Facets and Reformulations for Solving Production Planning with Changeover Costs

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OR 312-95 June 1995

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Abstract

We study a scheduling problem with changeover costs and capacity constraints. The problem is NP-complete and combinatorial algorithms for solving it have not performed well. We identify a general class of facets that subsumes as special cases some known facets from the literature. We also develop a cutting plane based procedure and reformulation for the problem, and obtain optimal solutions to problem instances with up to 1200 integer variables without resorting to branch and bound procedures.

A key issue in scheduling is the effective allocation of shared resources to multiple products, for instance, for a facility that incurs a changeover cost whenever it switches production from one product to another. For example, in producing printed circuit boards, a plant might include a machine that places a set of components on a board. Typically, the plant will produce different types of boards, each with a different set of components. If it switches from one product to another, the machine needs to change over to a new set of tools and thus incurs a fixed cost. Each time it produces, the machine might also incur an additional set up cost for placing components. The resource allocation problem in this product cycling model must trade off changeover and set up costs against production and inventory holding costs.

We study the polyhedral structure of a dynamic, deterministic version of the problem. This problem is NP-hard. As a result, the running time of all solution methods increases exponentially with the number of time periods and products. In the next section, we present an integer programming formulation of the problem. We then describe valid inequalities and facets for the problem, solve the separation problem, and present computational results for problems with up to 4 products.

Magnanti and Vachani (1990), who give many further references to the literature on the problem, developed a solution technique based on cutting planes for the constant capacity case. This approach performed well on problems having up to 300 integer variables. Our results generalize those of Magnanti and Vachani by providing a more extensive set of valid inequalities and facets for the problem. We are able to solve larger problems to optimality with up to 1200 integer variables. For single item versions of these problems, the linear programming gaps (i.e., ratio of 100x(IP value - LP value)/IP value for a 'natural' formulation of the problem is between 75% and 83% and for multi item problems, the gaps are between 6% and 20%. In each case, we are able to eliminate this gap completely by adding valid inequalities.

Several researchers have used a polyhedral cutting plane approach for the lotsizing problem with start up costs. Wolsey (1989) used a cutting plane method that performed well for an uncapacitated version of our model. Van Hoesel, Wagelmans and Wolsey (1994) described the convex hull of this uncapacitated model. Van Hoesel (1991) and Van Hoesel and Kolen (1993) studied a capacitated version of the problem with start-up costs, but without setup costs, which they call the discrete lot sizing problem (DLSP) with start up costs. They introduced a class of strong valid inequalities. Our results differ from those in Van Hoesel and Kolen in two ways (i) we consider set up as well as changeover costs, and (ii) we derive valid inequalities and facets with arbitrary integer coefficients whereas Van Hoesel and Kolen consider valid inequalities and facets with 0-1 coefficients. Van Hoesel and Kolen (1994) also provide a complete linear description of DLSP with start up costs and no set up costs using an enhanced set of variables.

Pochet and Wolsey (1994) provide a detailed survey of lot sizing algorithms and reformulations. They provide many citations to the literature which we will not repeat. They classify the problems into five categories (i) uncapacitated lot-sizing (ii) capacitated lot-sizing (iii) lot-sizing with start-ups (iv) discrete lot-sizing and (v) multi-level lot-sizing. In this taxonomy, the model we investigate is a discrete lot sizing problem.

1. Problem Formulation

We consider a single machine, multi-product, production planning model. Let T denote the finite time horizon over which the facility is scheduled, P the number of products, d_i^p the demand in period i, and n_p the total demand for item p. We assume a constant capacity and follow a discrete production policy, i.e, we either do not produce at all or produce to capacity in each time period. This policy is reasonable when it is expensive to run the facility at less than full capacity, or when demand is high and the facility is capacity constrained. It is also easily implemented. As shown in Magnanti and Vachani (1990), without loss of generality we can assume that capacity in each period is 1 unit and that demand is either 0 or 1.

We assume that the relevant costs for each product p in period i are the changeover cost $\mathbf{F}_{pi}$, the fixed cost or the setup cost $\mathbf{f_{pi}}$, and the inventory holding cost $\mathbf{g_{pi}}$. Let z_{pi}, y_{pi} and w_{pi} denote the 0-1 changeover, setup, and production variables. We assume that demands are nonnegative, initial production $w_{p0} = 0$, and no starting or ending inventory. The Changeover Cost Scheduling Problem (CSP) can be formulated as follows:

$$\textbf{(CSP)}\ \text{Minimize } U = S_{p=1}{}^{P}\, S_{i=1}{}^{T}\{g_{pi}w_{pi} + f_{pi}y_{pi} + F_{pi}z_{pi}\} \qquad (1)$$

subject to

$S_{j=1}{}^{i}w_{pj} \geq S_{j=1}{}^{i}d_j^{\,p}$	for all p,i	(2)
$S_{j=1}{}^{T}w_{pj} = n_p$	for all p	(3)
$w_{pi}\text{-}y_{pi} \leq 0$	for all p,i	(4)
$z_{pi}+y_{p,i-1} - y_{pi} \geq 0$	for all p,i	(5)
$S_{p=1}{}^{P}y_{pi} \leq 1$	for all i	(6)
$w_{pi} \leq 1,\ y_{pi} \leq 1,\ z_{pi} \leq 1$	for all p,i	(7)
$w_{pi},\ y_{pi},\ z_{pi} \geq 0$	and integer	(8).

Let **CSP(L)** denote the linear programming relaxation of **CSP** for this problem. Constraints (2) and (3) are the demand constraints. Constraints (4) ensure that we can produce only if the machine is set up. Constraints (5) ensure that if the machine is set up for product p in period i (i.e., $y_{pi} = 1$) but not in period i-1, then the changeover variable z_{pi} equals 1. Constraints (6) ensure that we produce only one product in any period. Magnanti and Vachani (1990) give a detailed formulation with all the underlying assumptions. They also show how to view this problem as a specially structured network design problem.

To facilitate our discussion, we focus on the single product version of the problem. Although a dynamic programming algorithm will solve this problem in polynomial time, we have studied valid inequalities for the problem. There were two motivations for doing so. First, generalizations of these inequalities apply to the multi-product problem (which is NP-complete) or for problem settings with arbitrary demands and varying production capacity over time. Second, the inequalities provide us with a better understanding of the polyhedral structure of the problem.

Let **SCSP** denote the single product version of the problem and **SCSP(L)** the linear programming relaxation of **SCSP**. Since this model has only one product, we drop the subscript and superscript p. Let $d(i,k) = S_{t=i}^{k} d_t$ denote the total demand in periods i through k, and t_k denote the kth time period in which demand $d_j = 1$. If k<i, we define $d(i,k) = 0$. Since we do not produce in periods after t_n, we assume that $t_n = T$. The constraint $S_{i=1}^{t_k} w_i \geq S_{i=1}^{t_k} d_i$ implies the constraints $S_{i=1}^{t} w_i \geq S_{i=1}^{t} d_i$ for $t = t_k+1$ through $t_{k+1}-1$ because $d_i = 0$ between these periods. Therefore, we can drop the demand constraints for all periods except the periods $t_1, t_2, \ldots, t_n$. If the demand equals 1 in periods 1 through j, then $y_i = w_i = 1$ for all $1 \leq i \leq j$. Consequently, the problem reduces to finding a schedule for periods j+1 through T. Therefore, to exclude uninteresting cases, we assume that $t_1 \geq 2$.

2. Valid Inequalities

We consider a general class of valid inequalities for **SCSP.** To motivate the discussion, consider the following example. Assume that the costs $F_i = F$, $f_i = 0$ and $g_i = 0$ are constants and that $t_1 = T$. Then $z_1 = 1/T$, $y_i = w_i = 1/T$ for all i is an optimal fractional solution for **SCSP(L)**. This solution has a fixed cost of F/T instead of the optimal integer cost of F. If we let T approach infinity, then the gap (ratio) between the optimal objective values of **SCSP** and **SCSP(L)** becomes arbitrarily large. Note that since we must produce at least once up to period t_1, we must turn on the machine at least once before t_1. Therefore, $\sum_{i=1}^{t1} z_i \geq 1$ is a valid inequality that cuts off the fractional solution. We obtain this inequality by replacing the variable w_i by z_i in the demand constraint $\sum_{i=1}^{t_1} w_i \geq 1$. In general, to develop valid inequalities we will substitute values of z_i and/or y_i for w_i in the demand constraints.

