, we obtain

$$A_m = -a_m$$

and

$$ma_m - b_m = (A-B)c_{m-1}, \tag{9}$$

$$-ma_m - B_m = (A-B)d_{m-1}. \tag{10}$$

By solving Equations (9) and (10) for a_m, determining the absolute values, and by the Caratheodory Lemma [1], we obtain

$$|a_m| \leq \frac{(A-B)}{m} + 1,$$

upon noticing that

$$|b_m| \leq m \text{ and } |B_m| \leq m.$$

This completes Theorem 1. □

For $A = 1 - 2\alpha$ and $B = -1$, $0 \leq \alpha < 1$, in Theorem 1, we obtain a well-known corollary that was proved in [14].

Corollary 1. *Let $f \in \mathcal{C}_\Sigma(\alpha)$ if $a_{k+1} = 0, 1 \leq k \leq m$. Then,*

$$|a_m| \leq 1 + \frac{2(1-\alpha)}{m}, \quad \text{for } m \geq 3.$$

Theorem 2. *If an analytic function f provided by (1) belongs to the class $\mathcal{C}_\Sigma(A, B)$, then*

$$|a_2| \leq \begin{cases} \sqrt{B(B-A)}, & B \leq 0 < A, \\ \\ (A-B), & A \leq 0, \end{cases}$$

$$|a_3| \leq \begin{cases} \frac{(A-B)}{2}(1+|B|(A-B)), & B \leq 0 < A, \\ B(B-A), & A \leq 0, \end{cases}$$

$$\left|a_3 - a_2^2\right| \leq \frac{(A-B)}{2}.$$

Proof. For the function $g = f$ in the proof of Theorem 1, we obtain $a_m = -b_m$. For $m = 2$, Equations (7) and (8), respectively, yield

$$\begin{aligned} a_2 &= (A-B)c_1 && (11) \\ -a_2 &= (A-B)d_1. && (12) \end{aligned}$$

From (11) and (12), we have

$$c_1 = -d_1. \tag{13}$$

If we adopt the absolute values of any of these two equations, for the coefficients of the Schwarz functions p and q, that is $|c_m| \leq 1$ and $|d_m| \leq 1$ (e.g., see Duren [1]), we obtain

$$|a_2| \leq (A-B).$$

For $m = 3$, Equations (7) and (8), respectively, yield

$$a_2^2 - 2a_3 = (A-B)\left(Bc_1^2 - c_2\right) \tag{14}$$

and

$$2a_3 - 3a_2^2 = (A-B)\left(Bd_1^2 - d_2\right). \tag{15}$$

Adding (14) and (15), we arrive at

$$\begin{aligned} -2a_2^2 &= -(A-B)\left[\left(c_2 - Bc_1^2\right) + \left(d_2 - Bd_1^2\right)\right] \\ &= -(A-B)\left[\left(c_2 + (-B)c_1^2\right) + \left(d_2 + (-B)d_1^2\right)\right]. \end{aligned}$$

For the coefficients of the Schwarz functions p and q, that is $|c_m| \leq 1$ and $|d_m| \leq 1$ (e.g., see Duren [1]), we have

$$2|a_2|^2 \leq (A-B)\left[\left|c_2 + (-B)c_1^2\right| + \left|d_2 + (-B)d_1^2\right|\right].$$

If $B \leq 0 < A$, then using Lemma 2, we have

$$2|a_2|^2 \leq (A-B)\left[1 + (-B-1)|c_1|^2 + 1 + (-B-1)|d_1|^2\right].$$

For the coefficients of the Schwarz functions p and q, that is $|c_m| \leq 1$ and $|d_m| \leq 1$ (e.g., see Duren [1]), we obtain

$$|a_2| \leq \sqrt{B(B-A)}.$$

Consequently, we note that, if $B \leq 0 < A$, then

$$\sqrt{B(B-A)} < B(B-A).$$

Multiplying Equation (14) by 3 and adding it to (15), we obtain

$$a_3 = \frac{A-B}{4}\left[3\left(c_2 + (-B)c_1^2\right) + \left(d_2 + (-B)d_1^2\right)\right],$$

