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FACTORABLE SURFACES IN 3-MINKOWSKI SPACE

Huihui Meng and Huili Liu*

Abstract. In this paper, we mainly discuss factorable surfaces in 3-dimensional Minkowski space and give classification of such surfaces whose mean curvature and Gauss curvature satisfy certain conditions.

1. Introduction

Let E_1^3 be 3-dimensional Minkowski space equipped with the inner product

$$g(x, y) = \langle x, y\rangle = x_1y_2 + x_2y_1 + x_3y_3 \tag{1.1}$$

and the vector product

$$x \times y = \left(\begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right),$$

where $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in E_1^3$.

Let M be a connected, oriented 2-dimensional manifold and $r : M \to E_1^3$ be a surface in E_1^3 with parameters (u, v). We denote the surface $S : r(u, v)$ by

$$r(u, v) = (x(u, v), y(u, v), z(u, v)).$$

A surface S in E_1^3 is called a *factorable surface* if S can be written as

$$r(u, v) = (x, y, f(x)g(y)) \quad \text{or} \quad r(u, v) = (x, f(x)g(z), z) \quad \text{or}$$
$$r(u, v) = (f(y)g(z), y, z).$$

According to the spacelike direction, timelike direction and lightlike direction, the factorable surfaces in E_1^3 can be considered as the following six types

- type 1: along spacelike direction and spacelike direction;
- type 2: along spacelike direction and timelike direction;
- type 3: along lightlike direction and lightlike direction;
- type 4: along lightlike direction and spacelike direction;
- type 5: along timelike direction and lightlike direction;
- type 6: along timelike direction and timelike direction.

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In [9], the authors studied the factorable surfaces along two spacelike directions and spacelike-timelike directions, and gave some classification theorems. In this paper, we will consider the factorable surface along two lightlike directions and spacelike-lightlike directions in E_1^3, and also give some classification results.

2. Factorable surfaces in E_1^3

Theorem 1. *Let S be a factorable surface of type 3 in E_1^3.*

(1) *If Gauss curvature K of surface S vanishes identity, S is one of the following surfaces or an open part of them:*

(a) $z(x,y) = c_1 g(y)$,

(b) $z(x,y) = c_1 f(x)$,

(c) $z(x,y) = \exp(c_1 x + c_2 y + c_3)$,

(d) $z(x,y) = (c_1 x + c_2)^{\frac{1}{1-k_1}} (c_3 y + c_4)^{\frac{k_1}{k_1-1}}$.

(2) *If S is minimal, it is one of the following surfaces or an open part of them:*

(a) $z(x,y) = c_1 g(y)$,

(b) $z(x,y) = c_1 f(x)$,

where $c_1, c_2, c_3, c_4, k_1, k_2$ are constants and $k_1 \neq 1$.

Proof. In 3-dimensional Minkowski space E_1^3 with metric $\mathrm{d}s^2 = 2\mathrm{d}x\mathrm{d}y + \mathrm{d}z^2$, the factorable surface of type 3 can be written as $r(x,y) = (x, y, z(x,y)) = (x, y, f(x)g(z))$. Then Gauss curvature K and mean curvature H of S are given by

$$
\begin{aligned}
K &= -\frac{f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2}{(1 + 2f(x)f'(x)g(y)g'(y))^2}, \\
H &= -\frac{H_1}{2[1 + 2f(x)f'(x)g(y)g'(y)]^{\frac{3}{2}}}, \\
H_1 &= f(x)(f'(x))^2(g(y))^2 g''(y) + (f(x))^2 f''(x)g(y)(g'(y))^2 \\
&\quad -2f(x)(f'(x))^2 g(y)(g'(y))^2 - 2f'(x)g'(y).
\end{aligned}
$$

(1) When $K = 0$, we have

$$f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2 = 0. \tag{2.1}$$

Let $p(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$ and $q(y) = \frac{\mathrm{d}g}{\mathrm{d}y}$. Then by (2.1), we get

$$f(x)p(x)\frac{\mathrm{d}p}{\mathrm{d}f} g(y)q(y)\frac{\mathrm{d}q}{\mathrm{d}g} - (p(x)q(y))^2 = 0. \tag{2.2}$$

(a) If $p(x) = 0$, we have $f(x) = c_1$. Thus we get $z(x,y) = c_1 g(y)$.

(b) Analogously, if $q(y) = 0$, we have $z(x,y) = c_1 f(x)$.

(c) If $p(x) \neq 0$ and $q(y) \neq 0$, from (2.2), we get

$$f(x)\frac{\mathrm{d}p}{\mathrm{d}f} g(y)\frac{\mathrm{d}q}{\mathrm{d}g} = p(x)q(y).$$

Then by $p(x)q(y) \neq 0$ we know that $g(y)\frac{\mathrm{d}q}{\mathrm{d}y} \neq 0$. So we get

$$f(x)\frac{\mathrm{d}p}{\mathrm{d}f} \Big/ p(x) = \frac{q(y)}{g(y)\frac{\mathrm{d}q}{\mathrm{d}g}} = k_1, \tag{2.3}$$

where k_1 is a constant.

(i) If $k_1 = 1$, from (2.3) we have

$$\begin{cases} f(x) = \exp(c_1x + c_2) \\ g(y) = \exp(c_3y + c_4). \end{cases}$$

Then we get the result (c) of Theorem 1(1).

(ii) If $k_1 \neq 1$, from (2.3) we have

$$\begin{cases} f(x) = (c_1x + c_2)^{\frac{1}{1-k_1}} \\ g(y) = (c_3y + c_4)^{\frac{k_1}{1-k_1}}. \end{cases}$$

Then we get the result (d) of Theorem 1(1).

(2) Let $p(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$ and $q(y) = \frac{\mathrm{d}g}{\mathrm{d}y}$. When $H = 0$, we have

$$\begin{aligned} f(x)(p(x))^2(g(y))^2q(y)\frac{\mathrm{d}q}{\mathrm{d}g} + (f(x))^2p(x)\frac{\mathrm{d}p}{\mathrm{d}f}g(y)(q(y))^2 \\ -2f(x)(p(x))^2g(y)(q(y))^2 - 2p(x)q(y) = 0. \end{aligned} \tag{2.4}$$

(a) If $p(x) = 0$, we get $f(x) = c_1$. Then S can be written as $z(x, y) = c_1g(y)$.

(b) If $q(y) = 0$, we have $z(x, y) = c_1f(x)$.