Suppose we replace any single term w_i by z_i in the inequality $w_1+w_2+\ldots+w_{t1} \geq 1$. The following feasible solution violates the inequality: turn the machine on in period i-1, keep it on for the next period and produce in period i. To satisfy demand beyond t_1, we produce in periods after t_1. However, if we replace w_{i-1} by z_{i-1} or y_{i-1}, the feasible solution satisfies the inequality. Similarly, if we use z_{i-1} and z_i in periods i-1 and i, then we need to replace w_{i-2} by y_{i-2} or z_{i-2}.

Consider the inequality $w_1+w_2+....+w_{t_2} \geq 2$. Suppose we replace w_i by z_i for some $i \leq t_1$, and impose the condition that period i-1 contains y_{i-1} or z_{i-1}. In this case, we need to produce twice to meet the demand up to period t_2. The inequality is not valid: we can produce in periods i-1 and i. To obtain a valid inequality, we could replace w_{i-2} by y_{i-2} or z_{i-2} and w_{i-1} by $(y_{i-1}+z_{i-1})$. Then if we produce twice in the interval {i-2,..,i}, the lefthand side of the inequality equals at least two units, and the inequality is valid.

In general, whenever any period i^* contains the term z_{i^*}, we need to compensate for this term by introducing appropriate terms in periods prior to this period. If we produce in period i^*, we need to turn the machine on in some period $i' \leq i^*$ and keep it on in the interval $\{i',...,i^*\}$. We want to ensure that if we produce r times in this interval, then for any feasible solution, the terms in periods i' through i^* in the inequality add up to at least r units. Recalling that t_j denotes the period at which the jth demand occurs, we next introduce some nomenclature that we will use throughout our discussion.

Demand interval j. Demand interval j is the interval $\{t_{j-1}+1, t_{j-1}+2, ..., t_j\}$.

Contribution. We say that the sum of the terms on the lefthand side of any inequality associated with some sequence of machine operations (or some set of time periods) is the contribution of that set of operations (or time periods).

For example, suppose we turn the machine on in period 2 and keep it on until period 5, producing in periods 3 and 4. Suppose the inequality in the interval from period 2 through 5 has the form:

$$.....+w_2+y_3+z_4+w_5+.....$$

This set of operations (or the periods 2 through 5) contributes 1 unit, since $w_2 = 0$, $y_3 = 1$, $z_4 = 0$ and $w_5 = 0$.

2.1 Partition Inequalities.

We begin by considering a class of valid inequalities, which we call the **partition inequalities (PI)**. Later, we introduce a more general class of inequalities and show how we can tighten them to

obtain facets. We consider inequalities of the form

$$S_{ieW}w_i+S_{ieY}y_i+S_{ieZ}c_iz_i+S_{ieYZ}(y_i+c_iz_i)+S_{ieWZ}(w_i+c_iz_i) \geq q$$

$$q = 1,...,n \quad (\text{ PI })$$

obtained by replacing the terms w_i in the demand constraints by the terms y_i, c_iz_i, $y_i+c_iz_i$, or $w_i+c_iz_i$. Subsets W, Y, Z, YZ and WZ consist of periods i that contain the terms w_i, y_i, z_i, $y_i+c_iz_i$, and $w_i+c_iz_i$, respectively, for some integer $c_i \geq 1$. Let L = $\{1,....,t_q\}$. Then W, Y, Z, YZ and WZ are disjoint subsets of L that partition L: that is, W»Y»Z»YZ»WZ = L.

Example

Suppose q=3 and $t_1=5$, $t_2=6$ and $t_3=7$. Then

$$y_1+(y_2+z_2)+(y_3+2z_3)+3z_4+2z_5+z_6+w_7 \geq 5$$

is a valid inequality.

Notice that if we produce in any three periods, the lefthand side equals at least 3. For instance, if we turn the machine on in period 3, and produce in periods 3, 4, and 5, then periods 4 and 5 do not contribute to the lefthand side. To compensate for this, period 3 contributes two extra units beyond the one unit for producing in that period. In general, we need to specify integer coefficients for the variables z_i in any partition inequality to ensure that it is valid.

2.2 Skip Inequalities

To generalize the partition inequalities (PI), we consider another class of inequalities, which we call "skip" inequalities (SI). We say that an inequality extending up to period t_q skips a time period $i \leq t_q$ if iœW»Y»Z»YZ»WZ. Let S denote the set of all time periods skipped up to t_q. Then W»Y»Z»YZ»WZ»S = L ∫ $\{1,....,t_q\}$. Let b = ΩSΩ denote the number of periods the inequality skips. For any $b \leq q$, the skip inequality is of the form:

$$\sum_{i\in W} w_i+\sum_{i\in Y} y_i+\sum_{i\in Z} c_i z_i+\sum_{i\in YZ}(y_i+c_i z_i)+\sum_{i\in WZ}(w_i+c_i z_i) \geq m(t_q) \qquad \text{(SI)}$$

$$q = 1,\ldots.,n.$$

The righthand side $m(t_q)$ of this inequality and the coefficients c_i are constants whose values we need to specify. We first introduce some notation. For $t \geq i$, we define an **[i,t] on-interval** as a sequence of periods i, i+1,..., t with $z_i = 1$, $y_i = y_{i+1} = \ldots = y_t = 1$, and $z_{i+1} = \ldots = z_t = y_{t+1} = 0$. By definition $y_{T+1} = 0$. For any period i and any period $t \geq i$, let $\mathbf{n_{yz}(i,t)}$, $\mathbf{n_{wz}(i,t)}$, $\mathbf{n_z(i,t)}$ **and** $\mathbf{n_s(i,t)}$ denote the number of periods in Y»YZ, W»WZ, Z and S in the interval (i,i+1,...,t). We define these quantities, as well as d(i,t), to be zero whenever t < i. For any on-interval [i;t], the $n_z(i,t)$ periods in Z other than period i and the $n_s(i,t)$ periods do not contribute anything to the inequality even if we produce in these periods. Therefore, we need to compensate for these periods by introducing a large enough coefficient c_i for z_i in period i. Let $\mathbf{n^w(i,t)}$ denote the number of periods in W»WZ«(i,i+1,...,t) with $w_j = 1$.

Note that if for all periods $1 \leq i \leq t_q$, $c_i \geq n_z(i,t_q)$, then (SI) is valid since the contribution $c_i+n^w(i,t)+n_{yz}(i,t)$ in any on-interval [i,t] is at least $n_z(i,t)+n^w(i,t)+n_{yz}(i,t)$, an upper bound on the number of productions in the interval. Summing over all on-intervals in any feasible solution shows that the inequality contributes at least q units and so if $m(t_q) \leq q$, it is valid. Choosing the c_i coefficients to satisfy the inequality $c_i \geq n_z(i,t_q)$ gives a valid inequality, but one with coefficients that are too large. We next define better bounds on these coefficients.

Consider any feasible solution, and let **P(i,j)** denote the number of productions in nonskipped periods in the periods i through j. P(1,j) is at least $d(1,j)-n_s(1,j)$. Since $P(1,j) \geq P(1,k)$ for $j \geq k$, $P(1,j) \geq$ **m(j)** ∫ max $\{d(1,k)-n_s(1,k): 0 \leq k \leq j\}$. Note that since $d(1,0) = n_s(1,0)$, $m(j) \geq 0$. The quantity $m(t_q)$ defines the righthand side of the skip inequality.

Notice that since $d(1,k) > d(1,k-1)$ only if $k = t_r$ for some $1 \leq r \leq n$, $m(j) = \max\{m(t_r)\colon t_r \leq$

j}. Morevover, if $m(t_q) = m(t_{q-1}) < d(1,t_q)-n_s(1,t_q) = q-b$, then the skip inequality for period t_{q-1} dominates the skip inequality for period t_q. Therefore, we assume $m(t_q) = q-b$.

