SURVEY ARTICLE

Factorizable inverse monoids

In memory of Douglas Munn

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February 20, 2009

Abstract

Factorizable inverse monoids constitute the algebraic theory of those partial symmetries which are restrictions of automorphisms; the formal definition is that each element is the product of an idempotent and an invertible. This class of monoids has theoretical significance, and includes concrete instances which are important in various contexts. This survey is organised around the idea of group acts on semilattices and contains a large range of examples. Topics also include methods for construction of factorizable inverse monoids, and aspects of their inner structure, morphisms, and presentations.

1 Introduction

One persuasive view of the theory of inverse monoids is that it constitutes the algebraic theory of partial symmetries. Correspondingly, we may distinguish the theory of *factorizable*[1] inverse monoids as the algebraic theory of those partial symmetries which are restrictions of total symmetries, that is, of automorphisms. Another view of inverse monoids is as a common generalisation of groups and semilattices, and so structurally as made up somehow of groups and semilattices. Factorizable inverse monoids form a relatively simple class of such a construction, the formal algebraic definition being that each element is expressible as the product of a unit and an idempotent. These two viewpoints come together in McAlister's P-theory—which expresses each inverse semigroup S as an idempotent-separating quotient of an E-unitary one—because strict embeddings of S in factorizable inverse monoids correspond to E-unitary covers of S (this is discussed in a little more detail below).

The role of total symmetries in understanding structures is important and well-studied; the case for partial symmetries (that is, isomorphisms between subordinate structures) has been forcefully advocated in [18]. The theory can be

[1] I follow the spelling used by Chen and Hsieh, but the electronic searcher may need to be aware that *factorisable* is often preferred.

approached in various ways: by inductive groupoids, monosetting categories, or inverse monoids, all of which have technical advantages in certain situations, but are ultimately equivalent. The situations in which factorizable inverse monoids arise correspond to relatively highly symmetric structures, in which every local or partial symmetry can be extended to a full symmetry. (This is in general a stronger condition than that of *homogeneity* in model theory and its applications, in which any isomorphism of finite substructures can be extended to an automorphism.) In the categorical setting, partial symmetry has been given a functorial standing in [16]; the authors set out the general context, giving references to earlier works, and one of the functors they define exhibits factorizable inverse monoids as endomorphism monoids in a range category.

The present article is prompted by the belief that a systematic organisation of the known material on factorizable inverse monoids can both expose some interesting and useful connections with other areas of algebra, and also generate and describe many important occurrences and examples. From the remarks above, the reader can see that the potential scope is very wide; so the author has chosen to focus on relevant formal algebraic properties of factorizable inverse monoids and to the listing of their manifestations.

2 Some history

Generalising the (inner) direct product construction, Guido Zappa [37] in 1940 considered groups G possessing subgroups A and B such that $G = AB$; in the case that $A \cap B = \{1\}$, these are now known as *Zappa-Szép products* and the factors are unique. Informative reviews by Hirsch of articles of Rédei and Szép—including [34] and [31]—provide an overview of the area. Naturally, similar constructions were sought for semigroups, and in particular for their most 'group-like' classes. Thus what seems that the first usage of the term *factorizable* in the theory of semigroups was by Tolo [36], to mean $S = AB = \{ab \mid a \in A, b \in B\}$, where A, B are subsemigroups of the semigroup S which are of special kinds, e.g., groups, completely semisimple, etc.[2] One may speculate that the reason Tolo's paper did not attract much attention is that his definition was broader than the one we use today—too general, one may say—even though his paper does consider the case of a *chain of groups,* which is a special case (group by chain) of the contemporary sense (group by semilattice). Another promising way to impose extra structure is, as in the group case, to have unique factorisation [12], [11] and hence a Zappa-Szép product.

A particular class of factorizable inverse monoids—coset monoids, which we discuss in a later section—had been studied by Schein [33] and, as ordered groupoids, by Joubert [14], both in 1966. However Chen and Hsieh in 1974 [1] were the first authors to draw explicit attention to the class of inverse semi-

[2] Tolo completed a PhD in semigroup theory, advised by D. W. Miller, before changing fields and enjoying a distinguished academic career in the field of public policy (especially the interaction of education, politics and society).

groups[3] considered here, study their properties, and set out their importance: they proved, for example, that the archetypal inverse semigroup $\mathcal{I}_X$ is factorizable if and only if X is finite, and that every inverse semigroup S embeds in a factorizable one, which may be chosen finite if S is finite.

The paper of Chen and Hsieh caught the attention of researchers, with a peak in citations around 1980 as the importance of factorizable inverse monoids emerged. Citations of [1] then slowed for more than a decade, though classes of inverse semigroups important for other reasons have turned out to be factorizable, notably the Renner monoids ([30] Prop. 11.1; [32], section 8.1) and latterly the reflection monoids of Everitt and Fountain [8]. Appropriate generalisations of the concept were developed: significantly, Lawson [17] identified the appropriate generalisation from monoids to semigroups as *almost factorizable* semigroups, which had been used in McAlister [25]; for an account, see section 7.1 of [18]. Tirasupa [35] examined the Clifford by semilattice case and Mills [29] the group by aperiodic case. Consideration of these generalisations is beyond the scope of this article.