(c) For $p(x) \neq 0$ and $q(y) \neq 0$, (2.4) can be written as

$$(f(x)p(x))g(y)(g(y)\frac{\mathrm{d}q}{\mathrm{d}g} - q(y)) + (g(y)q(y))f(x)(f(x)\frac{\mathrm{d}p}{\mathrm{d}f} - p(x)) = 2. \tag{2.5}$$

Differentiating (2.5) with respect to x and y respectively, we get

$$(f(x)p(x))'g(y)(g(y)\frac{\mathrm{d}q}{\mathrm{d}g} - q(y)) + (g(y)q(y))[f(x)(f(x)\frac{\mathrm{d}p}{\mathrm{d}f} - p(x))]' = 0, \tag{2.6}$$

$$(f(x)p(x))[g(y)(g(y)\frac{\mathrm{d}q}{\mathrm{d}g} - q(y))]' + (g(y)q(y))'f(x)(f(x)\frac{\mathrm{d}p}{\mathrm{d}f} - p(x)) = 0. \tag{2.7}$$

If $(f(x)p(x))' = 0$, from (2.6), by a calculation we can get $p(x) = 0$. This contradicts to $p(x)q(y) \neq 0$. So $(f(x)p(x))' \neq 0$ and analogously, $(g(y)q(y))' \neq 0$. Thus (2.6) and (2.7) can be written as

$$\frac{[f(x)(f(x)\frac{\mathrm{d}p}{\mathrm{d}f} - p(x))]'}{(f(x)p(x))'} = -\frac{g(y)\frac{\mathrm{d}q}{\mathrm{d}g} - q(y)}{q(y)} = k_1, \tag{2.8}$$

$$\frac{f(x)\frac{\mathrm{d}p}{\mathrm{d}f} - p(x)}{p(x)} = -\frac{[g(y)(g(y)\frac{\mathrm{d}q}{\mathrm{d}g} - q(y))]'}{(g(y)q(y))'} = k_2, \tag{2.9}$$

here k_1, k_2 are nonzero constants. From (2.8) and (2.9) we have $k_1 = k_2 = k$. With (2.5) we obtain

$$kf(x)p(x)g(y)q(y) = 1.$$

We get $f(x)p(x) =$ constant and $g(y)q(y) =$ constant. It is also a contradictory. This completes the proof of Theorem 1. □

Theorem 2. *Let S be a factorable surface of type 4 in E_1^3.*

(1) *If Gauss curvature K of S vanishes identity, S is one of the following surfaces or an open part of them:*

(a) $x(y,z) = c_1 f(y)$,
(b) $x(y,z) = c_1 g(z)$,
(c) $x(y,z) = \exp(c_1 y + c_2 z + c_3)$,
(d) $x(y,z) = (c_1 y + c_2)^{\frac{1}{1-k_1}} (c_3 z + c_4)^{\frac{k_1}{k_1-1}}$.

(2) *If S is minimal, it is one of the following surfaces or an open part of them:*

(a) $x(y,z) = c_1 g(z)$,
(b) $x(y,z) = (c_1 y + c_2)\exp(c_3 z + c_4)$,
(c) $f(y), g(z)$ *satisfy*

$$\begin{cases} f(y) &= c_1 \tan(\dfrac{c_1 k_2}{2} y + c_2), \\ g(z) &= \dfrac{k_2}{c_3} \sinh^2(\dfrac{\varepsilon\sqrt{c_3}}{2} z)) \end{cases}$$

where $c_1, c_2, c_3, c_4, k_1, k_2$ are constants and $k_1 \neq 1, k_2 \neq 0, \varepsilon = \pm 1$.

Proof. In 3-dimensional Minkowski space E_1^3 with the metric $\mathrm{d}s^2 = 2\mathrm{d}x\mathrm{d}y + \mathrm{d}z^2$, the factorable surface of type 4 can be written as $r(y,z) = (x(y,z), y, z) = (f(y)g(z), y, z)$.

Gauss curvature K and mean curvature H of S are given by

$$\begin{aligned} K &= -\frac{f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2}{[(f(y)g'(z))^2 - 2f'(y)g(z)]^2}, \\ H &= -\frac{2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2}{2[(f(y)g'(z))^2 - 2f'(y)g(z)]^{\frac{3}{2}}}. \end{aligned}$$

(1) When $K = 0$, we have

$$f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2 = 0.$$

Then we get the results of Theorem 2(1).

(2) When $H = 0$, we have

$$2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2 = 0. \tag{2.10}$$

Let $p(y) = \frac{\mathrm{d}f}{\mathrm{d}y}$ and $q(z) = \frac{\mathrm{d}g}{\mathrm{d}z}$. Then (2.10) can be written as

$$2f(y)p(y)g(z)q(z)\frac{\mathrm{d}q}{\mathrm{d}g} + p(y)\frac{\mathrm{d}p}{\mathrm{d}f}g(z) - 2f(y)p(y)(q(z))^2 = 0. \tag{2.11}$$

(a) If $p(y) = 0$, we have $x(y,z) = c_1 g(z)$.
(b) If $p(y) \neq 0$, (2.11) can be written as

$$2f(y)g(z)q(z)\frac{\mathrm{d}q}{\mathrm{d}g} + \frac{\mathrm{d}p}{\mathrm{d}f}g(z) - 2f(y)(q(z))^2 = 0,$$

that is

$$\frac{\frac{\mathrm{d}p}{\mathrm{d}f}}{f(y)} = 2\frac{q(z)^2 - g(z)q(z)\frac{\mathrm{d}q}{\mathrm{d}g}}{g(z)} = k_2,$$

where k_2 is a constant.

(i) If $k_2 = 0$, solving this equation we have

$$\begin{cases} f(y) = c_1 y + c_2 \\ g(z) = \exp(c_3 z + c_4). \end{cases}$$

This is the result (b) of Theorem 2(2).

(ii) If $k_2 \neq 0$, we get

$$\begin{cases} \frac{\mathrm{d}p}{\mathrm{d}f} = k_2 f(y) \\ (q(z))^2 - g(z)q(z)\frac{\mathrm{d}q}{\mathrm{d}g} = \frac{k_2 g(z)}{2}. \end{cases}$$

Solving this equations, we get the result (c) of Theorem 2(2). Then we complete the proof of Theorem 2. □

Theorem 3. *In 3-dimensional Minkowski space E_1^3, there is no factorable surface of type 3 with nonzero constant Gauss curvature or nonzero constant mean curvature.*

Proof. (1) If Gauss curvature K of a factorable surface S of type 3 is a nonzero constant C, we have

$$K = -\frac{f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2}{[1 + 2f(x)f'(x)g(y)g'(y)]^2} = C \neq 0. \tag{2.12}$$

$K \neq 0$ yields that $f'(x)g'(y) \neq 0$.
Put $\alpha_1 = g(y)g''(y)$, $\alpha_2 = (g'(y))^2$ and $\alpha_3 = g(y)g'(y)$, where $\alpha_2\alpha_3 \neq 0$. Then

$$K = -\frac{\alpha_1 f(x)f''(x) - \alpha_2(f'(x))^2}{[1 + 2\alpha_3 f(x)f'(x)]^2} = C. \tag{2.13}$$

