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Facets for the cut cone I

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We study facets of the cut cone C_n, i.e., the cone of dimension $\frac{1}{2}n(n-1)$ generated by the cuts of the complete graph on n vertices. Actually, the study of the facets of the cut cone is equivalent in some sense to the study of the facets of the cut polytope. We present several operations on facets and, in particular, a "lifting" procedure for constructing facets of C_{n+1} from given facets of the lower dimensional cone C_n. After reviewing hypermetric valid inequalities, we describe the new class of cycle inequalities and prove the facet property for several subclasses. The new class of parachute facets is developed and other known facets and valid inequalities are presented.

Key words: Max-cut problem, cone, polytope, facet, lifting, hypermetric inequality.

1. Introduction

1.1. *The general max-cut problem*

One of the main motivations of this work is to contribute to the polyhedral approach for the following max-cut problem. Given a graph $G=(V, E)$ with nodeset V and edgeset E and given a subset S of V, the set $D(S)$ consisting of the edges of E having exactly one endnode in S is called the *cut* (or *split*, or *dichotomy*) determined by S, or more precisely by the partition of V into S and $V-S$. When nonnegative weights c_e are assigned to the edges e of E, the *max-cut problem* consists of finding a cut $D(S)$ whose weight (defined as the sum of the weights of its elements) is as large as possible; the max-cut problem is NP-hard [26]. However, if we replace "as large" by "as small", we obtained the *min-cut problem* which is known to be polynomially solvable, using network-flow techniques. On the other hand, polynomial algorithms exist for the max-cut problem for some classes of graphs. This is the case, for instance, for planar graphs [30], for graphs not contractible to K_5 [6], for weakly bipartite graphs [28], the last result being based on a polyhedral approach; the class of weakly bipartite graphs includes, in fact, planar graphs and graphs not contractible to K_5 [25]. We refer to the paper by Barahona et al. [8] for a description of possible applications of the max-cut problem to statistical physics and some circuit layout design problems with numerical results.

A way to attack the max-cut problem is the following *polyhedral* approach which is classical in combinatorial optimization. For any subset S of V, let $\delta(S)$ denote the incidence vector of the cut defined by S, i.e., $\delta(S)_e = 1$ if $e \in D(S)$ and $\delta(S)_e = 0$ otherwise; $\delta(S)$ is also called the *cut vector* defined by S. The polytope $P_c(G) = \mathrm{Conv}(\delta(S)\colon S \subseteq V)$ is the *cut polytope* of the graph G. The max-cut problem can then be rephrased as the linear programming problem:

$$\begin{array}{ll} \max & c \cdot x \\ \text{such that} & x \in P_c(G). \end{array}$$

It is therefore crucial to be able to find the linear description of the cut polytope and characterize its facets. The study of the cut polytope for general graphs has been initiated in [6] and continued in [11]. It was proved in [11] that the cut polytope has the following nice property; namely, a description of the facets that contain any particular extreme point gives the description of the whole polytope. For this reason, it is enough to study the facets that contain the origin, i.e., the facets of the *cut cone* $C(G)$ generated by the cut vectors. Actually, this property is, more generally, a property of cycle polytopes of binary matroids (see [7]).

1.2. The cut cone C_n

The goal of this paper is to study facets of the cut cone $C_n = C(K_n)$, i.e., the cone generated by the cuts of the complete graph K_n on n vertices. There are several motivations for restricting our attention to the case of complete graphs. One is that the max-cut problem on a general graph G with n vertices can be represented as the max-cut problem on the complete graph K_n by assigning weight zero to the missing edges in G. Of course, if the graph G is sparse, working with the complete graph K_n instead of G may increase the size of the problem beyond computer limits; also, there are classes of sparse graphs for which one can have a simple complete description of the cut polytope, e.g., for graphs not contractible to K_5 [11]. On the other hand, the study of the cut polytope $P_c(K_n)$ of the complete graph gives some insight for general cut polytopes $P_c(G)$; for instance, every facet defining inequality of $P_c(K_n)$ also defines a facet of $P_c(G)$ if G is any subgraph of K_n containing the supporting graph of the inequality or if G is any graph containing K_n [17]. Another motivation comes from the fact that elements of the cut cone C_n can be interpreted as semi-metrics on n points. In fact, C_n coincides with the family of semi-metrics on n points which are embeddable into L^1; in these terms, the study of the cut cone was started by Deza in 1960 in [18] and continued e.g., in [3, 5, 20, 21, 38]. There are also some strong connections between the study of the cut cone and the following subjects: cone of all metrics and multicommodity flows (see, for instance, [5]), description of lattices (i.e., Z-modules) in terms of metrics on pointsets on the boundary of their holes [1, 38, 23]. In this paper, we concentrate on polyhedral aspects of the cut cone C_n; some connections with other polyhedral problems are mentioned in Section 1.5.

1.3. Basic notations

We denote by N the set $[1, n]=\{1, 2, \ldots, n\}$ and we set $n'=\frac{1}{2}n(n-1)$. If S is a subset of N, $\delta(S)\in\{0,1\}^{n'}$ denotes the incidence vector of the cut determined by S, i.e., $\delta(S)_{ij}=1$ if $|S\cap\{i,j\}|=1$ and $\delta(S)_{ij}=0$ otherwise for $1\le i<j\le n$. The complete graph K_n with nodeset N admits exactly $2^{n-1}-1$ nonzero distinct cuts $D(S)$ determined by all subsets S of N for which we can assume, for instance, that $1\notin S$, since $D(S)=D(N-S)$. The cut cone C_n is a full-dimensional polyhedral cone in $\mathbb{R}^{n'}$ which contains the origin [20]. Given a vector $v\in\mathbb{R}^{n'}$, the inequality $v\cdot x\le 0$ is called *valid* for the cone C_n if it is satisfied by all vectors x of C_n or, equivalently, by all cut vectors $\delta(S)$. Then, the set $F_v=\{x\in C_n : v\cdot x=0\}$ is the *face* generated by the valid inequality $v\cdot x\le 0$, denoted simply as v. The nonzero cut vectors $\delta(S)$ which belong to F_v are called the *roots* of v, for short, we sometimes say that S itself is a root of v. The set of roots of v is denoted as $R(v)$. The *dimension* of the face F_v, denoted by $\dim(v)$, is the maximum number of affinely independent points in F_v minus one, or, equivalently, since F_v contains the origin, the maximum number of linearly independent roots of v; any set of $\dim(v)$ linearly independent roots is called a *basis* of v. The face F_v is called *simplicial* when $\dim(v)$ coincides with the cardinality of $R(v)$, i.e., when F_v is a polyhedral (unbounded) simplex. A *facet* is a face of dimension $n'-1=\frac{1}{2}n(n-1)-1$; one says then that the valid inequality v is facet defining.

There are several ways of describing a valid inequality $v\cdot x\le 0$. First, one can simply give explicitly the vector v whose coordinates are then ordered lexicographically as $v=(v_{12},\ldots,v_{1n}; v_{23},\ldots,v_{2n};\ldots; v_{n-1n})$. A more attractive way is to represent v by its *supporting graph* $G(v)$; $G(v)$ is the weighted graph with nodeset N whose edges are the pairs (i,j) for which v_{ij} is not zero, the edge (i,j) being then assigned weight v_{ij}. When the coefficients v_{ij} take only the values $0, 1, -1$, the inequality $v\cdot x\le 0$ is called *pure* and $G(v)$ is a bicolored graph (edges with weight $+1$ will be represented by a plain line while edges with weight -1 by a dotted line). Finally, our graph notations are classical; for instance, we define the cycle $C(i_1,\ldots,i_f)$ as the graph with nodes $i_1,\ldots,i_f$ and with edges (i_k, i_{k+1}) for $1\le k\le f$ (setting $i_{f+1}=i_1$) and the path $P(i_1,\ldots,i_f)$ has edges (i_k, i_{k+1}) for $1\le k\le f-1$.

1.4. Methods for checking facets

We use various techniques for proving the facet property for a given valid inequality $v\cdot x\le 0$.

(a) *The "polyhedral" method.* It consists of proving that, if $b\cdot x\le 0$ is another valid inequality of C_n such that the face F_v is contained in the face F_b, i.e., $b\cdot x=0$ whenever $v\cdot x=0$, then $b=\alpha v$ for some positive scalar α. We state two lemmas that will be thoroughly used in this type of proof; they follow from Lemmas 2.5 in [9] and [11].

Lemma 1.1. *Let $b \cdot x \le 0$ be a valid inequality of C_n. Let p, q be distinct elements of N and S be a subset of $N - \{p, q\}$ (possibly empty) such that the cut vectors $\delta(S)$, $\delta(S \cup \{p\})$, $\delta(S \cup \{q\})$ and $\delta(S \cup \{p, q\})$ define roots of b. Then, $b_{pq} = 0$ holds.* □

Lemma 1.2. *Let $b \cdot x \le 0$ be a valid inequality of C_n. Let p, q, r be distinct points of N and A be a subset of $N - \{p, q, r\}$. If the cut vectors $\delta(A \cup \{r\})$, $\delta(A \cup \{p, r\})$, $\delta(A \cup \{q\})$, $\delta(A \cup \{p, q\})$ define roots of b, then $b_{pq} = b_{pr}$ holds.* □

(b) *The "lifting" technique* that we shall describe in Section 2.2, for constructing iteratively facets of C_{n+1} from facets of C_n.

(c) *The "direct" method* which consists of finding a set of $\frac{1}{2}n(n-1)-1$ roots of v and proving that they are linearly independent; for small values of n: $n = 7, 8, 9$, linear independence can be tested by computer and, for general n, it is usually done by determinant manipulation.

1.5. Related polytopes and intersection pattern

It will sometimes be useful to represent cuts of K_n not only by their cut vectors $\delta(S)$, but also by their intersection vectors $\pi(S)$; actually, Deza [20] initiated its study of C_n within this framework of "intersection pattern" that we now describe (see also [5]).

Given vectors $z = (z_{ij})_{1 \le i < j \le n}$ and $y = (y_{ij})_{2 \le i \le j \le n}$, the function $y = f_1(z)$ is defined by

$$\begin{aligned} y_{ij} &= \tfrac{1}{2}(z_{1i} + z_{1j} - z_{ij}) \quad \text{for } 2 \le i < j \le n, \\ y_{ij} &= z_{1i} \quad \text{for } 2 \le i \le n. \end{aligned} \tag{1.3}$$

If S is a subset of N, the vector $\pi(S) = f_1(\delta(S))$ is called the *intersection vector* of *S pointed at position* 1; in this definition, we specialized position 1, but any other position k of N can be specialized as well with function f_k being correspondingly defined. The function f_1 is a bijective linear transformation. A first useful corollary is that, for subsets $S_1, \ldots, S_k$ of N, the families $\{\delta(S_1), \ldots, \delta(S_k)\}$ and $\{\pi(S_1), \ldots, \pi(S_k)\}$ are simultaneously linearly independent; we sometimes prefer to deal with the latter family, e.g., in the lifting procedure (see Section 2.2), since intersection vectors contain "more" zeros.

Another important implication is the connection between the cut polytope and the boolean quadric polytope considered by Padberg [34]. The *Boolean quadric polytope* is the polytope $\mathrm{QP}^n = \mathrm{Conv}(\{(x, y)\colon x \in \{0,1\}^n,\ y \in \{0,1\}^{n'}$ and $y_{ij} = x_i x_j$ for $1 \le i < j \le n\})$. It models the following general unconstrained quadratic zero-one program: $\max(c \cdot x + x^{\mathrm{T}} Q x\colon x \in \{0,1\}^n)$ where $c \in \mathbb{R}^n$ and Q is an $n \times n$ symmetric matrix (see [10, 35]). Let us introduce a new element, say 0, and consider the complete graph K_{n+1} with nodeset $N \cup \{0\}$; its cut polytope is $P_c(K_{n+1}) = \mathrm{Conv}(\delta(S)\colon S \subseteq N)$. It is easily observed that the vertices of QP^n are exactly the intersection vectors $\pi(S)$ pointed at position 0 for $S \subseteq N$ (after setting $x=(\pi(S)_{ii})_{1\le i\le n}$ and $y=(\pi(S)_{ij})_{1\le i<j\le n}$). Therefore, the mapping f_0 is a linear bijective transformation mapping the cut polytope $P_c(K_{n+1})$ onto the boolean quadric polytope QP^n. This simple but interesting connection was independently discovered, in different terms, by several authors (see [19, 20, 31, 10, 15, 16]). Consequently, any result concerning the cut polytope can be translated into a result on the boolean quadric polytope and conversely. For instance, the inequality

$$\sum_{0\le i<j\le n} c_{ij}x_{ij} \le d \tag{1.4}$$

defines a valid inequality (resp. facet) of $P_c(K_{n+1})$ if and only if the inequality

$$\sum_{1\le i\le n} a_i x_i + \sum_{1\le i<j\le n} b_{ij}y_{ij} \le d \tag{1.5}$$

defines a valid inequality (resp. facet) of QP^n, where a, b, c are related by

$$\begin{aligned} c_{0i} &= a_i + \tfrac{1}{2}\sum_{1\le j\le n,\, j\ne i} b_{ij} \quad \text{for } 1\le i\le n,\\ c_{ij} &= -\tfrac{1}{2}b_{ij} \quad \text{for } 1\le i<j\le n. \end{aligned} \tag{1.6}$$

This connection will be used in Remark 3.15. Another closely related polytope is the bipartite subgraph polytope which is the "monotonization" of the cut polytope; it is the convex hull of the incidence vectors of the bipartite subgraphs, the maximal ones corresponding to the cuts (see [9]). Other related polytopes are the clique-partitioning polytope [29], the equipartition polytope [14], and, in the more general framework of binary matroids, the cycle polytope [7].

1.6. Contents of the paper

Section 2 contains the permutation and switching operations which permit derivation of new facets of the cut cone from existing ones. We also describe a "lifting" procedure for constructing facets of the cone C_{n+1} on $n+1$ points from a given facet of the cone C_n on n points.

In Section 3, we describe classes of valid inequalities: hypermetric inequalities and new inequalities which we call cycle inequalities. We wish to point out that these cycle inequalities are distinct from those considered in [7, 9, 11]. The hypermetric inequalities are of the form $\sum_{1\le i<j\le n} b_ib_jx_{ij}\le 0$, where $b_1,\ldots,b_n$ are integers whose sum is equal to 1, while cycle inequalities are of the form $\sum_{1\le i<j\le n} b_ib_jx_{ij} - \sum_{(i,j)\in C} x_{ij}\le 0$, where the sum of the integers b_i is now equal to 3 and C is a suitable cycle. Our lifting technique provides an essential tool for showing that large classes of hypermetric and cycle inequalities are facet inducing. We feel, however, that hypermetric and cycle inequalities belong, in fact, to a much larger class of valid inequalities which may arise from integers b_i with suitably chosen sum; we suggest some possible extensions in this direction, but these ideas will be further developed in a follow-up work [24].

In Section 4, after presenting the new class of parachute facets, we discuss other known classes, in particular those of Barahona, Grötschel and Mahjoub and of Poljak and Turzik and we investigate a class of faces introduced by Kelly. After summing up known facts for the cut cone on seven points, we conclude the section by mentioning some results on simplicial faces and some open questions.

Section 5 contains the proofs of the results from the preceding sections which, in view of their length, are delayed in order to improve the flow of the text.

2. Operations on facets

We describe several operations: permutation, switching, lifting which produce "new" facets from "old" ones for the cut cone.

2.1. Permutation and switching

Let $v \cdot x \leq 0$ be a valid inequality of the cone C_n. Let σ be a permutation of the set N. The coordinates of the vector $x \in \mathbb{R}^{n'}$ being ordered lexicographically, we define the vector x^σ by $x^\sigma_{ij} = x_{\sigma(i)\sigma(j)}$ for $1 \leq i < j \leq n$ after setting $x_{\sigma(i)\sigma(j)} = x_{\sigma(j)\sigma(i)}$ when $\sigma(i) > \sigma(j)$. The inequality $v^\sigma \cdot x \leq 0$, obtained by *permutation of v by σ*, is valid for C_n and both inequalities v, v^σ are simultaneously facet defining. Hence, the permutation operation preserves valid inequalities and facets of C_n.

