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FACTORISATION IN NEST ALGEBRAS

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Abstract. We give a necessary and sufficient condition on an operator A for the existence of an operator B in the nest algebra Alg$N$ of a continuous nest N satisfying $AA^* = BB^*$ (resp. $A^*A = B^*B$). We also characterise the operators A in $B(H)$ which have the following property: For every continuous nest N there exists an operator B_N in Alg$N$ satisfying $AA^* = B_N B_N^*$ (resp. $A^*A = B_N^* B_N$).

1. Introduction–Preliminaries

The problem of factorisation of operators with respect to a nest algebra has been studied by many authors [8], [1], [13], [9], [11], [12], [10]. In this work we give a necessary and sufficient condition on an operator A for the existence of an operator B in the nest algebra Alg$N$ of a continuous nest N satisfying $AA^* = BB^*$ (resp. $A^*A = B^*B$). This result improves Theorem 4.9 in [9] for continuous nests. We also characterise the operators A in $B(H)$ which have the following property: For every continuous nest N there exists an operator B_N in Alg$N$ satisfying $AA^* = B_N B_N^*$ (resp. $A^*A = B_N^* B_N$).

Throughout this work H denotes a separable Hilbert space and $B(H)$ the space of all bounded operators from H into itself. If V is a subset of H we denote by $[V]$ the linear span of V. By subspace of H we mean a subset of H which is closed under addition of vectors and scalar multiplication. If $\{V_n\}_{n=1}^{\infty}$ is a sequence of closed mutually orthogonal subspaces of H we denote by $\sum_{n=1}^{\infty} \oplus V_n$ the closure of their linear span. If A is in $B(H)$ we denote by $r(A)$ the range of A and by coker$A$ the orthogonal complement of the kernel of A. An operator range is the range of a bounded operator in H. A nest in H is a totally ordered set of closed subspaces of H containing $\{0\}$ and H which is closed under intersection and closed span. If N is a nest in H and P is in N we will denote by the same symbol the orthogonal projection on the subspace P. If N is a nest we denote by $N^{\perp}$ the nest $\{P^{\perp} : P \in N\}$. A nest N is continuous if $P = \overline{[\bigcup_{Q<P} Q]}$ for every P in N. Given a nest N the associated nest algebra Alg$N$ is the set of operators A in $B(H)$ satisfying $PAP = AP$ for every P in N. For a general discussion of nest algebras the reader is referred to [3].

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2. Proper subspaces

We introduce in this section the notion of N-proper subspace for a nest N. We show that a closed subspace of H of co-finite dimension is N-proper for every continuous nest N.

Definition 1. Let N be a nest on H. A vector x in H is called N-proper if $x = Px$ for some P in N, $P \neq I$.

Definition 2. Let N be a nest on H. A subspace V of H is called N-proper if $[V \cap P : P \in N, P \neq I]$ is dense in V.

Lemma 3. *Let N be a continuous nest on H. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of elements of N such that: $P_n \neq I$, $P_{n+1} \geq P_n$, and P_n converges strongly to I. Let $x_1, x_2, \ldots, x_m$ be orthonormal vectors in H. Set $V = [x_1, x_2, \ldots, x_m]^{\perp}$. Then:*

(a) *There exists n_0 such that $P_n x_1, P_n x_2, \ldots, P_n x_m$ are linearly independent for $n \geq n_0$.*

(b) *We set $V_1 = P_1 H \ominus P_1 V^{\perp}$ and we define inductively*

$$V_n = P_n H \ominus \left(\sum_{i=1}^{n-1} \oplus V_i \oplus P_n V^{\perp} \right).$$

Then $V = \sum_{i=1}^{\infty} \oplus V_i$.

Proof. (a) The Grammian of the vectors $P_n x_1, P_n x_2, \ldots, P_n x_m$ converges to the Grammian of the vectors $x_1, x_2, \ldots, x_m$ which equals 1.