then, using Lemma 2, we have

$$|a_3| \le \frac{A-B}{4}\Big[3(1+(-B-1))|c_1|^2 + 1 + (-B-1)|d_1|^2\Big].$$

Since $|c_m| \le 1$ and $|d_m| \le 1$ (e.g., see Duren [1]), we obtain

$$|a_3| \le B(B-A).$$

Substituting $a_2 = (A-B)c_1$ in (14), we obtain

$$2a_3 = (A-B)\Big(c_2 - B(A-B)c_1^2\Big).$$

Using the triangle inequality, we have

$$2|a_3| \le (A-B)\Big(|c_2| + |B|(A-B)\Big|c_1^2\Big|\Big).$$

Since $|c_m| \le 1$ and $|d_m| \le 1$, we obtain

$$|a_3| \le \frac{(A-B)}{2}(1+|B|(A-B)).$$

Lastly, subtracting Equations (14) from (15), and using the fact (13)

$$a_3 - a_2^2 = \frac{(A-B)}{4}|c_2 - d_2|.$$

Since $|c_m| \le 1$ and $|d_m| \le 1$, we obtain

$$\Big|a_3 - a_2^2\Big| \le \frac{(A-B)}{2}.$$

□

For $A = 1-2\alpha$ and $B = -1$, $0 \le \alpha < 1$, in Theorem 2, we obtain the following known result provided in [14].

Corollary 2. *Let $f \in \mathcal{C}_\Sigma(\alpha)$ be provided by (1). Then,*

$$|a_2| \le \begin{cases} \sqrt{2(1-\alpha)} & \text{if } 0 \le \alpha < \frac{1}{2}, \\ 2(1-\alpha) & \text{if } \frac{1}{2} \le \alpha < 1, \end{cases}$$

$$|a_3| \le \begin{cases} (1-\alpha) & \text{if } 0 \le \alpha < \frac{1}{2}, \\ (1-\alpha)(3-2\alpha) & \text{if } \frac{1}{2} \le \alpha < 1. \end{cases}$$

Theorem 3. *Let $f \in \mathcal{C}^*_\Sigma(A,B)$ be provided by (1), if $a_k = 0$, $2 \le k \le m-1$. Then,*

$$|a_m| \le \frac{1}{m}\left(\frac{A-B}{m} + 1\right), \quad \textit{for} \quad m \ge 3.$$

Proof. For $f \in \mathcal{C}^*_\Sigma(A,B)$, there exists a function $g \in \mathcal{C}$, then the Faber polynomial expansion for $\frac{\left(zf'(z)\right)'}{g'(z)}$ is

$$\frac{\left(zf'(z)\right)'}{g'(z)} = 1 + \sum_{m=2}^{\infty}\left[m(ma_m - b_m)\sum_{l=1}^{m-2}\mathfrak{G}_l^{-1}(b_2, b_3, \ldots b_{l+1})((m-l)[(m-l)a_{m-l} - b_{m-l}])\right]z^{m-1}. \tag{16}$$

For the inverse mapping $F = f^{-1}$ and $G = g^{-1}$, we obtain the Faber polynomial expansion for $\frac{\left(wF'(w)\right)'}{G'(w)}$, which is

$$\frac{\left(wF'(w)\right)'}{G'(w)} = 1 + \sum_{m=2}^{\infty} \left[\begin{array}{c} m(mA_m - B_m) \sum_{l=1}^{m-2} \mathfrak{S}_l^{-1}(B_2, B_3, \ldots B_{l+1}) \\ \times((m-l)[(m-l)A_{m-l} - B_{m-l}]) \end{array} \right] w^{m-1}. \quad (17)$$

Opposite that, since $\frac{\left(zf'(z)\right)'}{g'(z)} \prec \frac{1+Az}{1+Bz}$ in E, by the definition of subordination, there exist a Schwarz function

$$w(z) = \sum_{m=1}^{\infty} c_m z^m, \ \ z \in E$$

such that

$$\begin{aligned} \frac{\left(zf'(z)\right)'}{g'(z)} &= \frac{1 + Aw(z))}{1 + Bw(z))} = \\ &= 1 + \sum_{m=1}^{\infty} (A - B)\mathfrak{S}_m^{-1}(c_1, c_2, \ldots c_m, B) z^m. \end{aligned} \quad (18)$$