Let $v \cdot x \leq \alpha$ be a valid inequality of the cut polytope $P_c(K_n)$. Given a subset A of N, we define the vector v^A by $v^A_{ij} = -v_{ij}$ if $(i, j) \in D(A)$ and $v^A_{ij} = v_{ij}$ if $(i, j) \notin D(A)$ and we set $\alpha^A = \alpha - v \cdot \delta(A)$. Then, the inequality $v^A \cdot x \leq \alpha^A$ is valid for $P_c(K_n)$; one says that it is obtained by *switching the inequality $v \cdot x \leq \alpha$ by the cut $\delta(A)$*. Furthermore, inequality $v \cdot x \leq \alpha$ is facet defining if and only if inequality $v^A \cdot x \leq \alpha^A$ is facet defining. This fact follows from the observation that the roots of $v^A \cdot x \leq \alpha^A$ are exactly the cut vectors $\delta(S \triangle A)$ for which $\delta(S)$ is root of $v \cdot x \leq \alpha$ and that the families $\{\delta(S_1), \ldots, \delta(S_k)\}$ and $\{\delta(S_1 \triangle A), \ldots, \delta(S_k \triangle A)\}$ are simultaneously affinely independent. When we switch the inequality $v \cdot x \leq \alpha$ by a root, i.e., by a cut such that $v \cdot \delta(A) = \alpha$, we obtain a valid inequality $v^A \cdot x \leq 0$ of the cut cone C_n. Consequently, the "switching by roots" operation preserves valid inequalities and facets of C_n. Furthermore, if $C_n = \{x\colon Mx \leq 0\}$, then $P_c(K_n) = \{x\colon Mx \leq 0$ and $M'x \leq b\}$ where vector b and matrix M' are derived from M through the "switching by cuts" operation [11]. The switching by roots operation was introduced in [20] for the cut cone C_n; the general switching by cut operation for the cut polytope of an arbitrary graph was given in [11] where it is called "changing the sign of a cut".

Remark 2.1. One can represent the switching operation using matrices as follows. Given a vector $v = (v_{ij})_{1 \leq i < j \leq n}$, define the $n \times n$ symmetric matrix $M(v)$ with zeros on its diagonal and $M(v)_{ij} = M(v)_{ji} = v_{ij}$ for $1 \leq i < j \leq n$ and, given a subset S of $[1, n]$, define the $n \times n$ diagonal matrix $D(S)$ by $D(S)_{ii} = -1$ if $i \in S$ and $D(S)_{ii} = 1$ otherwise. Then, the vector v^S obtained by switching of v by $\delta(S)$ is equivalently defined by relation $M(v^S)=D(S)\ M(v)\ D(S)$. In the case when $v_{ij}=1$ or -1 for all $1\le i<j\le n$, the matrix $M(v)$ can be interpreted as the $(1,-1)$-adjacency matrix of a graph H on nodeset $[1,n]$ whose edges are the pairs (i,j) for which $v_{ij}=-1$ and, then, the graph whose $(1,-1)$-adjacency matrix is $M(v^S)$ is a switching of H in the sense of Seidel (see, e.g., [13]).

Call two inequalities v, v' *equivalent* if v' is obtained from v by permutation and/or switching (by root). This defines an equivalence relation on valid inequalities; for this, observe that, for σ, σ' permutations of N, one has $(v^{\sigma})^{\sigma'}=v^{\sigma'\sigma}$ and, for A, B subsets of N, one has $(v^A)^B=v^{A\triangle B}$. This equivalence relation preserves facets of C_n; therefore, at least from a theoretical point of view, for describing all facets of C_n, it is, in fact, enough to give a list of *canonical* facets of C_n, i.e., a list containing a facet of each equivalence class. We will further specify how this equivalence relation behaves for the special classes of hypermetric and cycle inequalities.

2.2. *The lifting procedure*

Let $v\in\mathbb{R}^{n'}$, $n'=\frac{1}{2}n(n-1)$, and suppose that $v\cdot x\le 0$ defines a facet of C_n. Our goal is to "lift" this facet of C_n to a facet of C_{n+1}. For this, we want to find n additional coefficients: v_{in+1} for $1\le i\le n$ such that, if v' denotes the vector of length $\frac{1}{2}n(n+1)$ obtained by concatenating v with these n new coefficients, then $v'\cdot x\le 0$ defines a facet of C_{n+1}. The next theorem shows that lifting by zero, i.e., adding only zero coefficients, is always possible.

Theorem 2.2 [20]. *Let v be a vector of length $\frac{1}{2}n(n-1)$ and $v'=(v,0,\ldots,0)$ of length $\frac{1}{2}n(n+1)$. The following assertations are equivalent:*
(i) *$v\cdot x\le 0$ defines a facet of C_n.*
(ii) *$v'\cdot x\le 0$ defines a facet of C_{n+1}.*

Therefore, any facet of C_n extends to a facet of C_m for all $n\le m$. The proof of this result has not been published, so we give it here; it will help us at the same time to present the basic ideas of the lifting procedure. We must first state a technical lemma. Let F be a subset of the set $E(n)=\{(i,j)\colon 1\le i<j\le n\}$ and $F'=E(n)-F$ denote its complement. For a vector $x\in\mathbb{R}^{E(n)}$, we denote by x_F its projection onto $\mathbb{R}^F$ and, for a subset X of $\mathbb{R}^{E(n)}$, set $X_F=\{x_F: x\in X\}$ and $X^F=\{x\in X\colon x_F=0\}$. Let v be a valid inequality of C_n with set of roots $R(v)$; then, $\mathrm{r}(v,F)$ denotes the rank of the set $R(v)_F$ and $\mathrm{r}[v,F]$ denotes the rank of the set $R(v)^F$.

Lemma 2.3. *The following assertions hold:*
(i) *If $\mathrm{r}(v,F)=|F|$ and $\mathrm{r}[v,F]=|F'|-1$, then v is facet defining.*
(ii) *If v is facet defining and $v_{F'}\ne 0$, then $\mathrm{r}(v,F)=|F|$.*
(iii) *If v is facet defining and $v_{F'}=0$, then $\mathrm{r}(v,F)=|F|-1$.*

Proof. We first show (i). By assumption, we can find a set $A \subseteq R(v)$ of $|F|$ vectors whose projections on F are linearly independent and a set $B \subseteq R(v)$ of $|F'|-1$ linearly independent vectors whose projections on F are zero. It is easy to verify that $A \cup B$ is linearly independent, which implies that v is facet defining since $|F|+|F'|=n'-1=\frac{1}{2}n(n-1)-1$.

We prove now (ii). Since v is facet defining, we can find a set $A \subseteq R(v)$ of $n'-1$ linearly independent roots. Let M denote the $(n'-1)\times n'$ matrix whose rows are the vectors of A, its columns being indexed by $F \cup F'$. Hence, all columns but one are linearly independent. We distinguish two cases:

- either, all columns indexed by F are linearly independent, i.e., $\mathrm{r}(v, F)=|F|$,
- or all columns indexed by F' are linearly independent and, then, $\mathrm{rank}(A_F)=|F|-1$ from which one easily deduces that $\mathrm{r}(v, F)=|F|-1$.

Suppose we are in the second case, so $\mathrm{r}(v, F)=|F|-1$. Denote by T_1 a subset of $|F|-1$ vectors of A whose projections on F are linearly independent, $T_2=A^F$ and T_3 is the set of remaining rows of M; thus $|T_2 \cup T_3|=|F'|$. Given a vector x of T_3, x_F can be written as linear combination of the projections on F of the vectors of T_1:

$$x_F=\sum_{a\in T_1}\beta_a a_F;$$

$$\text{set}\quad x'=x-\sum_{a\in T_1}\beta_a a,\qquad \text{so}\quad x'_F=0.$$

It is easy to verify that $T_2 \cup T'_3$ is a set of $|F'|$ linearly independent vectors, where $T'_3=\{x' \colon x \in T_3\}$. Observe now that the vectors x of the set $T_2 \cup T'_3$ satisfy: $v \cdot x=0$ and $x_F=0$, which implies that $v_{F'}=0$, concluding the proof of (ii).

For proving (iii), observe that, if $r(v, F)=|F|$, then $r(v, F')=|F'|-1$ which, using (ii), implies that $v_F=0$ and therefore $v_{F'}\neq 0$. □

Proof of Theorem 2.2. We assume first that (ii) holds. Consider the index set $F=\{(1, n+1), \ldots, (n, n+1)\}$ and its complement in $E(n+1)$, $F'=\{(i,j)\colon 1\leq i<j\leq n\}$. By construction, we have that $v_F=0$; hence Lemma 2.3 (iii) implies that $r(v, F')=|F'|-1$ from which we deduce that v defines a facet of C_n.

We suppose now that v defines a facet of C_n; hence we can find $n'-1$ linearly independent roots of v of the form $\delta(S_j)$ with $1 \notin S_j$ and $S_j \subseteq N$ for $1\leq j\leq n'-1$. For $i \in N$, set $F_i=\{(1, i), \ldots, (i-1, i), (i, i+1), \ldots, (i, n)\}$. Since $v \neq 0$, the projection of v on $F'_i=E(n)-F_i$ is nonzero for some $i \in N$; we can suppose w.l.o.g. that $i=1$. Hence, we deduce from Lemma 2.3(ii) that $\mathrm{r}(v, F_1)=|F_1|=n-1$; therefore, there exist $n-1$ roots of v: $\delta(T_k)$ with $1 \notin T_k \subseteq N$ for $1 \leq k \leq n-1$, whose projections on F_1 are linearly independent. We construct $\frac{1}{2}n(n+1)-1=\frac{1}{2}n(n-1)+n-1$ roots of v' as follows: for $1\leq j\leq n'-1$, define the subsets $S'_j=S_j$ of $N \cup \{n+1\}$ and, for $1 \leq k \leq n-1$, set: $T'_k=T_k \cup \{n+1\}$ and $T'_n=\{n+1\}$; hence $1 \notin S'_j, T'_k$; $n+1 \notin S'_j$ and $n+1 \in T'_k$, for all j, k. We prove that the $\frac{1}{2}n(n+1)-1$ cut vectors defined by the sets S'_j, T'_k are linearly independent; it is in fact easier to verify that their intersection vectors (pointed at position 1) are linearly independent. For this, let M be the matrix whose rows are the vectors $\pi(S'_j)$, $\pi(T'_k)$, its columns being indexed by

$G \cup H \cup \{(n+1, n+1)\}$ where $G=\{(i,j): 2 \le i \le j \le n\}$ and $H=\{(i, n+1): 2 \le i \le n\}$. The fact that M is nonsingular follows by examining its block configuration using the easy observations:

$$\pi(S_j')_G = \pi(S_j) \text{ and } \pi(S_j')_{H \cup (n+1, n+1)} = 0 \text{ for all } 1 \le j \le n'-1,$$

$$\pi(T_k')_H = \delta(T_k)_F \text{ (setting } F=F_1) \text{ and } \pi(T_k')_{n+1,n+1} = 1 \text{ for } 1 \le k \le n-1,$$

$$\pi(T_n')_H = 0 \text{ and } \pi(T_n')_{n+1, n+1} = 1. \quad \square$$

Generally, suppose v defines a facet of C_n. We wish to lift v to a facet of C_{n+1}, i.e., to find a vector v' of length $\frac{1}{2}n(n+1)$ defining a facet of C_{n+1}; the vector v' is obtained by concatenating the vector v — after eventually, altering its coefficients in a suitable way — with n new well chosen coefficients. We now describe a set of conditions which, when they are satisfied, ensure that lifting is possible and produce a new facet v' of C_{n+1}. Since v defines a facet of C_n, we can find $n'-1$ linearly independent roots: $\delta(S_j)$ with $1 \notin S_j \subseteq N$ for $1 \le j \le n'-1$. Define the subsets $S_j' = S_j$ of $N \cup \{n+1\}$; then the intersection vectors (pointed at position 1) $\pi(S_j')$ are $n'-1$ linearly independent vectors of length $\frac{1}{2}n(n+1)$ whose projections on the index set $\{(2, n+1), \ldots, (n+1, n+1)\}$ are the zero vector. Consider the conditions:

$$v' \text{ defines a valid inequality of } C_{n+1}, \tag{2.4}$$

$$\text{the cut vectors } \delta(S_j') \text{ are roots of } v', \quad \text{for } 1 \le j \le n'-1, \tag{2.5}$$

There exist n cut vectors $\delta(T_k)$, with $1 \notin T_k$, $n+1 \in T_k \subseteq N \cup \{n+1\}$ for $1 \le k \le n$, which are roots of v' and such that the incidence vectors of the sets T_k are linearly independent. (2.6)

Proposition 2.7. *With the above notation, if conditions* (2.4), (2.5), (2.6) *hold, then v' defines a facet of C_{n+1}.*

Proof. The proof follows closely that for Theorem 2.2 and consists of verifying that the vectors $\pi(S_j')$, $1 \le j \le n'-1$, and $\pi(T_k)$, $1 \le k \le n$, are linearly independent. Set $G=\{(i,j): 2 \le i \le j \le n\}$, $H=\{(i, n+1): 2 \le i \le n+1\}$. Let M denote the matrix whose columns are indexed by $G \cup H$, its first $n'-1$ rows are the vectors $\pi(S_j')$ and its last n rows are the vectors $\pi(T_k)$.

Then M has the following block configuration:

P	0
X	Q

where P is the $(n'-1) \times n'$ matrix whose rows are the vectors $\pi(S_j)$, its rank is $n'-1$ by assumption and Q is the $n \times n$ matrix whose rows are the projections on $\{2, \ldots, n+1\}$ of the incidence vectors of the sets T_k, its rank is n from condition (2.6). Therefore matrix M has rank $n'-1+n$, implying that v' is facet defining. $\square$

We describe now a condition on v, v' which is sufficient for ensuring that (2.5) holds. Suppose that the vectors v, v' satisfy $v_{ij} = v'_{ij}$ for all $2 \leq i < j \leq n$ and the following relation:

$$v_{1i} = v'_{1i} + v'_{in+1} \quad \text{for } 2 \leq i \leq n. \tag{2.8}$$

This amounts to saying that the supporting graph $G(v')$ of v' is obtained from the supporting graph $G(v)$ of v by *splitting* node 1 into nodes 1, $n+1$ and correspondingly splitting the edge weights v_{1i} into v'_{1i}, v'_{in+1} for $2 \leq i \leq n$, all other coefficients v_{ij} remaining unchanged. It is easily verified that $v \cdot x = v' \cdot x$ for all cut vectors $x = \delta(S)$ with $S \subseteq [2, n]$; hence any root of v defines a root of v' and, therefore, condition (2.5) holds. We wish to point out that this node-splitting operation just described is distinct from the node-splitting procedure from [11].

We will see in the next section how the lifting procedure provides a very powerful tool for generating classes of facets, in particular when applied to hypermetric and cycle inequalities; we shall use in fact, the more specific node-splitting operation, so condition (2.5) holds and, since condition (2.4) will be automatically satisfied, the crucial point consists of satisfying (2.6).

3. Hypermetric and cycle inequalities

The first nontrivial known class of valid inequalities of the cut cone is the class of hypermetric inequalities, introduced in 1960 by Deza [18] and later, independently, by Kelly [33]. For small values of n, $n = 3, 4, 5, 6$, hypermetric facets are in fact sufficient for describing C_n; this was shown for $n \leq 5$ by Deza [18, 20] and for $n = 6$, using computer check, by Avis and Mutt [4]. However, for $n \geq 7$, there exist non-hypermetric facets. After examining in Section 3.1 hypermetric inequalities, we introduce in Section 3.2 the new class of cycle inequalities; we prove the facet property for some subclasses of the above two classes. We also discuss some possible extensions of hypermetric and cycle inequalities. In Section 3.3, we exhibit some upper bounds for the coefficients of hypermetric and cycle facets.