Differentiating (2.13) with respect to y, we have

$$\begin{aligned} &f(x)f''(x)[\alpha_1' + f(x)f'(x)(2\alpha_1'\alpha_3 - 4\alpha_1\alpha_3')] \\ &= (f'(x))^2[\alpha_2' - f(x)f'(x)(4\alpha_2\alpha_3' - 2\alpha_2'\alpha_3)]. \end{aligned} \tag{2.14}$$

(a) If $\alpha_1' + f(x)f'(x)(2\alpha_1'\alpha_3 - 4\alpha_1\alpha_3') = 0$, we have

$$\alpha_2' - f(x)f'(x)(4\alpha_2\alpha_3' - 2\alpha_2'\alpha_3) = 0. \tag{2.15}$$

By (2.15) $\alpha_2' = 0$ yields $g'(y) = 0$. Combining $\alpha_2' \neq 0$ and (2.15), we have

$$\frac{4\alpha_2\alpha_3' - 2\alpha_2'\alpha_3}{\alpha_2'} = \frac{1}{f(x)f'(x)}. \tag{2.16}$$

Then we get $f(x)f'(x)$ =constant. Together with (2.12), we obtain

$$(f(x))^{-2}\frac{g(y)g''(y) + (g'(y))^2}{[1 + c_1 g(y)g'(y)]^2} = \text{constant}.$$

That means $f(x) = $ constant. Thus $\alpha_1' + f(x)f'(x)(2\alpha_1'\alpha_3 - 4\alpha_1\alpha_3') \neq 0$.

(b) When

$$\alpha_1' + f(x)f'(x)(2\alpha_1'\alpha_3 - 4\alpha_1\alpha_3') \neq 0, \tag{2.17}$$

formula (2.14) can be written as

$$f(x)f''(x) = (f'(x))^2\frac{\alpha_2' - f(x)f'(x)\beta_1}{\alpha_1' + f(x)f'(x)\beta_2}, \tag{2.18}$$

here $\beta_1 = 4\alpha_2\alpha_3' - 2\alpha_2'\alpha_3$, $\beta_2 = 2\alpha_1'\alpha_3 - 4\alpha_1\alpha_3'$. Differentiating (2.18) with respect to y we have

(2.19)

$$(f(x)f'(x))^2(\beta_1'\beta_2 - \beta_1\beta_2') + f(x)f'(x)(\alpha_1'\beta_1' - \alpha_1''\beta_1 - \alpha_2''\beta_2 + \alpha_2'\beta_2') + \alpha_1''\alpha_2' - \alpha_1'\alpha_2'' = 0.$$

Thus we get

$$\begin{cases} \beta_1'\beta_2 - \beta_1\beta_2' &= 0, \\ \alpha_1'\beta_1' - \alpha_1''\beta_1 - \alpha_2''\beta_2 + \alpha_2'\beta_2' &= 0, \\ \alpha_1''\alpha_2' - \alpha_1'\alpha_2'' &= 0, \end{cases} \tag{2.20}$$

here $\alpha_2\beta_1 \neq 0$, otherwise $g'(y) = 0$.

(i) If $\alpha_1' = 0$, from (2.17) we have $\beta_2 \neq 0$. Applying this to (2.20) with $\alpha_1' = 0$ we get $g'(y) = 0$.

(ii) If $\alpha_1' \neq 0$, $\beta_2 = 0$, from (2.20) we get $g'(y)$ =constant. With $\beta_2 = 0$ we obtain $g'(y) = 0$ which contradicts to $K \neq 0$.

(iii) When $\alpha_1'\beta_2 \neq 0$, we get $\alpha_3' = 0$ which contradicts to $K \neq 0$ by solving (2.20).

Therefore, there is no factorable surface $r(x, y) = (x, y, f(x)g(y))$ with nonzero constant Gauss curvature.

(2) If mean curvature H of a factorable surface S is a nonzero constant C, we have

$$H = -\frac{A_1}{2[1 + 2f(x)f'(x)g(y)g'(y)]^{\frac{3}{2}}} = C, \tag{2.21}$$

$$\begin{aligned} A_1 = {} & f(x)(f'(x))^2(g(y))^2g''(y) + (f(x))^2f''(x)g(y)(g'(y))^2 \\ & -2f(x)(f'(x))^2g(y)(g'(y))^2 - 2f'(x)g'(y). \end{aligned}$$

Put $\alpha_1 = f(x)f'(x)$, $\alpha_2 = (f(x))^2 f''(x)$ and $\alpha_3 = f'(x)$. $H \neq 0$ yields that $\alpha_1\alpha_3 \neq 0$. Differentiating (2.12) with respect to x, we have

$$\begin{aligned}\beta_1(g(y))^3g'(y)g''(y) + \beta_2(g(y))^2(g'(y))^3 + \beta_3 g(y)(g'(y))^2 \\ +\beta_4(g(y))^2g''(y) - 2\alpha_3'g'(y) = 0,\end{aligned} \tag{2.22}$$

here $\beta_1 = 2\alpha_1^2\alpha_3' - \alpha_1\alpha_1'\alpha_3$, $\beta_2 = 2\alpha_1\alpha_2' - 3\alpha_1'\alpha_2 - 4\alpha_1^2\alpha_3' + 2\alpha_1\alpha_1'\alpha_3$, $\beta_3 = \alpha_2' + 4\alpha_1'\alpha_3 - 6\alpha_1\alpha_3'$, $\beta_4 = \alpha_1'\alpha_3 + \alpha_1\alpha_3'$. Let $p(y) = g'(y)$. Then (2.22) can be written as

$$(g(y))^2\frac{\mathrm{d}p}{\mathrm{d}g}(\beta_1 g(y)p(y) + \beta_4) = 2\alpha_3' - \beta_2(g(y)p(y))^2 - \beta_3 g(y)p(y). \tag{2.23}$$

(a) If $\beta_1 g(y)p(y) + \beta_4 = 0$, we get

$$g(y)p(y) = \text{constant}, \tag{2.24}$$

or

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \alpha_3' = 0. \tag{2.25}$$

Combining (2.24) with (2.21), we get $g'(y) = 0$; solving (2.25), we have $f'(x) = 0$. These contradict to $H \neq 0$.