Still the theory of factorizable inverse monoids was not exhausted, as shown in further citations dating from around 2000. Thus a brief survey of the topic is perhaps timely. The approach given here uses a definition formally different from, though equivalent with, that of Chen and Hsieh, and organises the theory of factorizable inverse monoids around the pervasive mathematical concept of group actions. Proofs are generally omitted, but are either straightforward, or referenced; unexplained concepts and notation follow [2] and [13].

3 Notation, definitions and basic results

A fixed system of notation proves difficult to sustain when we discuss different contexts in which the concepts appear. That chosen here is intended to be flexible, while maintaining a consistency in spirit. Functions will be written in various forms sensitive to custom and the context. We exploit the protean nature of semilattices as both commutative idempotent semigroups and as ordered sets with particular properties; frequently we need to distinguish *upper* semilattices, in which the order is given in terms of the operation by

$$a \leq b \text{ if and only if } ab = b,$$

from *lower* semilattices, in which the order is given by

$$a \leq b \text{ if and only if } ab = a.$$

The $\leq$ symbol is overloaded: depending on context it may signify the natural partial order in a semilattice or in an inverse semigroup, the given order in a semilattice Y, or the containment relationship between subgroups of a group. If

[3] Actually, as we shall see, this class consists of inverse monoids, and so we refer to them as such.

X is a poset, and $Y \subseteq X$, we shall denote the order ideal

$$\{x \in X \mid x \leq y \text{ for some } y \in Y\}$$

by $Y^{\downarrow}$. For any monoid S, we write $U(S)$, or simply U, for its group of units (two-sided invertibles), and $E(S)$, or simply E, for its set of idempotents. When S is inverse, the natural order in $E(S)$ will always be treated as that of a lower semilattice.

In the spirit of Tolo, and varying slightly from Chen & Hsieh [1], we suppose throughout this section that the semigroup S, not necessarily inverse, may be expressed as $S = GY$, where G and Y are subsemigroups of S, G is a group, and Y is a semilattice (so that $Y \subseteq E(S)$). The starting assumptions of [1] were different: the authors took S to be inverse throughout, and assumed $S = GY$, but had Y as merely a *set* of idempotents. In this definition, Y of course may be taken to be the set of *all* idempotents, and when S is inverse, as the *semilattice* of all idempotents of S. In a later section, we also consider a weaker hypothesis, investigating inverse monoids generated by units and idempotents. What is common between the present approach and [1] is that the members of Y commute (even if Y is not necessarily closed in [1]).

For any $g \in G$, we write g^{-1} for its inverse in G, and $g^0 = gg^{-1} = g^{-1}g$ for the identity element of G. Even if it is not especially mentioned, we assume $g, h \in G$ and $y, z \in Y$ throughout. In essence, the following basic results are from [1], but arranged to clarify the role of the condition that S be inverse.

Proposition 3.1 *Let $S = GY$, where G and Y are subsemigroups of S, G is a group, and Y is a semilattice. Then*
(i) S is a monoid with identity $g^0 \in Y \cap G$, and $G = U(S)$;
(ii) if $y = gz$ (for $y, z \in Y$ and $g \in G$) then $y = z$;
(iii) S is regular;
(iv) $S = GY = E(S)G$;
(v) for each $s \in S$, there is $u \in U(S)$ such that $s \leq u$.

Observe that (ii) implies that the idempotent factor in $s = gy$ is unique: if $gy = hz$ then $y = g^{-1}hz$ and then $y = z$. We noted above that $Y \subseteq E(S)$; something special happens when $Y = E(S)$.

Theorem 3.2 *Let $S = GY$. The following are equivalent:*
(i) $E(S) \subseteq Y$;
(ii) $E(S)$ is a semilattice;
(iii) S is inverse;
(iv) $g^{-1}yg \in Y$ for all $y \in Y$, $g \in G$.

Definition. Any of the equivalent conditions of Theorem 3.2 thus define S as a *factorizable* inverse monoid. More generally, if S is any inverse monoid, with group of units U and semilattice of idempotents E, then $F(S) = UE = U^{\downarrow}$ defines the unique largest factorizable inverse submonoid of S; we call $F(S)$ the *factorizable part* of S.

Remarks 3.3 (i) The monoid S of self-maps (written on the left of their arguments) of a two-element set $\{x, y\}$ provides a simple example of a semigroup meeting the conditions of Proposition 3.1 but not those of Theorem 3.2. Let $G = \{1, u\}$ be its group of units, where u transposes x and y, and let e be the idempotent (constant) map taking value x. Then with $Y = \{1, e\}$ we have $S = GY$, but S is not inverse.