3.1. *Hypermetric inequalities* $\mathrm{Hyp}_n(b)$

Let $b = (b_1, \ldots, b_n)$ where the b_i's are integers satisfying

$$\sum_{1 \leq i \leq n} b_i = 1. \tag{3.1}$$

The inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0 \tag{3.2}$$

is valid for C_n; it is called the *hypermetric inequality defined by* b and denoted by $\mathrm{Hyp}_n(b)$. If we set $k = \sum_{b_i < 0} |b_i|$, then $\sum_{1 \leq i \leq n} |b_i| = 2k+1$ holds and one says that the hypermetric inequality is $(2k+1)$-*gonal*. Pure hypermetric inequalities are obtained when $b_i = +1$ or -1 for all i; when all (resp. all but one) negative coefficients b_i are equal to -1, the hypermetric inequality is called *linear* (resp. *quasilinear*). Validity of (3.2) follows from the fact that, for any subset S of N, we have: $\sum_{1 \le i<j \le n} b_i b_j \delta(S)_{ij} = b(S)(1-b(S)) \le 0$, since $b(S) = \sum_{i \in S} b_i$ is an integer. Furthermore, the roots of $\mathrm{Hyp}_n(b)$ are the cut vectors $\delta(S)$ for which $b(S) = 0$ or 1.

The lifting by zero operation from Section 2.2 amounts to adding new coefficients b_i which are equal to zero; hence, $\mathrm{Hyp}_n(b)$ and $\mathrm{Hyp}_{n+1}(b, 0)$ are simultaneously facet inducing. Both permutation and switching (by roots) operations preserve the class of hypermetric inequalities. In fact, permutation of $\mathrm{Hyp}_n(b)$ amounts to permuting the b_i's: if σ is a permutation on n points, the inequality obtained from $\mathrm{Hyp}_n(b)$ by permutation by σ is $\mathrm{Hyp}_n(b_{\sigma(1)}, \dots, b_{\sigma(n)})$. Also, if S is a subset of N with $b(S) = 0$, then the inequality obtained from $\mathrm{Hyp}_n(b)$ by switching by the root $\delta(S)$ is $\mathrm{Hyp}_n(b')$ where $b'_i = -b_i$ if $i \in S$ and $b'_i = b_i$ otherwise.

We present some known hypermetric facets:

$$\mathrm{Hyp}_3(1, 1, -1) \quad (\textit{triangle} \text{ facet}), \tag{3.3}$$

$$\mathrm{Hyp}_5(1, 1, 1, -1, -1) \quad (\textit{pentagonal} \text{ facet}), \tag{3.4}$$

$$\mathrm{Hyp}_6(2, 1, 1, -1, -1, -1), \tag{3.5}$$

$$\mathrm{Hyp}_7(1, 1, 1, 1, -1, -1, -1), \tag{3.6}$$

$$\mathrm{Hyp}_7(3, 1, 1, -1, -1, -1, -1), \tag{3.7}$$

$$\mathrm{Hyp}_8(3, 2, 2, -1, -1, -1, -1, -2), \tag{3.8}$$

$$\mathrm{Hyp}_9(2, 2, 1, 1, -1, -1, -1, -1, -1). \tag{3.9}$$

One verifies trivially that (3.3) is facet defining; one then deduces that (3.4)-(3.9) define facets by applying the next Theorem 3.12 based on our lifting procedure. As an application, let us recall the linear description of C_n for $n \le 6$ which consists only of hypermetric facets. For $n = 3$, 4, the only canonical facet is (3.3) and for $n = 5$, the canonical facets are (3.3), (3.4) [21, 18]. For $n = 6$, the canonical facets are (3.3)-(3.5) and C_6 has exactly 210 facets obtained from permutation/switching of (3.3)-(3.5) [4].

The general lifting procedure from Section 2.2 can be specialized for hypermetric facets as follows. Let $b = (b_1, \dots, b_n)$ satisfying (3.1) and suppose $\mathrm{Hyp}_n(b)$ is a facet of C_n. Given an integer c, set $b' = (b_1 - c, b_2, \dots, b_n, c)$; hence b' satisfies (3.1). We say that $\mathrm{Hyp}_{n+1}(b')$ is obtained from $\mathrm{Hyp}_n(b)$ by *c-lifting*. Then, the conditions (2.4), (2.5) of the lifting procedure described in Proposition 2.7 always hold. We are left with the problem of finding a suitable value of c for which condition (2.6) holds; this question can be rephrased as follows:

Problem 3.10. Given any integers $b_2, \dots, b_n$, find an integer c such that there exists an $n \times n$ nonsingular binary matrix M satisfying:
- its last column consists of all ones,
- for all row vectors x of M, $b^* \cdot x = 0$ or 1, where $b^* = (b_2, \dots, b_n, c)$.

This problem seems quite hard in general. The following results show that, for quasilinear hypermetric facets, (-1)-lifting is always possible and c-lifting is possible for suitable positive c. These results were stated in [20] and a sketch of the proofs was given in the accompanying document (kept in the Academy of Sciences of Paris) which was never published; so, we give the full proofs in this paper.

Theorem 3.11 [20]. *Let $b_1, \ldots, b_n$ be integers satisfying* (3.1) *and suppose that $b_2 \geq b_3 \geq \cdots \geq b_f > 0$ and $b_i = -1$ for $f+1 \leq i \leq n$ with $f \geq 2$ and $n \geq 4$. Suppose furthermore that* $\mathrm{Hyp}_n(b_1, \ldots, b_n)$ *is a facet of C_n; then*:

(i) $\mathrm{Hyp}_{n+1}(b_1+1, b_2, \ldots, b_n, -1)$ *is a facet of C_{n+1}.*

(ii) $\mathrm{Hyp}_{n+1}(b_1-c, b_2, \ldots, b_n, c)$ *is a facet of C_{n+1}, for all c such that $0 < c \leq n-f-b_2$.*

Theorem 3.12 [20]. *Let $b = (b_1, \ldots, b_n)$ consist of integers satisfying* (3.1) *and suppose that $b_1 \geq b_2 \geq \cdots \geq b_f > 0 > b_{f+1} > \cdots \geq b_n$.*

(i) *If* $\mathrm{Hyp}_n(b)$ *is a facet of C_n, then, either $f=2$ and $b=(1,1,-1)$, or $f=n-2$ and $b_1=1$, or $3 \leq f \leq n-3$.*

(ii) *In the linear case, i.e., $b_n=-1$;* $\mathrm{Hyp}_n(b)$ *is facet inducing if and only if, either $b=(1,1,-1)$, or $b=(1,1,1,-1,-1)$, or $3 \leq f \leq n-3$.*

(iii) *In the quasilinear case, i.e., $b_{n-1}=-1$ if $f<n-1$; $Hyp_n(b)$ is facet inducing if and only if, either $b=(1,1,-1)$, or $b=(1,\ldots,1,-1,-n+4)$, or $3 \leq f \leq n-3$ and condition*: (QL) $b_1+b_2 \leq n-f-1+\mathrm{sign}|b_1-b_f|$ *holds.*

Observe that, for a linear hypermetric inequality, condition (QL) always holds whenever $3 \leq f \leq n-3$. Also, the inequality $\mathrm{Hyp}_n(1, \ldots, 1, b_{n-1}, b_n)$ from case (i), $f=n-2$, is facet inducing, since it is equivalent to the (linear) hypermetric facet $\mathrm{Hyp}_n(-b_n, -b_{n-1}, 1, -1, \ldots, -1)$.

Remark 3.13. Take $k \geq 3$ and positive integers $t_1, \ldots, t_n$ with $\sum_{1 \leq i \leq n} t_i = 2k+1$ and $\sum_{t_i > 1} t_i \leq k-1$; then the inequality

$$\sum_{1 \leq i < j \leq n} t_i t_j x_{ij} \leq k(k+1) \tag{3.14}$$

defines a facet of the cut polytope $P_c(K_n)$ [11, Theorem 2.4]. It is observed in [15] that this inequality identifies — via switching — with a subclass of hypermetric inequalities. For this, set $t_1 = \cdots = t_p = 1 < t_{p+1} \leq \cdots \leq t_n$, hence $p \geq k+2$; after switching the above inequality by the root $\{1, 2, \ldots, k\}$, we obtain the linear hypermetric inequality $\mathrm{Hyp}_n(1, \ldots, 1, t_{p+1}, \ldots, t_n, -1, \ldots, -1)$ consisting of $p-k \geq 2$ coefficients $+1$ and $k \geq 3$ coefficients -1, henceforth, using switching, the facet property for (3.14), can alternatively be derived from Theorem 3.12.

Remark 3.15. The clique and cut inequalities introduced by Padberg [35] for the boolean quadric polytope correspond, in fact, via the transformation between the cut polytope $P_c(K_{n+1})$ and the boolean quadric polytope QP^n discussed in 1.5 and via switching, to some class of hypermetric inequalities.

Given a subset S of N with $s=|S|\geq 2$ and $1\leq \alpha \leq s-2$, the *clique inequality*:

$$\alpha \sum_{i\in S} x_i - \sum_{(i,j)\in S\times S} y_{ij} \leq \tfrac{1}{2}\alpha(\alpha+1) \tag{3.16}$$

is a facet of QP^n [35, Theorem 4]. Using relation (1.6), (3.16) can be translated into the following facet of $P_c(K_{n+1})$:

$$(\alpha - \tfrac{1}{2}(s-1)) \sum_{i\in S} z_{0i} + \tfrac{1}{2} \sum_{(i,j)\in S\times S} z_{ij} \leq \tfrac{1}{2}\alpha(\alpha+1), \tag{3.17}$$

which is, in fact, a subcase of inequality (3.14) and, hence, from Remark 3.13, identifies — via switching — with some quasilinear hypermetric facet.

Similarly, the *cut inequality*

$$-\sum_{i\in S} x_i - \sum_{(i,j)\in S\times S} y_{ij} + \sum_{(i,j)\in S\times T} y_{ij} - \sum_{(i,j)\in T\times T} y_{ij} \leq 0, \tag{3.18}$$

where S, T are disjoint subsets of N of respective cardinalities $s\geq 1$, $t\geq 2$, is a facet of QP^n [35, Theorem 5] which corresponds to the facet

$$(t-s-1)\left(\sum_{i\in S} z_{0i} - \sum_{i\in T} z_{0i}\right) + \sum_{\substack{(i,j)\in S\times S\\ \text{or}\,(i,j)\in T\times T}} z_{ij} - \sum_{(i,j)\in S\times T} z_{ij} \leq 0 \tag{3.19}$$

of $P_c(K_{n+1})$; in fact, (3.19) coincides with the quasilinear hypermetric inequality $\mathrm{Hyp}_{n+1}(b)$ where $b_0=s-t+1$, $b_i=-1$ for $i\in S$, $b_i=1$ for $i\in T$ and $b_i=0$ otherwise.

Other examples of facets obtained with our lifting procedure will be given in [17, 24]. For instance, $\mathrm{Hyp}_n(w,\ldots,w,-w,\ldots,-w,1,\ldots,1,-1,\ldots,-1)$ consisting of $a+c$ coefficients $+w$, a coefficients $-w$, b coefficients $+1$ and $b+cw-1$ coefficients -1, is facet inducing whenever a, b, c, w are nonnegative integers such that $c\geq 0$, $b\geq w+1$ [24]; also, the inequality $\mathrm{Hyp}_n(2c+1,\ 3,\ 2,\ -1,-1,-1,-2,\ldots,-2)-\mathrm{Hyp}_n(c,1,1,0,0,0,-1,\ldots,-1)\leq 0$ (consisting of c coefficients -2 in the first part and c coefficients -1 in the second one) is facet defining for any positive integer c [17].

3.2. *Cycle inequalities* $\mathrm{Cyc}_n(b)$

Let $b=(b_1,\ldots,b_n)$ where the b_i's are integers satisfying

$$\sum_{1\leq i\leq n} b_i = 3. \tag{3.20}$$

The set $B_+=\{i\in N:\ b_i>0\}$ is called the *positive support* of b. Set $f=|B_+|$ and $B_+=\{i_1,\ldots,i_f\}$ with $1\leq i_1<\cdots<i_f\leq n$ and let C be a cycle with nodeset B_+. The inequality

$$\sum_{1\leq i<j\leq n} b_i b_j x_{ij} - \sum_{(i,j)\in C} x_{ij} \leq 0 \tag{3.21}$$

is called a *cycle inequality* and is denoted by $\mathrm{Cyc}_n(b, C)$ or, for short, by $\mathrm{Cyc}_n(b)$ when C is the cycle $(i_1,\ldots,i_f)$.

Take a cut vector $\delta(S)$ where S is a subset of N with $1 \notin S$ and set $b(S)=\sum(b_i: i \in S)$ and $C(S)=\sum(\delta(S)_{ij}: (i,j) \in C)$. Then, (3.21) computed at the cut vector $\delta(S)$ takes the value $b(S)(3-b(S))-C(S)$. The latter quantity is obviously negative if $b(S) \leq 0$ or $b(S) \geq 3$. In the remaining case: $b(S)=1$ or 2, $b(S)(3-b(S))=2$ and thus $S_+ = S \cap B_+$ is a proper subset of B_+ from which one deduces easily that $C(S)=C(S_+) \geq 2$. Therefore, we have proved:

Proposition 3.22. *Any cycle inequality* (3.21) *is valid for* C_n; *its roots are the cut vectors* $\delta(S)$ *for which* $b(S)=1$ *or* 2 *and* $C(S)=2$ *hold.* □

Let us analyze the effect of the permutation operation on cycle inequalities. Take a permutation σ on n points, $b=(b_1,\ldots,b_n)$ satisfying (3.20) with positive support $B_+=\{i_1,\ldots,i_f\}$ and let $C=(j_1,\ldots,j_f)$ be a cycle on B_+. Let $(\mathrm{Cyc}_n(b,C))^\sigma$ denote the inequality obtained by permutation by σ of the left-hand side of (3.21). We define the sequence $b^\sigma=(b_{\sigma(1)},\ldots,b_{\sigma(n)})$ and the cycle $\sigma(C)=(\sigma(j_1),\ldots,\sigma(j_f))$. It is not difficult to verify the following relation:

$$\mathrm{Cyc}_n(b^\sigma, \sigma^{-1}(C)) = \mathrm{Cyc}_n(b, C)^\sigma \tag{3.23}$$

i.e., the cycle inequality on the left-hand side of (3.23) is obtained from $\mathrm{Cyc}_n(b,C)$ by permutation by σ. Hence, the permutation operation preserves the class of cycle inequalities. Therefore, we can restrict our attention to the cycle inequalities of the form $\mathrm{Cyc}_n(b)$ where the positive support of b is $B_+=\{1,\ldots,f\}$ and the chosen cycle is $C=(1,2,\ldots f)$. We furthermore deduce from (3.23) that $\mathrm{Cyc}_n(b)$ and $\mathrm{Cyc}_n(b^\sigma)$ are permutation equivalent inequalities whenever σ is a permutation preserving the cycle $(1,2,\ldots,f)$. However, the following example shows that, if σ does not preserve the cycle $(1,2,\ldots,f)$, then $\mathrm{Cyc}_n(b)$, $\mathrm{Cyc}_n(b^\sigma)$ are not necessarily permutation equivalent; in fact, they are not necessarily simultaneously facet defining.

Example 3.24. Consider the sequence $b_1=(2,2,1,1,-1,-1,-1)$; there are five other sequences obtained by permuting the coefficients of b_1: $b_2=(2,1,2,1,-1,-1,-1)$, $b_3=(2,1,1,2,-1,-1,-1)$, $b_4=(1,1,2,2,-1,-1,-1)$, $b_5=(1,2,1,2,-1,-1,-1)$, $b_6=(1,2,2,1,-1,-1,-1)$. From the above observations, the inequalities $\mathrm{Cyc}_7(b_i)$ for $i=1,3,4,6$ are all permutation equivalent, while $\mathrm{Cyc}_7(b_2)$ is permutation equivalent to $\mathrm{Cyc}_7(b_5)$ and one can verify that $\mathrm{Cyc}_7(b_1)$, $\mathrm{Cyc}_7(b_2)$ are not permutation equivalent. Computer check indicates that $\mathrm{Cyc}_7(b_2)$ is not facet inducing while $\mathrm{Cyc}_7(b_1)$ is.