Since all the coefficients in the skip inequality are nonnegative, to show that they are valid, we can restrict our attention to production plans with exactly $P(1,t_q) = m(t_q) = q-b$ productions in nonskipped periods in the interval 1 to t_q. We produce the remaining quantity for satisfying demand in the periods 1 to t_q in the b skipped periods and for the demand in the periods t_q+1, t_q+2, ..., T after t_q. Since $P(1,t) = P(1,j-1)+P(j,t) \leq m(t_q)$ for any $i < j \leq t$, and $P(1,j-1) \geq m(j-1)$, $P(j,t) \leq m(t_q) - m(j-1)$. Therefore, the required production in periods j through t_q is at most $\mathbf{D(j)} \int m(t_q) - m(j-1)$, which is a derived "demand" for these periods.

We will use this observation to obtain a bound on the coefficients c_i in the skip inequality. For any two periods $i \leq k$, $P(i,k) \leq n_{yz}(i,k)+n_z(i,k)+n^w(i,k)$. Therefore, for any two periods $i \leq t$,

$$
\begin{aligned}
P(i,t) = {} & P(i,j-1)+P(j,t) \\
& \leq n_{yz}(i,j-1)+n_z(i,j-1)+n^w(i,j-1)+\min\{D(j), \\
& \quad n_{yz}(j,t)+n_z(j,t)+n^w(j,t)\} \\
& \leq n_{yz}(i,j-1)+n_z(i,j-1)+n^w(i,t)+\min\{D(j), \\
& \quad n_{yz}(j,t)+n_z(j,t)\}.
\end{aligned}
$$

As a result, the contribution $c_i+n_{yz}(i,t)+n^w(i,t)$ in periods i through t exceeds the production $P(i,t)$ if

$$c_i+n_{yz}(i,t) \geq n_{yz}(i,j-1)+n_z(i,j-1)+\min\{D(j), n_{yz}(j,t)+n_z(j,t)\}$$

for any $i \leq j \leq t$ and, consequently, if

$$
\begin{aligned}
& c_i+n_{yz}(i,t) \geq \mathbf{N(i,t)} \int \\
& \min_{i \leq j \leq t}\{n_{yz}(i,j-1)+n_z(i,j-1)+\min\{D(j), n_{yz}(j,t)+n_z(j,t)\}.
\end{aligned}
$$

The following three properties are consequences of the definition of N(i,t).

P1. For any $i \leq t$, $N(i,t) = n_{yz}(i,t')+n_z(i,t')$ for some $i \leq t' \leq t$.

P2. For all $t \geq i$, $N(i,t+1) \geq N(i,t)$ and $N(i+1,t)+1 \geq N(i,t) \geq N(i+1,t)$.

P3. Suppose $N(i,t) = n_{yz}(i,t')+n_z(i,t')$ for some $i \leq t' \leq t$. Then $N(i,r) = N(i,t)$ for all $t' \leq r \leq t$.

We establish these properties in Appendix 1.

We apply these properties to determine the coefficients c_i. Let $\mathbf{i^*}$ be the minimum value of t satisfying the equation $N(i,t_q) = n_{yz}(i,t)+n_z(i,t)$. If $N(i,t_q) = 0$, we define $i^* = i$. We choose $c_i = n_z(i,i^*)$. We refer to i^* as the **look ahead period** for period i. Notice that if we define $D(t_q+1) = 0$, then $N(i,t_q) = n_z(i,i^*)+n_{yz}(i,i^*) = n_z(i,j-1)+n_{yz}(i,j-1)+D(j)$ for some period $j \geq i$.

By property P3, for any $t > i^*$, $N(i,t) = N(i,t_q)$, and therefore $c_i+n_{yz}(i,t) = n_z(i,i^*)+n_{yz}(i,t) \geq n_z(i,i^*)+n_{yz}(i,i^*) = N(i,t)$. For $t < i^*$, $c_i+n_{yz}(i,t) = n_z(i,i^*)+n_{yz}(i,t) \geq n_z(i,t)+n_{yz}(i,t) \geq N(i,t)$. Therefore, $c_i+n_{yz}(i,t) \geq N(i,t)$ for all periods $t \geq i$.

We restrict the class of skip inequalities to those that satisfy the following condition.

Compensation Condition. For any period i, $c_i = n_z(i,i^*)$.

We can interpret this condition as follows. Any on iterval [i,t] contains $n_z(i,t)$ periods in Z, and except for period i, none of these periods contribute to the inequality even if we produce in them. Therefore, the coefficient c_i must compensate for these periods. However, we need to compensate only for periods up to i^*, the look ahead period.

Example

Suppose q= 7, $t_1 = 2$, $t_2 = 6$, $t_3 = 7$, $t_4 = 8$, $t_5 = 14$, $t_6 = 15$ and $t_7 = 16$, and we skip periods 3 and 4 so that b = 2. Then the following terms satisfy the compensation condition.

$...+y_5+(w_6+z_6)+(w_7+z_7)+(w_8+z_8)+(y_9+z_9)+(y_{10}+2z_{10})+3z_{11}+2z_{12}+z_{13}+...$

For instance, for i = 6, $N(6,16) = n_{yz}(i,8)+n_z(6,8)+D(9) = 3$. Therefore, $i^* = 11$, and $c_6 \geq n_z(i,i^*) =$

1.

Proposition 1. The skip inequality (SI) is valid if it satisfies the compensation condition.

Proof.

The compensation condition implies that $c_i+n_{yz}(i,t)+n^w(i,t) \geq N(i,t)+n^w(i,t)$ for any on interval [i,t]. The quantity $N(i,t)+n^w(i,t)$ is an upper bound on the production in nonskipped periods in the on interval [i,t]. Adding over all on intervals in any feasible solution shows that the total contribution of the skip inequality is at least as large as the total production in periods 1 through t_q in nonskipped periods. Since we skip b periods and produce at least q times up to period t_q in any feasible solution, we produce at least q-b times in nonskipped periods. Therefore, the skip inequality is valid.

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For example, suppose q=5 and $t_1=4$, $t_2=8$, $t_3=9$, $t_4=10$ and $t_5=12$. If we skip periods 8 and 9, then b=2. The following inequality

$$w_1+y_2+(y_3+z_3)+(w_4+z_4)+(y_5+z_5)+2z_6+z_7+w_{10}+y_{11}+z_{12} \geq 3$$

is valid.

Corrollary 1. Every feasible solution contributes at least m(t) units in the interval 1 through t.

Proof. The contribution of periods 1 through t is at least equal to the number of productions in nonskipped periods up to period t, which in turn is at least equal to m(t).

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The partition inequalities (PI) are special cases of the skip inequalities with b = 0.

The following partitioning inequalities (pi) of Magnanti and Vachani (1990) are special versions of our inequalities (PI)

$S_{i=1}^{j-1} w_i + S_{i \in W} w_i + S_{i \in Y} y_i + S_{i \in Z} z_i \geq q.$

In this expression, $t_{q-1}+1 \leq j \leq t_q$ (implying $D(k) = 1$ for all $k \geq j$) and the sets W, Y and Z are subsets of $\{j, j+1,..., t_q\}$. The inequalities (pi) confine periods ieZ to the interval $\{t_{q-1}+2, ..., t_q\}$ and do not contain any terms in YZ or WZ. Inequalities (PI) allow ieZ anywhere in the inequality. Magnanti and Vachani impose the following conditions on inequalities (pi):

i) period jœZ, and

ii) if ieW, then i+1œZ.

Since we include the periods up to j-1 in W, condition (ii) implies (i), which in turn is implied by the compensation condition for partition inequalities (PI): that is, if period i lies in the interval $\{t_{q-1}+2, ... , t_q\}$ and ieZ, then i-1œW. Otherwise, the look ahead period for period i-1 is period i and the sum of the coefficients of y_{i-1} and z_{i-1} is zero which is less than one, the number of periods in Z until the look ahead period.

Notice that the skip inequalities satisfy the property that in any on interval starting in period i, if we produce $k \leq N(i,t_q)$ units then the on interval contributes k units. However, Van Hoesel and Kolen (1993) have described a set of inequalities (hole and bucket inequalities) that do not satisfy this property (a hole is a skipped period). For instance, if q = 3, and $t_1 = 4$, $t_2 = 6$ and $t_3 = 7$, then $y_1+z_2+y_3+z_4+y_5+z_6+z_7 \geq 2$ is a valid hole and bucket inequality. If we produce twice in on interval [1,2], then $N(1,t_q) = 2$ but the on interval contributes only one unit. The hole and bucket inequalities restrict the coefficient of z_t to 0 or 1 in each period. Using the ideas developed in this paper, it is possible to generalize the hole and bucket inequalities to obtain more general inequalities with coefficients $c_i \geq 0$. We will pursue this development in a subsequent paper.