(b) It is easy to see that the V_n's are mutually orthogonal and that V_n is contained in V for every n. We show that $(\sum_{i=1}^{\infty} \oplus V_i) \oplus V^{\perp} = H$. Let x be a vector in H which is orthogonal to $(\sum_{i=1}^{\infty} \oplus V_i) \oplus V^{\perp}$. For each n the vector $P_n x$ is orthogonal to $\sum_{i=1}^{n} \oplus V_i$ so $P_n x$ is in $P_n V^{\perp}$. For $n \geq n_0$ we have $P_n x = P_n(\sum_{i=1}^{m} a_i x_i)$, where the a_i's are complex numbers not depending on n. So $x = \lim_{n \to \infty} P_n x = \sum_{i=1}^{m} a_i x_i$. But x is orthogonal to $V^{\perp}$, hence it is 0. □

Proposition 4. *Let N be a continuous nest and V a closed subspace of H of co-finite dimension. Then V is N-proper.*

Proof. It follows immediately from Lemma 3. □

Let N be a continuous nest on H and A an operator in $B(H)$. Consider the set $\bigcup_{P \in N, P \neq I} A^{-1}(P)$. This set is equal to $\bigcup_{P \in N, P \neq I} \mathrm{Ker}(P^{\perp} A)$. If A is an Alg$N$ the set $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ contains $\bigcup_{P \in N, P \neq I} P$; hence it is dense in H. There exist operators A in $B(H)$ for which $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is not dense in H. We construct such an operator in Example 9. We will prove in the next section that $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is dense in H if and only if there exists an operator B in Alg$N$ such that $AA^* = BB^*$. We first prove some preliminary results.

Lemma 5. *Let N be a nest on H and A an operator in $B(H)$. The following are equivalent:*

(a) *The set $\bigcup_{P \in N, P \neq 0} (A^*)^{-1}(P^{\perp})$ is dense in H.*

(b) $\bigcap_{P \in N, P \neq 0} \overline{r(AP)} = \{0\}$.

Proof. We have that $(A^*)^{-1}(P^{\perp}) = \{x \in H : A^* x \in P^{\perp}\} = \{x \in H : P^{\perp} A^* x = A^* x\} = \mathrm{Ker}(PA^*) = \overline{r(AP)}^{\perp}$ and $\bigcup_{P \in N, P \neq 0} \overline{r(AP)}^{\perp}$ is dense in $(\bigcap_{P \in N, P \neq 0} \overline{r(AP)})^{\perp}$. □

Proposition 6. *Let N be a nest on H and A an operator in $B(H)$.*

(a) *Suppose that the set $\bigcup_{P\in N,P\neq I} A^{-1}(P)$ is dense in H. Then $r(A)$ is N-proper.*

(b) *Suppose that $r(A)$ is N-proper and closed. Then the set $\bigcup_{P\in N,P\neq I} A^{-1}(P)$ is dense in H.*

Proof. (a) The set $A(\bigcup_{P\in N,P\neq I} A^{-1}(P))$ is contained in $[r(A)\cap P : P\in N, P\neq I]$ and is dense in $r(A)$.

(b) The restriction of A to coker$A$ is an isomorphism from coker$A$ onto $r(A)$. Hence $(\bigcup_{P\in N,P\neq I} A^{-1}(P))\cap \operatorname{coker}A = A^{-1}(\bigcup_{P\in N,P\neq I} P)\cap \operatorname{coker}A$ is dense in coker$A$. Therefore $\bigcup_{P\in N,P\neq I} A^{-1}(P)$ is dense in H. □

Proposition 7. *Let N be a nest on H and A an operator in $B(H)$.*

(a) *Suppose that $\bigcap_{P\in N,P\neq 0} \overline{r(AP)} = \{0\}$. Then* coker$A$ *is $N^{\perp}$-proper.*

(b) *Suppose that* coker$A$ *is $N^{\perp}$-proper and $r(A)$ is closed. Then $\bigcap_{P\in N,P\neq 0} \overline{r(AP)} = \{0\}$.*

Proof. (a) It follows from Lemma 5 that $\bigcup_{P\in N,P\neq 0}(A^*)^{-1}(P^{\perp})$ is dense in H. It follows from Proposition 6 that $r(A^*)$ is $N^{\perp}$-proper. Since the closure of an $N^{\perp}$-proper subspace is an $N^{\perp}$-proper subspace we conclude that coker$A$ is $N^{\perp}$-proper.