(ii) There exist factorizable inverse subsemigroups of some inverse monoids S which neither contain nor are contained by the factorizable part of S. The apparent paradox lies in the wording: a factorizable inverse submonoid of S must by definition contain the identity of S. So, for example, take a non-trivial group H, and adjoin an extra identity and a zero 0; call the result S. Then $F(S) = E(S)$, while $H \cup \{0\}$ is a (maximal) factorizable inverse monoid which is merely a subsemigroup of S. More generally, if H is any subgroup of an inverse semigroup S, then $H^{\downarrow}$ is factorizable, and is contained in $F(S)$ if and only if $H \subseteq U^{\downarrow}$.

We conclude this section by noting that the class of factorizable inverse monoids is closed under the taking of direct products and homomorphic images (quotient semigroups). It is not closed under the taking of subsemigroups; indeed, as we shall see (in Example 8.1), every semigroup appears as a subsemigroup of some factorizable inverse monoid. The class is closed under the taking of subsemigroups containing all the units, the so-called *cofull* submonoids.

4 General construction, morphisms, and a category equivalence

From now on, unless otherwise stated, we consider only factorizable inverse monoids S, and write $S = UE$ or $S = GY$, etc., without further explanation, and denote the identity element of S by 1. Again, if not mentioned, it is to be understood that $u, v, \dots \in U$ and $e, f, \dots \in E$. By Proposition 3.1(iv), $S = UE = EU$ and in fact $ue = \left(ueu^{-1}\right) u$. The conjugates of e, ueu^{-1} and $u^{-1}eu$, occur so frequently that it is convenient to write $e^u = u^{-1}eu$ and ${}^u e = ueu^{-1} = e^{\left(u^{-1}\right)}$.

For $e \in E$, the map $\gamma_u : e \mapsto e^u$ satisfies

$$(e^u)^v = e^{uv} \quad \text{and} \quad (ef)^u = e^u f^u,$$

and therefore defines a group action of U on the semilattice E, which we refer to as the *natural* or *conjugation* action. We may also say that U is an E-*act* or that E is a U-*semilattice;* or choose to think of $\gamma : U \longrightarrow \mathrm{Aut}(E)$, defined by $\gamma : u \mapsto \gamma_u$, as a representation of U in $\mathrm{Aut}(E)$, called the *natural* representation. In particular, $e^1 = e$ and $1^u = 1$.

For chosen e, we call the subgroup $C(e) = \{u \in U \mid e^u = e\}$ the *centralizer* of e. (It is also naturally thought of as the *stabilizer* of the conjugation action, but we have occasion to consider several actions, and with the chosen notation we avoid conflict—for the most part: in one context $C(e)$ even appears as a normalizer!) By definition of $S = UE$, there is a surjective map of sets,

$\phi : U \times E \longrightarrow S$, whereby $(u,e)\,\phi = ue \in S$. Now we have

$$ue\,(vf) = uv\left(v^{-1}ev\right)f = uv\,(e^v f)\,, \quad \text{i.e.,}$$
$$(u,e)\,\phi \cdot (v,f)\,\phi = (uv, e^v f)\,\phi,$$

and so ϕ becomes a homomorphism of semigroups when the set $U \times E$ is endowed with the product operation

$$(u,e)\,(v,f) = (uv, e^v f)\,.$$

This is precisely the definition of $U \ltimes E$, the *semidirect product* of U and E associated with the E-action of U. It is easily seen to be an inverse semigroup (with $(u,e)^{-1} = \left(u^{-1}, {}^u e\right)$) and to have group of units $\{(u,1) \mid u \in U\} \cong U$ and semilattice of idempotents $\{(1,e) \mid e \in E\} \cong E$. Since $(u,e) = (u,1)\,(1,e)$ in $U \ltimes E$, it follows that $U \ltimes E$ is a factorizable inverse monoid. The factorisation is unique, and so it is also a Zappa-Szép product.

By its definition, ϕ is always idempotent-separating, that is, $(1,e)\,\phi = (1,f)\,\phi$ implies $e = f$; so ϕ will be determined by its kernel or equivalently, by its kernel normal system. Thus we need to examine the elements (u,e) such that $(u,e)\,\phi = (1,e)\,\phi$, that is, $ue = e$. Let us define, for each $e \in E$, the set $K_e = \{u \in U \mid ue = e\}$. Then the kernel normal system for ϕ consists of the sets $K_e \times \{e\}$, and the congruence $\sim$ on $U \ltimes E$ induced by ϕ is given by

$$(u,e) \sim (v,f) \Longleftrightarrow e = f \text{ and } u^{-1}v \in K_e. \tag{4.1}$$

Writing $[u,e]$ for the $\sim$-class containing (u,e), the multiplication in S is

$$[u,e]\,[v,f] = [uv, e^v f]\,. \tag{4.2}$$

We summarise the properties of the sets K_e :

Lemma 4.1 *The sets K_e have the properties:*
(i) $K_1 = \{1\}$;
(ii) $ue = e$ if and only if $u^{-1}e = e$ if and only if $eu = e$, etc.;
(iii) $ue = vf$ if and only if $e = f$ and $uv^{-1} \in K_e$;
(iv) K_e is a subgroup of U;
(v) $e \leq f$ implies $K_f \leq K_e$;
(vi) $K_e^u := u^{-1}K_e u = K_{e^u}$ for all $u \in U$;
(vii) $K_e \trianglelefteq C(e)$.