3. Separation Problem

We solve the separation problem for special cases of the partition inequalities (PI) using a linear programming based approach. Previously, in other problem contexts, Eppen and Martin

(1987) and Pochet and Wolsey (1994) have used this approach, enabling them to state an expanded reformulation of a problem that implicitly includes all valid inequalities that they use in the separation problem.

There are two advantages of using a linear programming based approach. First, rather than explicitly considering an exponential number of inequalities, this approach enables us to reformulate the original problem (SCSP) as a linear program with a polynomial (in T) number of variables and constraints. Second, this approach might have computational advantages: it might be more efficient or simpler to solve the reformulated problem as one linear program rather than solve a sequence of linear programs in a cutting plane based procedure, solving a separation problem each time we encounter a fractional solution.

The separation problem can be described as follows. Given a fractional solution (w^*, y^*, z^*) to the linear programming relaxation of the problem SCSP, we want to determine if (w^*, y^*, z^*) violates a particular set of valid inequalities. Let **P** denote the set of coefficients for the valid inequalities in the sense that (g,d)Œ**P** if and only if $g(w, y, z) \geq d$ is one of the valid inequalities in the set. Suppose we solve the following optimization problem

$$n = \min g(w^*, y^*, z^*) - d$$

s.t. (g,d) Œ **P**

If $n \geq 0$, then $g(w^*, y^*, z^*) \geq d$ for all the valid inequalities in our set. If $n = g^*(w^*, y^*, z^*) < 0$, then $g^*(w^*, y^*, z^*) < d$ is the most violated inequality from this set.

In certain cases, we can describe the set **P** as a polyhedron (or a projection of a polyhedron with additional variables). For example, suppose we consider the subset of partition inequalities that partition only the last demand interval $(t_{q-1}+1,...,t_q)$ for some $q = 1,2,...,n$. In this case, $m(t_q) = q$ and $D(i) = 1$ for any period $t_{q-1}+1 \leq i \leq t_q$, and, therefore, the coefficients in the resulting partition inequality are all less than or equal to one. The compensation condition reduces to the following condition: any period iŒZ, $i \geq t_{q-1}+2$, in this interval must be preceded by period i-1ŒY»Z. For this set of inequalities, we can describe the set **P** = **{(g,d = (a,b,e,d))}** by the constraints in the following separation problem.

(SEP)

Min n = $\Sigma^{n}_{q=1}[\Sigma^{t_q}_{i=t_{q-1}+1}\{w_i^* a_{iq}+y_i^* b_{iq}+z_i^* e_{iq}\} - d_q]$ (9)

subject to:

$a_{iq}+b_{iq}+e_{iq} - d_q = 0 \qquad t_{q-1}+1 \leq i \leq t_q-1,\ 1 \leq q \leq n$ (10)

$b_{iq}+e_{iq} - e_{i+1,q} \geq 0 \qquad t_{q-1}+1 \leq i \leq t_q-1,\ 1 \leq q \leq n$ (11)

$\Sigma^{n}_{q=1} d_q \leq 1$ (12)

all a_{iq}, b_{iq}, e_{iq}, d_q Œ {0,1}.

Constraint (12) ensures that $d_q = 1$ for at most one value of q. If $d_q = 1$, constraint (10) ensures that exactly one of the three variables a_{iq}, b_{iq}, or e_{iq} equals 1 for each period $t_{q-1}+1 \leq i \leq t_q$. When $d_q = 1$ the last period in the inequality is t_q. Constraint (11) ensures that the inequality satisfies the compensation condition, i.e., if period i+1ŒZ, then period iŒY»Z.

If the solution (w^*, y^*, z^*) violates any inequality, then by setting set $a_{iq} = 1$ for iŒW, $b_{iq} = 1$ for iŒY and $e_{iq} = 1$ for iŒZ, and $d_q = 1$, we see that $\Sigma_{i=1}^{t_{q-1}} w_i^* + \Sigma_{i\text{ŒW}} w_i^* + \Sigma_{i\text{ŒY}} y_i^* + \Sigma_{i\text{ŒZ}} z_i^* < q$. Since $\Sigma_{i=1}^{t_{q-1}} w_i^* \geq q-1$ in any fractional solution, $\Sigma_{i\text{ŒW}} w_i^* + \Sigma_{i\text{ŒY}} y_i^* + \Sigma_{i\text{ŒZ}} z_i^* < 1$, and so n < 0. If the point (w^*, y^*, z^*) satisfies all the inequalities, then $n \geq 0$. Since the solution with all variables a, b, m and d equal to zero is feasible, $n \leq 0$. Therefore, n = 0 if and only if (w^*, y^*, z^*) satisfies all of the inequalities.

Suppose we drop the constraint $\Sigma^{n}_{q=1} d_q \leq 1$. If we subtract equation (10) from equation (11), then each variable a_{iq}, b_{iq} and c_{iq} appears in at most two constraints, and if in two, with opposite signs. Therefore, the constraint matirix is unimodular and we can eliminate the integer constraints and solve the integer program as a linear program. Notice that if we drop the constraint $\Sigma^{n}_{q=1}d_q \leq 1$, then n is unbounded from below if and only if the solution (w^*, y^*, z^*) violates any inequality since we can set d_q to an arbitrarily large value. Therefore, the following dual problem has a feasible solution if and only if the solution (w^*, y^*, z^*) satisfies all the inequalities.

(D) Max 0

subject to

$$p_{iq} \leq w_i^* \qquad t_{q-1}+1 \leq i \leq t_q,\ 1 \leq q \leq n$$

$$p_{iq}+r_{iq} \leq y_i^* \qquad t_{q-1}+1 \leq i \leq t_q,\ 1 \leq q \leq n$$

$$p_{iq}+r_{iq}-r_{i-1,q} \leq z_i^* \qquad t_{q-1}+1 \leq i \leq t_q,\ 1 \leq q \leq n$$

$$\Sigma^{t_q}_{i=tq-1+1}p_{iq} = 1 \qquad 1 \leq q \leq n$$

$$r \geq 0.$$

We have shown that a fractional solution (w^*, y^*, z^*) satisfies all of the inequalities that partition the last interval if and only if this problem is feasible. Therefore, if we append these constraints to SCSP(L), the linear programming relaxation of SCSP, the following reformulation implicitly includes all such valid inequalities.

(R) Minimize $U = S_{i=1}{}^{T}\{g_iw_i+f_iy_i+F_iz_i\}$

subject to

$$p_{iq} \leq w_i \qquad t_{q-1}+1 \leq i \leq t_q,\ 1 \leq q \leq n$$

$$p_{iq}+r_{iq} \leq y_i \qquad t_{q-1}+1 \leq i \leq t_q,\ 1 \leq q \leq n$$

$$p_{iq}+r_{iq}-r_{i-1,q} \leq z_i \qquad t_{q-1}+1 \leq i \leq t_q,\ 1 \leq q \leq n$$

$$\Sigma^{t_q}_{i=tq-1+1}p_{iq} = 1 \qquad 1 \leq q \leq n$$

$$\Sigma_{i=1}{}^{T}w_i = n,\ w_i \leq y_i,\ y_{i-1}+z_i \geq y_i,\ w, y, z, r \geq 0.$$

Let $l(q) = t_q-t_{q-1}+1$. The demand interval q has $O(2^{l(q)})$ partition inequalities since each time period can be in W, Y or Z subject to the condition that iŒZ implies that i-1ŒY»Z. If 2^l = max ($2^{l(q)}$: 1≤q≤n), then the problem has O(2^l) inequalities. However, the reformulation contains only O(T^2) variables and constraints.

The approach we have just developed applies to more general inequalities. We say that the sequence of periods i, i+1, ..., j is a **{i,j} yz structure** if it satisfies the following properties:

(i) period i-1ŒW»S and period iŒY,

(ii) period jŒZ, c_j = 1 and period j+1ŒW»Y»S,

(iii) period tŒYZ»WZ»Z for i+1 ≤ t ≤ j.

We consider a special class of partition inequalities called **single sequence partition inequalities**. These inequalities satisfy two properties.