(b) When $\beta_1 g(y)p(y) + \beta_4 \neq 0$, from (2.23) we get

$$(g(y))^2\frac{\mathrm{d}p}{\mathrm{d}g} = \frac{2\alpha_3' - \beta_2(g(y)p(y))^2 - \beta_3 g(y)p(y)}{\beta_1 g(y)p(y) + \beta_4}. \tag{2.26}$$

Differentiating (2.26) with respect to x, we have

$$\begin{aligned}(\beta_1'\beta_2 - \beta_1\beta_2')[g(y)p(y)]^3 + (\beta_1\beta_3' - \beta_1'\beta_3 + \beta_2'\beta_4 - \beta_2\beta_4')[g(y)p(y)]^2 \\ + (\beta_3'\beta_4 - \beta_3\beta_4' - 2\alpha_3''\beta_1 + 2\alpha_3'\beta_1')g(y)p(y) + 2(\alpha_3'\beta_4' - \alpha_3''\beta_4) = 0.\end{aligned} \tag{2.27}$$

Then we get

$$\left\{\begin{array}{rcl}\beta_1'\beta_2 - \beta_1\beta_2' &=& 0,\\ \beta_1\beta_3' - \beta_1'\beta_3 + \beta_2'\beta_4 - \beta_2\beta_4' &=& 0,\\ \beta_3'\beta_4 - \beta_3\beta_4' - 2\alpha_3''\beta_1 + 2\alpha_3'\beta_1' &=& 0,\\ \alpha_3'\beta_4' - \alpha_3''\beta_4 &=& 0.\end{array}\right. \tag{2.28}$$

Solving (2.28), we have $f'(x) = 0$, that means $H = 0$. Therefore, there is no factorable surface $r(x,y) = (x, y, f(x)g(z))$ with nonzero constant mean curvature. This completes the proof of Theorem 3. □

Theorem 4. *Let S be a factorable surface of type 4 in E_1^3.*

(1) *If Gauss curvature K of S is a nonzero constant C, S is following surface or an open part of it: $f(y), g(z)$ satisfy*

$$\begin{cases} f(y) = (c_1 y + c_2)^{-1} \\ z = \displaystyle\int \left(\frac{c_1^2 c_3 g(z)}{c_1^2 - Cc_3 g(z)} - 2c_1 g(y)\right)^{-\frac{1}{2}} \mathrm{d}g(z), \end{cases}$$

where c_1, c_2, c_3 are constants.

(2) *If mean curvature H of S is a nonzero constant C, S is following surface or an open part of it: $f(x), g(z)$ satisfy*

$$\begin{cases} f(y) = (c_1 y + c_2)^{-1} \\ z = \displaystyle\int \left[\left(\frac{2c_1 c_3 g(z)}{2c_1 + C c_3 g(z)}\right)^2 - 2c_1 g(z) \right]^{-\frac{1}{2}} \mathrm{d}g(z), \end{cases}$$

where c_1, c_2, c_3 are constants.

Proof. (1) We assume that Gauss curvature of $S : r(y, z) = (f(y)g(z), y, z)$ is

$$K = -\frac{f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2}{[(f(y)g'(z))^2 - 2f'(y)g(z)]^2} = C \neq 0. \tag{2.29}$$

$K \neq 0$ yields that $f'(y)g'(z) \neq 0$. Put $\alpha_1 = f(y)f''(y)$, $\alpha_2 = f'(y)$ and $\alpha_3 = (f(y))^2$, where $\alpha_2\alpha_3 \neq 0$. Differentiating (2.29) with respect to y, we have

$$\beta_1(g(z))^2 g''(z) - \beta_2 g(z)(g'(z))^2 g''(z) - \beta_3(g'(z))^4 = 0, \tag{2.30}$$

here $\beta_1 = 2\alpha_1'\alpha_2 - 4\alpha_1\alpha_2'$, $\beta_2 = \alpha_1'\alpha_3 - 2\alpha_1\alpha_3'$, $\beta_3 = 2\alpha_2\alpha_2'\alpha_3 - 2\alpha_2^2\alpha_3'$. As $g''(z) \neq 0$, from (2.29) we obtain $g(z) \neq 0$. Then (2.30) can be written as

$$\beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)} - \beta_3 \left(\frac{(g'(z))^2}{g(z)}\right)^2 \frac{1}{g''(z)} = 0. \tag{2.31}$$

If $\beta_3 = 0$, we get the result of Theorem 4(1). If $\beta_3 \neq 0$, from (2.31) we have

$$\left(\frac{(g'(z))^2}{g(z)}\right)^2 \frac{1}{g''(z)} = -\frac{\beta_2 \frac{(g'(z))^2}{g(z)} - \beta_1}{\beta_3}. \tag{2.32}$$

Differentiating above equation with respect to y, we get

$$(\beta_2'\beta_3 - \beta_2\beta_3') \frac{(g'(z))^2}{g(z)} - (\beta_1'\beta_3 - \beta_1\beta_3') = 0. \tag{2.33}$$

From (2.33) we obtain

$$\begin{cases} \beta_2'\beta_3 - \beta_2\beta_3' = 0, \\ \beta_1'\beta_3 - \beta_1\beta_3' = 0, \end{cases} \tag{2.34}$$

or

$$\frac{(g'(z))^2}{g(z)} = \frac{\beta_1'\beta_3 - \beta_1\beta_3'}{\beta_2\prime\beta_3 - \beta_2\beta_3'} = \text{constant}, \tag{2.35}$$

where $\beta_2'\beta_3 - \beta_2\beta_3' \neq 0$. Combining (2.34) with (2.29) we have $f'(y) = 0$; combining (2.35) with (2.29) we get $g'(z) = 0$; these yield $K = 0$. So when $\beta_3 \neq 0$ there is no factorable surface $r(y, z) = (f(y)g(z), y, z)$ with nonzero constant Gauss curvature.

(2) We assume that the mean curvature H of S is

(2.36)
$$H = -\frac{2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2}{2[(f(y)g'(z))^2 - 2f'(y)g(z)]^{\frac{3}{2}}} = C \neq 0.$$

$H \neq 0$ yields that $f'(y)g'(z) \neq 0$. Put $\alpha_1 = f(y)f'(y),\ \alpha_2 = f''(y),\ \alpha_3 = f'(y)$ and $\alpha_4 = (f(y))^2$, here $\alpha_1\alpha_3\alpha_4 \neq 0$. Differentiating (2.36) with respect to y we get

$$\begin{aligned}(2.37)\quad &(8\alpha_1'\alpha_3 - 12\alpha_1\alpha_3')(g(z))^2g''(z) - (4\alpha_1'\alpha_4 - 6\alpha_1\alpha_4')g(z)(g'(z))^2g''(z)\\ &+ (4\alpha_2'\alpha_3 - 6\alpha_2\alpha_3')(g(z))^2 - (2\alpha_2'\alpha_4 - 3\alpha_2\alpha_4'\\ &+ 8\alpha_1'\alpha_3 - 12\alpha_1\alpha_3')g(z)(g'(z))^2 + (4\alpha_1'\alpha_4 - 6\alpha_1\alpha_4')(g'(z))^4 = 0.\end{aligned}$$