The following cycle inequalities are all facet inducing:

$$\mathrm{Cyc}_7(3,2,2,-1,-1,-1,-1), \quad \mathrm{Cyc}_7(2,2,1,1,-1,-1,-1),$$
$$\mathrm{Cyc}_7(1,1,1,1,1,-1,-1), \quad \mathrm{Cyc}_8(2,2,2,1,-1,-1,-1,-1),$$
$$\mathrm{Cyc}_8(2,1,1,1,1,-1,-1,-1), \quad \mathrm{Cyc}_8(3,3,2,-1,-1,-1,-1,-1),$$
$$\mathrm{Cyc}_8(3,2,1,1,-1,-1,-1,-1), \quad \mathrm{Cyc}_9(1,1,1,1,1,1,-1,-1,-1).$$

The first three were discovered by Assouad and Delorme (in fact, they gave facets equivalent to them after permuting $(1234567) \to (7654321)$, cf. [1]); we checked all others by computer.

The definition of c-lifting given for hypermetric facets in 3.1 extends to cycle inequalities. Let $b = (b_1, \ldots, b_n)$ satisfying (3.20) and c be an integer; the cycle inequality obtained from $\mathrm{Cyc}_n(b)$ by c-lifting is $\mathrm{Cyc}_{n+1}(b_1 - c, b_2, \ldots, b_n, c)$. For instance, in the above list, the last four facets are obtained from the first three by (-1)-lifting. The following results show the existence of classes of cycle facets extending the facets mentioned above.

Theorem 3.25. $\mathrm{Cyc}_n(1, 1, \ldots, 1, -1, \ldots, -1)$, *consisting of k coefficients -1 and $k+3$ coefficients $+1$, is facet inducing for all $n = 2k+3 \geq 7$.*

Theorem 3.26. *Let b_1, b_2, b_3 be integers such that $b_1 + b_2 + b_3 = n$ and $b_i \geq 2$ for $i = 1$, 2, 3. Then,* $\mathrm{Cyc}_n(b_1, b_2, b_3, -1, \ldots, -1)$, *consisting of $n-3$ coefficients -1, is facet inducing for all $n \geq 7$.*

Theorem 3.27. $\mathrm{Cyc}_n(n-5, 2, 1, 1, -1, \ldots, -1)$, *consisting of $n-4$ coefficients -1, is facet inducing for all $n \geq 7$.*

We refer to Section 5 for the proofs. Theorems 3.26, 3.27 are proved by applying iteratively (-1)-lifting, starting respectively with the known facets $\mathrm{Cyc}_7(3, 2, 2, -1, -1, -1, -1)$ and $\mathrm{Cyc}_7(2, 2, 1, 1, -1, -1, -1)$; the proof of Theorem 3.25 is based on the polyhedral method.

We conclude this section by mentioning possible extensions of cycle inequalities. Given integers $b_1, \ldots, b_n$, set $\sum(b) = b_1 + \cdots + b_n$. We have seen that, for $\sum(b) = 1$ or 3, we can produce from the b_i's respectively the hypermetric and cycle valid inequalities with large subclasses of facets. A natural idea is to ask whether one can define a class of valid inequalities from all integers b_i with arbitrary sum $\sum(b)$. When $\sum(b) = 0$, it is known that the inequality $\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0$ is valid for C_n (this remains true for real valued b_i's); however, it is never facet inducing since it is implied by the hypermetric inequalities [18]. We will see in 4.2 that a class of facets discovered by Barahona, Grötschel and Mahjoub can be interpreted as a generalization of cycle inequalities with $\sum(b) = 2k+1$.

When $\sum(b) = 2$, one verifies easily the validity of the following inequality:

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} - \sum_{(i,j) \in P} x_{ij} \leq 0, \tag{3.28}$$

where P is a path whose nodeset is the positive support B_+ of b; (3.28) is called a *path inequality* and denoted by $\mathrm{Path}_n(b, P)$. Its roots are the cut vectors $\delta(S)$ for which $b(S) = 1$ and $|\delta(S) \cap P| = 1$. An anonymous referee pointed out that the path inequality (3.28) is not facet inducing for $f = |B_+| \geq 4$. Indeed if $\delta(S)$ is a root, then $S \cap B_+$ is one of the following $f-1$ intervals $[i, f]$ for $2 \leq i \leq f$, if P is the path $(1, \ldots, f)$, and $f - 1 < \binom{f}{2} - 1$ holds for $f \geq 4$.

A possible extension for arbitrary sum $\sum(b)$ is as follows. Suppose that $n > \lfloor\frac{1}{2}\sum(b)\rfloor\lceil\frac{1}{2}\sum(b)\rceil+3$ and let $K=K_{2,n-3}$ denote the complete bipartite graph on N with node partition into $\{1, 2\}$ and $\{3, \ldots, n\}$. Consider the inequality

$$\sum_{3\le i<j\le n} b_ib_jx_{ij}+x_{12}-\sum_{(i,j)\in K} x_{ij}\le 0. \tag{3.29}$$

Take a cut vector $\delta(S)$ with $1\notin S$; then (3.29) computed at $\delta(S)$ takes the nonpositive value: $b(S)(\sum(b)-b(S))+1-(n-3)$ when $2\in S$ and the value: $b(S)(\sum(b)-b(S))-2|S|$ when $2\notin S$. Hence, (3.29) is valid if $b(S)(\sum(b)-b(S))-2|S|\le 0$ holds for all subsets S of $\{3, \ldots, n\}$. For instance, if $b=(5, 4, -1, -1, -1, -1, -1)$, (3.29) is valid, but is not a facet since it has only 10 roots.

3.3. *Bounds for hypermetric and cycle facets*

If $v\cdot x\le 0$ is a valid inequality of C_n, we are interested in finding bounds for $\|v\|=\sum(|v_{ij}|\colon 1\le i<j\le n)$. When v defines a pure inequality, then $\|v\|\le\binom{n}{2}$ obviously holds. For the classes of hypermetric and cycle inequalities, we are able to derive upper bounds for $\|v\|$ which are exponential in n. Observe first that, if v denotes the hypermetric inequality $\mathrm{Hyp}_n(b)$, then $\|v\|=\frac{1}{2}((\sum_{1\le i\le n}|b_i|)^2-\sum_{1\le i\le n}|b_i|^2)$ and, if v denotes the cycle inequality $\mathrm{Cyc}_n(b)$, then $\|v\|$ is the preceding quantity minus f, where f is the number of positive b_i's; therefore, it suffices to study upper bounds for $\|b\|=\sum(|b_i|\colon 1\le i\le n)$. We set $g_{\mathrm{h}}(n)=\max(\|b\|\colon \mathrm{Hyp}_n(b)$ is facet of $C_n)$ and $g_{\mathrm{c}}(n)=\max(\|b\|\colon \mathrm{Cyc}_n(b)$ is facet of $C_n)$.

Proposition 3.30.
(i) $\frac{1}{4}n^2-4\le g_{\mathrm{h}}(n)\le n\beta_{n-1}$ *for* $n\ge 7$,
(ii) $2n-3\le g_{\mathrm{c}}(n)\le 3+4(n-1)^2\beta_{n-2}$ *for* $n\ge 7$,
where β_n *is the maximum value of an* $n\times n$ *determinant with binary entries.*

Proof. (i) was proved in [5]; the upper bound in (ii) is an extension to the cycle case of the proof given in [5] and the lower bound follows from the facet of Theorem 3.26. □

The upper bounds from Proposition 3.30 are exponential in n and probably very weak; an interesting open question is to decide whether one can find upper bounds for hypermetric and cycle facets which are polynomial in n.

4. Other known facets and some interesting faces

4.1. *The parachute facet* Par_n

Take an integer $k\ge 2$ and $n=2k+1$, $n\equiv 3 \bmod 4$. The *parachute graph* Par_n is the bicolored graph whose n nodes are denoted as $0, 1, \ldots, k, 1', \ldots, k'$ and whose edges consist of the path $P=(k, k-1, \dots, 1, 1', 2', \dots, k')$ and the pairs $(0, i)$, $(0, i')$ for $1 \le i \le k-1$ and the pairs (k, i'), (k', i) for $1 \le i \le k$; edges of the path P are assigned weight 1 (represented by a plain line) while all other edges are assigned weight -1 (represented by a dotted line). Figure 1 shows the parachute graph on 7 points. We also denote by Par_n the (pure) inequality, called *parachute inequality,* whose supporting graph is the graph Par_n.

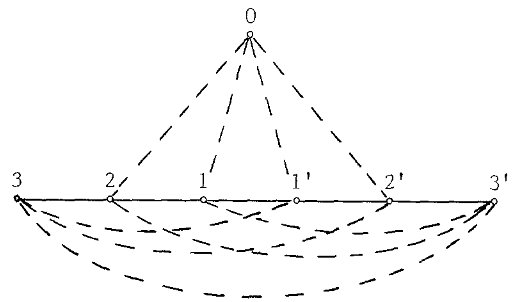

Fig. 1.

Theorem 4.1. *For all $n=2k+1$ with $k \ge 3$ odd, the parachute inequality defines a facet of C_n.*

The proof, based on the polyhedral method, is given in Section 5.

For $n=2k+1$ with k even, the parachute inequality is not valid; e.g., it is violated by the cut vector defined by $S=\{1, 3, \dots, k-1\} \cup \{2', 4', \dots, k'\}$.

For $n=7$, the facet (equivalent to) Par_7 was given by Assouad and Delorme (cf. [1]) and enumeration of the roots shows that Par_7 is a simplicial facet.

Remark 4.2. Both sets $S=\{k'\} \cup \{i \in [1, k]$: i is even$\}$ and $T=\{i \in [1, k]$: i is odd$\} \cup \{i' \in [1', k']$: i' is odd$\}$ define roots of the parachute inequality Par_n. Actually, for $n=7$, the parachute inequality Par_7 has only two (non-permutation equivalent) switchings obtained by switching by these two roots $\delta(S)$, $\delta(T)$ (see [17]).

4.2. Other facets

(a) Barahona-Grötschel-Mahjoub facet [9, 11]

A graph G is called a *bicycle p-wheel* if G consists of a cycle $C=(1, 2, \dots, p)$ of length p and two nodes $p+1$, $p+2$ that are adjacent to each other and to every node in the cycle; we assign weight 1 to the edges of the cycle C and to edge $(p+1, p+2)$ and weight -1 to all other edges. Figure 2 shows a bicycle 5-wheel.

We denote by BGM_n the pure inequality whose supporting graph is a bicycle $(n-2)$-wheel, i.e., described by

$$x_{n-1,\, n} + \sum_{1 \le i \le n-3} x_{i,i+1} + x_{1,n-2} - \sum_{1 \le i \le n-2} (x_{n-1,i} + x_{n,i}) \le 0. \tag{4.3}$$

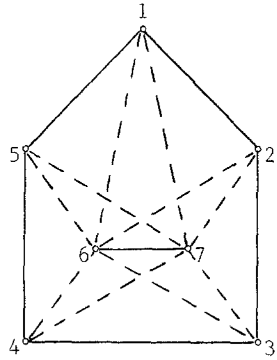

Fig. 2.

Theorem 4.4 [11, Theorem 2.3]. *For all odd $n \geq 5$, the inequality* BGM_n *defined by* (4.3) *is a facet of C_n.* □

Remark 4.5. In fact, Theorem 2.3 [11] presents a facet which is switching equivalent to BGM_n. For $n=5$, the inequality BGM_5 coincides with the pentagonal inequality $\mathrm{Hyp}_5(1,1,1,-1,-1)$ and for $n=7$, BGM_7 coincides with the cycle inequality $\mathrm{Cyc}_7(1,1,1,1,1,-1,-1)$. In fact, if we set $b=(1,\ldots,1,-1,-1)$ where the first $n-2$ b_i's take value $+1$ and the last two value -1 and if $K=K_{n-2}-C$ denotes the graph on $\{1,\ldots,n-2\}$ obtained by deleting the edges of the cycle $C=(1,\ldots,n-2)$ from the complete graph K_{n-2}, then, the inequality BGM_n can be alternatively described by

$$\sum_{1\leq i<j\leq n} b_ib_jx_{ij} - \sum_{(i,j)\in K} x_{ij} \leq 0. \tag{4.6}$$

Since n is odd, we can set $n=2k+3$ with $k\geq 1$; then, $\sum(b_i\colon 1\leq i\leq n)=2(k-1)+1$ and the graph K can be decomposed into $k-1$ edge disjoint cycles on $\{1,\ldots,n-2=2k+1\}$. Therefore, the inequality BGM_n can be interpreted as an extension of some hypermetric (when $k=1$) and cycle (when $k=2$) inequalities, which offers a partial answer to the question from Section 3.2 on how to define valid inequalities from any integers b_i.

Generally, if $b=(1,\ldots,1,-1,\ldots,-1)$ with $v\geq 2$ coefficients -1 and $v+2u+1$ coefficients $+1$, let K denote the antiweb on $m=v+2u+1$ nodes with parameter u, i.e., K is the circular graph on nodes $\{1,2,\ldots,v+2u+1\}$ in which each node i is adjacent to nodes $i+1, i+2, \ldots, i+u$; then inequality (4.6) is called a *clique-web inequality* (set $n=2u+2v+1$). Observe that, for $u=0$ or 1 and for $v=2$, then the clique-web inequality is facet inducing (it corresponds, respectively, to the pure hypermetric inequality, pure cycle inequality and BGM_n inequality). We can prove that, if the clique-web inequality is valid, then it is, in fact, facet inducing and that it is indeed valid for $u=2$ or when $m>(u-1)(u^2+u-2)$. We conjecture that the clique-web inequality is always valid; we will examine this conjecture in [24].

(b) Kelly's inequality

Consider a partition of N into $P \cup Q \cup \{n\}$ with $|P| = p$, $|Q| = q$, $p, q \geq 2$ and $p+q+1 = n$. Let K_p, K_q denote respectively the complete graph on P, Q. Set $t = qp - p^2 + 1$. The following inequality denoted by $\mathrm{Kel}_n(p)$ was mentioned by Kelly [34]:

$$(p-1) \sum_{(i,j) \in K_q} x_{ij} + (p+1) \sum_{(i,j) \in K_p} x_{ij} - p \sum_{i \in Q j \in P} x_{ij}$$
$$+ (q-p-t) \sum_{i \in Q} x_{in} + t \sum_{i \in P} x_{in} \leq 0. \tag{4.7}$$

Proposition 4.8. *For all $n \geq 5$, the inequality $\mathrm{Kel}_n(p)$ defined by (4.7) is a valid inequality of C_n.*

Proof. Consider a cut vector $\delta(S)$ with $n \notin S$, $\alpha = |S \cap Q|$, $\beta = |S \cap P|$. (4.7) computed at vector $\delta(S)$ takes the value

$$(p-1)\alpha(q-\alpha) + (p+1)\beta(p-\beta) - p[\alpha(p-\beta) + \beta(q-\alpha)] + (q-p-t)\alpha + t\beta.$$

One can verify that this quantity is equal to

$$(p+1)(\alpha-\beta)(\beta - 1 - \alpha(p-1)/(p+1)).$$

We now verify that the latter is nonpositive; for this, we distinguish two cases.

- Suppose first that $\alpha < \beta$. Then, we have $\alpha - \beta < 0$ and

$$\beta - 1 - \alpha(p-1)/(p+1) \geq \alpha - \alpha(p-1)/(p+1) = 2\alpha/(p+1) \geq 0.$$

- Suppose now that $\alpha > \beta$. We verify that $\beta - 1 - \alpha(p-1)/(p+1) \leq 0$. For this, note that $\beta \leq \min(\alpha - 1, p)$; when $\alpha - 1 \leq p$, then we have

$$\beta - 1 - \alpha(p-1)/(p+1) \leq \alpha - 2 - \alpha(p-1)/(p+1) = 2(\alpha - p - 1)/(p+1) \leq 0,$$

and when $p \leq \alpha - 1$, then we have

$$\beta - 1 - \alpha(p-1)/(p+1) \leq p - 1 - \alpha(p-1)/(p+1) = (p-1)(p+1-\alpha)/(p+1) \leq 0.$$

Therefore we have proved validity of (4.7). □

Remark 4.9. We deduce from the above proof that the roots of $\mathrm{Kel}_n(p)$ are exactly the cut vectors $\delta(S)$ with $n \notin S$ and $\alpha = |S \cap Q|$, $\beta = |S \cap P|$ satisfying

(a) Either $\alpha = \beta$; there are $\sum_{1 \leq \alpha \leq \min(p,q)} \binom{q}{\alpha}\binom{p}{\alpha}$ such roots.