(i) they contain no terms of the form $w_i+c_iz_i$ for c_i ≥ 1 and do not skip any periods (therefore, D(i) = q-r if iŒ{t_r+1, ..., t_{r+1}}); and

(ii) each yz structure contains one set of contiguous periods from Y»YZ followed by a sequence of periods in Z, and has a form like

$$y_i+(y_{i+1}+z_{i+1})+(y_{i+2}+z_{i+2})+(y_{i+3}+z_{i+3})+z_{i+4},$$

$$y_i+(y_{i+1}+z_{i+1})+(y_{i+2}+2z_{i+2})+(y_{i+3}+2z_{i+3})+2z_{i+4}+z_{i+5},$$

or $$y_i+(y_{i+1}+z_{i+1})+(y_{i+2}+2z_{i+2})+(y_{i+3}+3z_{i+3})+4z_{i+4}+3z_{i+5}+2z_{i+6}+z_{i+7}.$$

Note that the coefficients of z_t in the leading Y»YZ terms are increasing up to some period at which point they remain the same until the last period in Y»YZ. The coefficients of z_t in the Z terms are decreasing. If t is the last period in Y»YZ (and so t+1 is the first period in Z), then the coefficients also satisfy the one of the following conditions: (i) $c_{t+1} = c_t \le D(t)$, or (ii) $c_{t+1} = c_t+1 = D(t)$.

To model these inequalities, we must specify not only the coefficients a_{iq}, b_{iq} of w_i and y_i as before, but also the coefficients of the terms in the more elaborate yz structures. To do so, we let,

(i) $b_{iq}(k,j)$ be a 0-1 variable that indicates whether or not the inequality extending up to period t_q contains a term of the form $(y_i+c_iz_i)$ with $c_i = k$ in period i, which is in the jth position after the first period tŒY in the yz structure. In particular, $b_{iq}(0,1)$ indicates whether the inequality contains the term y_i.

(ii) $e_{iq}(k,j)$ be a 0-1 variable that indicates whether or not the inequality extending up to period t_q contains a term of the form c_iz_i with $c_i = k \geq 1$ in period i, which is in the jth position after the first period tŒY in the yz structure.

Given a fractional solution (w^*, y^*, z^*) we can solve the separation problem by solving the following linear program.

$$\text{Min } \Sigma^{n}_{q=1}[\Sigma^{t_q}_{i=1}\{w_i^*a_{iq}+y_i^*\Sigma_{k=0}^{D(i)}\Sigma_{j=k+1}^{D(i)}b_{iq}(k,j)+z_i^*\Sigma_{k=1}^{D(i)}\Sigma_{j=D(i)+1}^{2D(i)}e_{iq}(k,j)\}-d_q]$$

subject to:

$$a_{iq}+\Sigma_{k=0}^{D(i)}\Sigma_{j=k+1}^{D(i)}b_{iq}(k,j)+\Sigma_{k=1}^{D(i)}\Sigma_{j=D(i)+1}^{2D(i)}e_{iq}(k,j) - d_q = 0$$

$$1 \leq i \leq t_q;\ 1 \leq q \leq n \qquad (1)$$

$$b_{iq}(k-1,j-1)+b_{iq}(k,j-1) - b_{i+1,q}(k,j) \geq 0$$

$$1 \leq k \leq j-1 \leq D(i)-1 \qquad (2)$$

$$b_{iq}(k,D(i)) - e_{i+1,q}(k,D(i)+1) \geq 0$$

$$1 \leq k \leq D(i)-1 \qquad (3)$$

$$b_{iq}(D(i)-1,D(i)) - e_{i+1,q}(D(i),D(i)+1) \geq 0 \qquad (4)$$

$$e_{iq}(k+1,j-1) - e_{i+1,q}(k,j) \geq 0$$

$$1 \leq k \leq D(i)-1;\ D(i)+2 \leq j \leq 2D(i) \qquad (5)$$

$$\Sigma^{n}_{q=1}d_q \leq 1 \qquad (6)$$

a_{iq}, $b_{iq}(k,j)$, $e_{iq}(k,j)$, d_q Œ {0,1}.

Constraints (1) ensure that we choose at most one of the terms a_{iq}, $b_{iq}(k,j)$ or $e_{iq}(k,j)$ for each period i. Constraints (2) ensure that if period i+1 has the coefficient $b_{i+1,q}(k,j)$, i.e., the inequality has the term $(y_{i+1}+kz_{i+1})$ in the jth position, then it has the term (y_i+kz_i) or the term $(y_i+(k-1)z_i)$ in period i for $k \leq j-1 \leq D(i)-1$. Constraints (3) ensure that if period i+1ŒZ, j = D(i)+1 and $k \leq D(i)-1$, then period iŒY»YZ and the coefficients of z_i and z_{i+1} are the same. Constraints (4) ensure that if period i+1ŒZ and j-1 = k = D(i), then period iŒY»YZ and the coefficient of z_i is D(i)-1. Constraints (5) ensure that if period i+1ŒZ in position $j \geq D(i)+2$ has coefficient k, then period iŒZ has coefficient k+1 in position j-1. Constraint (6) ensures that g_q is one for at most one value of q, i.e., the linear program chooses at most one inequality.

Using an approach similar to the one we used before, we could formulate an integer program for the separation problem for these inequalities with a unimodular constraint matrix, i.e., a constraint matrix equivalent to a network flow problem. This enables us to reformulate a model that contains all of the single sequence partition inequalities as a linear program with a polynomial number of variables and constraints. That is, the model will contain additional dual variables and by projecting out the variables, we would obtain all these inequalities. The model contains $O(n^4)$ variables and $O(n^4)$ constraints.

4. Computational Results

Karmarkar and Schrage (1985) report computational experience for a continuous production policy version of the product cycling problem that allows production of any amount between zero and the production capacity. They use Lagrangean relaxation to solve problem instances of up to 4 products and 8 time periods. In our model, we use a discrete production policy in which we produce either zero or one unit in each period. Magnanti and Vachani (1990) report computational results for this model, and solving problem instances of up to 5 products and 15 time periods.

We use the same approach as Magnanti and Vachani (1990) to generate problem instances. For all problem instances, we assume that the initial inventory is zero, and that the machine is in the off state at the start of the time horizon.

For the multi-item problems, the cost parameters F_{pi} and f_{pi} are the same for all products, and are constant over the time horizon. The inventory holding cost function $g_i = 20(T\text{-}i)$ assumes a uniform inventory holding cost per unit per time period. We tested two categories of problems:

1) The single item problem. We tested problems of up to 100 time periods and 30 demands. The largest problem instance had 300 original (or natural) variables (100 each of the w_i production variables, y_i setup variables and z_i changeover variables).

2) The four item problem. We tested problems of up to 100 time periods and 15 demands for each item. The largest problem instance had 1200 natural variables (300 variables for each item).

For both problem categories, we used only a subset of the single sequence partition inequalities. For any inequality extending up to period t_q, we partition only the last 5 demand intervals t_{q-5} through t_q for inequalities with $c_i \leq 2$. We did not use any of the skip inequalities (SI). If we partition only the last r intervals, the reformulation has $O(r^2T^2)$ variables and constraints. The largest problem instance we solved, therefore, contained more than 250,000 variables and constraints. We performed our computations on a IBM 4341 computer using the GAMS package.

Let **v(IP)** and **v(LP)** denote the optimal objective function values of the original integer program SCSP and its linear programming relaxation SCSP(LP). Let **v(LAST)** denote the optimal objective function value of SCSP (LP) that includes only the inequalities partitioning the last demand interval and let **v(ss)** denote the optimal objective function value of SCSP (LP) that includes the single sequence partition inequalities. We define **gap (LP)** = (v(IP)-v(LP))*100/v(IP), **gap (LAST)** = (v(IP)-v(LAST))*100/v(IP) and **gap (0)** = (v(IP)-v(0))*100/v(IP). Tables I and II summarize the computational results.