(b) It follows from [2, Ch. VI, Th. 1.10] that $r(A^*)$ is closed. Hence $r(A^*) =$ coker$A$. It follows from Proposition 6 that $\bigcup_{P\in N,P\neq 0}(A^*)^{-1}(P^{\perp})$ is dense in H. Therefore from Lemma 5 we conclude that $\bigcap_{P\in N,P\neq 0} \overline{r(AP)} = \{0\}$. □

3. Factorisation

In this section we prove our main results and give some applications.

Theorem 8. *Let N be a continuous nest and A an operator in $B(H)$. The following are equivalent:*

(a) *There exists an operator B in* Alg$N$ *such that $AA^* = BB^*$.*

(b) *The set $\bigcup_{P\in N,P\neq I} A^{-1}(P)$ is dense in H.*

Proof. Assume (a) holds. In order to prove (b) it is enough to prove that the set $\bigcup_{P\in N,P\neq I}(A^{-1}(P)\cap \operatorname{coker}A)$is dense in coker$A$. Using polar decomposition one can see that there exists a partial isometry U with domain coker$A$ and range coker$B$ such that $A = BU$. We put $R = \overline{[\bigcup_{P\in N,P\neq I}(A^{-1}(P)\cap \operatorname{coker}A)]}$ and $M =$ coker$A\ominus R$. We will show that $M = \{0\}$. Take m in M and P in N, $P\neq I$. Since $r(A) = r(B)$ ([5, Th. 1]), we have $BPUm = Ax_P$ for some x_P in coker$A$. Since $BPUm$ is in P, x_P is in $A^{-1}(P)\cap \operatorname{coker}A$ and hence in R. We have $BPUm = Ax_P = BUx_P$ and so $PUm-Ux_P$ is in $\ker B$. We have $PUm = PUm-Ux_P+Ux_P$ which belongs to $\ker B\oplus UR$. Note that the decomposition $H = \ker B\oplus UR\oplus UM$ is orthogonal. Therefore $Um = \lim_{P\in N,P\neq I,P\to I} PUm$ is in $(\ker B\oplus UR)\cap UM = \{0\}$. We conclude that $m = 0$.

Assume (b) holds. It is then clear that the set $\bigcup_{P\in N,P\neq I}(A^{-1}(P)\cap \operatorname{coker}A)$ is dense in coker$A$. Take a sequence $\{P_n\}_{n=0}^{\infty}$ of elements of N such that: $P_0 = 0$, $P_{n+1} > P_n, P_n\neq I$ for every n and P_n converges strongly to I. We set: $R_1 = A^{-1}(P_1)\cap \operatorname{coker}A$, $R_n = (A^{-1}(P_n)\cap \operatorname{coker}A)\ominus R_{n-1}$ for $n > 1$.

It is clear that R_n is orthogonal to R_m for $n\neq m$ and that R_n is contained in coker$A$ for every n. We show that coker$A = \sum_{n=1}^{\infty}\oplus R_n$. Take y in coker$A$. If y is orthogonal to $\sum_{n=1}^{\infty}\oplus R_n$, then y is orthogonal to $A^{-1}(P_n)\cap \operatorname{coker}A$ for every

n; hence y is orthogonal to $(\bigcup_{n=1}^{\infty}(A^{-1}(P_n) \cap \operatorname{coker}A))$. Since $(\bigcup_{n=1}^{\infty}(A^{-1}(P_n) \cap \operatorname{coker}A))$ is dense in coker$A$, $y = 0$, and so coker$A = \sum_{n=1}^{\infty} \oplus R_n$.