(b) Or $\beta = 1 + \alpha(p-1)/(p+1)$; such roots exist only if $p+1$ divides $\alpha(p-1)$ and, if p is odd, we can suppose that $\alpha \neq \frac{1}{2}(p+1)$ (else $\alpha = \beta$).

Set $\Gamma = \{\alpha\colon\ 0 \leq \alpha \leq \min(q, p+1),\ \alpha \neq \frac{1}{2}(p+1)$ such that $p+1$ divides $\alpha(p-1)\}$, then there are $\sum_{\alpha \in \Gamma} \binom{q}{\alpha}\binom{p}{\beta}$ such roots.

It is an open question to characterize the parameters for which $\mathrm{Kel}_n(p)$ is facet inducing; however, we have the following results:

Proposition 4.10. *For $n \geq 7$, the following assertions hold:*

(i) $\mathrm{Kel}_n(2)$ *is permutation equivalent to* $\mathrm{Cyc}_n(n-4, 2, 2, -1, \ldots, -1)$ *and is therefore facet inducing.*

(ii) $\mathrm{Kel}_n(n-3)$ *is a simplicial face of dimension* $\frac{1}{2}n(n-1) - 3$.

Proof. We leave it to the reader to verify that, setting $P=\{1,2\}$, $Q=\{3,\ldots,n-1\}$, $\mathrm{Kel}_n(2)$ coincides with $\mathrm{Cyc}_n(2,2,-1,\ldots,-1,n-4)$. From Remark 4.9, the roots of $\mathrm{Kel}_n(n-3)$ are $\delta(S)$ for:

- either $\alpha=\beta=1$: $S=\{1,i\}$ or $\{2,i\}$ with $3\le i\le n-1$,
- or $\alpha=\beta=2$: $S=\{1,2,i,j\}$ with $3\le i<j\le n-1$,
- or $\alpha=0$, $\beta=1$: $S=\{i\}$ with $3\le i\le n-1$.

Hence, there are $\frac{1}{2}n(n-1)-3$ roots. We verify that their intersection vectors (pointed at position n) are linearly independent. For this, form the matrix whose rows are, first the vectors $\pi(\{i\})$ for $3\le i\le n-1$, then $\pi(\{1,i\})$ for $3\le i\le n-1$, then $\pi(\{2,i\})$ for $3\le i\le n-1$ and finally $\pi(\{1,2,i,j\})$ for $3\le i<j\le n-1$, and whose columns are indexed by $(1,1)$, $(2,2)$, $(1,2)$, (i,i) for $3\le i\le n$, $(1,i)$ for $3\le i\le n$, $(2,i)$ for $3\le i\le n$ and (i,j) for $3\le i<j\le n$. After deleting the columns indexed by $(1,1)$, $(2,2)$, $(1,2)$, the matrix has the configuration shown in Figure 3 and is clearly nonsingular (setting $m=n-2$, $s=\frac{1}{2}(n-2)(n-3)$). $\square$

$$
\begin{array}{|c|c|c|c|}
\hline
I_m & 0 & 0 & 0 \\
\hline
I_m & I_m & 0 & 0 \\
\hline
I_m & 0 & I_m & 0 \\
\hline
x & x & x & I_s \\
\hline
\end{array}
$$

Fig. 3.

(c) Poljak–Turzik inequality [36, 37]

Let k, r be even integers and $n=kr+1$. Let $C(n,r)$ denote the circular graph of order n with edges $(i,i+1)$, $(i,i+r)$ for $1\le i\le n$. Poljak and Turzik [36] proved that the inequality

$$\sum_{(i,j)\in C(n,r)} x_{ij}\le 2n-k-r \tag{4.11}$$

is valid for the cut polytope $P_c(K_n)$ and defines a facet of the bipartite subgraph polytope of K_n. Poljak and Turzik [37] proved that inequality (4.11) defines, in fact, a facet of $P_c(K_n)$ for $r \leq k+2$.

Figure 4 shows the graph C(9, 2). If we switch (4.11) by the root {1, 4, 7}, we obtain a facet of the cone C_9 whose supporting graph is shown in Figure 5.

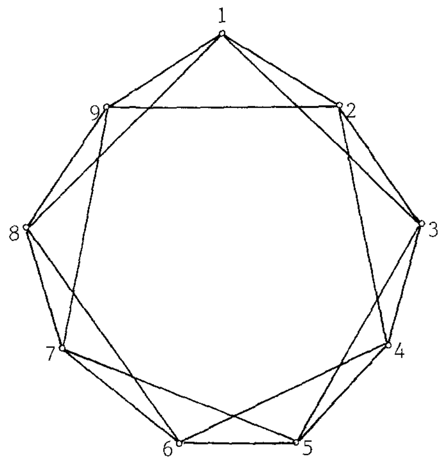

Fig. 4.

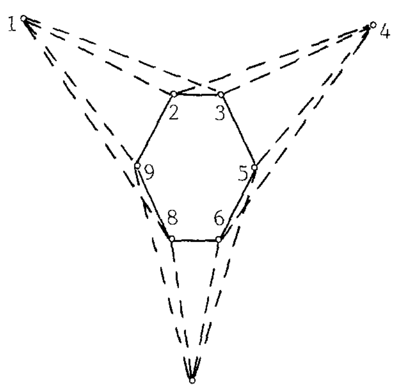

Fig. 5.

Remark 4.12. For $r=k=2$, $n=5$, $C(5, 2)=K_5$ and, if we then switch (4.11) by root {1, 3}, we obtain exactly the pentagonal hypermetric facet. For $k=4$, $r=2$, $n=9$, (4.11) is also facet defining; in fact, after switching by root {1, 4, 7}, we obtain an inequality which is permutation equivalent to that from Figure 5.

4.3. The cut cone on seven points

Let Gr_7 denote the graph on 7 points shown in Figure 6; its edges are weighted 1, -1 or -2 (the circle around nodes 1, 2, 3, 4 indicates that node 5 is adjacent to all

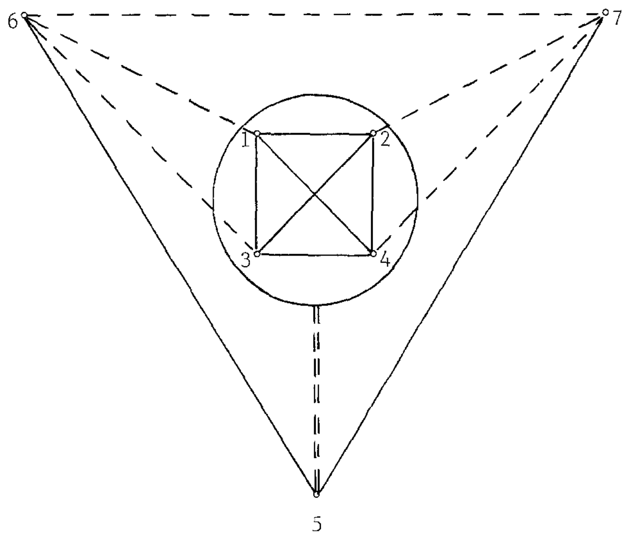

Fig. 6.

of them; weight -2 is indicated by a double dotted line). We also denote by Gr_7 the inequality supported by the graph Gr_7 and defined by

$$\sum_{1\le i<j\le 4} x_{ij} + x_{56} + x_{57} - x_{67} - x_{16} - x_{36} - x_{27} - x_{47} - 2 \sum_{1\le i\le 4} x_{5i} \le 0. \qquad (4.16)$$

This inequality was discovered by Grishukhin [27] who proved that it defines a simplicial facet of the cone C_7 (by computer check).

Remark 4.17. Figure 7 shows the graph obtained from Gr_7 after identifying nodes 6, 7; observe that the inequality supported by this graph is exactly the hypermetric facet $\mathrm{Hyp}_6(1, 1, 1, 1, -2, -1)$. Therefore, the facet Gr_7 can be seen as the result of

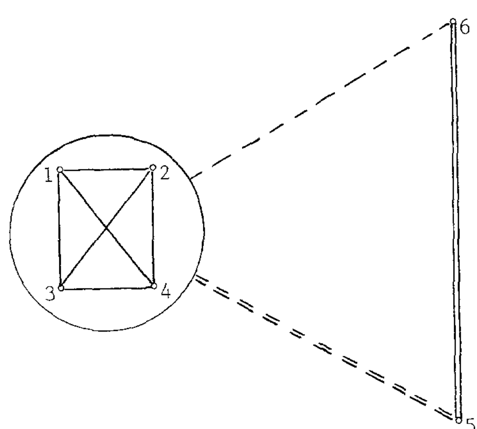

Fig. 7.

splitting node 6 in the above hypermetric facet; i.e., Gr_7 is a lifting of the hypermetric facet $Hyp_6(1, 1, 1, 1, -2, -1)$.

Up to permutation and switching, all known facets of the cut cone C_7 are:

- Six hypermetric facets $Hyp_7(b)$ for
  (1) $b=(1, 1, -1, 0, 0, 0, 0)$,
  (2) $b=(1, 1, 1, -1, -1, 0, 0)$,
  (3) $b=(1, 1, 1, 1, -1, -1, -1)$,
  (4) $b=(2, 1, 1, -1, -1, -1, 0)$,
  (5) $b=(2, 2, 1, -1, -1, -1, -1)$,
  (6) $b=(3, 1, 1, -1, -1, -1, -1)$.
- Three cycle facets $Cyc_7(b)$ for
  (7) $b=(1, 1, 1, 1, 1, -1, -1)$,
  (8) $b=(2, 2, 1, 1, -1, -1, -1)$,
  (9) $b=(3, 2, 2, -1, -1, -1, -1)$.

(10) The parachute facet Par_7.

(11) Grishukhin facet Gr_7.

Among these facets, the last five are non-hypermetric, the non-simplicial ones are the first five and five of them: (1), (2), (3), (7), (10) are pure, i.e., have 0, 1, −1 coefficients. Grishukhin [27] proved that the above list is, up to permutation and switching, complete, i.e., that every facet of the cone C_7 is permutation and/or switching equivalent to some facet of the above list of facets (1)-(11). The number of non-permutation equivalent switchings of facets (7), (8), (9), (10), (11) is, respectively, 3, 6, 4, 2, 6 [17].

Assouad and Delorme [2] studied graphs G whose suspension ∇G (obtained by adding a new node adjacent to all nodes of G) is hypermetric, but not embeddable into L^1, i.e., the graphic distance d induced by ∇G satisfies all hypermetric inequalities but does not belong to the cut cone, where $d_{ij}=1$ if (i, j) is an edge of G and $d_{ij}=2$ otherwise. They proved that ∇G is hypermetric but not embeddable into L^1 if and only if G is an induced subgraph of the Schläfli graph (see, e.g., [12]) and contains as an induced subgraph one of the following eight forbidden subgraphs:

(1) $G_1=K_7-C_5$, with C_5 is the cycle (3, 6, 4, 7, 5).
(2) $G_2=K_7-P_3$, with P_3 is the path (4, 6, 7, 5).
(3) $G_3=K_7-P_2$, with P_2 is the path (5, 7, 6).
(4) $G_4=\nabla B_8$ where B_8 is the graph shown in Figure 8.
(5) $G_5=\nabla B_7$ where B_7 is the graph shown in Figure 9.
(6) $G_6=\nabla B_5$ where B_5 is the graph shown in Figure 10.
(7) $G_7=\nabla\nabla H_3$ where H_3 is the graph shown in Figure 11.
(8) $G_8=\nabla H_4$ where H_4 is the graph shown in Figure 12.

Let d_i denote the graphic distance for graph G_i; since $d_i \notin C_7$ but d_i is hypermetric, there exists a non-hypermetric facet v of C_7 which separates d_i from C_7, i.e., $v \cdot d_i > 0$.

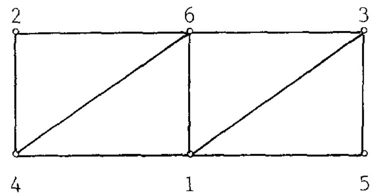

Fig. 8.

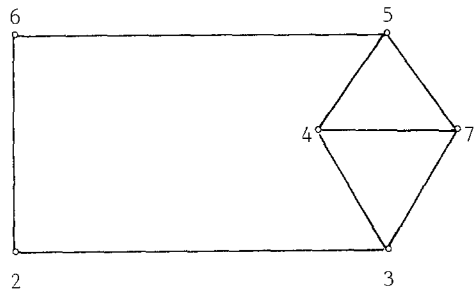

Fig. 9.

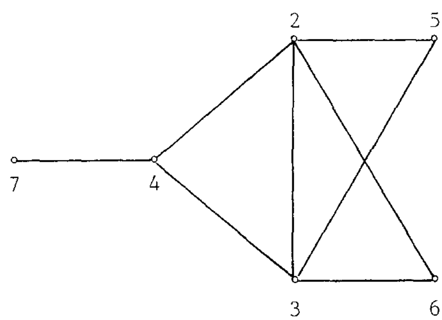

Fig. 10.

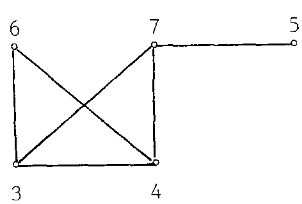

Fig. 11.

For the first five graphs, such separating facets were found by Assouad and Delorme; they are respectively for the first four graphs: $\mathrm{Cyc}_7(-1,-1,1,1,1,1,1)$, $\mathrm{Cyc}_7(-1,-1,-1,1,1,2,2)$, $\mathrm{Cyc}_7(-1,-1,-1,-1,2,2,3)$, the parachute facet Par_7 (after renumbering its nodes: $(0,3,2,1,1',2',3')$ as $(7,1,2,3,4,5,6)$). The distance d_5 is separated by the facet supported by the graph from Figure 13; it is, in fact,

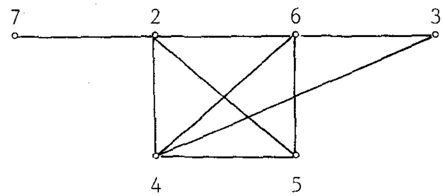

Fig. 12.

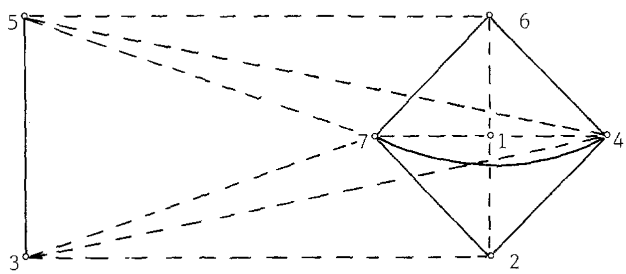

Fig. 13.

equivalent to the facet $\mathrm{Cyc}_7(-1, -1, 1, 1, 1, 1, 1)$ (after switching the latter by root $\{3, 4\}$ and then permuting the vertices $(1, 2, 3, 4, 5, 6, 7) \to (7, 4, 2, 6, 3, 1, 5)$). We verified that d_6 is separated by the facet $\mathrm{Cyc}_7(-1, -1, -1, 1, 2, 2, 1)$. Grishukhin (personal communication) observed that d_7 is separated by the facet equivalent to Gr_7 obtained by switching Gr_7 by the root $\delta(\{1, 3, 6\})$ and then permuting the vertices: $(1, 2, 3, 4, 5, 6, 7) \to (4, 2, 3, 1, 5, 7, 6)$; also that d_8 is separated by the facet equivalent to $\mathrm{Cyc}_7(2, 2, 1, 1, -1, -1, -1)$ obtained by switching it by root $\delta(\{1\})$ and then permuting the vertices: $(1, 2, 3, 4, 5, 6, 7) \to (7, 2, 1, 3, 5, 6, 4)$.