Table I						
Single machine, single item problems						
# of demands	v(LP)	v(LAST)	v(SS)	gap(LP)	gap(LAST)	gap(SS)
3	26.7	146.7	160	83.3	8.3	0.0
5	56.7	267	300	81.1	11.1	0.0
10	117.8	497	540	78.2	8.0	0.0
15	195.6	746.7	840	76.7	11.1	0.0
20	282.2	1047	1160	75.7	9.8	0.0
25	340	1277	1440	76.4	11.3	0.0
30	371	1496	1680	77.9	10.9	0.0
Notes: Constant turn on and setup cost Inventory holding cost = g*(T-i)						

Table II					
Four Item Problems					
Turn on cost F = 100					
# of demands	v(LP)	gap(LP)	v(SS)	v(IP)	gap(SS)
3	1453.3	15.51	1720	1720	0
5	4700	16.37	5620	5620	0
10	13569	5.77	14400	14400	0
15	32278	4.56	33820	33820	0
Turn on cost F = 200.					
3	1520	20.83	1920	1920	0
5	4820	20.72	6080	6080	0
10	13787	9.06	15160	15160	0
15	32633	6.07	34740	34740	0
Notes: Constant turn on and setup cost Inventory holding cost = g*(T-i)					

Using the single sequence partition inequalities, we obtained optimal integer solutions for all the test problem instances. The gaps between the optimal objective function value of the linear programming relaxation and the optimal integer program objective function value are large for the single item, single machine problem, varying between 75% and 83%. A small subset of the partitioning inequalities that partition only the last demand interval reduces the gap considerably to between 8% and 11%. However, we still obtain fractional solutions, and need to introduce the more complex single sequence partitioning inequalities to reduce the gaps to zero. For multi-item problems, gap(LP) is much smaller and varies between 6% and 21%. For this class of problems, the linear programming relaxation with the single sequence inequalities optimally solves problem instances with up to 1200 variables.

5. Facets

It is possible to tighten the skip inequalities if we impose restrictions on them in addition to the compensation condition. In this section and in Appendix 2, we describe these conditions and show that they are necessary for these inequalities to be facets of the underlying integer polyhedron. (The conditions are also sufficient for the inequalities to be facets, but we will not prove this fact).

Suppose for any period i, D(i) = q-b-m(i-1) = 0. Then N(i',t) = N(i',i-1) for any period t ≥ i. If t_r+1 ≤ i ≤ t_{r+1}, we can drop all the terms in periods t_r+1 through t_q from the inequality and obtain a tighter inequality. Therefore, assume that D(i) = q-b-m(i-1) > 0 for every period i ≤ t_q.

We impose the following conditions on the skip inequalities and show in Appendix 2 that the conditions 1 through 5 are neccessary for the inequality to be a facet.

Condition 1. If W ≠ {1, 2,..., t_q}, then every facet defining skip inequality consists of a set of yz structures separated by periods in W»S. In particular, period j_0 = min {j: jœS} belongs to W»Y, and period t_qœY. Moreover, if the look ahead period i^* for any period iŒY»YZ satisfies the equation $n_{yz}(i,i^*)$ = D(i), then period k = min {k' ≥ i^* : k'œWZ} belongs to Z.

Condition 2. The number of periods skipped from t_j+1 through t_q is strictly less than q-j, for j = 0,..., q-1. Moreover, if m(t_j) = j-b_j and b_j > 0, then iŒY for some i ≤ t_j (therefore the inequality contains at least one yz structure).

Condition 3. If q = n, then any facet defining inequality contains at least one yz structure, and if S = F, then it contains exactly one yz structure.

Condition 4. If $t_{q+1} = t_q+1$, then iŒY for some $i < t_q$.

Condition 5. If period iŒWZ»YZ, then $c_i < D(i)$. If $D(i) = 1$, then period iœWZ»YZ and if iŒZ, then $c_i = 1$. In addition, period t_qŒW»Z.

As described in Section 3.2, the hole and bucket inequalities do not satisfy the property that if we produce $k \leq N(i,t_q)$ units in any on interval starting in period i, then the interval contributes k units. Consider the following example with $q = 4$, $t_1 = 2$, $t_2 = 5$, $t_3 = 7$ and $t_4 = 8$. The following inequalities are valid.

$$y_2+z_3+y_4+z_5+y_6+z_7+z_8 \geq 2$$

and, $\quad y_3+z_4+z_5+z_6+z_7 \geq 1.$

The first inequality is a hole and bucket inequality and the second one can be viewed either as a skip inequality or as a hole and bucket inequality. If we add these inequalities, we obtain $y_2+(y_3+z_3)+(y_4+z_4)+2z_5+(y_6+z_6)+2z_7+z_8 \geq 3$, which is a valid skip inequality. It satisfies all the conditions 1 through 5. But it is not a facet since it can be expressed as a linear combination of two other inequalities. However, the similar inequality $y_2+(y_3+z_3)+(y_4+z_4)+2z_5+(y_6+z_6)+z_7+w_8 \geq 3$ is, as we show later, a facet. In order to rule out non facet skip inequalities that are combinations of hole and bucket inequalities and other skip inequalities, we impose the following condition.

Condition 6. Let j = min {i: iœS}, $r = \min \{u: m(t_u) = 1\}$, and suppose $q-b \geq 2$. If jŒY and j^* is the look ahead period for period j, and the last period k in this yz structure starting with period j satisfies the condition $k \geq t_r$, then $n_z(i,k)+n_{yz}(i,k) < D(i)$ for all periods $\min \{t_r, j^*+1\} \leq i \leq k$.

Notice that for the demand structure we introduced previously, the inequality $y_2+(y_3+z_3)+(y_4+z_4)+2z_5+(y_6+z_6)+2z_7+z_8 \geq 3$ does not satisfy this condition since $\min \{t_r, j^*+1\} = 4$, $k = 8$ and $n_z(i,8)+n_{yz}(i,8) \geq D(i)$ for all periods $4 \leq i \leq 8$. However, the inequality $y_2+(y_3+z_3)+(y_4+z_4)+2z_5+(y_6+z_6)+z_7+w_8 \geq 3$ satisfies the condition since $k = 7$ and

and it can be written as a linear combination of

$$S_{ieW»WZ}w_i+S_{ieY»YZ}y_i+S_{ieZ»WZ»YZ}c_iz_i \geq q\text{-}b \qquad \text{times b}$$

and $\quad S_{i=1}{}^{T}w_i = n \qquad \text{times a.}$

Therefore, it is a facet.

To establish these results, we construct solutions in C^* that produce in certain on intervals [i,t]. Therefore the inequality aw+by+gz = d contains the terms $a_t w_t + b_t y_t$ and g_i. Notice that if $i < t$, then $z_i = 1$ and $z_t = 0$, and so, g_t does not appear in the inequality. By shifting the production in period t (typically to period t_q), we produce another solution in C^*, and by comparing these solutions we relate the coefficients a_t and b_t in different periods. We also construct another solution in C^* that has shifted all the production from on interval [i,t] to another on interval [j,k]. By comparing these solutions, we relate the coefficients g_i in different periods. The complete proof, which shows how to select the on intervals [i,t] and [j,k], is fairly intricate and long and so we will not provide the details.

7. Further research

There are many ways to extend the results in this paper. One research direction would be to extend the hole and bucket inequalities by permitting coefficients other than zero or one - for example, finding a class of inequalities that include both the hole and bucket inequalities and the skip inequalities as special cases.

Although it is possible to solve the single item problem in polynomial time, the convex hull of feasible solutions is still unknown. So another direction for future research would be to determine the convex hull of this problem and for the related problems, e.g., those with start up costs but no set up costs. The results in this paper suggest that this polyhedron is quite complex.

In this paper, we have used facets for the single item problem in solving multi-item

$n_z(7,7)+n_{yz}(7,7) = 1 < D(7) = 2.$

The following proposition shows that skip inequalities satisfying conditions 1 through 6 are facet defining.

Proposition 2. A skip inequality (SI) that satisfies condition 6 is a facet of conv(W,Y,Z) if and only if it satisfies conditions 1 through 5.

Proof. (Sketch).

Let **conv(W,Y,Z)** denote the convex hull of feasible solutions of SCSP. For any valid inequality (SI), let C^* = {(w,y,z)Œconv(W,Y,Z): (w,y,z) satisfies (SI) as an equality}. To show that (SI) is a facet, let aw+by+gz = d represent an arbitrary equation that is satisfied by all (w, y, z)eC^*. We show that aw+by+gz = d is a linear combination of

$$S_{ieW}w_i+S_{ieY}y_i+S_{ieZ}c_iz_i+S_{ieYZ}(y_i+c_iz_i)+S_{ieWZ}(w_i+c_iz_i) = q\text{-}b.$$

and the only equality in SCSP, $S_{i=1}{}^{T}w_i = n$.

