FACETS AND NONFACETS OF CONVEX POLYTOPES

BY

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1. Introduction

Throughout this paper we shall follow, with very few exceptions, the notation and terminology introduced by Professor B. Grünbaum in [5], and the reader is referred to this work for further information on the properties of convex polytopes. By an *equifacetted d-polytope* we mean any d-dimensional convex polytope in Euclidean space whose facets (that is, faces of dimension $d-1$) are all of the same combinatorial type. Many equifacetted polytopes are known, and we mention, by way of example, three classes of polytopes which have been extensively studied: the regular polytopes [3], the simplicial polytopes [5, §§ 4.5 and 9.2] and the cubical polytopes [5, §§ 4.6 and 9.4]. This paper is concerned with problems of the following kind: If P is a given d-dimensional convex polytope, does there exist an equifacetted $(d+1)$-polytope Q whose facets are all combinatorially equivalent to P? If the answer to this question is in the affirmative, then P will be called a *d-facet* or a *facet*, and if the answer is in the negative, then P will be called a *d-nonfacet* or a *nonfacet*.

In the literature only the case $d=2$ has been mentioned, and the problem of characterising the 2-facets and 2-nonfacets is completely straightforward. It is well known (see, for example, [15, p. 149]) that if a three-dimensional convex polytope Q has p_n 2-faces which are n-gons ($n=3, 4, \ldots$) then

$$3p_3+2p_4+p_5 \geqslant 12. \tag{1}$$

It is therefore impossible for all the 2-faces of Q to be n-gons with $n \geqslant 6$. On the other hand, the tetrahedron, cube, and regular dodecahedron are equifacetted 3-polytopes bounded by triangles, quadrilaterals, and pentagons respectively, so we deduce:

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(2) *A convex n-gon is a 2-facet if* $n=3$, 4, *or* 5, *and is a 2-nonfacet if* $n \geqslant 6$.

For $d \geqslant 3$ the problem of characterising facets and non-facets, or finding criteria to distinguish between them, seems to be extremely difficult. Here we shall establish some partial results in this direction. Certainly there is no simple numerical criterion involving only the number of vertices as in the case $d=2$. In § 2 we shall give some general theorems concerning facets, and in particular, prove that every d-dimensional convex polytope with at most $d+2$ vertices is a facet. In § 3 we shall formulate some sufficient conditions for a d-dimensional convex polytope P to be a nonfacet in terms of the numbers of faces of the $(d-1)$-dimensional polytopes which arise by orthogonally projecting polytopes combinatorially equivalent to P on to hyperplanes. The case $d=3$, where the problem seems to be of a slightly different nature from that in higher dimensions, is discussed in § 4. In particular, we shall show that if no simple closed edge-circuit on P contains more than one third of the edges of P, then P is a 3-nonfacet. In § 5 we are concerned with the problem of determining whether regular polytopes are nonfacets, and it will be shown that the d-crosspolytope (generalised octahedron) is a nonfacet if $d \geqslant 6$. This is of interest since the other two regular polytopes in $d \geqslant 5$ dimensions (namely the d-simplex and the d-cube) are known to be facets. Since the octahedron is a 3-facet, the question whether the d-crosspolytope is a facet or not remains open only in the cases $d=4$ and $d=5$. In § 6 we shall consider the problem of finding, for given $d \geqslant 4$, the smallest number of vertices of a d-nonfacet. From the results of § 2 it is clear that if P is a d-nonfacet then it has at least $d+3$ vertices, and we conjecture that for all $d \geqslant 3$ there exists a d-nonfacet with exactly $d+3$ vertices. This we can prove in the cases $d=6$, 8, 9 and 10 only. We can show, however, that for any $d \geqslant 4$, a simplicial $[\frac{1}{2}d]$-neighbourly d-dimensional convex polytope with a large number of vertices is a nonfacet. In § 7 we shall find d-nonfacets with a large number of vertices and comparatively few j-faces $(1 \leqslant j \leqslant d-1)$, and the paper concludes in § 8 with some general remarks and the statement of some unsolved problems.

2. Theorems on facets

As stated in the introduction, many equifacetted polytopes have been mentioned in the literature, and so it is simple to compile an extensive list of polytopes which are known to be facets. We shall not do this here, but prove some general theorems on the construction of facets, and, in particular, prove that every d-dimensional convex polytope (more briefly, d-polytope) with at most $d+2$ vertices is a facet.

With the relation of inclusion, the set of all faces (proper and improper) of a polytope P forms a lattice $\mathfrak{F}(P)$ [5, Exercise 2.4.6]. If F^0 is a vertex of a $(d+1)$-polytope Q, then the faces of Q which include F^0 form a sublattice of $\mathfrak{F}(Q)$ which will be denoted by $\mathfrak{F}(Q, F^0)$. Let H be any hyperplane which strictly separates F^0 from the remaining vertices of Q, and put $R = H \cap Q$. Then the correspondence which maps each j-face F^j of Q containing F^0 on to the $(j-1)$-face $F^j \cap H$ of R is a lattice isomorphism between $\mathfrak{F}(Q, F^0)$ and $\mathfrak{F}(R)$. The d-polytope R (or, more precisely, any polytope combinatorially equivalent to R) will be called a *vertex figure* of Q at F^0 (compare [5, Exercise 3.4.8]) and will be denoted by $Q(F^0)$. Since the combinatorial type of a polytope is completely determined by its lattice of faces, we deduce from the above discussion that the combinatorial type of the vertex figure $Q(F^0)$ depends only on the combinatorial type of Q and the choice of F^0, and is independent of H.

Let Q^* be a $(d+1)$-polytope dual to Q [5, § 3.4]. Then there exists a one-to-one correspondence ψ between the lattices $\mathfrak{F}(Q)$ and $\mathfrak{F}(Q^*)$ which is inclusion reversing. Each j-face of Q corresponds to a $(d-j)$-face of Q^*, and so, in particular, a vertex F^0 of Q corresponds to a facet F^d of Q^*. We deduce that ψ maps the sublattice $\mathfrak{F}(Q, F^0)$ of $\mathfrak{F}(Q)$ onto the sublattice $\mathfrak{F}(F^d)$ of $\mathfrak{F}(Q^*)$, and so F^d is dual to $Q(F^0)$. Hence:

(3) *Let P be a d-polytope, Q be a $(d+1)$-polytope, and Q^* be a $(d+1)$-polytope dual to Q. Then the facets of Q^* are all combinatorially equivalent to P if and only if all the vertex figures of Q are dual to P.*

Statement (3) is useful since it is sometimes more convenient to consider polytopes with combinatorially equivalent vertex figures instead of polytopes with combinatorially equivalent facets.

(4) *Let G be any finite group of affine transformations of E^{d+1}, and $x \in E^{d+1}$ be any given point. Write $G(x)$ for the orbit of x under G, that is,*

$$G(x) = \{g(x) \mid g \in G\},$$

and conv $G(x)$ *for the convex hull of the finite set $G(x)$. Then any polytope dual to* conv $G(x)$ *is an equifacetted polytope.*

It is well known that for each finite group G of affine transformations of E^{d+1} there exists an affine transformation A of E^{d+1} such that the conjugate group $\{A^{-1}gA \mid g \in G\}$ is a group of orthogonal transformations (see, for example, [8, pp. 47–48]). Remembering also that the polar set Q^* of a $(d+1)$-polytope $Q \subset E^{d+1}$ containing the origin as an interior point is dual to Q [5, § 3.4], we see that (4) is an immediate consequence of the following:

(5) *Let G be a finite group of orthogonal transformations of E^{d+1} and let $x \in E^{d+1}$ be any point such that $Q =$* conv *$G(x)$ is a $(d+1)$-polytope. Then Q^* (the polar set of Q) is an equi-*

facetted $(d+1)$-polytope. Further, G is a group of symmetries of Q^* and acts transitively on the facets of Q^*, which are therefore congruent d-polytopes.

Proof of (5). From the assumption that Q is a $(d+1)$-polytope it follows easily that the origin is an interior point of Q, and therefore Q^* is a $(d+1)$-polytope. If $g \in G$ then $gQ = Q$ and G acts transitively on $G(x)$, which is the set of vertices of Q. Hence, by the properties of polarity, $gQ^* = Q^*$ for each $g \in G$, and G acts transitively on the set of facets of Q^*. Thus the facets of Q^* are congruent, Q^* is equifacetted, and (5) is proved.

Statement (5) motivates the following definition. A d-polytope P will be called a *superfacet* if there exists an equifacetted d-polytope Q, with all its facets combinatorially equivalent to P, such that the group of orthogonal symmetries of Q acts transitively on its facets. Although we shall be concerned almost entirely with the properties of facets, we shall mention briefly those cases where our assertions can be extended to superfacets.

Statement (5) enables us to construct arbitrarily many superfacets (and therefore facets) in any dimension $d \geq 3$. We illustrate this by an example which seems to be of considerable intrinsic interest.

Write $g(\theta)$ for the orthogonal transformation of E^{2n} with matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & . & . & . & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & \cos 2\theta & -\sin 2\theta & . & . & . & 0 & 0 \\ 0 & 0 & \sin 2\theta & \cos 2\theta & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & \cos n\theta & -\sin n\theta \\ 0 & 0 & 0 & 0 & . & . & . & \sin n\theta & \cos n\theta \end{bmatrix}$$

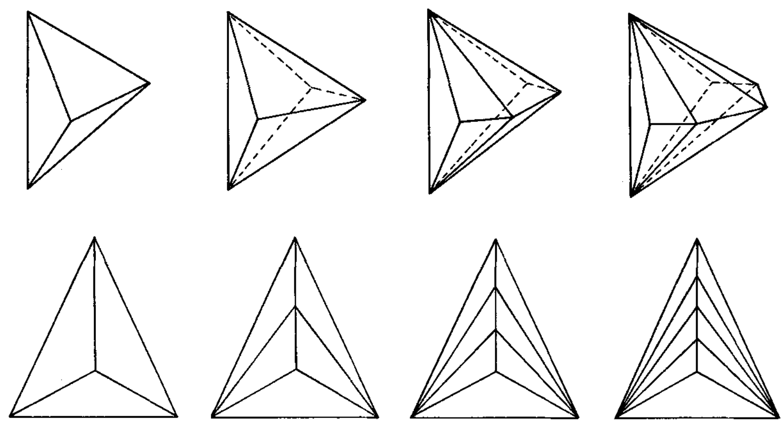
and $x(\theta)$ for the point with coordinate column vector

$$(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos n\theta, \sin n\theta)^T.$$

Let G_k^{2n} be the cyclic group of order k $(k \geq 2n+1)$ generated by $g(2\pi/k)$. Then conv $G_k^{2n}(x(0))$ is a cyclic polytope [5, § 4.7] with k vertices

$$x(0), x(2\pi/k), \ldots, x(2\pi(k-1)/k),$$

(see [4, § 3] where conv $G_k^{2n}(x(0))$ is called a regular cyclic polytope). From the proof of (5), all the vertex figures of this polytope are combinatorially equivalent. (This last statement can also be established for non-regular even-dimensional cyclic polytopes by combinatorial arguments based on Gale's evenness condition quoted below.)