Remark 4.18. In all above cases, if v is the facet separating the graphic distance d, then $v \cdot d = 1$ holds, i.e., $v \cdot d$ takes the minimum possible value over $\{v \cdot x$: x is an integer vector that violates inequality $v \cdot x \leq 0\}$.

4.4. Some counting results and open questions

(a) Some counting

Recall that a valid inequality $v \cdot x \leq 0$ is simplicial if all its roots are linearly independent. Permutation and switching by roots preserve the property of being simplicial. However, lifting by zero does not in general preserve this property. For this, suppose that $v \cdot x \leq 0$, $v' \cdot x \leq 0$ define respectively simplicial facets of C_n, C_{n+m} $(m \geq 1)$ where $v' = (v, 0, \ldots, 0)$; then, we have the relations: $|R(v)| = \binom{n}{2} - 1$, $|R(v')| = \binom{n+m}{2} - 1$ and

$$|R(v')| = 2^m |R(v)| + 2^m - 1, \tag{4.19}$$

from which we deduce that: $(n+m)(n+m-1)=2^m n(n-1)$, implying that $n=3$, $m=1$. Therefore, $\mathrm{Hyp}_3(1,1,-1)$ and its 0-lifting $\mathrm{Hyp}_4(1,1,-1,0)$ are the only case of simultaneous simplicial facets. On the other hand, we obtain from (4.19) that $\mathrm{Hyp}_n(1,1,-1,0,\ldots,0)$ has $2^{n-2}+2^{n-3}-1$ roots; therefore, it is simplicial when $n=3,4$ and Proposition 4.20 shows that it realizes the maximum possible number of roots for a hypermetric facet of C_n (the extreme opposite of being simplicial).

Proposition 4.20. *Any hypermetric facet of C_n has at most $2^{n-2}+2^{n-3}-1$ roots.*

Proof. Take a hypermetric facet $\mathrm{Hyp}_n(b)$ with $b_1 \geq \cdots \geq b_f > 0 > b_{f+1} \geq \cdots \geq b_n$ where $f \geq 2$. The set of roots can be partitioned into: $R(v)=R_1 \cup R_2$ where $R_1=\{\text{root } \delta(S\cup\{2\})\colon S\subseteq[3,n]\}$ and $R_2=\{\text{root } \delta(S)\colon S\subseteq[3,n]\}$. When $b_2\neq 1$, there exists no subset S of $[3,n]$ such that $\delta(S)\in R_2$ and $\delta(S\cup\{2\})\in R_1$; hence $|R(v)|\leq 2^{n-2}$. When $b_2=1$, i.e., $b_2=\cdots=b_f=1$, we set $A_1=\{S\subseteq[3,n]\colon b(S)=0\}$, $A_2=\{S\subseteq[3,n]\colon b(S)=1\}$ and $A_3=\{S\subseteq[3,n]\colon b(S)=-1\}$; then, $|R_1|=|A_1|+|A_3|$ and $|R_2|=|A_1|+|A_2|-1$, i.e., $|R(v)|=2|A_1|+|A_2|+|A_3|-1$. We have that: $|A_1|+|A_2|+|A_3|\leq 2^{n-2}$ and $|A_1|\leq 2^{n-3}$ (by partitioning again A_1 into those sets containing 3 and the others). The result extends to the case when some coefficients b_i are zero by using relation (4.19). □

The pentagonal facet: $\mathrm{Hyp}_5(1,1,1,-1,-1)$ is also simplicial; in fact, the number of roots of the pure hypermetric facet $\mathrm{Hyp}_n(1,\ldots,1,-1,\ldots,-1)$ (with $k+1$ ones and k minus ones) is equal to:

$$\sum_{1\leq i\leq k}\binom{k}{i}\left(\binom{k}{i}+\binom{k}{i-1}\right)=\sum_{1\leq i\leq k}\binom{k}{i}\binom{k+1}{i}\geq\binom{2k+1}{2}-1,$$

with equality if and only if $k=1,2$, i.e., for the triangle or pentagonal facets. Indeed, $\mathrm{Hyp}_3(1,1,-1)$, $\mathrm{Hyp}_4(1,1,-1,0)$, $\mathrm{Hyp}_5(1,1,1,-1,-1)$ belong to the larger class of simplicial facets: $\mathrm{Hyp}_n(n-4,1,1,-1,\ldots,-1)$ for $n\geq 3$ which follows from Proposition 4.21. We conjecture that this is the only (up to equivalence) class of simplicial hypermetric facets, at least for the linear or quasilinear case.

Proposition 4.21. *Let $b=(b_1,b_2,1,1,-1,\ldots,-1)$ with $b_1+b_2=n-5$, $b_1\geq b_2$ and $n\geq 7$.*

(i) *$\mathrm{Hyp}_n(b)$ is facet defining if and only if $b_1\leq n-4$.*

(ii) *$\mathrm{Hyp}_n(b)$ is a simplicial face if and only if $b_1\geq n-4$.*

Proof. We prove (i). When $b_2\leq -1$, from Proposition 3.12, $\mathrm{Hyp}_n(b)$ is a (quasilinear) facet if and only if: $n-4\geq b_1+1-\operatorname{sign}|b_1-1|$, i.e., $b_1\leq n-4$. When $b_2\geq 1$, then $b_1\leq n-6$ and, from Proposition 3.12, $\mathrm{Hyp}_n(b)$ is a (linear) facet. We prove now (ii). One verifies easily that $\mathrm{Hyp}_n(b)$ has $\binom{n}{2}-n$ roots of the form $\delta(S)$

with $S \subseteq [3, n]$; the number of roots $\delta(S)$ with $2 \in S$, $1 \notin S$ is equal to:

$$A = \binom{n-4}{b_2} + \binom{n-4}{b_2-1} + 2\binom{n-4}{b_2+1} + 2\binom{n-4}{b_2} + \binom{n-4}{b_2+2} + \binom{n-4}{b_2+1},$$

setting $\binom{n-4}{a}$ to zero whenever $a < 0$ or $a > n-4$.

When $b_1 = n-4$, i.e., $b_2 = -1$, then $A = n-1$ and the total number of roots is $\binom{n}{2} - 1$; $\mathrm{Hyp}_n(b)$ is then a simplicial facet. When $b_1 = n-3$, i.e., $b_2 = -2$, then $A = 1$ and the total number of roots is $\binom{n}{2} - n + 1$; we verify that these roots are all linearly independent. For this, consider the matrix whose rows are the projections on the index set $I = \{(i, j)\colon 3 \leq i < j \leq n\}$ of the intersection vectors pointed at position 1 of the roots $\delta(S)$ for $S = \{3\}, \{4\}, \{2, 3, 4\}, \{3, 4, i\}$ $(5 \leq i \leq n)$, $\{3, i\}$ $(5 \leq i \leq n)$, $\{4, i\}$ $(5 \leq i \leq n)$ and $\{3, 4, i, j\}$ $(5 \leq i < j \leq n)$. If one permutes the columns of this matrix by reordering the pairs in I as: $(3, 3)$, $(4, 4)$, $(3, 4)$, (i, i) for $5 \leq i \leq n$, $(3, i)$ for $5 \leq i \leq n$, $(4, i)$ for $5 \leq i \leq n$ and (i, j) for $5 \leq i < j \leq n$, one obtains a matrix whose block configuration is shown in Figure 14 and which is clearly non singular (setting $m = n-2$, $s = \frac{1}{2}(n-2)(n-3)$). Hence, $\mathrm{Hyp}_n(b)$ is a simplicial face. When $b_1 \geq n-2$, then $A = 0$ and, from the previous argument, $\mathrm{Hyp}_n(b)$ is again a simplicial face. When $b_1 \leq n-5$, i.e., $b_2 \geq 0$, then $A \geq n$ and there are at least $\binom{n}{2}$ roots, hence $\mathrm{Hyp}_n(b)$ is not simplicial. □

1 0 0 0 1 0 1 1 1	0	0	0	0
1 1 0 . . . 1 1 0	I_m	I_m	I_m	0
1 0 0 . . . 1 0 0	I_m	I_m	0	0
0 1 0 . . . 0 1 0	I_m	0	I_m	0
1 1 1 . . . 1 1 1	x	x	x	I_s

Fig. 14.

Proposition 4.22. $\mathrm{Hyp}_n(1, 1, 1, -1, -1, b_6, \ldots, b_n)$ *(with* $b_6 + \cdots + b_n = 0$*) is not simplicial whenever* $n \geq 6$.

Proof. Observe that there exist 19 distinct roots $\delta(S)$ with $S \subseteq [1, 5]$; they are not linearly independent, since their intersection vectors take nonzero value only on the 15 positions (i, j) with $1 \leq i < j \leq 5$. □

(b) Some open questions

We have described above classes of valid inequalities for the cut cone C_n containing large subclasses of facets. Almost all of them belong to the following three families: hypermetric, cycle and pure (i.e., with $0, 1, -1$ coefficients) inequalities. It is of interest to consider the cones defined by each of the above families: the hypermetric cone HYP_n defined by the hypermetric inequalities, the cycle cone CYC_n defined by the cycle inequalities and the pure cone P_n defined by all pure valid inequalities of C_n. The set of all semi-metrics on n points is the polyhedral cone M_n whose facets consist exactly of the triangle inequalities. We have the inclusions: $C_n \subseteq \mathrm{HYP}_n \subseteq M_n$ and $C_n \subseteq \mathrm{HYP}_n \cap \mathrm{CYC}_n \cap P_n$. There are many interesting open questions concerning these cones; we mention some which are most relevant to our work. Obviously, the cone P_n is polyhedral; is this true as well for the cones HYP_n, CYC_n? It is proven in [23] that the hypermetric cone is indeed polyhedral. It would be very interesting to determine the complexity of the separation problem over the cones HYP_n, CYC_n, P_n.

Another interesting question is whether the cones HYP_n, CYC_n, P_n realize a "good approximation" of C_n. If C is a cone containing C_n, one can consider the quantity: $d(C, C_n) = \max(v \cdot x \colon x \in C - C_n,\ v \text{ is facet of } C_n \text{ with } \|v\| \leq 1)$. It would be of interest to study whether $d(C, C_n)$ is bounded for $C = \mathrm{HYP}_n$, CYC_n or P_n (recall Remark 4.18).

Another development of this work concerns *restricted cut cones*, i.e., cones generated by a subset of the family of cuts of the complete graph, e.g., all cuts with given cardinalities; the applications to the related max-cut problem are obvious. In [14], the case for subfamilies consisting of all *equicuts*, i.e., cuts $\delta(S)$ with $|S| = \lfloor \frac{1}{2}n \rfloor$ or $\lceil \frac{1}{2}n \rceil$, was considered (in the polytope version). In [22], we consider equicuts and the complementary case of *inequicuts*, i.e., all cuts except equicuts.

5. Proofs

5.1. Proofs for Section 3.1 on hypermetric inequalities

Proof of Theorem 3.11. Given integers $b_1, \ldots, b_f$ such that $b_2 \geq b_3 \geq \cdots \geq b_f > 0$ and $b_1 + b_2 + \cdots + b_f = n - f + 1$ and given an integer c, we set $b = (b_1, \ldots, b_f, -1, \ldots, -1)$, $b' = (b_1 - c,\ b_2, \ldots, b_f, -1, \ldots, -1, c)$ (with $n-f$ coefficients -1) and we denote respectively by v, v' the hypermetric inequalities $\mathrm{Hyp}_n(b)$, $\mathrm{Hyp}_{n+1}(b')$. We assume that v is facet defining. We show that v' is facet defining for suitable choice of c by using our lifting technique from Section 2.2 and Proposition 2.7. We observe first that conditions (2.4), (2.5) hold; for this, note that if a subset S of $N = [1, n]$ such that $1 \notin S$ defines a root of v, it also defines a root of v', since the coefficients of b' differ from those of b only in positions 1, $n+1$ and $1, n+1 \notin S$. In order to complete the proof, we must show that condition (2.6) holds, i.e., that there exist n roots of $v' = \mathrm{Hyp}_{n+1}(b')$ of the form $\delta(S)$ with $1 \notin S$,

$n+1 \in S$ and the projections of their incidence vectors on $\{2, \ldots, n+1\}$ are linearly independent.

Case $c=-1$ and $b_2=1$. Then, we choose the following n roots $\delta(S)$:

$$S=\{i,\ n+1\} \quad \text{for } 2 \le i \le f,$$
$$S=\{2, 3, i, n+1\} \quad \text{for } f+1 \le i \le n,$$
$$S=\{2, 3, n+1\}.$$

Their incidence matrix, shown in Figure 15, is easily verified to be nonsingular (I_n denotes the $n \times n$ identity matrix, a matrix whose entries are all zeros (or ones) is indicated by 0 (or 1)).

Case $c=-1$ and $b_2 \ge 2$. Then, we choose the following n roots:

$$S=\{i, n+1\} \cup [f+1, f+b_i-1] \quad \text{for } 2 \le i \le f,$$
$$S=\{2, n+1\} \cup [f+1, f+b_2-1]-\{i\} \quad \text{for } f+1 \le i \le f+b_2-1,$$
$$S=\{2, n+1\} \cup [f+1, f+b_2-2] \cup \{i\} \quad \text{for } f+b_2 \le i \le n,$$
$$S=\{2, 3, n+1\} \cup [f+1, f+b_2+b_3-1].$$

Set $t=n-f-b_2+1$, $b=b_2$ and let K_n denote the $n \times n$ matrix of all ones except zero on the diagonal; then, the incidence matrix of the above n roots has the block configuration shown in Figure 16. We denote by I, J, K and $\{n\}$ the index sets for

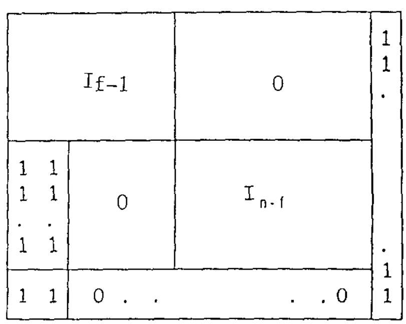

Fig. 15.

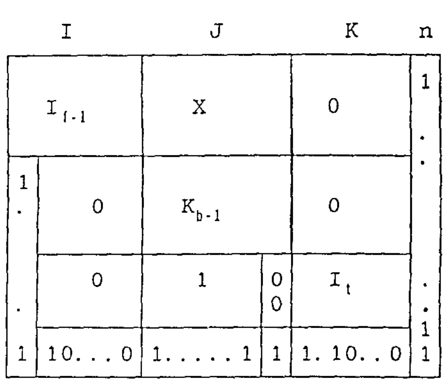

Fig. 16.

its columns and by C_i, $i \in I \cup J \cup K \cup \{n\}$, its columns. One obtains that the matrix has a nonzero determinant by performing the following manipulation on the columns:

- replace C_j by $C_j - C_1$ for $j \in J$,
- replace C_n by $C_n - \sum_{i \in I} C_i$,
- replace C_1 by $C_1 + \sum_{j \in J} C_j$,
- replace C_i by $C_i + \sum_{k \in K} C_k$ for the last element i of J.

Case $0 < c \leq n - f - b_2$. We consider the following n roots:

$$S = \{i, n+1\} \cup [f+1, f+b_i+c-1] \quad \text{for } 2 \leq i \leq f,$$

$$S = \{2, n+1\} \cup [f+1, f+b_2+c] - \{i\} \quad \text{for } f+1 \leq i \leq f+b_2+c-1,$$

$$S = \{2, n+1\} \cup [f+1, f+b_2+c-1] \cup \{i\} \quad \text{for } f+b_2+c \leq i \leq n,$$

$$S = \{2, 3, n+1\} \cup [f+1, f+b_2+b_3+c].$$

Their incidence matrix is shown below in Figure 17 (we set: $s = b_2 + c - 1$, $t = n - f - b_2 - c + 1$). As before, I, J, K and $\{n\}$ denote the index sets for the columns corresponding to the block configuration of the matrix and its columns are denoted by C_i. One observes that its determinant is nonzero by performing the following manipulation on the columns:

- replace C_j by $C_j - C_n$ for $j \in J$,
- replace C_n by $C_n - \sum_{i \in I} C_i$,
- replace C_1 by $C_1 - \sum_{k \in K} C_k$. □

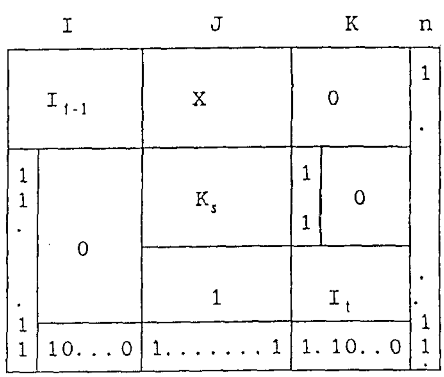

Fig. 17.