Let $\beta_1 = 8\alpha_1'\alpha_3 - 12\alpha_1\alpha_3',\ \beta_2 = 4\alpha_1'\alpha_4 - 6\alpha_1\alpha_4',\ \beta_3 = 4\alpha_2'\alpha_3 - 6\alpha_2\alpha_3',\ \beta_4 = 2\alpha_2'\alpha_4 - 3\alpha_2\alpha_4' + 8\alpha_1'\alpha_3 - 12\alpha_1\alpha_3'$. Then (2.37) can be written as

$$\begin{aligned}(2.38)\quad &\beta_1(g(z))^2g''(z) - \beta_2 g(z)(g'(z))^2g''(z) + \beta_3(g(z))^2\\ &- \beta_4 g(z)(g'(z))^2 + \beta_2(g'(z))^4 = 0,\end{aligned}$$

that is

$$(2.39)\qquad g''(z)\left(\beta_1 - \beta_2\frac{(g'(z))^2}{g(z)}\right) = -\beta_2\left[\frac{(g'(z))^2}{g(z)}\right]^2 + \beta_4\frac{(g'(z))^2}{g(z)} - \beta_3,$$

here $g''(z) \neq 0$. If $g''(z) = 0$, from (2.36) we have $g'(z) = 0$.

(a) If $\beta_1 - \beta_2\frac{(g'(z))^2}{g(z)} = 0$, we obtain $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ or $\frac{(g'(z))^2}{g(z)} =$ constant. But from $\frac{(g'(z))^2}{g(z)} =$ constant we have $g'(z) = 0$. Thus we get the result of Theorem 4,(2).

(b) If $g''(z)(\beta_1 - \beta_2\frac{(g'(z))^2}{g(z)}) \neq 0$, (2.39) can be written as

$$(2.40)\qquad g''(z) = \frac{-\beta_2\left[\frac{(g'(z))^2}{g(z)}\right]^2 + \beta_4\frac{(g'(z))^2}{g(z)} - \beta_3}{\beta_1 - \beta_2\frac{(g'(z))^2}{g(z)}}.$$

Differentiating (2.40) with respect to y, we get

$$\begin{aligned}(2.41)\quad &(\beta_1'\beta_2 - \beta_1\beta_2' + \beta_2'\beta_4 - \beta_2\beta_4')\left[\frac{(g'(z))^2}{g(z)}\right]^2\\ &-(\beta_1'\beta_4 - \beta_1\beta_4' + \beta_2'\beta_3 - \beta_2\beta_3')\frac{(g'(z))^2}{g(z)} - (\beta_1'\beta_3 - \beta_1\beta_3') = 0.\end{aligned}$$

Then we get

$$(2.42)\qquad \begin{cases}\beta_1'\beta_2 - \beta_1\beta_2' + \beta_2'\beta_4 - \beta_2\beta_4' &= 0,\\ \beta_1'\beta_4 - \beta_1\beta_4' + \beta_2'\beta_3 - \beta_2\beta_3' &= 0,\\ \beta_1'\beta_3 - \beta_1\beta_3' &= 0,\end{cases}$$

here $\beta_1\beta_2 \neq 0$ because $\beta_1 = 0$ yields $\beta_2 = 0$, this means $\beta_1 - \beta_2\frac{(g'(z))^2}{g(z)} = 0$. From above equations, we obtain $(\frac{\beta_1}{\beta_2})' = 0$. Assume $\beta_1 = c_0\beta_2$, where c_0 is a constant. From (2.42) we obtain also $\beta_1 = \beta_2 = 0$. Therefore, there is no factorable surface satisfying $g''(z)(\beta_1 - \beta_2\frac{(g'(z))^2}{g(z)}) \neq 0$. This completes the proof of Theorem 4. □

Theorem 5. *There is no factorable surface of type* 3 *satisfying the equation* $H^2 = K \neq 0$ *in* E_1^3.

Proof. When $H^2 = K \neq 0$, we have

(2.43)
$$[f(x)(f'(x))^2(g(y))^2g''(y) + (f(x))^2f''(x)g(y)(g'(y))^2 - 2f(x)(f'(x))^2g(y)(g'(y))^2 - 2f'(x)g'(y)]^2 + 4[1 + 2f(x)f'(x)g(y)g'(y)] [f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2] = 0.$$

Put $p(x) = f'(x)$, $\alpha_1 = f(x)p(x)$, $\alpha_2 = p(x)$, $\alpha_3 = f(x)\frac{\mathrm{d}p}{\mathrm{d}f}$. From (2.43) we have

$$[\alpha_1\alpha_2(g(y))^2g''(y) + \alpha_1(\alpha_3 - 2\alpha_2)g(y)(g'(y))^2 - 2\alpha_2g'(y)]^2 + 4[1 + 2\alpha_1g(y)(g'(y)][\alpha_2\alpha_3g(y)g''(y) - \alpha_2^2(g'(y))^2] = 0, \tag{2.44}$$

here $\alpha_1\alpha_2\alpha_3 \neq 0$. Denoting $t(y) = g'(y)$, then (2.44) can be written as

$$\begin{aligned}&\beta_1(g(y))^4(t(y))^2(\frac{\mathrm{d}t}{\mathrm{d}g})^2 + \beta_2(g(y))^2(t(y))^4 + \beta_3(g(y))^3(t(y))^3\frac{\mathrm{d}t}{\mathrm{d}g}\\ &+ \beta_4(g(y))^2(t(y))^2\frac{\mathrm{d}t}{\mathrm{d}g} - \beta_5g(y)(t(y))^3 + \beta_6g(y)t(y)\frac{\mathrm{d}t}{\mathrm{d}g} = 0,\end{aligned} \tag{2.45}$$

here $\beta_1 = \alpha_1^2\alpha_2^2$, $\beta_2 = \alpha_1^2(\alpha_3 - 2\alpha_2)$, $\beta_3 = 2\alpha_1^2\alpha_2(\alpha_3 - 2\alpha_2)$, $\beta_4 = 8\alpha_1\alpha_2\alpha_3 - 4\alpha_1\alpha_2^2$, $\beta_5 = 4\alpha_1\alpha_2\alpha_3$, $\beta_6 = 4\alpha_2\alpha_3$. Using $h = g(y)t(y)$ (2.45) can be written as

$$[(g(y))^2\frac{\mathrm{d}t}{\mathrm{d}g}]^2(\beta_1h) + [(g(y))^2\frac{\mathrm{d}t}{\mathrm{d}g}](\beta_3h^2 + \beta_4h + \beta_6) + \beta_2h^3 - \beta_5h^2 = 0. \tag{2.46}$$