The proof proceeds as follows:

(i) For periods iœZ»WZ»YZ, we show that $g_i = 0$, and for periods iœY»YZ, we show that $b_i = 0$.

(ii) For periods iœW»WZ, we show that $a_i = a$, and for periods iŒW»WZ, we show that $a_i = a^*$.

(iii) We then show that $b_i = b$ for all iŒY»YZ, and that $a^* = a+b$.

(iv) Finally, we show that $g_i = c_ib$ and that d = (q-b)b+na.

Therefore, the inequality has the form:

$$a^*S_{ieW»WZ}w_i+aS_{iœW»WZ}w_i+ßS_{ieY»YZ}y_i+bS_{ieZ»WZ»YZ}c_iz_i \geq d,$$

problems. Another possibility would be to use facets of the multi-item problem itself. Very little seems to be known about the polyhedral structure of this problem.

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Appendix 1

In this section, we establish properties P1, P2 and P3 from section 2.

P1. $N(i,t) = n_{yz}(i,t')+n_z(i,t')$ for some $i \leq t' \leq t$.

Proof. By definition, $N(i,t) \leq n_{yz}(i,t)+n_z(i,t)$. So either $t' = t$, or, since $n_{yz}(i,t)+n_z(i,t)$ increases by zero or one unit for each time period t, some period $i \leq t' \leq t$ satisfies the property.

P2. For all t, $N(i,t+1) \geq N(i,t) \geq N(i+1,t)$ and $N(i,t) \leq N(i+1,t)+1$.

Proof. The definition of $N(i,t)$ implies that $N(i,t) = \min \{D(i), n_{yz}(i,i)+n_z(i,i)+N(i+1,t)\}$. Therefore, $N(i,t) \leq n_{yz}(i,i)+n_z(i,i)+N(i+1,t) \leq 1+N(i+1,t)$. If $N(i,t) = n_{yz}(i,i)+n_z(i,i)+N(i+1,t)$, then clearly $N(i+1,t) \leq N(i,t)$. If $N(i,t) = D(i)$, then since $N(i+1,t) \leq D(i+1)$ and $D(i+1) \leq D(i)$, $N(i+1,t) \leq N(i,t)$.

If $N(i,t+1) < N(i,t) \leq n_{yz}(i,t)+n_z(i,t)$, then $N(i,t+1) = n_{yz}(i,k-1)+n_z(i,k-1)+D(k)$ for some $i \leq k \leq t$. However, the definition of $N(i,t)$ implies that $N(i,t) \leq n_{yz}(i,k-1)+n_z(i,k-1)+D(k)$, which contradicts our assumption that $N(i,t+1) < N(i,t)$. Therefore, $N(i,t+1) \geq N(i,t)$.

P3. Suppose $N(i,t) = n_{yz}(i,t')+n_z(i,t')$ for some $i \leq t' \leq t$. Then $N(i,r) = N(i,t)$ for all $t' \leq r \leq t$.

Proof. If $N(i,t') < n_{yz}(i,t')+n_z(i,t')$, then $N(i,t') = n_{yz}(i,k-1)+n_z(i,k-1)+D(k)$ for some $k < t'$, and $D(k) \leq n_{yz}(k,t')+n_z(k,t')$. Therefore, $D(k) \leq n_{yz}(k,r)+n_z(k,r)$ for all $t' < r \leq t$, and so $N(i,t) \leq n_{yz}(i,k-1)+n_z(i,k-1)+D(k) < n_{yz}(i,t')+n_z(i,t')$, which contradicts our assumption. Consequently, $N(i,t') = n_{yz}(i,t')+n_z(i,t') = N(i,t)$. Property P2 implies that $N(i,r) = N(i,t')$ for all $t' \leq r \leq t$.

Appendix 2

We consider a set of inequalities that satisfy the compensation condition and have 0-1 coefficients for the w and y variables. We show that among this class of inequalities, every facet satisfies conditions 1 through 6. We first establish the following result.

Lemma 1. For any two periods $t \leq i$, the look ahead periods satisfy the inequality $t^* \leq i^*$.

Proof.

We show the result is true for $t = i-1$, which implies the general result. The definition of $N(i-1,t_q)$ implies that $N(i-1,t_q) \leq n_{yz}(i-1,k-1)+n_z(i-1,k-1)+D(k)$ for all $k \geq i-1$. Recall that since we define $D(t_q+1) = 0$, $n_{yz}(i,i^*)+n_z(i,i^*) = N(i,t_q) = n_{yz}(i,j-1)+n_z(i,j-1)+D(j)$ for some period $j \geq D(i)$. However, $N(i-1,t_q) = n_{yz}(i-1,(i-1)^*)+n_z(i-1,(i-1)^*) \leq n_{yz}(i-1,j-1)+n_z(i-1,j-1)+D(j)$. Therefore, $n_{yz}(i,(i-1)^*)+n_z(i,(i-1)^*) \leq n_{yz}(i,i^*)+n_z(i,i^*)$, and so $(i-1)^* \leq i^*$.

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In the following arguments, we consider an arbitrary valid skip inequality and show that in order for it to be a facet, it must satisfy certain conditions.

Condition 1 (a). If iŒY»YZ»WZ or if iŒZ and $c_i \geq 2$, then i+1ŒZ»YZ»WZ. If the look ahead period i^* for any period i satisfies the equation $n_{yz}(i,i^*) = D(i)$, then period k = min {k' $\geq i^*$: k'œWZ} belongs to Z.

Proof.

Suppose period iŒY»YZ»WZ and period i+1ŒW»Y»S. We show that if we replace y_i by w_i, the inequality is still valid. To do so, we need to show that the compensation condition is satisfied for periods $t \leq i$ with look ahead period $t^* \geq i$. For periods $t \leq i$, Lemma 1 implies that $t^* \leq i^*$. The compensation condition implies that $n_z(i+1,(i+1)^*) = 0$. Therefore, $n_z(i,(i+1)^*) = 0$. Let t^0 denote the new look ahead period for periods $t \leq i$. Since $t^0 \leq (i+1)^*$, the inequality satisfies the compensation condition for all periods $t \leq i$ with $t^0 \geq i$.

If iŒZ, $c_i \geq 2$ and period i+1ŒW»Y»S, then $c_i > n_z(i,i^*) = 1$. We can reduce the coefficient of z_i to 1 and obtain a tighter inequality.

Finally, suppose the look ahead period i^* satisfies the equation $n_{yz}(i,i^*) = D(i) > 0$ for some period iŒY»YZ and that period k = min {k' ≥ i^* : k'œWZ} does not belong to Z. Note that since $N(i,t_q) = n_{yz}(i,i^*)+n_z(i,i^*) \leq D(i)$, $n_z(i,i^*) = 0$. If kŒY»YZ, then $i^0 \leq k$. Therefore, $t^0 \leq k$ and since $n_z(i,k) = 0$, the inequality still satisfies the compensation condition for periods $t \leq i$ with $t^0 \geq i$. If kŒS, then the compensation condition implies that $n_z(k,k^*) = 0$. Therefore, $n_z(i,k^*)= 0$. Since $t^0 \leq k^*$, the inequality satisfies the compensation condition for all periods $t \leq i$ with $t^0 \geq i$. Therefore, the modified inequality is valid and since $y_i \geq w_i$, it dominates the original inequality.

Condition 1 (b). If period i'ŒZ»WZ»YZ, then period i = max (t<i': tœZ»WZ»YZ) belongs to Y. In addition, period j_0 = min {j: jœS} belongs to W»Y.

Proof.

If period i'ŒWZ»YZ»Z and period i = max (t<i': tœWZ»YZ»Z) belongs to W»S, then we show that the inequality does not satisfy the compensation condition. Since iŒW»S, $c_i = 0$, and therefore $n_z(i,i^*) = 0$ and $n_{yz}(i,i^*) = n_{yz}(i+1,i^*) = n_{yz}(i+1,j-1)+D(j)$ for some period $j \geq i+1$. Since i+1ŒZ»WZ»YZ, $c_{i+1} \geq 1$, and hence $c_{i+1}+n_{yz}(i+1,i^*) > n_{yz}(i+1,j-1)+D(j)$. Lemma 1 implies that $(i+1)^* \geq i^*$, and so $c_{i+1}+n_{yz}(i+1,(i+1)^*) > n_{yz}(i+1,j-1)+D(j) \geq N(i+1,(i+1)^*) = n_{yz}(i+1,(i+1)^*)+n_z(i+1,(i+1)^*)$. But this result contradicts the compensation condition $c_{i+1} = n_z(i+1,(i+1)^*)$.