$C(4, 3)$ $C(5, 3)$ $C(6, 3)$ $C(7, 3)$

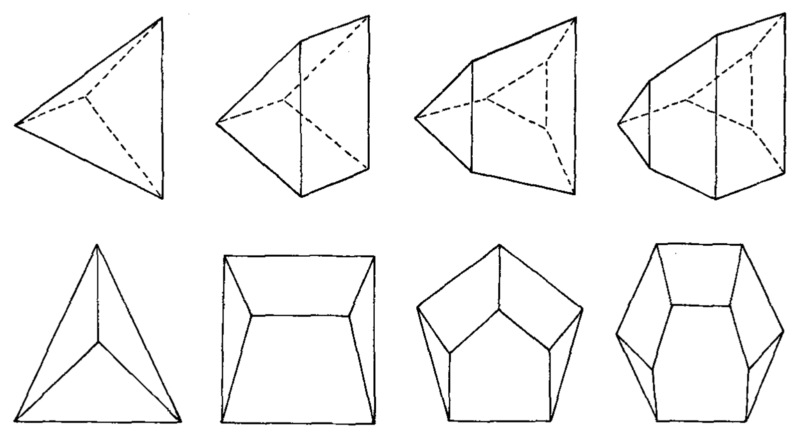
Fig. 1

(6) *Let $C(v, d)$ be a cyclic d-polytope with $v \geqslant d+1$ vertices. For any integer $n \geqslant 2$, the vertex figures of $C(v, 2n)$ are of the same combinatorial type as $C(v-1, 2n-1)$.*

Proof. We need not assume that $C(v, 2n)$ is regular, so take its vertices to be v points $p_1, \ldots, p_v$ in order on a suitable curve (for example the moment curve) in E^{2n} (see [4, p. 225] or [5, § 4.7]). Let H be a hyperplane which strictly separates the vertex p_v from the remaining vertices of $C(v, 2n)$, so that $C' = H \cap C(v, 2n)$ is a vertex figure of $C(v, 2n)$ at p_v.

Since $n \geqslant 2$, each closed line segment $[p_i, p_v]$ $(i=1, \ldots, v-1)$ is an edge of $C(v, 2n)$ and therefore the points $q_i = H \cap [p_i, p_v]$ $(i=1, \ldots, v-1)$ are the vertices of C'. Let I be any subset of $\{1, \ldots, v\}$. Then by Gale's evenness condition [5, Theorem 4.7.2] the points p_i $(i \in I)$ are the vertices of a facet of $C(v, 2n)$ if and only if I has $2n$ elements, and every two members of $\{1, \ldots, v\} \backslash I$ are separated by an even number of elements of I. The facets of C' are the intersections of H with those facets of $C(v, 2n)$ that are incident with p_v, in other words, with those facets for which the index set I contains v. We deduce that if I' is a subset of $\{1, \ldots, v-1\}$, then the points q_i $(i \in I')$ are the vertices of a facet of C' if and only if $I' \cup \{v\}$ has $2n$ members, and every two elements of $\{1, \ldots, v\} \backslash (I' \cup \{v\})$ are separated by an even number of elements of $I' \cup \{v\}$. This condition is equivalent to the statement that I' has $2n-1$ members and that every two members of $\{1, \ldots, v-1\} \backslash I'$ are separated by an even number of elements of I'. Again by Gale's evenness condition, and [5, Exercise 3.2.3], this implies that C' is a cyclic polytope $C(v-1, 2n-1)$. Since we have already shown that the vertex figures at all the vertices of $C(v, d)$ are combinatorially equivalent, (6) is proved.

From (6) and (3) we deduce that any polytope dual to $C(v, 2n)$ is equifacetted, and



$(C(4, 3))^*$ $(C(5, 3))^*$ $(C(6, 3))^*$ $(C(7, 3))^*$

Fig. 2

all its facets are dual to $C(v-1, 2n-1)$. In the case $n=2$ $(d=4)$ Gale's evenness condition enables us to determine the polytope $C(v-1, 2n-1)=C(v-1, 3)$, and therefore $(C(v-1, 3))^*$, without difficulty. (See Figures 1 and 2 which represent the cases $v=5$, 6, 7 and 8. In each case a drawing of the polytope and its Schlegel diagram [5, § 3.3] are given.) $C(v, 3)$ may be described as the convex hull of a line segment and a plane $(v-1)$-gon situated in such a way that an interior point of the segment coincides with a vertex of the polygon. $(C(v, 3))^*$ is called a *wedge*; two of its 2-faces are $(v-1)$-gons, two are triangles, and $v-4$ are quadrilaterals. This construction therefore leads to an interesting infinite sequence of simple 3-facets which starts with the tetrahedron $(v=4)$ and the triangular prism $(v=5)$. It shows, incidentally, that 3-facets with arbitrarily many vertices exist, a fact that is also established easily by other means. The construction of equifacetted 4-polytopes bounded entirely by wedges can be traced back to Brückner [2a, pp. 12–13]. The three dimensional diagrams constructed by Brückner are Schlegel diagrams of 4-polytopes dual to the cyclic polytopes $P(v, 4)$.

In the next theorem we shall describe methods by which two or more facets can be combined to yield further facets. We require the following definitions.

Let $P_1 \subset E_1^r$ and $P_2 \subset E_2^s$ be, respectively, a given r-polytope and a given s-polytope. If E_1^r and E_2^s are affine subspaces of E^{r+s} which intersect in a single point z, then each point of E^{r+s} can be represented uniquely in the form x_1+x_2 with $x_1 \in E_1^r$ and $x_2 \in E_2^s$. The $(r+s)$-polytope

$$P_1 \times P_2 = \{x_1+x_2 \mid x_1 \in P_1,\ x_2 \in P_2\}$$

is called the *cartesian product* of P_1 and P_2. If, further, z belongs to the relative interiors of P_1 and of P_2, then the $(r+s)$-polytope

$$P_1 \oplus P_2 = \operatorname{conv}(P_1 \cup P_2)$$

is called the *direct sum* of P_1 and P_2 (see [5, Exercise 4.8.4]). If, on the other hand, E_1^r and E_2^s are independent affine subspaces of E^{r+s+1} (that is to say, they are disjoint and contain no parallel lines), then the $(r+s+1)$-polytope

$$P_1 \bigovee P_2 = \operatorname{conv}(P_1 \cup P_2)$$

is called the *free join* of P_1 and P_2 (see [5, Exercise 4.8.1] where $P_1 \bigovee P_2$ is called a *pyramidoid*). If P_1 is a line segment then $P_1 \times P_2$ is a *prism*, and $P_1 \oplus P_2$ is a *bipyramid*. If P_1 is a single point, $P_1 \bigovee P_2$ is a *pyramid*. In each case P_2 is called the *base* of the figure.

In the following, whenever we write $P_1 \times P_2$, $P_1 \oplus P_2$ or $P_1 \bigovee P_2$, we shall implicitly assume that P_1 and P_2 are situated in such a way that the polytopes $P_1 \times P_2$, $P_1 \oplus P_2$ or $P_1 \bigovee P_2$ exist. This convention is used in the statement and proof of the next theorem.

(7) (i) *If P_1 is an r-facet and P_2 is an s-facet, then $P_1 \bigovee P_2$ is an $(r+s+1)$-facet.*

(ii) *If P_1 is combinatorially equivalent to each facet of an equifacetted $(r+1)$-polytope P_2, then $P_1 \times P_2$ is a $(2r+1)$-facet and $P_1 \bigovee P_2$ is a $(2r+2)$-facet.*

(iii) *If P_1 is an r-simplex and P_2 is either an s-simplex or an s-cube, then $P_1 \oplus P_2$ is an $(r+s)$-facet.*

Proofs. (i) If Q_1 and Q_2 are equifacetted $(r+1)$- and $(s+1)$-polytopes with facets combinatorially equivalent to P_1 and P_2 respectively, then it is easy to verify that the $(r+s+2)$-polytope $Q_1 \oplus Q_2$ is equifacetted and its facets are combinatorially equivalent to $P_1 \bigovee P_2$.

(ii) Let P_2' and P_2'' be $(r+1)$-polytopes combinatorially equivalent to P_2 and lying in linearly independent $(r+1)$-dimensional linear subspaces of E^{2r+2}. Then it is easily verified that the $(2r+2)$-polytope $P_2' \times P_2''$ is equifacetted, and its facets are combinatorially equivalent to $P_1 \times P_2$.

If, on the other hand, P_2' and P_2'' are $(r+1)$-polytopes combinatorially equivalent to P_2 and lying in independent affine subspaces of E^{2r+3}, then it is easy to see that $P_2' \bigovee P_2''$ is equifacetted and its facets are combinatorially equivalent to $P_1 \bigovee P_2$. (An obvious extension of our argument shows that if

$$P = P_1 \bigovee P_2 \bigovee P_2' \bigovee \dots \bigovee P_2^{(k)},$$

or

$$P = P_1 \times P_2 \times P_2' \times \dots \times P_2^{(k)},$$

and $P_2', P_2'', \dots, P_2^{(k)}$ are combinatorially equivalent to P_2, then P is a facet.)

(iii) Let Q be the rth truncation of an $(r+s+1)$-simplex T^{r+s+1}, that is to say, the convex hull of the barycentres of the r-faces of T^{r+s+1}. In [3, § 8.7] Q is denoted by $\begin{Bmatrix} 3^r \\ 3^s \end{Bmatrix}$ and in Coxeter's graphical notation [3, §§ 11.6 and 11.7] it is represented by a simple chain of $r+s+1$ nodes with the $(r+1)$st node ringed. The coordinates vector of the vertices of Q may be taken to be the $\binom{r+s+2}{r+1}$ permutations of

$$(1, 1, \ldots, 1, 0, 0, \ldots, 0)$$

in which 1 occurs $r+1$ times and 0 occurs $s+1$ times [3, § 8.7]. From these coordinates it is easily verified that each vertex figure of Q is a cartesian product $T^r \times T^s$ (compare [3, § 11.7]) and so, by (3), Q^* is an equifacetted polytope whose facets are combinatorially equivalent to

$$(T^r \times T^s)^* = T^r \oplus T^s.$$

This proves the first part of statement (iii).