Put $A_1 = \beta_1h$, $A_2 = \beta_3h^2 + \beta_4h + \beta_6$, $A_3 = \beta_2h^3 - \beta_5h^2$. From (2.46) we have

$$(g(y))^2\frac{\mathrm{d}t}{\mathrm{d}g} = \frac{-A_2 + \varepsilon\sqrt{A_2^2 - 4A_1A_3}}{2A_1},$$

here $\varepsilon = \pm 1$. Differentiating above equation with respect to f, we get

$$(A_1A_2' - A_1'A_2)(A_2'A_3 - A_2A_3') + (A_1A_3' - A_1'A_3)^2 = 0, \tag{2.47}$$

here $A_i' = \frac{dA_i}{df}$, $(i = 1, 2, 3)$. That is

(2.48)
$$[(\beta_1\beta_3' - \beta_1'\beta_3)h^3 + (\beta_1\beta_4' - \beta_1'\beta_4)h^2 + (\beta_1\beta_6' - \beta_1'\beta_6)h][(\beta_2\beta_3' - \beta_2'\beta_3)h^3 + (\beta_3\beta_5' - \beta_3'\beta_5 + \beta_2\beta_4' - \beta_2'\beta_4)h^2 + (\beta_2\beta_6' - \beta_2'\beta_6 + \beta_4\beta_5' - \beta_4'\beta_5)h - (\beta_5\beta_6' - \beta_5'\beta_6)] + [(\beta_1\beta_2' - \beta_1'\beta_2)h^3 + (\beta_1'\beta_5 - \beta_1\beta_5')h^2]^2 = 0,$$

where $\beta_i' = \frac{d\beta_i}{df}$, $(i = 1, \ldots, 6)$. If $h \neq$ constant, observing (2.48), we have

$$(\beta_1\beta_6' - \beta_1'\beta_6)(\beta_5'\beta_6 - \beta_5\beta_6') = 0.$$

As $\beta_1\beta_5\beta_6 \neq 0$, we have $\beta_1\beta_6' - \beta_1'\beta_6 = 0$ or $\beta_5'\beta_6 - \beta_5\beta_6' = 0$.

(a) If $\beta_1\beta_6' - \beta_1'\beta_6 = 0$, we have $f'(x) = 0$ from (2.43).

(b) If $\beta_5'\beta_6 - \beta_5\beta_6' = 0$, we have $f'(x) = 0$. Thus $h =$ constant, that is $g(y)t(y) =$ constant. Analogously, we have $f(x)f'(x)$ =constant. From (2.43) we have $f'(x)g'(y) = 0$ which is a contradictory. Finally we obtain that there is no factorable surface $r(x, y) = (x, y, f(x)g(y))$ satisfying $H^2 = K \neq 0$ in E_1^3. This completes the proof of Theorem 5. □

Theorem 6. *Let $S : r(y, z) = (x(y, z), y, z) = (f(y)g(z), y, z)$ be a factorable surface of type 4 in E_1^3. If Gauss curvature K and mean curvature H satisfy $H^2 = K \neq 0$, it is following surface or an open part of it:*

$$x(y, z) = c_1(y + c_2)^{-1}(c_3 - \frac{c_1}{2}(\frac{z}{c_1} - c_4)^2),$$

where c_1, c_2, c_3, c_4 are constants.

Proof. If Gauss curvature K and mean curvature H of surface S satisfy $H^2 = K \neq 0$, we have

(2.49)
$$[2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2]^2 + 4[f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2][(f(y)g'(z))^2 - 2f'(y)g(z)] = 0.$$

Put $\alpha_1 = g(z)g''(z)$ and $\alpha_2 = (g'(z))^2$. Then (2.49) can be written as

$$\beta_1(t(y)\frac{dt}{df})^2 - \beta_2 f(y)(t(y))^2\frac{dt}{df} + \beta_3(f(y))^2(t(y))^2 + \beta_4(f(y))^3 t(y)\frac{dt}{df} + \beta_5(t(y))^3 = 0,$$

here $t(y) = f'(y)$, $\beta_1 = (g(z))^2$, $\beta_2 = 4(\alpha_1 + \alpha_2)g(z)$, $\beta_3 = 4\alpha_1^2 - 8\alpha_1\alpha_2$, $\beta_4 = 4\alpha_1\alpha_2$, $\beta_5 = 8g(z)\alpha_2$. From above equation we obtain

$$\left(\frac{\frac{dt}{df}}{f(y)}\right)^2\left(\beta_1\frac{t(y)}{(f(y))^2}\right) + \left(\frac{\frac{dt}{df}}{f(y)}\right)\left(-\beta_2\frac{t(y)}{(f(y))^2} + \beta_4\right) + \left(\beta_5(\frac{t(y)}{(f(y))^2})^2 + \beta_3\frac{t(y)}{(f(y))^2}\right) = 0. \tag{2.50}$$

(a) Put $h = \frac{t(y)}{(f(y))^2}$. If $h \neq$ constant, we get

$$\begin{aligned}&[h(\beta_1'\beta_2 - \beta_1\beta_2') + (\beta_1\beta_4' - \beta_1'\beta_4)][(\beta_2\beta_5' - \beta_2'\beta_5)h^2 \\ &+ (\beta_2\beta_3' - \beta_2'\beta_3 + \beta_4'\beta_5 - \beta_4\beta_5')h + (\beta_3\beta_4' - \beta_3'\beta_4)] \\ &+ [(\beta_1\beta_5' - \beta_1'\beta_5)h^2 + (\beta_1\beta_3' - \beta_1'\beta_3)h]^2 = 0.\end{aligned} \tag{2.51}$$

Observing (2.51), we have $\beta_1\beta_5' - \beta_1'\beta_5 = 0$. With (2.49) we get $g'(z) = 0$, thus $H = K = 0$.

(b) If $h = C \neq 0$, we have $\frac{t(y)}{(f(y))^2} =$ constant, here C is a constant. Solving it we obtain $f(y) = c_1(y + c_2)^{-1}$. With (2.49) we get

$$\left[-g(z)g''(z) + \frac{g(z)}{c_1} + (g'(z))^2\right]^2 + [2g(z)g''(z) - (g'(z))^2]\left[(g'(z))^2 + 2\frac{g(z)}{c_1}\right] = 0.$$

Solving above equation we have $g(z) = c_3 - \frac{c_1}{2}(\frac{z}{c_1} - c_4)^2$, that is

$$\begin{cases} f(y) = c_1(y + c_2)^{-1} \\ g(z) = c_3 - \frac{c_1}{2}(\frac{z}{c_1} - c_4)^2. \end{cases}$$

This completes the proof of Theorem 6. □

Theorem 7. *In E_1^3, there is no factorable surface S: $r(x, y) = (x, y, z(x, y))$ of type 3 satisfying $aH + bK = 0$, where $HK \neq 0$ and $a, b \in R - \{0\}$.*