Proof of Theorem 3.12. We take integers $b_1 \geq \cdots \geq b_f > 0 > b_{f+1} \geq \cdots \geq b_n$.

Proof of (i). Suppose that $v = \mathrm{Hyp}_n(b)$ is facet inducing and denote by R its set of roots. If $f = 1$, then $b(S) < 0$ holds for all $S \subseteq N$; if $f = n-1$, the number of roots is equal to the number of indices i such that $b_i = 1$; hence both cases $f = 1$, $n-1$ are excluded. Suppose now that $f = 2$; for all roots $\delta(S)$, we can assume that $1 \notin S$, $2 \in S$. Set $F = \{(1,2), (2,3), (1,3)\}$; then the set R_F (of projections on F of the roots) consists exactly of the two vectors $(1,1,0)$, $(1,0,1)$; hence, $r(v, F) = 2 < |F| = 3$, which, from Lemma 2.3(ii), implies that $v_{F'}=0$, i.e., $n=3$ and thus $b=(1,1,-1)$. We now suppose that $f=n-2$; for all roots $\delta(S)$, we can assume that $n\notin S$. Suppose for contradiction that $b_1>1$. Then, for all roots $\delta(S)$, $n-1\in S$ whenever $1\in S$; therefore, setting $F=\{(1,n-1),(1,n),(n-1,n)\}$, the set R_F consists of vectors $(0,1,1)$, $(1,0,1)$, $(0,0,0)$ and thus $r(v,F)=2$ which, from Lemma 2.3(ii), yields a contradiction.

Proof of (ii). We take $b=(b_1,\ldots,b_f,-1,\ldots,-1)$. The "only if" part follows from (i) and the "if" part by applying iteratively the (-1)-lifting procedure from Theorem 3.11(i) starting with the facet $\mathrm{Hyp}_3(1,1,-1)$. (Note that if, at some step, one knows that $\mathrm{Hyp}_m(b_1,\ldots,b_k,-1,\ldots,-1)$ (with $m=b_1+\cdots+b_k+k-1$ and $k\leq f-1$) is facet inducing, then one can apply repeated (-1)-lifting starting with the facet $\mathrm{Hyp}_{m+1}(0,b_1,\ldots,b_k,-1,\ldots,-1)$ in order to obtain the facet $\mathrm{Hyp}_1(b_{k+1},b_1,\ldots,b_k,-1,\ldots,-1)$ with $1=b_1+\cdots+b_{k+1}+k$).

Proof of (iii). We take $b=(b_1,\ldots,b_f,-1,\ldots,-1,b_n)$ with $b_n\leq -2$ and $n-f-1$ coefficients -1. $\mathrm{Hyp}_n(1,\ldots,1,-1,-(n-4))$ is (switching and permutation) equivalent to $\mathrm{Hyp}_n(n-4,1,1,-1,\ldots,-1)$, the latter being a facet from (ii). Hence we can suppose that $3\leq f\leq n-3$.

Assume first that $\mathrm{Hyp}_n(b)$ is facet defining. We prove that condition (QL) holds. We can suppose that, for all roots $\delta(S)$, $n\notin S$. If $b_1+b_2\geq n+1-f$, then S does not contain $\{1,2\}$ if $\delta(S)$ is root; set $F=\{(1,2),(1,n),(2,n)\}$, then R_F consists of vectors $(1,1,0)$, $(1,0,1)$ and thus $r(v,F)=2<|F|$, contradicting Lemma 2.3(ii). Therefore, $b_1+b_2\leq n-f$ holds and, if $b_1>b_f$, then condition (QL) holds. We suppose now that $b_1=b_f$ and we prove that the case $b_1+b_2=n-f$ is excluded, by counting roots. If $b_1+b_2=n-f$, then, for $1\leq i<j\leq f$, there exists exactly one root containing both i,j. Denote by A the family of intersection vectors (pointed at position n) $\pi(S)$ for which $\delta(S)$ is root with $|S\cap[1,f]|=1$. For any vector $\pi(S)$ of A, its nonzero coordinates occur at positions (i,j) for $(i,j)=(1,1),\ldots,(f,f)$ or, $1\leq i\leq f$ and $f+1\leq j\leq n-1$, or $f+1\leq i\leq j\leq n-1$; yielding that $\mathrm{rank}(A)\leq f+f(n-1-f)+\binom{n-1-f}{2}$. Therefore, $\mathrm{rank}(R)\leq \mathrm{rank}(A)+\binom{f}{2}<\binom{n}{2}-1$, contradicting the fact that $\mathrm{Hyp}_n(b)$ is a facet. We prove now that, conversely, if condition (QL) holds and $3\leq f\leq n-3$, then $\mathrm{Hyp}_n(b)$ is a facet. We distinguish two cases:

Case $b_1=b_f$; then condition (QL) becomes $b_1+b_2\leq n-f-1$. Applying 0-lifting and (-1)-lifting from Theorem 3.11(i) starting with facet $\mathrm{Hyp}_3(1,1,-1)$, we obtain the facet $\mathrm{Hyp}_m(1,b_1,b_f,-1,\ldots,-1)$ $(m=b_1+b_f+3=b_1+b_2+3)$. Applying Theorem 3.11(ii) with $c=b_2$ (which is possible since $b_2\leq m-3-b_1$), we obtain that $\mathrm{Hyp}_{m+1}(1-b_2,b_1,b_2,b_f,-1,\ldots,-1)$ is a facet. Similarly, applying successively Theorem 3.11(ii) with $c=b_3,\ldots,b_{f-1}$, we deduce that $\mathrm{Hyp}_{m+f-2}(1-b_2-\cdots-b_{f-1},b_1,b_2,\ldots,b_f,-1,\ldots,-1)$ is a facet with $m+f-2=b_1+b_2+f+1\leq n$. Finally apply (-1)-lifting until obtaining the facet $\mathrm{Hyp}_n(b_n,b_1,\ldots,b_f,-1,\ldots,-1)$ where $b_n=1-b_2-\cdots-b_{f-1}+n-(m+f-2)=n-f-b_1-\cdots-b_f$.

Case $b_1>b_f$; then condition (QL) becomes $b_1+b_2\leq n-f$. As before, by (-1)-lifting, we obtain the facet $\mathrm{Hyp}_k(b_2-b_f,b_1,b_f,-1,\ldots,-1)$ with $k=b_1+b_2+2$. We can apply Theorem 3.11(ii) with $c=b_2$ and obtain facet

$\mathrm{Hyp}_{k+1}(-b_f, b_1, b_2, b_f, -1, \ldots, -1)$, then with $c = b_3, \ldots, b_{f-1}$ until deducing facet $\mathrm{Hyp}_{k+f-2}(-b_3 - \cdots - b_f, b_1, \ldots, b_f, -1, \ldots, -1)$ where $k+f-2 = b_1 + b_2 + f \leq n$. Finally, apply (-1)-lifting until obtaining facet $\mathrm{Hyp}_n(b_n, b_1, \ldots, b_f, -1, \ldots, -1)$ where $b_n = -b_3 - \cdots - b_f + n - (k+f-2) = n - f - b_1 - \cdots - b_f$. □

5.2. Proofs for Section 3.2 on cycle inequalities

Proof of Theorem 3.26. We use again our lifting technique. We prove Theorem 3.26 by induction on $n \geq 7$; for $n = 7$, the result holds since $\{b_1, b_2, b_3\} = \{3, 2, 2\}$. We denote respectively by v, v' the inequalities $\mathrm{Cyc}_n(b_1, b_2, b_3, -1, \ldots, -1)$ and $\mathrm{Cyc}_{n+1}(b_1+1, b_2, b_3, -1, \ldots, -1, -1)$. By the inductive assumption, we know that v is facet defining; we prove that v' is facet defining by using Proposition 2.7. Condition (2.4) always holds; condition (2.5) holds because, if S is a subset of $N = [1, n]$ with $1 \notin S$ defining a root of v, then S also defines a root of v' since both cycle inequalities v, v' have the same positive support: $\{1, 2, 3\}$ and $1, n+1 \notin S$. In order to satisfy condition (2.6), we must find n roots of v' with $n+1 \in S$ whose incidence vectors projected on $\{2, \ldots, n+1\}$ are linearly independent; these roots must be chosen from the following list:

$$S = \{2, n+1\} \cup \{b_2 - 2 \text{ or } b_2 - 3 \text{ points from } [4, n]\},$$

$$S = \{3, n+1\} \cup \{b_3 - 2 \text{ or } b_3 - 3 \text{ points from } [4, n]\},$$

$$S = \{2, 3, n+1\} \cup \{b_2 + b_3 - 2 \text{ or } b_2 + b_3 - 3 \text{ points from } [4, n]\}.$$

We distinguish 3 cases:

Case $b_2, b_3 \geq 3$. Then, we choose the following n roots:

$$S = \{3, n+1\} \cup [n - b_3 + 4, n],$$

$$S = \{2, n+1\} \cup [4, b_2],$$

$$S = \{2, 3, n+1\} \cup [4, b_2 + 3] \cup [n - b_3 + 4, n],$$

$$S = \{3, i, n+1\} \cup [n - b_3 + 4, n] \quad \text{for } 4 \leq i \leq n - b_3 + 3,$$

$$S = \{2, i, n+1\} \cup [4, b_2] \quad \text{for } n - b_3 + 4 \leq i \leq n,$$

(setting $[a, b] = \emptyset$ if $b < a$). Their incidence matrix is shown below in Figure 18 (setting: $u = n - b_3$, $v = b_3 - 3$), Denote by $\{1\}$, $\{2\}$, I, J, K, $\{n\}$ the partition of the index set of the columns corresponding to the block configuration of the matrix and denote by C_i its columns. One verifies that the matrix has nonzero determinant by performing the following manipulations on its columns:

- replace C_n by $C_n - C_1 - C_2$,
- replace C_i by $C_i - C_1$ for $i \in I$,
- replace C_k by $C_k - C_2$ for $k \in K$.

Case $b_2 = b_3 = 2$. Then, choose the following n roots:

$$S = \{2, 3, 4, 5, n+1\}, \{2, n+1\}, \{3, n+1\}, \{2, 3, i, n+1\} \quad \text{for } 4 \leq i \leq n.$$

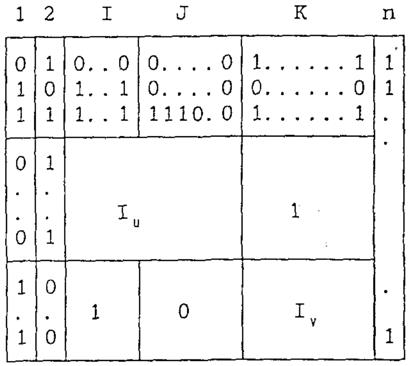

Fig. 18.

Case $b_2 = 2$, $b_3 \geq 3$. We choose the n roots:

$$S = \{3, i, n+1\} \cup [n-b_3+4, n] \quad \text{for } 4 \leq i \leq n-b_3+1,$$

$$S = \{2, 3, n+1, n-b_3+1\} \cup [n-b_3+2, n] - \{i\} \quad \text{for } n-b_3+2 \leq i \leq n,$$

$$S = \{2, 3, n+1\} \cup [n-b_3+2, n],$$

$$S = \{2, 3, n+1\} \cup [n-b_3+1, n],$$

$$S = \{2, n+1\}.$$

Their incidence matrix is shown in Figure 19 (we set: $u = n - b_3 - 2$, $v = b_3 - 1$). Denote by $\{1\}$, $\{2\}$, I, $\{a\}$, J and $\{n\}$ the partition of the index set of the columns corresponding to the block configuration of the matrix. One verifies that its determinant is nonzero by performing the following manipulations on the rows and columns:

- replace C_n by $C_n - C_2$,
- replace C_j by $C_j - C_1$ for $j \in J$,

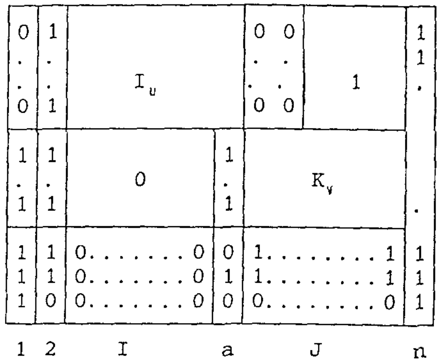

Fig. 19.

- replace C_2 by $C_2 - C_1 - C_a - \sum_{i\in I} C_i$,
- replace C_a by $C_a - C_1$, C_1 by $C_1 + C_2$,
- replace L_3 by $L_3 - L_2$, where L_2, L_3 denote the second and third rows (starting from the bottom of the matrix). $\square$

Proof of Theorem 3.27. As for Theorem 3.26, the proof goes by induction on $n \geq 7$. It is similar, so we simply indicate which additional n roots must be chosen: $S = \{2, n+1\}$, $\{2, 3, n+1\}$, $\{3, 4, n+1\}$, $\{2, 3, 4, 5, n+1\}$ and $\{2, 3, i, n+1\}$ for $5 \leq i \leq n$. One verifies easily that their incidence vectors are linearly independent. $\square$

Proof of Theorem 3.25. We prove that $\mathrm{Cyc}_n(1, \ldots, 1, -1, \ldots, -1)$ is facet defining by using the "polyhedral" method discussed in 1.4(a). We denote by $1, 2, \ldots, k+3$ the $k+3$ points corresponding to coefficients $b_i = 1$ and by $1', 2', \ldots, k'$ the k points corresponding to coefficients -1, so $n = 2k+3$ with $k \geq 2$. We denote by v the cycle inequality $\mathrm{Cyc}_n(1, \ldots, 1, -1, \ldots, -1)$ and we consider a valid inequality $b \cdot x \leq 0$ of C_n such that $b \cdot x = 0$ holds whenever $v \cdot x = 0$. In order to show that v is facet defining, it suffices to prove the following statements:

(a) $b_{ij'} = \beta$ for all $1 \leq i \leq k+3$, $1 \leq j \leq k$,

(b) $b_{i'j'} = -\beta$ for all $1 \leq i < j \leq k$,

(c) $b_{ij} = -\beta$ for all $1 \leq i < j \leq k+3$ where (i, j) is not an edge of the cycle $(1, 2, \ldots, k+3)$,

(d) $b_{ii+1} = 0$ for all $1 \leq i \leq k+3$ (setting $k+4 = 1$),

where β is some scalar; negativity of β will then follow from relation: $b \cdot \delta(\{1, 1'\}) = \beta < 0$.

We first observe that the roots of v, which are then also roots of b, are of the form $\delta(S)$ with $S = T \cup T'$ where T is a circular interval of $[1, k+3]$, T' is a subset of $[1', k']$ and $|T| = |T'|+1$ or $|T'|+2$.

(1) Condition (d) follows from Lemma 1.1, since the sets $\{i\}$, $\{i+1\}$, $\{i, i+1\}$ all define roots (of v, hence of b) for any $1 \leq i \leq k+3$.

(2) For proving that condition (a) holds, observe that, for $A = [4, k+3] \cup [3', k']$, the sets $A \cup \{1'\}$, $A \cup \{2'\}$, $A \cup \{1, 1'\}$, $A \cup \{1, 2'\}$ all define roots; hence we deduce from Lemma 1.2 that $b_{11'} = b_{12'}$ and the general result follows by symmetry. We set $b_{ij'} = \beta$ for any i, j.