Consider for $n \geq 1$ a partial isometry V_n with domain contained in $(P_{n+1} - P_n)H$ and range R_n. Put $V = \sum_{n=1}^{\infty} \oplus V_n$. Then V is a partial isometry with range coker$A$. Note that $A = AVV^*$. We show that AV belongs to Alg$N$. Let P be in N and x be a vector in P. We show that AVx is in P. If $P \leq P_1$ we have $AVx = 0$. If $P > P_1$ there exists $m \geq 1$ such that $P_m < P \leq P_{m+1}$. Then $AVx = A(\sum_{n=1}^{m} \oplus V_n)x$ and $(\sum_{n=1}^{m} \oplus V_n)x$ is contained in $(\sum_{n=1}^{m} \oplus R_n)$. Therefore AVx is in $A(\sum_{n=1}^{m} \oplus R_n)$ which is contained in P_m. Since $P_m < P$ we conclude that AVx is in P.

Put $B = AV$. Then $BB^* = AVV^*A^* = AA^*$ and B is in Alg$N$. □

Remark. Theorem 8 remains true under the weaker assumption that N is a nest which satisfies $H = \overline{[\bigcup_{Q<H} Q]}$.

Let N be a continuous nest. We give an example of an operator with N-proper range which does not satisfy condition (b) of Theorem 8.

Example 9. Let N be a continuous nest. Take a sequence $\{P_n\}_{n=0}^{\infty}$ of elements of N such that:

$$P_0 = 0, \quad P_{n+1} > P_n, \quad P_n \neq I \quad \text{for every } n \text{ and } P_n \text{ converges strongly to } I.$$

For each n consider a vector e_n of norm 1 and such that $(P_{n+1} - P_n)e_n = e_n$. Put $y = \sum_{i=1}^{\infty} n^{-1}e_n$. Let A be the operator defined by: $Ae_n = n^{-1}e_n$ for $n \geq 1$, $Ae_0 = y$ and A is 0 on $[e_n : n = 0, 1, 2, \dots]^{\perp}$. Then $r(A)$ is N-proper and it is easy to see that A does not satisfy condition (b) of Theorem 8. In fact, e_0 is orthogonal to $\bigcup_{P \in N, P \neq I} A^{-1}(P)$. So A does not satisfy condition (a) of Theorem 8.

Corollary 10. *Let N be a continuous nest and A an operator in $B(H)$. The following are equivalent:*

(a) *There exists an operator B in* Alg$N$ *such that $A^*A = B^*B$.*

(b) $\bigcap_{P \in N, P \neq 0} \overline{r(AP)} = \{0\}$.

Proof. There exists an operator B in Alg$N$ such that $A^*A = B^*B$ if and only if there exists an operator C in Alg$N^{\perp}$ such that $A^*A = CC^*$. The corollary follows now from Theorem 8 and Lemma 5. □

Corollary 11. *Let N be a continuous nest and A an operator in $B(H)$. Suppose A is onto (resp. one-to-one and $r(A)$ is closed). Then there exists an operator B in* Alg$N$ *such that $AA^* = BB^*$ (resp. $A^*A = B^*B$).*

Proof. It follows from Proposition 6 and Theorem 8 (resp. from Proposition 7 and Corollary 10). □

Corollary 12. *Let N be a continuous nest and Q a projection in $B(H)$. Then there exists an operator B in* Alg$N$ *such that $Q = BB^*$ (resp. $Q = B^*B$) if and only if QH is N-proper (resp. $N^{\perp}$-proper).*

Proof. It follows from Proposition 6 and Theorem 8 (resp. from Proposition 7 and Corollary 10). □

The following corollary answers a question posed by Shields in [13].

Corollary 13. *Let N be a continuous nest and A a positive operator in $B(H)$. Assume there exists an operator B in* Alg$N$ *such that $A^2 = B^*B$. Then there exists an operator C in* Alg$N$ *such that $A = C^*C$.*

Proof. We have to show that if $\bigcap_{P\in N,P\neq 0} \overline{r(AP)} = \{0\}$, then $\bigcap_{P\in N,P\neq 0} \overline{r(A^{1/2}P)} = \{0\}$. Let y be in $\bigcap_{P\in N,P\neq 0} \overline{r(A^{1/2}P)}$. Then $A^{1/2}y$ is in $\bigcap_{P\in N,P\neq 0} \overline{r(AP)} = \{0\}$; hence $A^{1/2}y = 0$. So y is in Ker $A^{1/2}$. Since y is also in $\overline{r(A^{1/2})}$ we see that $y = 0$. □

We will characterise now the operators that satisfy condition (a) of Theorem 8 (resp. condition (a) of Corollary 10) for every continuous nest.