Suppose period 1 belongs to YZ»WZ»Z. Then $c_1 \geq 1$. If $z_1 = 1$ and $y_1 = 0$ in any feasible solution, then $w_1 = 0$. Therefore, setting z_1 to zero gives another feasible solution that satisfies the inequality. Therefore, if $c_1 \geq 1$, we can replace c_1z_1 by c_1y_1 and since $z_1 \geq y_1$, we obtain a tighter valid inequality, and so, period 1œZ»YZ»WZ.

If period 1ŒY and the coefficient of y_1 is $c_1 \geq 2$, then arguments similar to those used earlier establish that period 2ŒYZ»WZ and so the coefficient of z_2, $c_2 \geq 1$. We can replace the quantity $c_1y_1+c_2z_2$ by $(c_1-1)y_1+(c_2-1)z_2+y_2$ and obtain a new inequality. The quantity $n_z(1,t)+n_{yz}(1,t)$ does not change and so the inequality satisfies the compensation condition for period 1. The contribution from period 2 for any on interval [2,t] also remains unchanged. Therefore the inequality also satisfies the compensation condition for period 2. Consequently, the new inequality is valid, and since $y_1+z_2 \geq y_2$, it dominates the original inequality. Therefore, if period 1ŒY, then the coefficient of y_1 must be 1 if the inequality is a facet.

Conditions 1 (a) and (b) establish Condition 1.

We next derive conditions on the number of skip periods and their location.

Condition 2 (a). The number of periods skipped from t_j+1 through t_q is strictly less than q-j, for j = 0,..., q-1.

Proof.

Recall that b_j is the number of periods skipped in the interval 1 through t_j. If $b-b_j$, the number of skipped periods after period t_j equals q-j or more, then $j-b_j \geq q-b$. Therefore, $m(t_j) \geq j-b_j \geq q-b$. Corollary 1 of Proposition 1 implies that every feasible solution contributes at least $m(t_j)$ units up to period t_j. Therefore, we can drop all the terms in the inequality with indices greater than or equal to t_j+1 and obtain a stronger valid inequality.

Condition 2 (b). If $m(t_j) = j-b_j$ and $b_j > 0$, then ieY for some period $i \leq t_j$.

Proof.

If iœY for any period $i \leq t_j$, then condition 1 implies that all periods 1 through t_j belong to W»S. Let t = min {$i \leq t_j$: iŒS}. Since $b_j > 0$, such a period always exists. Let W' = (W«{1,..., t_j})»{t} and let $S(t_j+1,t_q)$ denote the terms in the inequality in the interval {t_j+1,..., t_q}. Then the inequality can be expressed as a linear combination of the inequalities

$$S_{i\text{Œ}W'}w_i+S(t_j+1,t_q) \geq q-b+1 \tag{*}$$

and $1 \geq w_t$. We show that the inequality (*) is valid. Let $n'_s(1,i)$ denote the number of periods that the inequality (*) skips in the periods 1 through i, let m'(i) = max {$d(1,k)-n'_s(1,k)$: $1 \leq k \leq i$}, and

$D'(i) = q-b+1-m'(i-1)$. Since $m(t_j) = j-b_j$, $d(1,i)-n_s(1,i) \le j-b_j$ for periods $i \le t_j$. Therefore, $d(1,i)-n'_s(1,i) \le d(1,i)-n_s(1,i)+1 \le j-b_j+1 = d(1,t_j)-n'_s(1,t_j)$ and so $m'(t_j) = j-b_j+1 = m(t_j)+1$. Consequently, for periods $t_j < i \le t_q$,

$$\begin{aligned} m'(i) &= \max \{m'(t_j), d(1,i')-n'_s(1,i'): t_j+1 \le i' \le i\} \\ &= \max \{m(t_j)+1, d(1,i')-n_s(1,i')+1: t_j+1 \le i' \le i\} \\ &= m(i)+1. \end{aligned}$$

Therefore, $D'(i) = q-b+1-m'(i) = D(i)$ and the definition of $N(i,t_q)$ implies that for periods $i \ge t_j+1$, the look ahead period i^* and the quantity $N(i,t_q)$ remain unchanged after we introduce the variable w_t. Therefore, the inequality (*) satsifies the compensation condition for all periods $i \ge t_j+1$. For periods $i \le t_j$, $N(i,t_q) = N(t_j+1,t_q)$ since the periods 1 through t_j belong to W»S, and so the periods $i \le t_j$ satsify the compensation condition as well.

Consequently, the new inequality is valid. Since our original skip inequality is a linear combination of two inequalities, it is not a facet.

Conditions 2 (a) and (b) establish condition 2.

Condition 3. If $q = n$, then any facet defining inequality contains at least one yz structure, and if S = F, then it contains exactly one yz structure.

Proof.

Consider the case $q = n$. If the inequality contains no yz structure, then Y = f and then by the previous condition, it skips no periods. Therefore, the inequality reduces to $S_{i=1}{}^T w_i \ge n$, which is implied by $S_{i=1}{}^T w_i = n$ and $w_i \ge 0$. Therefore, if $q = n$, then $Y \ne f$. Condition 1 implies that the inequality contains at least one yz structure.

Suppose ≥Y≥>1 and S = f. By Condition 1, the inequality has at least two yz structures. For each {i,t} yz structure, let SI(i,t) denote the terms in periods i through t. We can write the inequality as the sum of the following inequalities (and therefore it cannot be a facet):

$n = S_{j=1}{}^{T}w_j$ written ΩYΩ-1 times.

$S^{T}{}_{j=1,\, j\text{œ}\{i, \ldots t\}}w_j + (SI(i,t)) \geq n$ for each ieY.

Condition 4. If $t_{q+1} = t_q+1$, then iŒY for some $i < t_q$.

Proof.

If Y = f, then condition 1 implies that Z»YZ»WZ = f, and so the inequality reduces to

$$\overset{t_q}{\underset{i=1}{\hat{A}}} w_i \geq q. \text{ But this is inequality is implied by } \overset{t_q+1}{\underset{i=1}{\hat{A}}} w_i \geq q+1 \;\; 1 \geq w_{t_q+1}.$$

Condition 5. If period iŒWZ»YZ, then $c_i < D(i)$. If D(i) = 1, then period iœWZ»YZ, and if iŒZ, then $c_i = 1$. In addition, t_qŒW»Z.

Proof.

Suppose period iŒYZ and $c_i \geq D(i)$. Since $N(i,t) \leq N(i,t_q) \leq D(i)$, and $n_{yz}(i,i) = 1$, $c_i+n_{yz}(i,t) > N(i,t)$ for all periods $t \geq i$. Therefore, we can reduce c_i by 1 and obtain a tighter valid inequality. Similarly, if iŒWZ and $c_i > D(i)$, we can reduce c_i by 1. Suppose iŒWZ and $c_i = D(i)$. The compensation condition implies that $c_i = n_z(i,i^*) = D(i) > 0$. Since by definition of $N(i,i^*)$, $n_{yz}(i,i^*)+n_z(i,i^*) \leq D(i)$, $n_{yz}(i,i^*) = 0$. Therefore, for any period $t \leq i$ with look ahead period $t^* \geq i$, since $t^* \leq i^*$, $c_t+n_{yz}(t,t^*) = c_i+n_{yz}(t,i-1) \geq N(t,t^*)$. Note that this condition is satisfied even if we drop w_i from the inequality, and so the inequality is valid. Therefore, the original inequality cannot be a facet. In particular, if D(i) = 1, then $c_i = 0$, and therefore, iœWZ»YZ. If iŒZ, then $c_i \leq D(i) =$ 1. Since iŒZ, $c_i \geq 1$. Therefore, $c_i = 1$. Condition 2 implies that t_qœS. Therefore, t_qŒW»Y»Z. If period t_qŒY, we can shift t_q to W and obtain a tighter valid inequality. For any period i, $c_i = n_z(i,i^*)$ and by shifting t_q to W, $n_z(i,i^*)$ does not change (even if $i^* = t_q$ does change). Therefore, t_qŒW»Z.