For the second part we take Q as the rth truncation of the regular $(r+s+1)$-cross-polytope. In [3, § 8.7] Q is denoted by $\begin{Bmatrix} 3^r \\ 3^{s-1}, 4 \end{Bmatrix}$. The coordinate vectors of the vertices of Q may be taken to be the $2^{r+1}\binom{r+s+1}{r+1}$ permutations of

$$(\pm 1, \pm 1, \ldots, \pm 1, 0, 0, \ldots, 0)$$

(in which ± 1 occurs $r+1$ times and 0 occurs s times) with all possible sets of ambiguous signs. Using these coordinates it is easily verified that each vertex figure of Q is a cartesian product $T^r \times X^s$ where X^s is an s-crosspolytope (compare [3, § 11.7]). Thus by (3), Q^* is an equifacetted polytope whose facets are combinatorially equivalent to

$$(T^r \times X^s)^* = T^r \oplus C^s,$$

where C^s is an s-cube. This completes the proof of (iii) and also of (7). The following is an important corollary:

(8) *Every d-polytope P with at most $d+2$ vertices is a d-facet.*

If P has $d+1$ vertices then it is a simplex and so is trivially a facet. If P has $d+2$ vertices then it is known [5, § 6.1] that either

(a) $P = T^r \oplus T^s$ $(r+s=d)$, or

(b) P is a pyramid whose base is a $(d-1)$-polytope with $d+1$ vertices.

If (a) holds then P is a facet by (7) (iii). In case (b) we use an obvious inductive argument on the dimension, recalling that by (7) (i) every pyramid whose base is a facet is itself a facet.

Statements (7) and (8) remain true if we replace 'facet' by 'superfacet' throughout. (In 7 (ii) we have to add the condition that the group of orthogonal symmetries of P_2 is transitive on the set of facets of P_2.)

We conclude this section with a statement, the proof of which involves a process called *adjoining* one polytope to a facet of another.

(9) *Let P be any d-facet $(d > 0)$. Then there exists an enumerable infinity of combinatorial types of equifacetted $(d+1)$-polytopes whose facets are combinatorially equivalent to P.*

Proof. Let Q be any equifacetted $(d+1)$-polytope in E^{d+1} whose facets are combinatorially equivalent to P. Let $\bar{E}^{d+1}$ be the projective space formed from E^{d+1} by adjoining a hyperplane at infinity, and let H be the hyperplane containing one of the facets P_1 of Q. Let z be any point beyond P_1 and beneath all the other facets of Q (see [5, § 5.2]). For any $p < 0$ define a p-transformation τ_p of $\bar{E}^{d+1}$ as follows (see [13], where however, o is written in place of z). If $x \neq z$, $x \notin H$, and the line l_x joining x to z meets H in x', then $x^* \in l_x$ is chosen so that the cross-ratio

$$cr(z, x'; x, x^*) = p,$$

and τ_p is defined by

$$\tau_p(z) = z, \quad \tau_p(x') = x' \quad \text{and} \quad \tau_p(x) = x^*.$$

The transformation τ_p is a non-singular projective transformation leaving z and H pointwise fixed (it is a homology) so $\tau_p Q = Q_p$ is combinatorially equivalent to Q [5, Theorem 3.2.3], and $P_1 = Q \cap H = Q_p \cap H$. Since $p < 0$, Q_p lies in the pyramid with base P_1 and apex z, from which we easily deduce that $Q_p \cup Q$ is convex. Further it is equifacetted and all its facets are combinatorially equivalent to P. If Q has $f_d(Q)$ facets then $Q_p \cup Q$ has $2f_d(Q) - 2 > f_d(Q)$ facets, and so, starting from any equifacetted polytope Q we can obtain arbitrarily many such polytopes by repeated application of the process just described.

Whenever, as in the above proof, we have two convex polytopes Q and Q' such that $Q \cup Q'$ is convex, $Q \cap Q' = P_1$ is a facet of each, and every proper face of P_1 is a face of $Q \cup Q'$, then we shall say that $Q \cup Q'$ arises by *adjoining* Q' to the facet P_1 of Q. This process, which has already been mentioned implicitly in the works of many authors, will turn out to be of considerable importance in the construction of nonfacets in the following sections.

3. Theorems on nonfacets

The purpose of this section is to establish criteria (sufficient conditions) for a given d-polytope P to be a nonfacet. These criteria will be stated in general terms, and will then be applied to interesting special cases in the remaining sections. The discussion will depend heavily on [11], and we shall assume that the reader is familiar with the results of that paper. For the most part, the same notation will be adopted. Thus for $0 \leqslant j \leqslant d-1$, the j-faces of a d-polytope P will be denoted by F_i^j ($i=1, ..., f_j(P)$), $\phi(P, F^j)$ will denote the interior angle of P at the face F^j, and $\phi_j(P)$ will denote the sum of the interior angles of P at all its j-faces. We require the following lemma:

(10) *Let Q be a convex $(d+1)$-polytope $(d \geqslant 2)$ and F^j be a j-face of Q $(0 \leqslant j \leqslant d-2)$. If $P_1, ..., P_s$ are the facets of Q containing F^j, then the sum of the interior angles of $P_1, ..., P_s$ at F^j is strictly less than* 1, *that is,*

$$\sum_{r=1}^{s} \phi(P_r, F^j) < 1. \tag{11}$$

The case $d=2$, $j=0$ of this lemma is well known and widely quoted. It is simple to prove using the formula for the area of a spherical polygon in terms of its interior angles (see, for example, [1, pp. 37–38]). For general values of j and d, two independent proofs will be published shortly (see [12] and [14]). Although we shall discuss the implications of (10) in the general case, in most proofs it will only be necessary to use the special value $j=d-2$. Here the statement can be established easily by applying the known case $d=2$, $j=0$ to the three-dimensional section of Q by a 3-flat normal to F^j $(j=d-2)$ and passing through a relative interior point of F^j.

Let the facets of the $(d+1)$-polytope Q be denoted by $P_1, ..., P_t$ $(t=f_d(Q))$. Then, if we sum (11) over all the j-faces F_i^j ($i=1, ..., f_j(Q)$) of Q, we obtain

$$\sum_{r=1}^{t} \phi_j(P_r) = \sum_{i=1}^{f_j(Q)} \sum_{r=1}^{t} \phi(P_r, F_i^j) < \sum_{i=1}^{f_j(Q)} 1 = f_j(Q), \tag{12}$$

where j is any integer satisfying $0 \leqslant j \leqslant d-2$, and $\phi_j(P_r)$ is the sum of the interior angles of the d-polytope P_r at its j-faces. (By definition $\phi(P_r, F^j)=0$ if F^j is not a face of P_r.)

Let $g_{jd}(Q)$ be the number of incidences of j-faces of Q with facets of Q, that is, the number of distinct ordered pairs (F_i^j, P_r) for which $F_i^j \subset P_r$. Since each P_r is incident with exactly $f_j(P_r)$ j-faces we obtain

$$g_{jd}(Q) = \sum_{r=1}^{t} f_j(P_r),$$

and since each j-face of a $(d+1)$-polytope is incident with at least $d+1-j$ d-faces [5, Theorem 3.1.7],

$$(d+1-j) f_j(Q) \leqslant g_{jd}(Q).$$

Combining these relations we obtain the inequality

$$(d+1-j) f_j(Q) \leqslant \sum_{r=1}^{t} f_j(P_r),$$

and hence, using (12):

(13) *For any $(d+1)$-polytope Q with facets $P_1, \ldots, P_t$, and for any integer j satisfying $0 \leqslant j \leqslant d-2$,*

$$\sum_{r=1}^{t} \phi_j(P_r) < \frac{1}{d+1-j} \sum_{r=1}^{t} f_j(P_r). \tag{14}$$

This may be expressed verbally as follows: the average interior angle at a j-face of a $(d+1)$-polytope (the average being taken over all incident pairs of j-faces and facets) has a value strictly less than $(d+1-j)^{-1}$. We shall be particularly concerned with the case $j=d-2$, and in this case the average angle is strictly less than $1/3$.

As an example, consider (13) with $d=2$, $j=0$. If the convex 3-polytope Q has p_n n-gons $(n=3, 4, \ldots)$ as facets, then, remembering that the sum of the interior angles at the vertices of an n-gon is $\frac{1}{2}(n-2)$, inequality (14) becomes

$$\sum \tfrac{1}{2}(n-2)\, p_n < \tfrac{1}{3} \sum n p_n,$$

which may be written
$$\sum (6-n)\, p_n > 0.$$

This implies
$$3p_3 + 2p_4 + p_5 > 0.$$

Although this inequality is considerably weaker than (1), it is still sufficient to imply statement (2). In higher dimensions it seems as though inequality (14) becomes relatively stronger, and in any case, no analogue of (1) is known for $d \geqslant 3$.

In the remaining sections of this paper we shall repeatedly apply the contrapositive form of (13) to establish that a given d-polytope P is a nonfacet. For this purpose it is necessary to estimate a lower bound for $\phi_j(P)$, and the methods of [11, § 3] enable us to do this. Let P be any d-polytope and x be any unit vector parallel to the d-flat containing P but not parallel to any proper face of P. If P_x is the $(d-1)$-polytope that arises by orthogonal projection of P on to a hyperplane normal to x, then P_x is called a *regular projection* of P. It can be shown [11, Theorem (10)] that the angle sum $\phi_j(P)$ is a positive convex combination of the numbers $\frac{1}{2}(f_j(P) - f_j(P_x))$ where P_x runs through all the regular projections of P. Hence (compare [11, (23)]):

$$\phi_j(P) \geqslant \tfrac{1}{2}(f_j(P) - \max_x f_j(P_x)). \tag{15}$$

Thus, if we write P_{rx} for the regular projection of P_r in direction x, we obtain from (14) and (15),

(16) *If $P_1, \ldots, P_t$ are the facets of a $(d+1)$-polytope Q and $0 \leqslant j \leqslant d-2$, then*

$$\tfrac{1}{2} \sum_{r=1}^{t} (f_j(P_r) - \max_x f_j(P_{rx})) < \frac{1}{d+1-j} \sum_{r=1}^{t} f_j(P_r), \tag{17}$$

or, equivalently,

$$\sum_{r=1}^{t} f_j(P_r) < \frac{d+1-j}{d-1-j} \sum_{r=1}^{t} \max_x f_j(P_{rx}). \tag{18}$$

In both (17) and (18), $\max_x f_j(P_{rx})$ means the maximum value of $f_j(P_{rx})$ as x ranges over all unit vectors parallel to the d-flat containing P_r, but to none of the proper faces of P_r.