Similarly $\frac{\left(wF'(w)\right)'}{G'(w)} \prec \frac{1+Az}{1+Bz}$ in E, by the definition of subordination, there exist a Schwarz function

$$\phi(w) = \sum_{m=1}^{\infty} d_m w^m$$

such that

$$\begin{aligned} \frac{\left(wF'(w)\right)'}{G'(w)} &= \frac{1 + A\phi(w))}{1 + B\phi(w))} \\ &= 1 + \sum_{m=1}^{\infty} (A - B)\mathfrak{S}_m^{-1}(d_1, d_2, \ldots d_m, B) w^m. \end{aligned} \quad (19)$$

In general (e.g., see [21,28]), the coefficients $\mathfrak{S}_m^p(k_1, k_2, \ldots k_m, B)$ are provided as:

$$\begin{aligned} \mathfrak{S}_m^p(k_1, k_2, \ldots k_m, B) = {} & \frac{p!}{(p-m)!m!} k_1^m B^{m-1} + \frac{p!}{(p-m+1)!(m-2)!} k_1^{m-2} k_2 B^{m-2} \\ & \ldots + \frac{p!}{(p-m+2)!(m-3)!} k_1^{m-3} k_3 B^{m-3} \\ & + \frac{p!}{(p-m+3)!(m-4)!} k_1^{m-4} \left[k_4 B^{m-4} + \frac{p-m+3}{2} k_3^2 B \right] \\ & + \frac{p!}{(p-m+4)!(m-5)!} k_1^{m-5} \left[k_5 B^{m-5} + (p-m+4) k_3 k_4 B \right] \\ & + \sum_{j \geq 6}^{\infty} k_1^{m-j} X_j, \end{aligned}$$

where, X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, \ldots k_m$.

Evaluating the coefficients of Equations (16) and (18), for any $m \geq 2$, yields

$$\left\{ \begin{array}{c} m(ma_m - b_m) \sum_{l=1}^{m-2} \mathfrak{S}_l^{-1}(b_2, b_3, \ldots b_{l+1}) \\ \times((m-l)[(m-l)a_{m-l} - b_{m-l}]) \end{array} \right\} = (A-B)\mathfrak{S}_{m-1}^{-1}(c_1, c_2, \ldots c_{m-1}, B). \tag{20}$$

Evaluating the coefficients of Equations (17) and (19), for any $m \geq 2$, yields

$$\left\{ \begin{array}{c} m(mA_m - B_m) \sum_{l=1}^{m-2} \mathfrak{S}_l^{-1}(B_2, B_3, \ldots B_{l+1}) \\ \\ \times((m-l)[(m-l)A_{m-l} - B_{m-l}]) \end{array} \right\} = (A-B)\mathfrak{S}_{m-1}^{-1}(d_1, d_2, \ldots d_{m-1}, B). \tag{21}$$

For special case $m = 2$, from Equations (20) and (21), we obtain

$$\begin{aligned} 2(2a_2 - b_2) &= (A-B)c_1, \\ 2(2A_2 - B_2) &= (A-B)d_1. \end{aligned}$$

Solving for a_2 and adopting the absolute values for the coefficients of the Schwarz functions p and q, that is $|c_m| \leq 1$ and $|d_m| \leq 1$ (e.g., see Duren [1]), we can obtain

$$|a_2| \leq \frac{1}{2}\left(\frac{(A-B)}{2} + 1\right).$$

Furthermore, from the assumption, $2 \leq k \leq m-1$, and $a_k = 0$, respectively, yields:

$$A_m = -a_m,$$

$$m(ma_m - b_m) = (A-B)c_{m-1}, \tag{22}$$

and

$$m(-ma_m - B_m) = (A-B)d_{m-1}. \tag{23}$$

By solving Equations (22) and (23) for a_m and determining the absolute values and using the Caratheodory Lemma provided in [1], we obtain

$$|a_m| \leq \frac{1}{m}\left(\frac{(A-B)}{m} + 1\right),$$

upon noticing that

$$|b_m| \leq m \text{ and } |B_m| \leq m.$$

This completes Theorem 3. □

Theorem 4. *Let $f \in \mathcal{C}_\Sigma^*(A, B)$ be provided by (1). Then,*

$$|a_2| \leq \left\{ \begin{array}{ll} \sqrt{\frac{B(B-A)}{2}} & \text{if } B \leq 0 < A, \\ \frac{(A-B)}{2} & \text{if } A \leq 0, \end{array} \right.$$

$$|a_3| \leq \left\{ \begin{array}{ll} \frac{B(B-A)}{2} & \text{if } B \leq 0 < A, \\ \frac{(A-B)}{6}(1 + |A - 2B|) & \text{if } A \leq 0 \end{array} \right.$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{A-B}{6}.$$