Remarks. These properties have been noted in the literature more than once. Part (v) is equivalent to $K_e \vee K_f \leq K_{ef}$.

Let us rephrase the statements of Lemma 4.1 in terms of representations: (iv) and (v) say that the map $K : e \mapsto K_e$ is an order-reversing or antitone map of E to $\operatorname{Sub}(U)$, the lattice of subgroups of U, which by (i) sends the top element of E to the bottom element of $\operatorname{Sub}(U)$; we call it the *natural* representation of E. Actually it is often more natural to use the *reverse* of the usual order in $\operatorname{Sub}(U)$, i.e. think of it as an upper or join semilattice, and describe K as an *order representation* of E in $\mathrm{Sub}\,(U)$ which preserves the identity. In general, K need not preserve the semilattice operation, so it is not a representation of semilattices; because it satisfies the weaker condition $K_{ef} \geq K_e \vee K_f$, when taken with respect to this reverse order, K is a *lax* homomorphism or *prehomomorphism* (of semilattices or of inverse semigroups).

In summary, we have representations γ and K of U in $\mathrm{Aut}\,(E)$ and of E in $\mathrm{Sub}\,(U)$, which are *linked* in the sense that there hold for all $e \in E, u \in U$

$$K_1 = \{1\}\,, \tag{4.3(a)}$$

$$K_e \trianglelefteq C\,(e)\,, \quad \text{i.e., } e^{K_e} = \{e\}\,, \tag{4.3(b)}$$

$$K_e^u = K_{e^u}\,. \tag{4.3(c)}$$

We also refer to the subgroups K_e as *pre-kernels* and the map K as the pre-kernel map. If K is a true homomorphism of monoid semilattices, i.e., $K_{ef} = K_e \vee K_f$ for all $e, f \in E$, we shall call K *exact*; and by extension, we call a factorizable inverse monoid $S = EU$ *exact* if its pre-kernel map K is exact. We return to a closer examination of the case in which K is exact later in the article.

The following recipe for factorizable inverse monoids provides a fertile source of examples and a context for describing their structure and relationships.

Theorem 4.2 *Any factorizable inverse monoid $S = UE$ determines natural linked representations of U in $\mathrm{Aut}\,(E)$ and of E in $\mathrm{Sub}\,(U)$. Conversely any pair of linked representations of a group G and semilattice Y determine a factorizable inverse monoid S such that $G \cong U\,(S)$ and $Y \cong E\,(S)$, whose natural linked representations are (up to relabelling) the given ones.*

Now we consider the homomorphisms between factorizable inverse monoids, reminding the reader of a distinction between semigroup homomorphisms, which preserve multiplication, and monoid homomorphisms, which preserve also the identity element (earlier, there was a related distinction between subsemigroups and submonoids). Consider factorizable inverse monoids $S = GY$ and $T = UE$. Let the linked representations γ and K defining S and T be distinguished, if necessary, by superscripts, thus: γ^S and K^S for S, etc.

Theorem 4.3 *A monoid morphism $\Phi : S \longrightarrow T$ restricts to a group morphism $\phi = \Phi|_G : G \longrightarrow U$ and to a monoid morphism of semilattices $\psi = \Phi|_Y : Y \longrightarrow E$. Then the natural representations in S and T are related by*

$$(y\psi)^{g\phi} = (y^g)\,\psi, \text{ i.e., } \psi \circ \gamma^T_{g\phi} = \gamma^S_g \circ \psi, \quad \text{and} \tag{4.4(a)}$$

$$K^S_y \phi \leq K^T_{y\psi}. \tag{4.4(b)}$$

Suppose, conversely, we have a group morphism $\phi : G \longrightarrow U$ and a monoid morphism of semilattices $\psi : Y \longrightarrow E$, satisfying (4.4) for given pairs of linked representations $y \mapsto y^g$ and K^{GY}, and $e \mapsto e^u$ and K^{UE} respectively. Then the assignment

$$\Phi : gy \mapsto (g\phi)\,(y\psi)$$

is well-defined and a (monoid) homomorphism $GY \longrightarrow UE$ of factorizable inverse monoids.

We may rephrase Theorems 4.2 and 4.3 together as a statement of a categorical equivalence. First we introduce a category $\mathbf{Q}$ which has

> *objects:* all quadruples $(G, Y; \gamma, K)$, where G is a group, Y a semilattice, and γ and K are a pair of linked representations as defined in equations (4.3), and
>
> *morphisms* $\left(G, Y; \gamma^S, K^S\right) \longrightarrow \left(U, E; \gamma^T, K^T\right)$: pairs (ϕ, ψ), where $\phi : G \longrightarrow U$ and $\psi : Y \longrightarrow E$ are morphisms of groups and monoid semilattices respectively, and satisfy conditions (4.4) for all $g \in G$ and $y \in Y$.

Corollary 4.4 *The category of factorizable inverse monoids and monoid morphisms is equivalent to the category* $\mathbf{Q}$.

It is of use for identifying relationships between factorizable inverse monoids to characterise injective, surjective and bijective homomorphisms, in terms of $\mathbf{Q}$.