In the case of an equifacetted polytope Q, with facets combinatorially equivalent to P, inequality (18) may be written in the simplified but slightly weaker form

$$f_j(P) < \frac{d+1-j}{d-1-j} m_j(P), \tag{19}$$

where $m_j(P)$ means the maximum number of j-faces that occur among all the regular projections of all the d-polytopes combinatorially equivalent to P. In § 4, inequality (19) will be used repeatedly in the particular case $d=3$, $j=1$:

(20) *If P is a 3-facet, then there exists a polytope P' combinatorially equivalent to P with the property that at least one of the regular projections P'_x of P' is a polygon with more than $\frac{1}{3}f_1(P)$ edges.*

Another useful form of (18) can be obtained as follows. Write $m_j(v, d)$ for the maximum number of j-faces of a polytope, the maximum being taken over all convex d-polytopes with v vertices. (This number is denoted by $\mu_j(v, d)$ in [5, Chapter 10].) Then since P_{rx} has at most $f_0(P_r)$ vertices, we deduce that

$$\max_x f_j(P_{rx}) \leqslant m_j(f_0(P_r), d-1),$$

and so, from (18),
$$\sum_{r=1}^{t} f_j(P_r) < \frac{d+1-j}{d-1-j} \sum_{r=1}^{t} m_j(f_0(P_r), d-1). \tag{21}$$

This form of (18) will be used in §§ 5, 6 and 7, usually with the value $j=d-2$. If Q is equifacetted, then from (21) we obtain a relation identical with (19) except that $m_j(f_0(P), d-1)$ is written in place of $m_j(P)$. This new inequality is weaker than (19), and in the three-dimensional case it is completely useless. Our discussion in §§ 4 and 7 will be concerned essentially with showing that certain polytopes P have the property that every regular projection P_x has *strictly* less than $f_0(P)$ vertices. In higher dimensions, however, (21) and deductions from it provide a very useful tool for finding nonfacets.

We conclude with a theorem that, under certain circumstances, enables us to prove that a polytope is a nonfacet by induction on the dimension.

(22) *Let P be a d-polytope dual to an equifacetted polytope, and let S be a $(d-1)$-polytope combinatorially equivalent to each vertex figure of P. If S is a nonfacet, then P is a nonfacet.*

Proof. Suppose that P is combinatorially equivalent to each facet of an equifacetted $(d+1)$-polytope Q. Let H be any hyperplane strictly separating one vertex F^0 of Q from the remainder, and let $R=H\cap Q$. Then R is a d-polytope whose facets are the intersections of H with the facets of Q meeting at F^0. Consequently R is equifacetted, and its facets are combinatorially equivalent to S. But this is a contradiction since S is a nonfacet, and so we deduce that P is a nonfacet. This proves (22).

4. Three-dimensional nonfacets

Let P be a 3-polytope and suppose that a certain regular projection P_x of P is an n-gon $(3\leqslant n\leqslant f_0(P))$. The vertices and edges of P_x are projections of some of the vertices and edges of P, and the incidences of these vertices and edges are preserved under the projection. We deduce that the inverse image of the boundary of P_x under the projection is a simple closed path of n edges on P, that is, a simple closed circuit of length n in the 1-skeleton, or graph, of P. Hence if we write $h(P)$ for the number of edges in the longest simple closed edge-path on P, the inequality $n\leqslant h(P)$ must hold. Consequently, in the notation of (19), $m_1(P)\leqslant h(P)$ and we deduce (see (19)):

(23) *If P is a* 3*-polytope and*

$$h(P)\leqslant \tfrac{1}{3}f_1(P), \tag{24}$$

then P is a nonfacet.

If P is simplicial (all its 2-faces are triangles) then $f_1(P)=3(f_0(P)-2)$, and (24) is equivalent to the statement that no simple closed edge-path on P contains more than $f_0(P)-2$ vertices.

In order to find 3-nonfacets, therefore, we look for 3-polytopes with short maximal simple closed edge-paths. In constructing the following examples we have made use of the methods and results of T. A. Brown [2], J. W. Moon and L. Moser [10] and B. Grünbaum and T. Motzkin [6] concerning edge-paths on 3-polytopes.

Example 1. A 3*-nonfacet with* 14 *vertices and* 24 *triangular* 2*-faces.*

Let X be a regular octahedron in E^3, and let X' be a polytope formed by adjoining to each 2-face of X a triangular pyramid as described in § 2 (see Figure 3). X' may be regarded as the convex hull of X and eight points, each of which lies beyond one 2-face of X and beneath all the rest. These eight points must be chosen in such a manner that the line segment joining any two of them intersects the interior of X. Then $f_0(X')=14$, $f_1(X')=36$ and $f_2(X')=24$. We shall now show that $h(X')=12$. Following the terminology

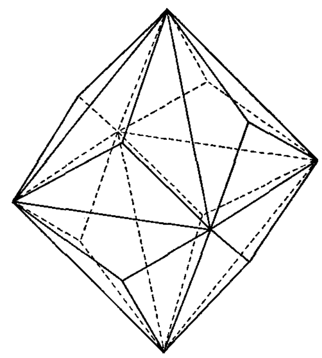


Fig. 3

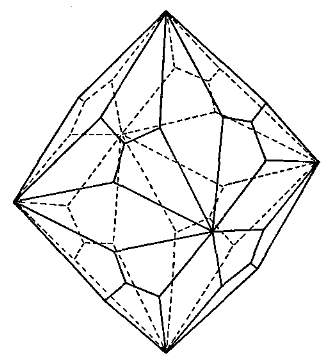


Fig. 4

of [10], we refer to the six vertices of X as 0th stage vertices, and to the eight remaining vertices of X' as 1st stage vertices. Since no path can join two 1st stage vertices without passing through an intermediate 0th stage vertex, and there are only six such vertices available, at most six 1st stage vertices can occur in any simple closed edge-path. We deduce that $h(X') \leqslant 12$. It is, in fact, easy to construct a simple closed edge-path containing 12 edges, and so $h(X') = 12$ as claimed. Thus

$$h(X') = 12 \leqslant \tfrac{1}{3} \cdot 36 = \tfrac{1}{3} f_1(X'),$$

and therefore, by (23), X' is a nonfacet.

The next two examples show the existence of 3-nonfacets with quadrilateral and pentagonal 2-faces.

Example 2. A 3-*nonfacet with* 38 *vertices and* 36 *quadrilateral* 2-*faces.*

Through each edge of X' (Example 1) choose a supporting plane which meets X' only in that edge. Each of these 36 planes bounds a closed half-space containing X', and the polytope X'' is defined as the intersection of these 36 half-spaces. If X' and the supporting planes are properly chosen, then it is easy to see (Figure 4) that $f_0(X'') = 38$, $f_1(X'') = 72$ and $f_2(X'') = 36$, the 2-faces being quadrilaterals. We shall now show that $h(X'') = 24$. Fourteen of the vertices of X'' are also vertices of X', and we call these 0th stage vertices and 1st stage vertices as before. The 24 remaining vertices of X'' will be called 2nd stage vertices. By an argument similar to that used in Example 1 we can show that a simple closed edge-path on X'' can contain at most 12 vertices of stages 0 and 1. Further, of the 24 2nd stage vertices, at most 12 can be included in a simple closed edge-path since, as before, no edge-path can join two 2nd stage vertices without passing through

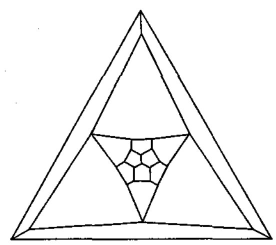


Fig. 5

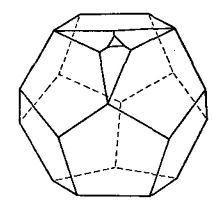


Fig. 6

an intermediate vertex of a lower stage, and there are only 12 such available. Thus $h(X'') \leqslant 24$. In fact, equality holds since it is easy to find a simple closed edge-path containing 24 vertices, and so

$$h(X'') = 24 \leqslant \tfrac{1}{3} \cdot 72 = \tfrac{1}{3} f_1(X''),$$

and X'' is a nonfacet by (23).

Example 3. A 3-nonfacet with 542 vertices and 360 pentagonal 2-faces.

To construct the polytope X^0 of this example we adjoin to each of the 24 2-faces of X' an affine image of a polytope R' defined below.

Figure 5 represents a planar 3-connected graph. Define R to be any 3-polytope whose 1-skeleton is combinatorially equivalent to this graph. Figure 5 can be thought of as representing a Schlegel diagram of such a polytope. That R exists follows immediately from Steinitz' Theorem [5, Theorem 13.1.1], but even without using this theorem it is easy to construct R by paring off three concurrent edges of a regular dodecahedron, and then cutting off one of the new vertices that are formed (see Figure 6).

It is apparent from either Figure 5 or 6 that $f_0(R) = 25$, $f_1(R) = 39$ and that R has sixteen 2-faces of which fifteen are pentagons and one is a triangle. Let z be any point beyond the triangular face of R and beneath the remaining 2-faces, and H be the plane containing the triangular face. Then if we apply a suitable p-transformation (see the proof of (9)) we obtain a polytope R' of the same combinatorial type as R, which includes the triangular face $H \cap R$ and is entirely included in the triangular pyramid with base $H \cap R$ and apex z. By a suitable affine transformation, the triangular face of R' can be made to coincide with any one of the triangular faces F^2 of X', and the point z can be made to lie beyond F^2 and beneath all the other 2-faces of X'. Thus we may adjoin this copy of R' to F^2. Repeating for each 2-face of X' we obtain X^0, and it is clear from the construction that $f_0(X^0) = 542$, $f_1(X^0) = 900$ and $f_2(X^0) = 360$, each 2-face being pentagonal. We shall now show that $h(X^0) \leqslant 276$.

Fourteen of the vertices of X^0 belong to X' and we call these 0th stage and 1st stage vertices as before. The remaining $24 \times 22 = 528$ vertices of X^0 will be called 2nd stage vertices, and these lie in 24 sets (called *clumps*) of 22, each clump consisting of all those vertices of X^0 which lie beyond a particular 2-face of X'. By an argument similar to that used in Example 1 we can show that a simple closed edge-path on X^0 contains at most 12 vertices of stages 0 and 1. Any edge-path on X^0 which connects two 2nd stage vertices belonging to different clumps must necessarily pass through an intermediate vertex of a lower stage, and there are only twelve such vertices available. Hence we deduce that no simple closed edge-path on X^0 contains 2nd stage vertices belonging to more than 12 different clumps, and therefore $h(X^0) \leqslant 12 + 12 \cdot 22 = 276$. But then

$$h(X^0) \leqslant 276 < \tfrac{1}{3} \cdot 900 = \tfrac{1}{3} f_1(X^0),$$

and so, by (23), X^0 is a nonfacet.

Example 4. A 3-*nonfacet which is simple.*

A d-polytope is said to be *simple* if exactly d facets are incident with each vertex. This example is of particular interest since no simple nonfacets are known in $d > 3$ dimensions (see §§ 7 and 8).(1)

Grünbaum and Motzkin have established [6, Theorem 1], for each even integer $n \geqslant 4$, the existence of a simple 3-polytope P_n with n vertices having the following property: every simple (open) edge-path on P_n contains less than $2n^\alpha$ vertices, where $\alpha = 1 - 2^{-19}$. This implies that $h(P_n) < 2n^\alpha$, and so, if we take n large enough,

$$h(P_n) < 2n^\alpha \leqslant \tfrac{1}{2}n = \tfrac{1}{3} \cdot \tfrac{3}{2}n = \tfrac{1}{3} f_1(P_n),$$

and P_n is a nonfacet by (23). The smallest value of n for which $2n^\alpha \leqslant \frac{1}{2}n$ is $2^{2^{20}} \simeq 10^{315,653}$. Using the slightly smaller value $\alpha = 1 - 2^{-16}$ mentioned in [6, p. 156] we can prove the existence of simple 3-nonfacets with as few as $2^{2^{17}}$ vertices, but this is still more than $10^{39,456}$, so these polytopes are extremely 'large'.