Proof. For the function $g^{'} = f^{'}$ in the proof of Theorem 1, we obtain $a_m = -b_m$. For $m = 2$, Equations (20) and (21), respectively, yield

$$\begin{aligned} 2a_2 &= (A-B)c_1 && (24)\\ -2a_2 &= (A-B)d_1. && (25)\end{aligned}$$

From (24) and (25), we have

$$c_1 = -d_1. \tag{26}$$

If we adopt the absolute values of any of these two equations, for the coefficients of the Schwarz functions p and q, that is $|c_m| \leq 1$and $|d_m| \leq 1$ (e.g., see Duren [1]), we obtain

$$|a_2| \leq \frac{(A-B)}{2}.$$

For $m = 3$, Equations (20) and (21), respectively, yield

$$4a_2^2 - 6a_3 = (A-B)\left(Bc_1^2 - c_2\right) \tag{27}$$

and

$$6a_3 - 8a_2^2 = (A-B)\left(Bd_1^2 - d_2\right). \tag{28}$$

Adding (27) and (28), we arrive at

$$\begin{aligned} -4a_2^2 &= -(A-B)\left[\left(c_2 - Bc_1^2\right) + \left(d_2 - Bd_1^2\right)\right]\\ &= -(A-B)\left[\left(c_2 + (-B)c_1^2\right) + \left(d_2 + (-B)d_1^2\right)\right].\end{aligned}$$

Since $|c_m| \leq 1$ and $|d_m| \leq 1$, we have

$$4|a_2|^2 \leq (A-B)\left[\left|c_2 + (-B)c_1^2\right| + \left|d_2 + (-B)d_1^2\right|\right].$$

If $B \leq 0 < A$, then using Lemma 2, we have

$$4|a_2|^2 \leq (A-B)\left[1 + (-B-1)|c_1|^2 + 1 + (-B-1)|d_1|^2\right].$$

Since $|c_m| \leq 1$ and $|d_m| \leq 1$, we obtain

$$|a_2| \leq \sqrt{\frac{B(B-A)}{2}}.$$

Consequently, we note that, if $B \leq 0 < A$,

$$\sqrt{\frac{B(B-A)}{2}} < \frac{B(B-A)}{2}.$$

Multiplying Equation (14) by 2 and adding it to (15), we obtain

$$-6a_3 = (A-B)\left[2\left(c_2 + (-B)c_1^2\right) + \left(d_2 + (-B)d_1^2\right)\right],$$

then using Lemma 2, we have

$$|a_3| \leq \frac{A-B}{6}\left[2\left(1 + (-B-1)|c_1|^2\right) + 1 + (-B-1)|d_1|^2\right].$$

Since $|c_m| \leq 1$ and $|d_m| \leq 1$, we obtain

$$|a_3| \leq \frac{B(B-A)}{2}.$$

Substituting $a_2 = \frac{(A-B)}{2}c_1$ in (27) and using the triangle inequality, we have

$$6|a_3| \leq (A-B)\left(|c_2| + |A-2B|\left|c_1^2\right|\right).$$

Since $|c_m| \leq 1$ and $|d_m| \leq 1$, we obtain

$$|a_3| \leq \frac{(A-B)}{6}(1 + |A-2B|).$$

Lastly, subtract Equations (27) from (28) and use the fact (26)

$$a_3 - a_2^2 = \frac{(A-B)}{12}|c_2 - d_2|.$$

Since $|c_m| \leq 1$ and $|d_m| \leq 1$, we obtain

$$\left|a_3 - a_2^2\right| \leq \frac{A-B}{6}.$$

□

4. Conclusions

The Faber polynomial expansion method is a useful tool that has been widely used to find the coefficient bounds of analytic functions. In this article, we have defined two new subclasses of bi-univalent functions associated with Janowski functions. We derived bounds on the initial as well as on the general coefficients for each of the defined classes. In addition, we have provided some intriguing corollaries as special cases of our obtained results. Furthermore, for future work, certain coefficient problems, such as Hankel determinants, Zalcman inequalities, Krushkal inequalities, etc., can be found for these classes of functions. For more about said coefficient problems, see [36–40].

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