Theorem 4.5 *A* $\mathbf{Q}$*-morphism* $(\phi, \psi) : \left(G, Y; \gamma^S, K^S\right) \longrightarrow \left(U, E; \gamma^T, K^T\right)$ *is*
(i) injective if and only if ϕ *and* ψ *are injective and* $\left(K^T_{y\psi}\right)\phi^{-1} \leq K^S_y$;
(ii) surjective if and only if ϕ *and* ψ *are surjective;*
(iii) bijective if and only if ϕ *and* ψ *are bijective and* $K^S_y \phi = K^T_{y\psi}$.
Proof. (i) 'If': suppose $g\phi \cdot y\psi = h\phi \cdot z\psi$. Then $y\psi = z\psi$, whence $y = z$, and $\left(g^{-1}h\right)\phi \in K_y\phi$, whence $gy = hy$ by injectivity of ϕ. 'Only if': (ϕ, ψ) injective clearly implies ϕ, ψ are injective. Take u such that $u\phi \in K_{y\psi}$; then $u\phi \cdot y\psi = 1\phi \cdot y\psi$, whence injectivity implies $uy = y$, i.e $u \in K_y$.

(ii) 'If': for all $u \in U$ and $e \in E$ there are $g \in G$ and $y \in Y$ such that $ue = g\phi \cdot y\psi$; the 'only if' is clear.

(iii) Follows from (i) and (ii).

We remark that $\left(K^T_{y\psi}\right)\phi^{-1} \leq K^S_y$ is equivalent to $G\phi \cap K^T_{y\psi} \leq K^S_y\phi$. We now specialize to the case when Φ is an inclusion, i.e., when S is a factorizable inverse submonoid of T. Then ϕ and ψ are restrictions of the inclusion map on S, and the conditions (4.4) translate to requirements that $\gamma^S_g := \gamma^T_g|_Y \in \mathrm{Aut}\,(Y)$ and $K^S_y := K^T_y \cap G$. Since $\gamma^T|_G$ remains a (group) representation, of G in $\mathrm{Aut}\,(E)$, it is thus necessary and sufficient for Φ to be an inclusion that $y^g \in Y$ for all $y \in Y, g \in G$; that is, that Y is invariant under the action of G induced from the action of U. ($K_y \cap G$ automatically gives a representation of Y in $\mathrm{Sub}\,(G)$.) The linkage conditions then hold by restriction.

Next, if $\Phi : S \longrightarrow T$ is merely a semigroup morphism, put $f = 1\Phi \in E$. Φ induces a morphism of semilattices $\psi = \Phi|_Y : Y \longrightarrow E$ and a morphism of groups $\phi = \Phi|_G : G \longrightarrow H = H_f$ (the maximal subgroup containing f). The image of ψ is contained in $Z := f^{\downarrow}$, an order-ideal of E, and HZ is a factorizable inverse monoid which is merely a subsemigroup (not submonoid) of $T = UE$.

However by simply restricting the range of Φ to HZ, a trivial modification, we have a monoid homomorphism of $S = GY$ to HZ and we may use the criteria above.

Since a group action $G \times E \to E$ may be viewed equivalently as an ordered groupoid, all the above generalities could also be worked out in the groupoid context, providing another linkage to inverse semigroups; a particular instance is found in Example 2.2.3 of [19] (cf. section 8.5 below). For the sake of trying to keep things simple, we eschew this possibility.

5 Structural features

In this section we use the properties of the linked natural representations to examine structural features of factorizable inverse monoids. This yields formulæ for the cardinalities of S and its $\mathcal{D}$-classes, and enables us to describe the maximal subgroups of S and to characterise special classes of factorizable inverse monoids.

5.1 Natural order and Green's relations

First we characterize the natural partial order and Green's relations by the components of the recipe.

Proposition 5.1 *In a factorizable inverse monoid,*
(i) $ue\,\mathcal{L}\,vf$ *if and only if* $e = f$*;*
(ii) $ue\,\mathcal{R}\,vf$ *if and only if* ${}^{u}e = {}^{v}f$*;*
(iii) $ue\,\mathcal{H}\,vf$ *if and only if* $e = f$ *and* $u^{-1}v \in C(e)$*;*
(iv) $ue\,\mathcal{D}\,vf$ *if and only if* $e \in f^G$*, i.e.,* e *and* f *are in the same orbit of the* U*-act;*
(v) $ue\,\mathcal{J}\,vf$ *if and only if there exist* $i, j \in E$ *such that* $e \leq i \in f^U$ *and* $f \leq j \in e^U$*;*
(vi) $ue \leq vf$ *if and only if* $e \leq f$ *and* $u^{-1}v \in K_e$*.*

If H is any subgroup of an inverse semigroup, then $H^{\downarrow}$ is factorizable in its own right. (Chen and Hsieh give over part of their paper [1] to a discussion of factorizable inverse monoids which are subsemigroups of an arbitrary inverse monoid.) So the factorizable part of S may also be defined in terms of the natural partial order by $F(S) = \{s \in S \mid s \leq u \text{ for some } u \in U\} = U^{\downarrow}$; and S is a factorizable inverse monoid if and only if every element of S is bounded above by a unit. This dovetails with an interpretation of the natural partial order as a restriction relation, and $s = eu$ as a partial symmetry which is the restriction of the automorphism u to the 'domain' e $\left(= ss^{-1}\right)$ or to a 'range' e^u $\left(= s^{-1}s\right)$.