Example 5. To construct a 3-*nonfacet arbitrarily close to a given 3-polytope P.*

Suppose that we wish to find a nonfacet P_2 whose Hausdorff distance ϱ from P is less than ε $(\varepsilon > 0)$. First construct a simplicial 3-polytope P_1 with at least eight triangular faces such that $\varrho(P, P_1) < \frac{1}{2}\varepsilon$. Adjoin to each triangular face of P_1 a tetrahedron as in Example 1, and denote the resulting polytope by P_2. It is clear that this can be done so that $\varrho(P_1, P_2) < \frac{1}{2}\varepsilon$, and then $\varrho(P, P_2) < \varepsilon$. We shall now show that P_2 is a nonfacet.

(1) See the note at the end of this paper.

Since P_1 is simplicial we have

$$f_0(P_1) = n+2, \quad f_1(P_1) = 3n, \quad f_2(P_1) = 2n$$

for some integer $n \geqslant 4$, and

$$f_0(P_2) = 3n+2, \quad f_1(P_2) = 9n, \quad f_2(P_2) = 6n.$$

By an argument similar to that of Example 1 (in which the vertices of P_1 are used as 0th stage vertices and the remaining vertices of P_2 as 1st stage vertices), we can show that $h(P_2) \leqslant 2f_0(P_1) = 2(n+2)$. But, for $n \geqslant 4$,

$$h(P_2) \leqslant 2(n+2) \leqslant \tfrac{1}{3} \cdot 9n = \tfrac{1}{3} f_1(P_2),$$

and so P_2 is a nonfacet by (23). This example leads immediately to the following statement:

(25) *In the set* $\mathfrak{P}^3$ *of all* 3-*polytopes in* E^3, *the subset consisting of simplicial nonfacets is dense with respect to the Hausdorff metric.*

Example 5 can be modified (following Examples 2 and 3) to prove similar assertions regarding the density of nonfacets with quadrilateral 2-faces or with pentagonal 2-faces.

5. Regular polytopes which are nonfacets

In the following list of regular d-polytopes we have marked those known to be facets by a star (*) and those known to be nonfacets by a dagger (†).

d = *2:* triangle*; quadrilateral*; pentagon*; n-gon† ($n \geqslant 6$).

d = *3:* tetrahedron*; cube*; octahedron*; dodecahedron*; icosahedron.

d = *4:* 4-simplex*; 4-cube*; 4-crosspolytope; 24-cell; 120-cell; 600-cell.

d ⩾ *5:* d-simplex*; d-cube*; d-crosspolytope.

Here we prove two additional results, namely:

(26) *The d-crosspolytope* X^d *is a nonfacet if* $d \geqslant 6$.

(27) *The* 600-*cell is a* 4-*nonfacet.*

The first of these statements is of particular interest for the following reason. The equifacetted $(d+1)$-polytopes whose facets are d-simplexes and those whose facets are combinatorially equivalent to d-cubes have been widely studied and have many interesting properties. Statement (26) shows that there is no analogous theory for $(d+1)$-polytopes with facets combinatorially equivalent to d-crosspolytopes, at least for $d \geqslant 6$. We conjecture that X^4 and X^5 are also nonfacets but our methods do not seem to be powerful enough to prove this. More generally it seems plausible that every regular polytope in the above list that is not asterisked is a nonfacet, but whether or not this is so is certainly a difficult question. By (9) we know that if P is any regular d-polytope asterisked in the list, then there exist infinitely many combinatorial types of equifacetted $(d+1)$-polytopes Q whose facets are combinatorially equivalent to P. Recently a stronger result has been proved [13], namely that every $(d+1)$-polytope can be approximated arbitrarily closely by such a polytope Q.

We now prove (26). Since all the vertex figures of a d-crosspolytope are $(d-1)$-crosspolytopes, we deduce from (22) that in order to establish (26) it is sufficient to prove:

(28) *The* 6-*crosspolytope* X^6 *is a nonfacet.*

The proof is by contradiction. Assume that X^6 is a 6-facet, then by (19) with $d=6$ and $j=d-2=4$,

$$f_4(X^6) < 3m_4(X^6). \tag{29}$$

Now, for $0 \leqslant j \leqslant d-1$, $f_j(X^d) = 2^{j+1}\binom{d}{j+1}$ (see, for example, [5, § 4.3]), so that $f_0(X^6)=12$, $f_1(X^6)=60$ and $f_4(X^6)=192$. Let X^6_x be any regular projection of X^6. Since, for $0 \leqslant j \leqslant 4$, each j-face of X^6_x is the image of a j-face of X^6 under the projection, X^6_x is simplicial. By the solution of the Dehn–Sommerville equations for simplicial 5-polytopes [5, § 9.5], we deduce

$$f_4(X^6_x) = 2f_1(X^6_x) - 6f_0(X^6_x) + 12. \tag{30}$$

This equation will enable us to estimate $f_4(X^6_x)$ and therefore $m_4(X^6)$. The number of vertices of X^6_x is at least six (since it is a 5-polytope) and at most 12 $(=f_0(X^6))$. We distinguish three cases:

(a) $f_0(X^6_x)=12$. Then $f_1(X^6_x) \leqslant f_1(X^6)=60$, and so by (30),

$$f_4(X^6_x) \leqslant 2\cdot 60 - 6\cdot 12 + 12 = 60.$$

(b) $f_0(X^6_x)=11$. One vertex of X^6 projects into the interior of X^6_x, and the projections of the ten edges incident with this vertex do not lie on the boundary of X^6_x, and so are not edges of X^6_x. We deduce that $f_1(X^6_x) \leqslant 50$, and so by (30),

$$f_4(X^6_x) \leqslant 2\cdot 50 - 6\cdot 11 + 12 = 46.$$

(c) $6 \leqslant f_0(X^6_x) \leqslant 10$. At least two vertices of X^6 project into the interior of X^6_x and there are at least 19 edges incident with these vertices. The projections of these edges of X^6 are not edges of X^6_x, and so $f_1(X^6_x) \leqslant 41$. Hence by (30),

$$f_4(X^6_x) \leqslant 2\cdot 41 - 6\cdot 6 + 12 = 58.$$

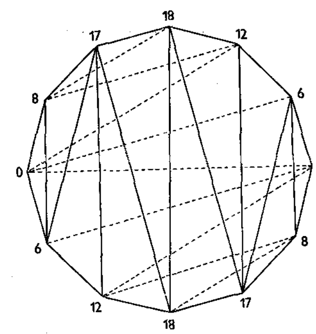


Fig. 7

In all three cases $f_4(X_x^6) \leqslant 60$, and so, if P is any polytope combinatorially equivalent to X^6, every regular projection of P has at most 60 4-faces. Therefore $m_4(X^6) \leqslant 60$. But

$$f_4(X^6) = 192 > 180 \geqslant 3m_4(X^6),$$

which contradicts (29). We deduce that (28) is true, and so (26) is established.

The argument we have used does not seem capable of modification to deal with the values $d=4$ and $d=5$, and so these two cases remain open.

We now prove (27). Let P be any polytope combinatorially equivalent to the regular 600-cell, so that $f_0(P)=120$, $f_1(P)=720$, $f_2(P)=1200$ and $f_3(P)=600$ (see [3, p. 292]). Since each 2-face of P is a triangle, every regular projection P_x of P is simplicial, and so $f_2(P_x)=2(f_0(P_x)-2)$. But $f_0(P_x) \leqslant f_0(P)=120$, therefore $f_2(P_x) \leqslant 2\cdot(120-2)=236$ and consequently $m_2(P) \leqslant 236$. But then

$$f_2(P) = 1200 > 3\cdot 236 \geqslant 3m_2(P),$$

so (19) does not hold, and P is a nonfacet. This establishes (27).

Consulting the list of regular polytopes given at the beginning of this section, it will be seen that there are still five undecided cases. The first is that of the icosahedron in E^3, and in this case we can obtain a partial result only. We can show that if K is a polytope *projectively* equivalent to the regular icosahedron, then every regular projection of K has at most ten edges. (As the proof of this statement is rather long it is omitted). Since $f_1(K)=30$, it follows from (18) that there exists no 4-polytope Q all of whose facets are projectively equivalent to the regular icosahedron. On the other hand, Dr. L. Danzer constructed an example of a polytope combinatorially equivalent to the regular icosahedron, one of whose regular projections is a regular 12-gon (This is represented in Figure 7. The dodecagonal projection is shown and the number by each vertex is the height of that vertex above the plane.) Thus (19) holds, and our criterion breaks down. We are therefore unable to decide whether the regular icosahedron is a facet or a nonfacet.

6. Nonfacets in $d \geqslant 4$ dimensions with a minimal number of vertices

A d-polytope P is said to be k-neighbourly if every k vertices of P are the vertices of a $(k-1)$-face of P. It can be shown that, unless P is a simplex, it cannot be k-neighbourly for $k > [\frac{1}{2}d]$, and that, when d is even, every $(\frac{1}{2}d)$-neighbourly d-polytope is simplicial. The cyclic polytopes are examples of $[\frac{1}{2}d]$-neighbourly d-polytopes. For proofs of these assertions, as well as other properties of neighbourly polytopes, the reader is referred to [5, § 4.7 and Chapter 7]. It is conjectured that within the class of all d-polytopes with v vertices, the simplicial $[\frac{1}{2}d]$-neighbourly d-polytopes with v vertices have the maximum possible number of j-faces for $1 \leqslant j \leqslant d-1$. It is for this reason that we consider such polytopes here; by maximising $f_j(P)$ it is simpler to establish that inequalities such as (19) and (21) fail to hold, and therefore P is a nonfacet. In addition there is the technical advantage that for simplicial $[\frac{1}{2}d]$-neighbourly d-polytopes P with v vertices, the numbers $f_j(P)$ are known explicitly as functions of v and d (see [5, Theorem 9.6.1] and (34) below).

Denote by $v(d)$ the smallest integer v for which the following statement is true: If Q is a $(d+1)$-polytope whose facets $P_1, \ldots, P_t$ $(t = f_d(Q))$ are simplicial $[\frac{1}{2}d]$-neighbourly d-polytopes, then $f_0(P_r) \leqslant v$ for at least one r $(1 \leqslant r \leqslant t)$. We put $v(d) = \infty$ if this statement is false for all v. We already know that $v(1) = 2$, $v(2) = 5$ and $v(3) \geqslant 6$ (since there exist 4-polytopes bounded entirely by octahedra having 6 vertices). Also, by (8), $v(d) \geqslant d+2$ for $d \geqslant 4$. From the definition of $v(d)$ we deduce:

(31) *A simplicial $[\frac{1}{2}d]$-neighbourly d-polytope P with $f_0(P) > v(d)$ is a nonfacet.*

Because of (31) it is of interest to determine upper bounds for $v(d)$, and we shall now do this. We conjecture that for all $d \geqslant 4$, $v(d) = d+2$. If this conjecture is true, then (31) will imply the existence of d-nonfacets with $d+3$ vertices. In fact we shall be able to establish that $v(d) = d+2$ only for $d = 6$, 8, 9 and 10; for all other values of $d \geqslant 4$, the problem of determining $v(d)$ remains open. We summarise our results in the next theorem.