Proof. If Gauss curvature K and mean curvature H of S satisfy $aH + bK = 0$, we have $H/K = -b/a =$ constant, that is

$$\frac{H_1H_2}{f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2} = -\frac{2b}{a}, \tag{2.52}$$

$$\begin{aligned} H_1 &= f(x)(f'(x))^2(g(y))^2g''(y) + (f(x))^2f''(x)g(y)(g'(y))^2 \\ &\quad -2f(x)(f'(x))^2g(y)(g'(y))^2 - 2f'(x)g'(y), \\ H_2 &= (1 + 2f(x)f'(x)g(y)g'(y))^{\frac{1}{2}}. \end{aligned}$$

We assume that $p(x) = f'(x)$, $\alpha_1 = f(x)p(x)$, $\alpha_2 = p(x)$ and $\alpha_3 = f(x)\frac{dp}{df}$. $K \neq 0$ yields $\alpha_1\alpha_2\alpha_3 \neq 0$. Differentiating (2.52) with respect to f, we get

$$\begin{aligned}&\beta_1(g(y))^3(g''(y))^2 - \beta_2(g(y)g'(y))^2g''(y) + \beta_3(g(y))^4g'(y)(g''(y))^2 \\ &+ \beta_4(g(y)g'(y))^3g''(y) + \beta_5g(y)(g'(y))^4 - \beta_6(g(y))^2(g'(y))^5 \\ &+ \beta_7g(y)g'(y)g''(y) - \beta_8(g'(y))^3 = 0,\end{aligned} \tag{2.53}$$

$$\begin{cases}\beta_1 = \alpha_2^2(\alpha_1'\alpha_3 - \alpha_1\alpha_3'),\\ \beta_2 = \alpha_1'\alpha_2^3 - \alpha_1\alpha_2^2\alpha_2' - \alpha_1'\alpha_2\alpha_3^2 + 4\alpha_1'\alpha_2^2\alpha_3' + \alpha_1\alpha_2\prime\alpha_3^2 - 6\alpha_1\alpha_2^2\alpha_3',\\ \beta_3 = 3\alpha_1\alpha_1'\alpha_2^2\alpha_3 - 2\alpha_1^2\alpha_2^2\alpha_3',\\ \beta_4 = 3\alpha_1\alpha_1'\alpha_2\alpha_3^2 - 6\alpha_1\alpha_1'\alpha_2^2\alpha_3 - 2\alpha_1\alpha_2'\alpha_3^2 + 4\alpha_1^2\alpha_2^2\alpha_3' - 3\alpha_1\alpha_1'\alpha_2^3 + 2\alpha_1^2\alpha_2^2\alpha_2',\\ \beta_5 = -4\alpha_1\alpha_2^2\alpha_2' + 2\alpha_1'\alpha_2^2\alpha_3 - 2\alpha_1\alpha_2^2\alpha_3' - \alpha_1'\alpha_2\alpha_3^2 + \alpha_1\alpha_2'\alpha_3^2 + 2\alpha_1'\alpha_2^3,\\ \beta_6 = 3\alpha_1\alpha_1'\alpha_2^2\alpha_3 + 2\alpha_1^2\alpha_2^2\alpha_3' - 6\alpha_1\alpha_1'\alpha_2^3 + 4\alpha_1^2\alpha_2^2\alpha_2' - 4\alpha_1^2\alpha_2\alpha_2'\alpha_3,\\ \beta_7 = 2\alpha_2^2\alpha_3',\\ \beta_8 = 2\alpha_2^2\alpha_2',\\ \alpha_i' = \frac{\mathrm{d}\alpha_i}{\mathrm{d}f} \quad (i = 1, 2, 3).\end{cases}$$

Denoting $t(y) = g'(y)$, then (2.52) is

$$[(g(y))^2\frac{\mathrm{d}t}{\mathrm{d}g}]^2(\beta_1 + \beta_3 g(y)t(y)) + [(g(y))^2\frac{\mathrm{d}t}{\mathrm{d}g}][\beta_7 - \beta_2 g(y)t(y) + \beta_4(g(y)t(y))^2] + [-\beta_8 g(y)t(y) + \beta_5(g(y)t(y))^2 - \beta_6(g(y)t(y))^3] = 0.$$

Using the same method as Theorem 5, we get

(2.54)

$$\begin{aligned}&[(\beta_3'\beta_4 - \beta_3\beta_4')(g(y)t(y))^3 + (\beta_1'\beta_4 - \beta_1\beta_4' + \beta_2'\beta_3 - \beta_2\beta_3')(g(y)t(y))^2\\ &+ (\beta_1\beta_2' - \beta_1'\beta_2 + \beta_3'\beta_7 - \beta_3\beta_7')g(y)t(y) - (\beta_1\beta_7' - \beta_1'\beta_7)]\\ &\times [(\beta_4'\beta_6 - \beta_4\beta_6')(g(y)t(y))^5 + (\beta_2\beta_6' - \beta_2'\beta_6 + \beta_4\beta_5' - \beta_4'\beta_5)(g(y)t(y))^4\\ &+ (\beta_6\beta_7' - \beta_6'\beta_7 + \beta_2'\beta_5 - \beta_2\beta_5' + \beta_4'\beta_8 - \beta_4\beta_8')(g(y)t(y))^3\\ &+ (\beta_5'\beta_7 - \beta_5\beta_7' + \beta_2\beta_8' - \beta_2'\beta_8)(g(y)t(y))^2 - (\beta_7\beta_8' - \beta_7'\beta_8)g(y)t(y)]\\ &\times [(\beta_3\beta_6' - \beta_3'\beta_6)(g(y)t(y))^4 + (\beta_3'\beta_5 - \beta_3\beta_5' + \beta_1\beta_6' - \beta_1'\beta_6)(g(y)t(y))^3\\ &+ (\beta_1'\beta_5 - \beta_1\beta_5' + \beta_3\beta_8' - \beta_3'\beta_8)(g(y)t(y))^2 + (\beta_1\beta_8' - \beta_1'\beta_8)g(y)t(y)]^2 = 0.\end{aligned}$$

From above equations we get

$$(\beta_1\beta_7' - \beta_1'\beta_7)(\beta_7\beta_8' - \beta_7'\beta_8) = 0. \tag{2.55}$$

At first we have $\beta_1 \neq 0$. If $\beta_1 = 0$, we obtain that the coefficient of $(g(y)t(y))^2$ doesn't equal to zero in above equation. Then we have $\beta_7 \neq 0$ and $\beta_8 \neq 0$. If $\beta_7 = 0$, $f'(x) = 0$ and from $\beta_7 \neq 0$ we can get $\beta_8 \neq 0$. As $\beta_1\beta_7\beta_8 \neq 0$, we obtain $f'(x) = 0$, this means $K = 0$ from (2.55). This completes the proof of Theorem 7. □

Theorem 8. *Let S: $r(y,z) = (x(y,z), y, z) = (f(y)g(z), y, z)$ be a factorable surface of type 4 in E_1^3. If Gauss curvature K and mean curvature H of S satisfy $aH + bK = 0$ with $HK \neq 0$ and $a, b \in R - \{0\}$, S is following surface or an open part of it:*

$$\begin{cases} f(y) = c_1(y + c_2)^{-1}\\ z = \displaystyle\int \left[2c_3^2(g(z))^2\left(-\frac{2}{c_1} - \frac{2bc_3}{a|c_1|}\right)g(z) + 2\varepsilon c_3 g(z)\left(c_3^2(g(z))^2 - \frac{2bc_3}{a|c_1|}g(z)\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}} \mathrm{d}g,\end{cases}$$

where $\varepsilon = \pm1, c_1, c_2, c_3$ are constants.