Proposition 14. *Let V be an operator range. Assume V is not of co-finite dimension in H. Then there exists a continuous nest N in H such that $V \cap P = \{0\}$ for every P in N, $P \neq I$.*

Proof. (i) We first show that there exists a non-closed operator range W which contains V. We will use the following fact: If V_1, V_2 are operator ranges, then $V_1 + V_2$ is an operator range [7, Ch. I, 1]. If V is closed we consider an operator range U which is non-closed and is contained in $V^\perp$. We set $W = V + U$. Then W is an operator range which is non-closed and contains V.

(ii) It follows from (i) above that we may assume that V is non-closed. An operator range R is called of type J_S (Dixmier's notation) if it is dense and there exists a sequence $\{H_n\}_{n=0}^\infty$ of closed mutually orthogonal infinite dimensional subspaces of H such that $R = \{\sum_{n=0}^\infty x_n : x_n \in H_n \text{ and } \sum_{n=0}^\infty (2^n\|x_n\|)^2 < \infty\}$. It is shown in the proof of Theorem 3.6 in [6] that any non-closed operator range is contained in an operator range of type J_S. It follows that there exists an operator range S of type J_S such that $V \subset S$. It follows from Theorem 3.6 in [6] that there exists a unitary operator U on H such that $US \cap S = \{0\}$. We conclude that there exists an operator range T of type J_S such that $V \cap T = \{0\}$. Now it is easy to see that there exists a continuous nest N in H such that $P \subset T$ for every P in N, $P \neq I$. It follows that $P \cap V = \{0\}$ for every P in N, $P \neq I$. □

Theorem 15. *Let A be an operator in $B(H)$.*

(a) *There exists for every continuous nest N an operator B_N in* Alg$N$ *satisfying $AA^* = B_N B_N^*$ if and only if A is a right Fredholm operator.*

(b) *There exists for every continuous nest N an operator B_N in* Alg$N$ *satisfying $A^*A = B_N^* B_N$ if and only if A is a left Fredholm operator.*

Proof. (a) Assume that for every continuous nest N there exists an operator B_N in Alg$N$ satisfying $AA^* = B_N B_N^*$. It follows from Theorem 8 and Proposition 6 that $r(A)$ is N-proper for very continuous nest N. Proposition 14 implies that $r(A)$ is of co-finite dimension in H. If the range of an operator is of co-finite dimension, then it is closed [4, Prop. 3.7]. Therefore A is a right Fredholm operator. Assume now that A is a right Fredholm operator. Then $r(A)$ is closed and of co-finite dimension in H. By Proposition 4, $r(A)$ is N-proper for every continuous nest N. It follows then from Proposition 6 and Theorem 8 that for every continuous nest N there exists an operator B_N in Alg$N$ satisfying $AA^* = B_N B_N^*$.

(b) Consider the following properties of an operator A:

(i) There exists for every continuous nest N an operator B_N in Alg$N$ satisfying $AA^* = B_N B_N^*$.

(ii) There exists for every continuous nest N an operator B_N in Alg$N$ satisfying $A^*A = B_N^* B_N$.

Since a nest N is continuous if and only if the nest $N^\perp$ is continuous we see that an operator A has property (i) if and only if the operator A^* has property (ii). The assertion follows now from (a). □

Added in proof

After this work was submitted a paper of G. T. Adams, J. Froelich, P. J. McGuire, and V. I. Paulsen entitled *Analytic reproducing kernels and factorisation*, Indiana Univ. Math. J. **43** (1994), came to our attention. Condition (b) of our Theorem 8 is essentially the same with the density condition given in Theorem 3.1 of this paper in a different but related context.

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