Remark 5.2 We may paraphrase the condition of part (v) as "the orbits of e and f are cofinal". An example where e^U and f^U are cofinal but not equal, that is, where $\mathcal{J} \neq \mathcal{D}$, arises in the action of $G = (\mathbb{Z}, +)$ on $Y = (\mathbb{Z} \cup \{\infty\}, \min)$ defined by $\infty^g = \infty$ and $y^g = y + ng$ for $y \neq \infty, g \in G$ and $n \geq 2$. The orbits of the action are $\{\infty\}$ and the congruence classes modulo n, so the non-units of $G \ltimes Y$ fall into n distinct $\mathcal{D}$-classes in the one $\mathcal{J}$-class. In fact if $\mathcal{J} \neq \mathcal{D}$ in $S = UE$, then U has an element of infinite order which acts non-trivially, and E contains a doubly infinite chain.

Remark 5.3 Let us recall that an *inverse algebra* is defined as an inverse monoid in which the natural order is a semilattice order [20]; equivalently, in which beneath each element s there is a maximum idempotent, denoted $f[s]$ and called the *fixed-point idempotent* of s. Now from Proposition 5.1 (vi), $i \leq eu$ if and only if $i \leq e$ and $u \in K_i$, i.e. $i \leq u$. So a factorizable inverse monoid S is an inverse algebra if and only if for each unit u there is a fixed-point idempotent $f[u]$, that is, a maximum element of $\{i \in E \mid u \in K_i\}$; in which case, $f[eu] = ef[u]$.

5.2 Subgroups, central idempotents and zeros

The following corollary treating central idempotents and subgroups is now evident from the preceding section 5.1. We use $Z(S)$ to mean the centre of S, viz., $\{a \in S \mid as = sa \text{ for all } s \in S\}$.

Corollary 5.4 *(i)* $ue\,\mathcal{H}\,f = f^2 \iff e = f = {}^u e$. *Thus, ue is completely regular if and only if $ue = eu$, i.e., $u \in C(e)$.*
(ii) $H_e = \{ue \mid u \in C(e)\} \cong C(e)/K_e$.
(iii) $e \in Z(S) \iff e \in Z(G) \iff C(e) = U \iff \{e\}$ *is a singleton orbit* $\iff H_e = D_e$ *in* S.

In particular, if E has a zero (bottom element) 0_E, then $C(0_E) = U$; and 0_E is also a zero of S if and only if $K_{0_E} = U$.

5.3 Clifford, E-unitary, etc.

By Prop. 5.1 (iv), the orbits of the action determine the $\mathcal{D}$-classes of S. The identity 1 is always in a singleton orbit, so we see immediately that S cannot be bisimple unless it is a group. We proceed to deduce other structural relationships from sections 5.1 and 5.2.

Corollary 5.5 *The following are equivalent for a factorizable inverse monoid $S = UE$:*
(i) S is a Clifford semigroup;
(ii) the action is trivial ($\gamma_u = id$ for all u);
(iii) $C(e) = U$ for all $e \in E$;
(iv) the multiplication in $U \ltimes E$ is that of the direct product;
(v) S is the homomorphic image of a direct product $U \ltimes E$.

In the case described, the pre-kernels are all normal in U, and the Cliffordian structure maps

$$\phi_{e,f} : U/K_e \longrightarrow U/K_f \quad (e \geq f)$$

are induced by the inclusion $K_e \trianglelefteq K_f$ ($\phi_{e,f} : K_e u \mapsto K_f u$) and are all surjective. Chen and Hsieh include a substantial discussion of factorizable Clifford monoids in [1], including conditions on the structure maps.

For the next result, we use the definition that $S = UE$ is E-unitary if and only if $e \in E$ and $e \leq s$ in S imply $s \in E$, and note that by Proposition 5.1(vi), $e \leq v$ if and only if $v \in K_e$.

Proposition 5.6 *The following are equivalent for $S = UE$:*
(i) S is E-unitary;
(ii) $K_e = \{1\}$ for all e;
(iii) $K_e = K_f$ for all e, f;
(iv) $S \cong U \ltimes E$.

Combining the preceding results, one sees that the direct product case $S \cong U \times E$ is precisely the Clifford E-unitary case.