(32) *Theorem. For $1 \leqslant d \leqslant 10$, and $d \neq 3$, the number $v(d)$ defined above satisfies the following equalities and inequalities:*

$d =$	1	2	4	5	6	7	8	9	10
$v(d) =$	2	5			8		10	11	12
$v(d) \leqslant$			7	9		10			

(33)

For $d = 3$ we have no information except that $v(3) \geqslant 6$. The cases $d \geqslant 11$ will be discussed later.

Proof. As in § 3, we write $m_j(v, d)$ for the maximum value of $f_j(P)$, the maximum being taken over all d-polytopes P with v vertices. Let $f_j(v, d)$ be the number of j-faces of a cyclic d-polytope with v vertices (or of any $[\frac{1}{2}d]$-neighbourly d-polytope with v vertices, see [5, § 9.6]). Then by [5, Theorem 9.6.1],

$$\left.\begin{aligned} f_j(v, 2n) &= \sum_{i=1}^{n} \frac{v}{i}\binom{v-i-1}{i-1}\binom{i}{j-i+1}, \\ \text{and} \qquad f_j(v, 2n+1) &= \sum_{i=0}^{n} \frac{j+2}{i+1}\binom{v-i-1}{i}\binom{i+1}{j-i+1}, \end{aligned}\right\} \tag{34}$$

for $0 \leqslant j \leqslant d-1$ and $d = 2n$ or $2n+1$ $(n \geqslant 1)$. In particular, for $j = d-1$ we obtain

$$\left.\begin{aligned} f_{2n-1}(v, 2n) &= \frac{v}{n}\binom{v-n-1}{n-1}, \\ f_{2n}(v, 2n+1) &= 2\binom{v-n-1}{n}. \end{aligned}\right\} \tag{35}$$

From (34) it follows that

$$f_j(v, 2n-1) = f_j(v, 2n+1) - \frac{j+2}{n+1}\binom{v-n-1}{n}\binom{n+1}{j-n+1}$$

for $1 \leqslant j \leqslant 2n-2$, and so, in particular,

$$f_{n-1}(v, 2n-1) = \binom{v}{n} - \binom{v-n-1}{n}, \tag{36}$$

since $f_{n-1}(v, 2n+1) = \binom{v}{n}$.

The *upper bound conjecture* [5, § 10.1] states that

$$m_j(v, d) = f_j(v, d) \tag{37}$$

for all j, v and d satisfying $0 < j < d < v$. Let us denote by UBC(j, v, d) the assertion that (37) holds. Then UBC(j, v, d) has been proved [5, Theorem 10.1.3] in the following cases:

$$\left.\begin{aligned} &\text{(a)}\ \ d \leqslant 8, \\ &\text{(b)}\ \ d = 2n, \quad j = d-1 \quad \text{and} \quad v \geqslant n^2 - 1, \quad \text{or} \\ &\qquad d = 2n+1, \quad j = d-1 \quad \text{and} \quad v \geqslant n^2 + 2n - 1, \\ &\text{(c)}\ \ d = 2n+1, \quad j = n \quad \text{and} \quad v \geqslant \tfrac{1}{2}(n^2 + 5n - 2), \end{aligned}\right\} \tag{38}$$

as well as for certain other values of j, v and d.

Let Q be a convex $(d+1)$-polytope whose facets $P_1, \ldots, P_t$ $(t=f_d(Q))$ are simplicial $[\frac{1}{2}d]$-neighbourly d-polytopes, and write $f_0(P_r)=v_r$ for $r=1, \ldots, t$. By (21) with $j=d-2$ we obtain

$$\sum_{r=1}^{t} f_{d-2}(v_r, d)=\sum_{r=1}^{t} f_{d-2}(P_r)<3\sum_{r=1}^{t} m_{d-2}(v_r, d-1).$$

Hence, for at least one value of r we have

$$f_{d-2}(v_r, d)<3m_{d-2}(v_r, d-1).$$

Choose such an r and write $v=v_r$. Each $(d-2)$-face of P_r is incident with two facets of P_r, and since each facet is a $(d-1)$-simplex, it is incident with exactly d $(d-2)$-faces. Thus,

$$2f_{d-2}(v, d)=g_{d-2,\, d-1}(P_r)=df_{d-1}(v, d),$$

and so, from these equalities and inequalities,

$$df_{d-1}(v, d)<6m_{d-2}(v, d-1).$$

If $d\leqslant 9$ then by (38a) UBC$(d-2, v, d-1)$ is true, that is

$$m_{d-2}(v, d-1)=f_{d-2}(v, d-1),$$

and therefore,
$$df_{d-1}(v, d)<6f_{d-2}(v, d-1). \tag{39}$$

We now consider separately the cases where d is even and where d is odd.

(a) $d=2n$ *is even*. Substituting in (39) from (35) we obtain

$$2n\cdot\frac{v}{n}\binom{v-n-1}{n-1}<6\cdot 2\binom{v-(n-1)-1}{n-1},$$

or, simplifying,
$$v(v-2n-5)+6n<0.$$

This inequality holds whenever $v=2n+1$ or $2n+2$. It also holds for $v=2n+3$ if and only if $n\leqslant 2$. It never holds for $v\geqslant 2n+4$. We deduce that

$$\min_{1\le r\le t} v_r\leqslant v\leqslant\begin{cases} 2n+3=d+3 & \text{if} \quad n=1,2,\\ 2n+2=d+2 & \text{if} \quad n=3,4.\end{cases}$$

These inequalities hold for every $(2n+1)$-polytope Q whose facets are simplicial n-neighbourly $2n$-polytopes, and so

$$v(2n)\leqslant\begin{cases}2n+3 & \text{if} \quad n=1,2,\\ 2n+2 & \text{if} \quad n=3,4.\end{cases}$$

Hence we obtain the entries in table (33) for $d=2$, 4, 6 and 8. (Equality holds when $d=2$, 6 and 8 since $v(2)\geqslant 5$, and $v(d)\geqslant d+2$ for $d\geqslant 4$ by (8).) The value of $v(10)$ cannot be determined in this manner since UBC(8, v, 9) has not been proved for all $v\geqslant 10$.

(b) $d=2n+1$ *is odd*. Substituting in (39) from (35) we obtain

$$(2n+1)\,2\binom{v-n-1}{n} < 6\,\frac{v}{n}\binom{v-n-1}{n-1},$$

or, simplifying,
$$v < 2n+3+3(n-1)^{-1}$$

for $n>1$. Hence for $n=2$ we obtain $v\leqslant 9$, for $n=3$ we obtain $v\leqslant 10$, and for $n=4$ we obtain $v\leqslant 11$. These values lead to the entries in table (33) corresponding to $d=5$, 7 and 9.

The value $v(1)=2$ is obvious and so we have proved all the assertions of the theorem except that $v(10)=12$. This will follow from:

(40) *If* $n\geqslant 2$, *then* $v(2n)\leqslant v(2n-1)+1$.

Proof. Let Q be a $(2n+1)$-polytope with facets $P_1, \ldots, P_t$ $(t=f_{2n}(Q))$ which are simplicial and n-neighbourly. Suppose that $f_0(P_r)=v_r$ for $1\leqslant r\leqslant t$. Let F^0 be a vertex of Q, and suppose that $P_1, \ldots, P_s$ are the facets of Q incident with F^0. Any vertex figure $Q(F^0)$ of Q at F^0 is a $2n$-polytope whose facets ($(2n-1)$-faces) are vertex figures $P_r(F^0)$ of P_r at F^0 $(r=1, \ldots, s)$. Now it is easy to show (see (6)) that each $P_r(F^0)$ is a simplicial $(n-1)$-neighbourly $(2n-1)$-polytope with v_r-1 vertices (since $n\geqslant 2$). Hence

$$v(2n-1)\geqslant \min_{1\leq r\leq s}(v_r-1)\geqslant \min_{1\leq r\leq t}(v_r-1)=\min_{1\leq r\leq t} v_r-1.$$

But Q is an arbitrary $(2n+1)$-polytope bounded by simplicial n-neighbourly facets. Therefore, by the definition of $v(d)$,

$$v(2n)\leqslant v(2n-1)+1,$$

and (40) is proved.

If we put $n=5$, statement (40) leads to the value $v(10)=12$ since we already know that $v(9)=11$, and the proof of Theorem (32) is completed.

Examination of the above proof shows that if UBC($d-1$, v, d) were known to be true for all $v>d\geqslant 10$, then our methods would establish that $v(d)=d+2$ for all $d\geqslant 8$. (We only need the upper bound conjecture for even d because of (40).) The conjecture stated earlier in this section would then be proved for $d=6$ and all $d\geqslant 8$, and only the cases $d=4$, 5 and 7 would remain open. Unfortunately our methods seem to be too weak to establish the result for these three values of d. (The case $d=3$, which is not included in the conjecture, is completely open.)

If, on the other hand, we use only the parts of the upper bound conjecture that have been proved, in particular (38b), then it is simple to establish, by reasoning similar to that used in the proof of (32), that

$$v(2n) \leqslant n^2 - 3, \quad \text{and} \quad v(2n+1) \leqslant n^2 - 2,$$

for all $n \geqslant 4$. Using the case $d = 2n$, $j = n-1$ of (21), (38c) and (36) we can show in a similar manner that

$$v(2n) \leqslant \tfrac{1}{2}(n^2 + 3n - 8)$$

for $n \geqslant 14$. The computations in this case are somewhat more involved, however, and to obtain the numerical results in the cases $n = 14$, 15, 16, 17 we had to consult tables of binomial coefficients [9].

If $d = 2n \geqslant 10$ and $2n+3 < v < n^2 - 1$, then we cannot prove that $m_{d-1}(v, d) = f_{d-1}(v, d)$, but at least we know that $m_{d-1}(v, d) < m_{d-1}(n^2-1, d) = f_{d-1}(n^2-1, d)$. Using this and similar facts one can improve the upper bounds for $v(d)$ slightly. This involves a considerable amount of computation, which we omit, and we merely summarise the results as follows:

For $11 \leqslant d \leqslant 30$, the table below gives the best inequalities that we have been able to prove for $v(d)$.