(3) Take $i \in [3, k]$ and set $A = [1, k] \cup [3', k'] - \{i'\}$; the sets $A \cup \{2'\}$, $A \cup \{i'\}$, $A \cup \{1', 2'\}$, $A \cup \{1', i'\}$ all define roots; hence we deduce from Lemma 1.2 that $b_{1'2'} = b_{1'i'}$. By symmetry, we conclude that, for some scalar α, $b_{i'j'} = \alpha$ for all $1 \leq i \leq j \leq k$.

(4) Take v, $1 \leq v \leq k+3$; then $\delta(\{v\})$ is a root. From the preceding statements and the equality: $b \cdot \delta(\{v\}) = 0$, we can deduce the following relation:

$$(\mathrm{S}_v) \qquad \sum_{\substack{1 \leq i \leq k+3 \\ i \neq v-1, v, v+1}} b_{vi} + k\beta = 0.$$

(5) *Claim.* $\beta=-\alpha$.

Proof. Since the set $\{1,2,1'\}$ defines a root, equality $b\cdot\delta(\{1,2,1'\})=0$ yields

$$(6)\qquad b_{13}+\sum_{4\le i\le k+3} b_{1i}+b_{2i}+\beta(3k-1)+\alpha(k-1)=0.$$

By adding relations (S_1) and (S_2), we obtain

$$(7)\qquad b_{13}+\sum_{4\le i\le k+3} b_{1i}+b_{2i}+2k\beta=0.$$

Subtracting (6) from (7), we deduce that $\beta=-\alpha$.

(8) *Claim.* $b_{13}=-\beta$.

Proof. Using the fact that $\{1,2,3,1'\}$ is a root, we deduce the relation

$$(9)\qquad \sum_{4\le i\le k+3} b_{1i}+b_{2i}+b_{3i}+(3k-2)\beta=0.$$

Adding relations (S_1), (S_2), (S_3) and then subtracting the resulting relation from (9) yields equality $b_{13}=-\beta$.

In order to finish the proof, we must show that condition (c) holds. For this, we prove by induction on u, $3\le u\le k+3$, the following statement:

$$(\mathrm{H}_u)\qquad b_{vw}=-\beta\quad\text{for all } 1\le v<w\le u \text{ and } w\ne v+1.$$

From (8), the inductive assumption holds for $u=3$. Take $u\ge 4$ and assume that (H_{u-1}) holds; we prove that (H_u) holds, i.e., $b_{1u}=b_{2u}=\cdots=b_{u-2u}=-\beta$. We show the latter again by induction on v, $1\le v\le u-2$, in the following claims (10), (14).

(10) *Claim.* $b_{1u}=-\beta$.

Proof. Using the fact that both sets $[1,u]\cup[1',(u-2)']$ and $[2,u]\cup[1',(u-3)']$ are roots, we deduce respectively

$$(11)\qquad \sum_{u+1\le i\le k+3} b_{1i}+b_{2i}+\cdots+b_{ui}+2\beta(k-u+2)+\beta(u-2)(k-u+3)=0,$$

$$(12)\qquad b_{1u}+\sum_{u+1\le i\le k+3} b_{2i}+b_{3i}+\cdots+b_{ui}+\beta(k-u+3)(u-1)=0.$$

Relation (S_1) together with the inductive assumption becomes

$$(13)\qquad b_{1u}+\sum_{u+1\le i\le k+3} b_{1i}+\beta(k-u+3)=0.$$

By computing $(12)-(11)+(13)$, we deduce that $b_{1u}=-\beta$.

(14) *Claim.* Assume that $b_{1u}=b_{2u}=\cdots=b_{v-1u}=-\beta$ where $2\le v\le u-3$. Then, $b_{vu}=-\beta$.

Proof. Using the fact that both sets $[v+1,\ u]\cup[1',(u-v-2)']$, $[v,u]\cup[1',(u-v-1)']$ are roots and the inductive assumptions $b_{sw}=-\beta$ if $1\le s<w\le u-1$, $w\ne s+1$ and $b_{su}=-\beta$ if $1\le s\le v-1$, we deduce respectively

$$(15)\qquad b_{vu}+\sum_{u+1\le i\le k+3} b_{v+1i}+\cdots+b_{ui}+\beta(u-v)(k-u+3)=0,$$

(16) $\sum_{u+1\le i\le k+3} b_{vi}+\cdots+b_{ui}-\beta+\beta(u-v+1)(k-u+3)=0.$

Relation (S_v) becomes

(17) $b_{vu}+\sum_{u+1\le i\le k+3} b_{vi}+\beta(k-u+4)=0.$

Now, computing (15) − (16) + (17) yields $b_{vu}=-\beta$. □

5.3. *Proof of Theorem 4.1 on the parachute facet*

The nodes of the parachute graph are denoted as $0,1,2,\ldots,k,1',2',\ldots,k'$; E_+ denotes the set of edges with weight +1 consisting of the path $P=(k,k-1,\ldots,1,1',\ldots,(k-1)',k')$ while E_- denotes the set of edges with weight −1 consisting of the pairs $(0,i)$, $(0,i')$ for $1\le i\le k-1$ and the pairs (k,i'), (k',i) for $1\le i\le k$. We suppose that k is odd. We subdivide the proof into two parts: first, we show that the parachute inequality Par_n, denoted by v, which can be written as $v\cdot x=\sum_{(i,j)\in E_+}x_{ij}-\sum_{(i,j)\in E_-}x_{ij}\le 0$, is valid for the cut cone and, then, that it is facet defining.

(i) *The parachute inequality is valid.* Consider a cut vector $\delta(S)$; we can assume that $0\notin S$. Set $\alpha=|S\cap[1,k-1]|$ and $\alpha'=|S\cap[1',(k-1)']|$, $s_+=|\delta(S)\cap E_+|$ and $s_-=|\delta(S)\cap E_-|$. In order to prove validity, we must show that $s_+\le s_-$ holds. We first compute the value of s_- by distinguishing four cases (whether $k,k'\in S$):

(a) $k,k'\in S$. Then, $s_-=2k-2$.

(b) $k,k'\notin S$. Then, $s_-=2\alpha+2\alpha'$.

(c) $k\in S$, $k'\notin S$. Then, $s_-=2\alpha+k$.

(d) $k\notin S$, $k'\in S$. Then, $s_-=k+2\alpha'$.

(1) *Claim.* Let $P=(1,2,\ldots,n)$ be a path, S be a subset of $[1,n]$ and set $\beta=|S\cap[2,n-1]|$. Then, $|\delta(S)\cap P|\le 2\beta+|S\cap\{1,n\}|$.

The proof is easy. Validity is now checked:

- In case (a), $s_+\le|P|-1=2k-2$, since both endpoints of P belong to S and k is odd.

- In case (b), $s_+\le 2\alpha+2\alpha'$ from Claim (1).

- In case (c) (idem for (d)), decomposing P into paths $P_1=(1,\ldots,k)$ and $P_2=(1,1',\ldots,k')$ and using claim (1), we have: $s_+\le|S\cap\{1,k\}|+2|S\cap[2,k-1]|+|\delta(S)\cap P_2|=2\alpha+1-|S\cap\{1\}|+|\delta(S)\cap P_2|$; hence $s_+\le s_-$ holds whenever $|\delta(S)\cap P_2|\le k-1$; if $|\delta(S)\cap P_2|=k$, then, since $k'\notin S$ and k is odd, $1\in S$ and we have again $s_+\le s_-$.

(ii) *The parachute inequality is facet inducing.* Our proof for facetness is based on the polyhedral method. Let $b\cdot x\le 0$ be a valid inequality of C_n such that $b\cdot x=0$ whenever $v\cdot x=0$. In order to show that the parachute inequality v is facet inducing, it is enough to prove the following statements:

(a) $b_{ij}=0$ for all $(i,j)\notin E_+\cup E_-$,

(b) $b_{ij}=\beta$ for all $(i,j)\in E_+$,

(c) $b_{ij}=\alpha$ for all $(i,j)\in E_-$, for some scalars α, β. Then, using the fact that $\{1\}$ defines a root of v, hence of b, one deduces that $\alpha=-\beta$ holds; positivity of β will then follow from relation $b\cdot\delta(\{0\})=2(k-1)\alpha=-2(k-1)\beta<0$, implying that v is indeed facet defining.

We now give a sketch of proof for assertions (a), (b), (c), the detailed verifications (which are easy but tedious) being left to the reader.

(2) *Claim.* Assertion (a) holds.

Proof. Given i, i', $1\le i\le k-1$ with $(i,i')\ne(1,1')$, the sets $\{i\}$, $\{i'\}$, $\{i,i'\}$ all define roots; hence, Lemma 1.1 implies that $b_{ii'}=0$.

Given $S=\{1,3,5,\ldots,k\}\cup\{2',4',\ldots,(k-1)'\}$, the sets S, $S\cup\{0\}$, $S\cup\{k'\}$ and $S\cup\{0,k'\}$ all define roots, hence Lemma 1.1 implies that $b_{0k'}=0$.

(3) *Claim.* For some scalar α, $b_{0i}=b_{0i'}=\alpha$ for all $1\le i\le k-1$.

Proof. Take i, $1\le i\le k-2$, and set $A=\{3',5',\ldots,k'\}\cup\{1,3,\ldots,i-1\}\cup\{i+2,i+4,\ldots,k-1\}$ when i is even and set $B=\{1',3',\ldots,k'\}\cup\{2,4,\ldots,i-1\}\cup\{i+2,i+4,\ldots,k\}$ when i is odd. Using Lemma 1.2 applied to the set A when i is even, or B when i is odd, and to the points $p=0$, $q=i$, $r=i+1$, we deduce that $b_{0i}=b_{0i+1}$. Applying Lemma 1.2 to set $A=\{3,5,\ldots,k\}\cup\{3',5',\ldots,k'\}$ and points $p=0$, $q=1$, $r=1'$, we deduce that $b_{01}=b_{01'}$. This concludes the proof.

(4) *Claim.* $b_{11'}=-b_{k1'}=-b_{k'1}\overset{\text{def}}{=}\beta_1$ and $b_{12}=b_{1'2'}=-\alpha$.

Proof. Set $A=\{1,3,\ldots,k\}\cup\{3',5',\ldots,k'\}$; both A and $A\cup\{1'\}$ define roots, which yields $0=b\cdot\delta(A)-b\cdot\delta(A\cup\{1'\})$ and thus

$$0=-b_{11'}+b_{1'2'}+\alpha-b_{k1'}. \tag{5}$$

Using the fact that $\{1'\}$ defines a root, we obtain

$$0=b_{11'}+b_{1'2'}+\alpha+b_{k1'}. \tag{6}$$

Combining (5), (6), we have: $b_{1'2'}=-\alpha$ and $b_{11'}=-b_{k1'}$ and claim (4) follows by symmetry.

We now proceed to compute the value of b_{ij} along the path P and on edges (k,i'), (k',i). For this, we prove by induction on i the following relations:

$$(\mathrm{O}_i)\quad b_{ii+1}=b_{i'(i+1)'}=-\alpha\quad\text{for } i \text{ odd, } i=1,3,\ldots,k-2.$$

$$(\mathrm{E}_i)\quad b_{ii+1}=-b_{k'i}=-b_{k'i+1}\overset{\text{def}}{=}\beta_i\quad\text{for } i \text{ even, } i=2,4,\ldots,k-3.$$

$$(\mathrm{E}_{i'})\quad b_{i'(i+1)'}=-b_{ki'}=-b_{k(i+1)'}\overset{\text{def}}{=}\beta_i'\quad\text{for } i \text{ even, } i=2,4,\ldots,k-3.$$

By symmetry, it is enough to show (E_i) or (E_i'). For $i=1$, relation (O_1) follows from claim (4). The next claim shows that relation (E_2) holds.

(7) *Claim.* $b_{23}=-b_{k'2}=-b_{k'3}$.

Proof. Since $\{2\}$ is root, $0=b\cdot\delta(\{2\})=b_{02}+b_{12}+b_{23}+b_{k'2}$ which, from the precedings claims, implies that $b_{23}=-b_{k'2}$. Set $A=\{1,3,\ldots,k\}\cup\{5',7',\ldots,k'\}$; since both $A\cup\{2'\}$ and $A\cup\{1',3'\}$ are roots, we deduce

$$0=b\cdot\delta(A\cup\{1',3'\})-b\cdot\delta(A\cup\{2'\})$$

and therefore

(8) $\quad 0=b_{3'4'}+\alpha-b_{k3'}+b_{k2'}.$

From the fact that $\{3'\}$ is root, we deduce

(9) $\quad 0=\alpha+b_{2'3'}+b_{3'4'}+b_{k3'}.$

Combining (8), (9) and using $b_{2'3'}=-b_{k2'}$, we obtain $b_{3'4'}=-\alpha$ and then, from (8), $b_{k3'}=b_{k2'}$, which concludes the proof.

In claim (10), we proceed to show that induction is possible. Take i even, $4\le i\le k-2$, and assume that (E_j), (E'_j) hold for all j even, $j\le i-2$, and (O_k) holds for all k odd, $k\le i-3$.

(10) *Claim.* (E_i), (E'_i), (O_{i-1}) hold.

Proof. The sets $A=\{1,3,\ldots,k\}\cup\{1',3',\ldots,k'\}$, $B=\{1,3,\ldots,k\}\cup\{2',4',\ldots,(i-2)'\}\cup\{(i+1)',(i+3)',\ldots,k'\}$ and $C=\{1,3,\ldots,k\}\cup\{2',4',\ldots,i'\}\cup\{(i+3)',(i+5)',\ldots,k'\}$ are all roots. Hence $0=b\cdot\delta(A)-b\cdot\delta(B)$ and $0=b\cdot\delta(A)-b\cdot\delta(C)$, from which we deduce respectively, using the inductive assumption,

(11) $\quad 0=\alpha+b_{(i-1)'i'}-b_{k(i-1)'}+b_{k(i-2)'},$

(12) $\quad 0=\alpha+b_{(i+1)'(i+2)'}-b_{k(i+1)'}+b_{ki'}.$

Using (E'_{i-2}) and (11), we deduce $b_{(i-1)'i'}=-\alpha$, i.e., (O_{i-1}) holds. From the fact that $\{i'\}$ is root, we have

(13) $\quad 0=\alpha+b_{(i-1)'i'}+b_{i'(i+1)'}+b_{ki'},$

from which we deduce

(14) $\quad b_{i'(i+1)'}=-b_{ki'}.$

From the fact that $\{(i+1)'\}$ is root, we have

(15) $\quad 0=\alpha+b_{i'(i+1)'}+b_{(i+1)'(i+2)'}+b_{k(i+1)'}.$

Adding (12), (15) and using (14) yields $b_{(i+1)'(i+2)'}=-\alpha$ and then (15) implies $b_{i'(i+1)'}=-b_{k(i+1)'}$, i.e., (E'_i) holds, which concludes the proof.

(16) *Claim.* $b_{k-1k}=-b_{b'k-1}\overset{\text{def}}{=}\beta_{k-1}$ and $b_{(k-1)'k'}=-b_{k(k-1)'}\overset{\text{def}}{=}\beta'_{k-1}$.

Proof. Both sets $A=\{1,3,\ldots,k\}\cup\{1',3',\ldots,k'\}$ and $B=\{1,3,\ldots,k\}\cup\{2',4',\ldots,(k-1)',k'\}$ give roots, implying $0=b\cdot\delta(A)-b\cdot\delta(B)$ which, using preceding results, yields claim (16).

(17) *Claim.* $b_{kk'}=\beta_1-\beta_{k-1}-\beta'_{k-1}$.

Proof. Use relation $0=b\cdot\delta(A)$ where $A=\{3,5,\ldots,k\}\cup\{2',4',\ldots,(k-1)'\}$ is a root.

We conclude the whole proof by showing that $\beta_1=\cdots=\beta_{k-1}=\beta'_1=\cdots=\beta'_{k-1}=-\alpha$.

(18) *Claim.* $\beta_i=\beta'_i=-\alpha$ for all $1\le i\le k-1$.

Proof. For i even, $2\le i\le k-2$, set $B=\{k\}\cup\{1',3',\ldots,(i-1)'\}\cup\{(i+2)',\ldots,(k-1)'\}$, B and $B\cup\{i'\}$ are both roots, yielding $\beta'_i=-\alpha$. $\square$

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