5.4 Cardinality

Here, of course, we assume U and E to be finite. We let $\mathcal{O}$ be the set of orbits of the action, in bijective correspondence with $S/\mathcal{D}$, the set of $\mathcal{D}$-classes of S. Since S is finite, $\mathcal{D} = \mathcal{J}$ and $\mathcal{O}$ also corresponds to the set of principal two-sided ideals of S. Let $\Omega = \Omega_e = e^G$ be the orbit containing e. There are $|\Omega|$ idempotents in the $\mathcal{D}$-class D_e containing e, hence $|\Omega|$ $\mathcal{L}$- and $\mathcal{R}$-classes each in D_e. By Prop. 5.5 (ii), $|H_e| = |C(e)/K_e|$. It is familiar knowledge in the theory of group actions that elements of the orbit Ω bijectively correspond with cosets of $C(e)$ in $U : e^u = e^v$ if and only if $uv^{-1} \in C(e)$, so the correspondence $y \mapsto uC(e) \iff y = e^u$ is well-defined and bijective. So the cardinality of the $\mathcal{D}$-class of e is

$$|D_e| = \frac{|C(e)|}{|K_e|}|\Omega|^2 = \frac{|U|^2}{|K_e||C(e)|},$$

and the total number of elements of S is

$$|S| = |U|^2 \sum (|K_e||C(e)|)^{-1},$$

the sum being taken over a cross-section of idempotents e from the orbits of the action. Lipscomb and Konieczny [23] treat a fairly wide class of examples of this situation with special orbit properties.

6 Submonoids and congruences

This section describes some submonoids and key congruences of a factorizable inverse monoid in terms of the elements of the construction given in section 4.

6.1 Full and cofull factorizable inverse submonoids

We say that an inverse submonoid T of S is *full* (or *wide*) if $E(T) = E(S)$ and *cofull* (or *top-heavy*) if $U(T) = U(S)$. The discussion of section 4 now shows that for *any* subgroup G of U there is a full factorizable inverse submonoid GE of S; and so such submonoids are in bijective correspondence with subgroups of U. However for a cofull factorizable inverse submonoid we are constrained to choose subsemilattices Y of E containing 1 and invariant under the action of U; the cofull factorizable inverse submonoids are in bijective correspondence with submonoid subsemilattices of E which are also invariant under U.

6.2 Minimum group congruence

The minimum group congruence σ may be expressed in terms of the pre-kernels. Let K_E denote $\cup_{i\in E} K_i$, and note that $K_E \trianglelefteq U$.

Proposition 6.1 $(ue, vf) \in \sigma$ *if and only if* $u^{-1}v \in K_E$, and $S/\sigma \cong U/K_E$.

Again we see that the case $S = U \ltimes E$ (i.e., $K_i = \{1\}$ for all $i \in E$) is equivalent to $K_E \subseteq \cap_{i\in E} K_i$, and hence to the condition $(ue, f) \in \sigma \iff ue = e$, which is an equivalent definition of S being E-unitary. In this case the maximum group image S/σ is U, but this can also happen when S is not E-unitary. For example, let G be a group with non-trivial normal subgroup N such that $G/N \cong G$. Let G act trivially on $Y = \{0, 1\}$ with $K_0 = N$ and $K_1 = \{1\}$. The resulting (Clifford) $S = GY$ has $S/\sigma \cong G$ but S is not E-unitary (by Prop. 5.6). But if $G/N \cong G$ implies $N = \{1\}$ (in particular, if G is finite) then any factorizable inverse monoid $S = GY$ is E-unitary if and only if $S/\sigma \cong G$.

6.3 Fundamentality and exactness

The representation γ of U in $\mathrm{Aut}(E)$ may be extended to the standard Munn representation (here we also denote it by γ) of the whole of S in $\mathcal{T}_E$, the subsemigroup of $\mathcal{I}_E$ consisting of partial isomorphisms between principal ideals of E:

$$\gamma : s \mapsto \gamma_s, \quad \gamma_s : \left(ss^{-1}\right)^{\downarrow} \longrightarrow \left(s^{-1}s\right)^{\downarrow}, \quad \gamma_s : i \mapsto s^{-1}is \quad \left(i \leq ss^{-1}\right).$$

This is defined for any inverse semigroup S, but for a factorizable inverse monoid $S = UE = EU$, it simplifies, when we use the form $s = eu$ and note that $i \leq e$ implies $i^u \leq e^u$, to

$$\gamma_{eu} : e^{\downarrow} \longrightarrow (e^u)^{\downarrow}, \quad \gamma_{eu} : i \mapsto i^u \quad (i \leq e).$$

The image is contained in $F(\mathcal{T}_E)$. The corresponding congruence is the maximal idempotent-separating congruence μ and is characterized on $S = EU$ by

$$(eu, fv) \in \mu \text{ if and only if } e = f \text{ and } uv^{-1} \in C(i) \text{ for all } i \leq e.$$

So

Proposition 6.2 *S is fundamental if and only if $\cap_{i\leq e} C(i) \leq K_e$ for all $e \in E$.*

Similarly, the (lax) representation K of E in $\mathrm{Sub}(U)$ may also be extended to all of $S = UE$, as we shall now see. Recall (from the Remark 5.3) that $K_e = \{u \in U \mid u \geq e\}$; and so define, for $s = eu \in S$,

$$\begin{aligned} K_{eu} &= \{v \in U \mid v \geq eu\} = \left\{v \in U \mid vu^{-1} \in K_e\right\} \\ &= \{v \in U \mid v \in K_e u\} = K_e u. \end{aligned}$$

This means that K_s is a member of the coset monoid $\mathbb{K}(U)$ of U, about which we say more later (section 8.5).