$d=$	11	12	13	14	15	16	17	18	19	20
$v(d) \leqslant$	21	22	30	31	41	42	54	55	69	70

$d=$	21	22	23	24	25	26	27	28	29	30
$v(d) \leqslant$	85	81	104	92	124	103	146	115	170	131

Also, for $n \geqslant 14$, $v(2n) \leqslant \frac{1}{2}(n^2 + 3n - 8)$, and for every $\varepsilon > 0$ there exists an $n_0(\varepsilon)$ such that for $n > n_0(\varepsilon)$, $v(2n) \leqslant \frac{1}{2} n(n + 3 + \log(1 - e^{-2}) + \varepsilon)$ where $\log(1 - e^{-2}) \simeq -0.1454$. For $n \geqslant 5$,

$$v(2n+1) < n^2 - n((n-2)\log \tfrac{1}{3}(2n+1) - 1)(n + \log \tfrac{1}{3}(2n+1))^{-1} = n^2 - n \log n + o(n \log n).$$

We conclude this section with a theorem which is the analogue of (25) for $d \geqslant 4$ dimensions. It is included here because the method of proof is very similar to that of Theorem (32).

(41) *Theorem. In the set* $\mathcal{P}^d$ *of all d-polytopes in* E^d $(d \geqslant 4)$, *the subset consisting of simplicial nonfacets is dense with respect to the Hausdorff metric.*

Proof. Since the set of all simplicial d-polytopes in E^d is dense in $\mathcal{P}^d$, it suffices to show that for every simplicial d-polytope $P \subset E^d$ and for any given $\varepsilon > 0$ there exists a simplicial d-nonfacet $P_1 \subset E^d$ such that $\varrho(P, P_1) < \varepsilon$.

As in § 2 we denote by $C(v, d)$ a cyclic d-polytope with v vertices. Using the procedure described in the proof of (9) we adjoint to one facet of P a projective copy of $C(v, d)$. This copy may be chosen in such a way that the resulting polytope, which we denote by $P(v)$, satisfies the condition $\varrho(P, P(v)) < \varepsilon$. $P(v)$ is clearly simplicial. If $P(v)$ is a facet, then by (21) with $j = d-2$,

$$f_{d-2}(P(v)) < 3m_{d-2}(f_0(P(v)), d-1). \tag{42}$$

Writing $f_0(P) = a$ and $f_{d-1}(P) = b$, we obtain

$$f_0(P(v)) = f_0(C(v, d)) + f_0(P) - d = v + a - d,$$

and

$$\begin{aligned} f_{d-2}(P(v)) &= \tfrac{1}{2} d f_{d-1}(P(v)) \\ &= \tfrac{1}{2} d(f_{d-1}(C(v, d)) + f_{d-1}(P) - 2) \\ &= \tfrac{1}{2} d(f_{d-1}(v, d) + b - 2). \end{aligned}$$

Also, if v is large enough (say $v > \frac{1}{4} d^2$), by (38b) $\mathrm{UBC}(d-2, f_0(P(v)), d-1)$ holds and so

$$m_{d-2}(f_0(P(v)), d-1) = f_{d-2}(f_0(P(v)), d-1).$$

Substituting these values in (42) we obtain the equivalent inequality

$$d(f_{d-1}(v, d) + b - 2) < 6f_{d-2}(v + a - d, d-1). \tag{43}$$

Using the expressions for $f_{d-1}(v, d)$ given in (35) we observe that if $d = 2n$ is even, then the left side of (43) is a polynomial of degree n in v with a positive leading coefficient, and the right side is a polynomial of degree $n-1$. On the other hand, if $d = 2n+1$ is odd, then both sides of (43) are polynomials of degree n in v, the leading coefficient on the left being $2(2n+1)(n!)^{-1}$, and that on the right being $6(n!)^{-1}$. But

$$2(2n+1)(n!)^{-1} > 6(n!)^{-1}$$

since $n \geq 2$. Therefore in either case, if v is chosen sufficiently large, say $v = v_1$, then (43) fails to hold. We deduce that $P(v_1)$ is a nonfacet, so that $P_1 = P(v_1)$ has the required properties, and Theorem (41) is proved.

7. Nonfacets with a small number of j-faces

In the last section we considered the problem of constructing d-nonfacets which were 'minimal' in the sense that they had the smallest possible number of vertices. On the other hand, these nonfacets P usually had the (conjectured) maximum possible number of j-faces for the given number of vertices, and our proofs depended essentially on the fact that $f_{d-1}(P)$ was relatively large. We begin this section by constructing simplicial d-nonfacets P with a large number of vertices, for which the ratio $f_j(P)/f_0(P)$ is arbitrarily close to $\binom{d}{j}$ (for $1 \leqslant j \leqslant d-2$) and to $d-1$ (for $j=d-1$). These ratios are, in a sense, the smallest possible in view of the *lower bound conjecture* (LBC) which states the following:

If P is a simplicial d-polytope with v vertices, then

$$f_j(P) \geqslant \binom{d}{j} f_0(P) - \binom{d+1}{j+1} j \quad \text{for} \quad 1 \leqslant j \leqslant d-2,$$

and

$$f_{d-1}(P) \geqslant (d-1) f_0(P) - (d+1)(d-2). \tag{44}$$

(See [5, § 10.2] for the history of this conjecture and an account of those cases for which it has been proved.) Thus the LBC implies that for any $\varepsilon > 0$,

$$f_j(P)/f_0(P) \geqslant \binom{d}{j} - \varepsilon \quad \text{for} \quad 1 \leqslant j \leqslant d-2,$$

and

$$f_{d-1}(P)/f_0(P) \geqslant d-1-\varepsilon,$$

whenever P is simplicial and $f_0(P)$ is sufficiently large.

A d-polytope P is said to be of type $A(k)$ $(k \geqslant d+1)$ if it is simplicial, has k vertices, and equality holds in each of the relations (44). That d-polytopes of type $A(k)$ exist for all $d \geqslant 2$ and all $k \geqslant d+1$ is clear from the following construction. Firstly it is immediate that a d-simplex is of type $A(d+1)$. Secondly, if P is a d-polytope of type $A(k)$ and $d \geqslant 2$, then the convex hull of P and a point which lies beyond one of its facets and beneath all the others is a d-polytope of type $A(k+1)$. (In other words, adjoining a d-simplex to any facet of a d-polytope $(d \geqslant 2)$ of type $A(k)$ in the manner described in the proof of (9) produces a polytope of type $A(k+1)$.)

(45) *Theorem. For each $d \geqslant 4$ there exists an infinite sequence of simplicial d-nonfacets $\{P_k : d+1 \leqslant k < \infty\}$ such that*

$$\lim_{k \to \infty} f_0(P_k) = \infty,$$

and
$$\lim_{k\to\infty}(f_j(P_k)/f_0(P_k))=\binom{d}{j}\quad\text{for}\quad 0\leqslant j\leqslant d-2,$$
$$\lim_{k\to\infty}(f_{d-1}(P_k)/f_0(P_k))=d-1.$$

Proof. Let P be a simplicial $[\frac{1}{2}d]$-neighbourly d-polytope with v vertices (an appropriate value of v will be determined later), and let $m=f_{d-1}(P)$ be the number of facets of P. Construct a simplicial polytope P_k by adjoining d-polytopes $A_1, \ldots, A_m$, each of type $A(k)$, successively to all the m facets of P in the manner described in § 2. Then, using T^{d-1} to denote the $(d-1)$-simplex,

$$f_j(P_k)=f_j(P)+m(f_j(A_1)-f_j(T^{d-1}))=f_j(P)+m\left(\binom{d}{j}k-\binom{d+1}{j+1}j-\binom{d}{j+1}\right)$$

for $0\leqslant j\leqslant d-2$, and

$$f_{d-1}(P_k)=m(f_{d-1}(A_1)-1)=m((d-1)k-(d+1)(d-2)-1).$$

In particular $f_0(P_k)=v+m(k-d)$. Thus, for any fixed value of v,

$$\lim_{k\to\infty}(f_j(P_k)/f_0(P_k))=\binom{d}{j}\quad\text{for}\quad 0\leqslant j\leqslant d-2,$$

and
$$\lim_{k\to\infty}(f_{d-1}(P_k)/f_0(P_k))=d-1.$$

To complete the proof of (45) we need only show that if v is chosen appropriately, each polytope P_k is a nonfacet. We use the notation vert K for the set of vertices of a polytope K, and write

$$V=\operatorname{vert}P,$$
$$W_i=\operatorname{vert}A_i\setminus\operatorname{vert}P\quad(i=1,\ldots,m),$$

so that vert P_k is the disjoint union of the sets $V, W_1, \ldots, W_m$. It is clear that any edge-path connecting a point of W_i to a point of W_j $(j\neq i)$ must contain a point of V, in other words, V separates the sets W_i in the graph of P_k.

Let P_{kx} be a regular projection of P_k. Each vertex or edge of P_{kx} is the projection of a unique vertex or edge of P_k, and we denote by $V_x, W_{1x}, \ldots, W_{mx}$ the sets of vertices of P_x which are images under the projection of vertices in $V, W_1, \ldots, W_m$ respectively. Then V_x separates the sets W_{ix} in the graph of P_{kx}. Let v, v_x denote the number of vertices in the sets V, V_x respectively, and let w_x be the number of sets W_{ix} which are not empty. Then, by [5, Theorem 11.4.1] or [7, p. 1040],

$$w_x \begin{cases} \leqslant m_{d-2}(v_x, d-1) \leqslant m_{d-2}(v, d-1) & \text{if} \quad v_x \geqslant d, \\ \leqslant 2 & \text{if} \quad v_x = d-1, \\ = 1 & \text{if} \quad v_x < d-1, \end{cases}$$

using the notation $m_j(v, d)$ introduced in the proof of (32). Since each set W_{ix} contains at most $k-d$ vertices of P_{kx}, it follows that

$$f_0(P_{kx}) \leqslant v_x + (k-d)\, w_x \leqslant v + (k-d)\, m_{d-2}(v, d-1).$$

This inequality holds not only for every regular projection P_{kx} of the polytope P_k, but, by the same argument, for every regular projection of any d-polytope combinatorially equivalent to P_k. Hence we deduce that

$$m_0(P_k) \leqslant v + (k-d)\, m_{d-2}(v, d-1).$$

In order to show that P_k is a nonfacet it is sufficient, by (19), to establish that

$$(d+1)\, m_0(P_k) \leqslant (d-1)\, f_0(P_k), \tag{46}$$

or, substituting the values we have obtained above,

$$(d+1)\,(v + (k-d)\, m_{d-2}(v, d-1)) \leqslant (d-1)\,(v + m(k-d)) = (d-1)\,(v + (k-d)\, f_{d-1}(v, d)). \tag{47}$$

By (38b) we may choose v sufficiently large for UBC$(d-2, v, d-1)$ to hold. Then $m_{d-2}(v, d-1) = f_{d-2}(v, d-1)$ and inequality (47) is equivalent to

$$\frac{2v}{(k-d)(d+1)} + f_{d-2}(v, d-1) \leqslant \frac{d-1}{d+1} f_{d-1}(v, d). \tag{48}$$

We shall show that this inequality holds for sufficiently large v in the special case $k = d+1$; it will then hold generally since the left side is a decreasing function of k for fixed v and $k > d$. Substitute in (48) the values of $f_{d-2}(v, d-1)$ and $f_{d-1}(v, d)$ from (35). If d is even ($d = 2n \geqslant 4$) then the left side is a polynomial of degree $n-1$ in v and the right side is a polynomial of degree n with positive leading coefficient. If d is odd ($d = 2n+1 \geqslant 5$) then both sides of inequality (48) are polynomials of degree n in v, the coefficient of v^n on the left being $(n!)^{-1}$ and on the right $2(d-1)/(d+1)n!$. Since $d \geqslant 4$, $(n!)^{-1} < 2(d-1)/(d+1)n!$. In both cases we conclude that inequality (48) holds if v is sufficiently large. This implies (46) and so P_k is a nonfacet. (Closer inspection shows that (47) always holds if $d = 2n \geqslant 12$ and $v \geqslant n^2 - 2$, or $d = 2n+1 \geqslant 13$ and $v \geqslant n^2 - 1$; slightly larger values of v are required if $d \leqslant 11$.) This completes the proof of Theorem (45).