Proof. If Gauss curvature K and mean curvature H of S satisfy $aH+bK=0$, we have

$$(2.56)\quad -\frac{2b}{a} = \frac{2f(y)f'(y)g(z)g''(z)+f''(y)g(z)-2f(y)f'(y)(g'(z))^2}{f(y)f''(y)g(z)g''(z)-(f'(y)g'(z))^2} \times[(f(y)g'(z))^2-2f'(y)g(z)]^{\frac{1}{2}}.$$

Differentiating (2.56) with respect to g we have

$$(2.57)\quad \begin{aligned}&\beta_1(f(y)f'(y))^2f''(y)-\beta_2 f(y)(f'(y))^4\\ &+\beta_3(f(y))^4f'(y)f''(y)+\beta_4(f(y)f'(y))^3\\ &+\beta_5 f(y)f'(y)(f''(y))^2-\beta_6(f'(y))^3f''(y)-\beta_7(f(y))^3(f''(y))^2=0,\end{aligned}$$

$$\begin{cases}\beta_1=-2\alpha_1^2+4\alpha_1\alpha_2'\alpha_3+2\alpha_1\alpha_2-4\alpha_1'\alpha_2\alpha_3-\alpha_2^2+\frac{1}{2}\alpha_2\alpha_2'\alpha_3,\\ \beta_2=4\alpha_1\alpha_2'\alpha_3+2\alpha_2^2-4\alpha_1'\alpha_2\alpha_3-2\alpha_1\alpha_2,\\ \beta_3=\alpha_1^2\alpha_2'-3\alpha_1\alpha_2\alpha_2'+2\alpha_1'\alpha_2^2,\\ \beta_4=\alpha_1\alpha_2\alpha_2'-2\alpha_1'\alpha_2^2+\alpha_2^2\alpha_2',\\ \beta_5=2\alpha_1'\alpha_3^2-3\alpha_1\alpha_3,\\ \beta_6=2\alpha_2'\alpha_3^2-3\alpha_2\alpha_3,\\ \beta_7=\alpha_1'\alpha_2\alpha_3-\alpha_1\alpha_2-\frac{1}{2}\alpha_1\alpha_2'\alpha_3,\\ \alpha_1=g(z)g''(z),\\ \alpha_2=(g'(z))^2,\\ \alpha_3=g(z),\\ \alpha_i'=\frac{\mathrm{d}\alpha_i}{\mathrm{d}g}.\end{cases}$$

Denoting $t(y)=f'(y)$, then we have

$$(2.58)\quad \begin{aligned}&\left(\frac{\frac{\mathrm{d}t}{\mathrm{d}f}}{f(y)}\right)^2\left(\beta_5\frac{t(y)}{(f(y))^2}-\beta_7\right)\\ &+\frac{\frac{\mathrm{d}t}{\mathrm{d}f}}{f(y)}\left(-\beta_6\left(\frac{t(y)}{(f(y))^2}\right)^2+\beta_1\frac{t(y)}{(f(y))^2}+\beta_3\right)\\ &-\beta_2\left(\frac{t(y)}{(f(y))^2}\right)^2+\beta_4\frac{t(y)}{(f(y))^2}=0.\end{aligned}$$

Put $h=\frac{t(y)}{(f(y))^2}$, we get

$$\begin{aligned}&[h^3(\beta_5\beta_6'-\beta_5'\beta_6)+h^2(\beta_6\beta_7'-\beta_6'\beta_7+\beta_1\beta_5'-\beta_1'\beta_5)+h(\beta_1'\beta_7-\beta_1\beta_7'+\beta_3\beta_5'\\ &-\beta_3'\beta_5)-(\beta_3\beta_7'-\beta_3'\beta_7)][h^4(\beta_2'\beta_6-\beta_2\beta_6')+h^3(\beta_1'\beta_2-\beta_1\beta_2'+\beta_4\beta_6'-\beta_4'\beta_6)\\ &+h^2(\beta_2\beta_3'-\beta_2'\beta_3+\beta_1\beta_4'-\beta_4'\beta_4)+h(\beta_3\beta_4'-\beta_3'\beta_4)]+[h^3(\beta_2'\beta_5-\beta_2\beta_5')\\ &+h^2(\beta_4\beta_5'-\beta_4'\beta_5+\beta_2\beta_7'-\beta_2'\beta_7)+h(\beta_4'\beta_7-\beta_4\beta_7')]^2=0.\end{aligned}$$

If $h\neq$ constant, we have $(\beta_3\beta_7'-\beta_3'\beta_7)(\beta_3\beta_4'-\beta_3'\beta_4)=0$.

(a) If $\beta_3\beta_7'-\beta_3'\beta_7=0$, we get $g(z)=c_1(y+c_2)^{c_3}$. With (2.56) we have $c_1=0$, that is $g'(z)=0$.

(b) If $\beta_3\beta_4' - \beta_3'\beta_4 = 0$, we get $\frac{g(z)}{(g'(z))^2}$ = constant. With (2.56) we have $g'(z) = 0$.

Thus we know h = constant, that is $\frac{t(y)}{(f(y))^2}$ = constant. Solving it we get $f(y) = c_1(y + c_2)^{-1}$. Applying $f(y) = c_1(y + c_2)^{-1}$ and (2.57) we get

$$z = \int \left[2c_3^2(g(z))^2 \left(-\frac{2}{c_1} - \frac{2bc_3}{a|c_1|} \right) g(z) + 2c_3\varepsilon g(z) \left(c_3^2(g(z))^2 - \frac{2bc_3}{a|c_1|} g(z) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}} dg.$$

This completes the proof of Theorem 8. □

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Huihui Meng
Department of Mathematics
Northeastern University
Shenyang 110004, P. R. China
E-mail address: mmhuihui0827@126.com

Huili Liu
Department of Mathematics
Northeastern University
Shenyang 110004, P. R. China
E-mail address: liuhl@mail.neu.edu.cn