The method of constructing nonfacets P_k used in the above proof may be modified as follows. Instead of adjoining to the facets of P d-polytopes $A_1, \ldots, A_m$ of type $A(k)$, we may adjoin d-polytopes $B_1, \ldots, B_m$ of *any* combinatorial type, so long as one facet (at least) of each B_i is a $(d-1)$-simplex. Then an argument similar to that given above will establish that if all the B_i have the same number of vertices and v is sufficiently large, the resulting polytope is a nonfacet.

For $d=4$ we get sharper results. Let Q be a 5-polytope with facets $P_1, \ldots, P_t$ $(t=f_4(Q))$ and let $f_0(P_r)=v_r$ for $1 \leqslant r \leqslant t$. Then by (18) with $j=1, 2$ and $d=4$ we obtain

$$\left.\begin{aligned} \sum_{r=1}^{t} f_1(P_r) &< 6\sum_{r=1}^{t}(v_r-2), \\ \sum_{r=1}^{t} f_2(P_r) &< 6\sum_{r=1}^{t}(v_r-2). \end{aligned}\right\} \tag{49}$$

If all the facets of each P_r are simplexes, then we can substitute $f_2(P_r)=2(f_1(P_r)-v_r)$ in (49) (see [5, §§ 9.5 and 10.1]) and obtain, after simplification,

$$\sum_{r=1}^{t} f_1(P_r) < \sum_{r=1}^{t}(4v_r-6).$$

This implies the following theorem (which includes (27) as a special case):

(50) *Every simplicial* 4-*polytope with* v *vertices and more than* $4v-7$ *edges is a nonfacet.*

Since no simplicial 4-polytopes with v vertices and less than $4v-10$ edges are known, statement (50) is rather strong. Further, using the techniques described earlier, it is possible to find simplicial 4-nonfacets with v vertices and as few as $4v-10$ edges. For example, a simplicial 4-nonfacet with 22 vertices and 78 edges may be constructed as follows:

Let P_0 be a simplicial 4-polytope of type $A(8)$, so that $f_0(P_0)=8$, $f_1(P_0)=22$, $f_2(P_0)=28$ and $f_3(P_0)=14$. Let P be the result of adjoining successively 14 4-simplexes, one to each facet of P_0, in the manner described in § 2. Then P is a simplicial 4-polytope of type $A(22)$ with $f_0(P)=22$, $f_1(P)=78$, $f_2(P)=112$ and $f_3(P)=56$. We shall now show that P is a nonfacet. Removing the eight vertices of P_0 from the graph of P completely separates the remaining 14 vertices of P. It follows, as in the proof of (45), that a regular projection of P (or of any polytope combinatorially equivalent to P) has at most $8+m_2(8,3)=8+12=20$ vertices and therefore $m_0(P) \leqslant 20$. But all the regular projections of a simplicial 4-polytope are simplicial 3-polytopes, so that

$$m_2(P) = 2(m_0(P)-2) \leqslant 2(20-2) = 36 < \tfrac{1}{3}\cdot 112 = \tfrac{1}{3}f_2(P),$$

and we conclude that P is a nonfacet by (19).

Using similar methods we can construct 4-nonfacets of type $A(k)$ for every $k \geq 22$. (If $k = 8 + 14s + r \geq 22$, where $s \geq 1$ and $0 \leq r \leq 13$, then a 4-nonfacet P_k of type $A(k)$ can be obtained, for example, by adjoining r 4-polytopes of type $A(s+5)$ and $14-r$ 4-polytopes of type $A(s+4)$ to the 14 facets of a 4-polytope of type $A(8)$.)

As remarked earlier, we have been unable to find any examples of simple nonfacets in $d \geq 4$ dimensions. However we can find d-nonfacets which are simple at most of their vertices, that is to say, most of their vertices are included in exactly d facets. Such polytopes may be constructed as follows:

Let P be a (fixed) simplicial $[\frac{1}{2}d]$-neighbourly d-polytope ($d \geq 4$) with v vertices and $m = f_{d-1}(v, d)$ facets. Let R_k be a simple d-polytope with k facets, and R'_k be a d-polytope obtained by truncating R_k at one of its vertices. Then R'_k is simple and has $k+1$ facets, one of which is a simplex. Let Q_k be a polytope that results from adjoining successively, to the m facets of P, projective images of R'_k as described in § 2. Then Q_k is a d-nonfacet if v is chosen large enough. (The required size of v does not depend upon k but is a function of d only.) Q_k has mk facets and is simple at all but v of its vertices, these exceptional vertices being the vertices of P. In fact, however large k may be, at most

$$(d-1)\, m_{d-2}(v-1, d-1)$$

facets of Q_k are incident at each exceptional vertex.

In particular, if we choose R_k to be a polytope dual to a simplicial $[\frac{1}{2}d]$-neighbourly d-polytope with k vertices, we obtain

$$f_0(Q_k) = v + m(f_{d-1}(k, d) - 1),$$

$$f_j(Q_k) = f_j(v, d) + m f_{d-1-j}(k, d) \quad \text{for} \quad 1 \leq j \leq d-2,$$

and $$f_{d-1}(Q_k) = mk.$$

Hence, by taking k large compared with v, it is possible to find a d-nonfacet Q_k which is simple at all but a small fraction of its vertices and for which the ratio $f_j(Q_k)/f_{d-1}(Q_k)$ $(0 \leq j \leq d-2)$ is extremely large.

8. Remarks and open problems

In the previous sections we have already mentioned a number of open problems concerning the characterisation of facets and nonfacets. Here we collect these together and indicate possible extensions and generalisations of our results. We remark, however, that many of these questions are likely to remain unanswered until the discovery of techniques more powerful than those based on angle sums that have been used here.

We begin with the problem discussed in § 6:

(i) *What is the smallest number of vertices possessed by a d-nonfacet, for each value of* $d \geq 3$*?*

We conjectured that the answer to this question is $d+3$, and could prove this if the upper bound conjecture were true, except in the cases $d=3$, 4, 5 and 7. The case $d=3$ deserves particular mention. Of the seven combinatorial types of 3-polytopes with six vertices [5, Figure 6.3.1], only three are known to be facets; it is unknown whether the other four are facets or nonfacets. Consequently, although we have been unable to find 3-nonfacets with less than 14 vertices (§ 4, Example 1), it is possible that 3-nonfacets with as few as six vertices may exist.

Statements (25) and (41) suggest the following question:

(ii) *Is the set of all d-facets in* E^d *dense in the set* $\mathcal{P}^d$ *of all d-polytopes in* E^d*?*

Clearly (2) implies that the answer is in the negative for $d=2$, and we guess that it is also in the negative for all $d>2$. However, this will probably not be established using the methods of this paper.

Other problems arise if we restrict attention to certain classes of polytopes. Consider, for example, simplicial polytopes. For each $d \geq 2$ we know one example of a simplicial d-facet with $d+3$ vertices, namely the pentagon if $d=2$, and the direct sum $C^2 \oplus T^{d-2}$ of a square and a $(d-2)$-simplex if $d \geq 3$ (see (7) (iii)). We know of no simplicial d-facets with more than $d+3$ vertices, so it is natural to ask the following question:

(iii) *For every value of* $d \geq 3$*, does there exist a finite number* $s(d)$ *with the property that every simplicial d-polytope with* $s(d)$ *or more vertices is a d-nonfacet?*

In particular, is $s(d) = d+4$ *such a number?*

Again the case $d=3$ deserves mention. On the one hand it is possible that $s(3)$ may not exist, so that there are simplicial 3-facets with arbitrarily many vertices. On the other hand, if, in the notation of § 6, $v(3)=6$, then this would imply that $s(3)=7$. Both extreme possibilities are compatible with the results we have obtained so far.

We know even less about the class of simple nonfacets. We have asked (see § 4 and § 7) the following question:

(iv) *Are there any simple d-nonfacets for* $d \geq 4$*?*

The existence of simple 3-nonfacets is shown in § 4. Generally we guess that the answer to (iv) is in the affirmative for all $d \geq 4$, but it seems that the construction of such polytopes will be extremely difficult until more powerful techniques become available.(1)

A related problem (suggested by the referee) is as follows:

(1) See the note at the end of this paper.

(v) *If P is a simple d-facet, does there always exist an equifacetted simple $(d+1)$-polytope whose facets are combinatorially equivalent to P?*

The answer is obviously in the affirmative in the case $d=2$.

The discussion of § 2 suggests the following question:

(vi) *Are there any facets which are not superfacets?*

Again it seems likely that the answer to this question is in the affirmative, though so far we have been unable to discover any facet which is not also a superfacet.

Finally we mention a generalisation of the concept of a facet which may be of some interest. Define a d-polytope P to be a (j, d)-*face* $(j \geqslant 1)$ if there exists a $(j+d)$-polytope Q all of whose d-faces are combinatorially equivalent to P. Thus the property of being a $(1, d)$-face is the same as that of being a d-facet. Many results of this paper concerning facets lead to analogous questions concerning (j, d)-faces. In particular, the following seem to be of interest:

(vii) *If $j \geqslant 2$, every (j, d)-face is obviously a $(j-1, d)$-face. Formulate necessary and sufficient conditions for the converse statement to be true.*

(viii) *d-simplexes and d-cubes are (j, d)-faces for all $j \geqslant 1$. Are there any other polytopes having this same property?*

(ix) *The 2-faces of a regular 120-cell are pentagons, so that the pentagon is a $(2, 2)$-face. Is it a $(j, 2)$-face for any $j > 2$?*

(x) *Is the 3-octahedron a $(2, 3)$-face?*

The reader will be able to formulate many similar problems for himself in what is, at present, a completely unexplored field of research.

Added in proof. Recently, using an extension of the methods described in § 4, D. W. Bamette has established the existence of an enumerable infinity of simple 4-nonfacets. Details will be published.

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