In general, $u \geq s$ and $v \geq t$ imply $uv \geq st$, so $K_s K_t \subseteq K_{st}$. Taking into account the reverse order in $\mathbb{K}(U)$, this means that $K : S = UE \longrightarrow \mathbb{K}(U)$ is a lax or pre-homomorphism. This map seems to have first been explored extensively by McAlister; the next lemma is a special case of Corollary 1.4 of his article [26]. Recall from section 4 that S is *exact* when the pre-kernel map $E \longrightarrow \mathrm{Sub}(U)$ is a semilattice homomorphism.

Proposition 6.3 *The map $K : S = EU \longrightarrow \mathbb{K}(U)$ is a homomorphism if and only if S is exact.*

7 Generators and relations

In this section we consider a slightly weakened form of Chen and Hsieh's concept of factorizable inverse monoids, namely a monoid S generated by units and idempotents. For a set X, $\langle X\rangle$ denotes the monoid generated by X and $\mathrm{gp}\langle X\rangle$ the group generated by X. The next lemma gives a test for the case in which S is inverse; it and its proof are slight extensions of Lemma 2 in [9]. For an alternative criterion, see also [27].

Proposition 7.1 *Let $S = \langle H \cup I\rangle$ be a monoid, where H is a set of units and I a set of idempotents. Let $G = \mathrm{gp}\langle H\rangle$ and $Y = \left\{g^{-1}ig \mid g \in G, i \in I\right\}$. Then S is inverse if, and only if, $eg^{-1}fg = g^{-1}fge$ for all $g \in G$ and all $e, f \in I$. In this case S is factorizable with $E(S) = \langle Y\rangle$ and $U(S) = G$.*

The construction method for a factorizable inverse monoid S (Theorem 4.2) also allows a relatively mechanical way of setting up a presentation of S by generators and relations, given presentations for the group of units and the semilattice of idempotents, including normal forms, and knowledge of the action and of the pre-kernels. The resulting presentation is usually rather prolix, but may be simplified in concrete cases. For a proof, see [7].

Theorem 7.2 *Suppose that U and E have monoid presentations $\langle X_U \mid R_U\rangle$ and $\langle X_E \mid R_E\rangle$ respectively; write their members as equivalence classes of words, $[u] \in U$ and $[e] \in E$ with $u \in X_U^*$, etc. For each $[e] \in E$ and $[u] \in U$, let $\hat{e} \in [e]$ and $\hat{u} \in [u]$ be chosen and in such a way that $\hat{x} = x$ for each $x \in X_U \sqcup X_E$. Suppose further that for each $[e] \in E$, $K_e = \langle \Sigma_e\rangle$ for a convenient $\Sigma_e \subseteq K_e$*

(we do not have to know a presentation). Let

$$R_{\ltimes} = \left\{ \left(xy, y\widehat{(x^y)} \right) \mid x \in X_E,\ y \in X_U \right\} \qquad \text{and}$$
$$R_K = \{ (\widehat{e}\widehat{u}, \hat{e}) \mid [e] \in E, [u] \in \Sigma_e \} .$$

Then $S = EU$ *has presentation*

$$\langle X_U \sqcup X_E \mid R_U \sqcup R_E \sqcup R_{\ltimes} \sqcup R_K \rangle .$$

8 Examples and special cases

In this section we examine some special ways of obtaining linked representations and hence factorizable inverse monoids. Group actions are nearly ubiquitous in mathematics, and in many cases the action may be viewed as action on a semilattice. We take several such actions, and look for natural ways to set up pre-kernel maps of the semilattice which are linked with the given action. Generally we leave verification of the linking conditions, and certain other statements, to the motivated reader.

8.1 The factorizable part of the symmetric inverse monoid

The symmetric group $\mathcal{S}_X$ is defined by its action on the set X, which we write as $x \mapsto x \cdot u$ for $x \in X$ and $u \in \mathcal{S}_X$. This action may be naturally extended to an action on the lower semilattice 2^X of all subsets of X, thus: for $e \in 2^X$ and $u \in \mathcal{S}_X$, $e^u := \{x \cdot u \mid x \in e\}$. Let us define pre-kernel maps as 'isotropy' groups,

$$K_e := \{u \in \mathcal{S}_X \mid x \cdot u = x \text{ for all } x \in e\} .$$

Then one easily verifies the linking conditions, equations (4.3), are satisfied and that there is a map from the resulting factorizable inverse monoid S to the symmetric inverse monoid $\mathcal{I}_X$; in this map the element ue ($= [u, e]$ in the notation of equation 4.2) maps to the partial bijection $u|_D$, which is u restricted to domain $D = {}^u e = \left\{x \cdot u^{-1} \mid x \in e\right\}$ (its range is e). This map is easily seen to be an injective homomorphism, by using Theorem 4.5 or direct proof. Thus $S \cong F(\mathcal{I}_X)$, the factorizable part of the symmetric inverse monoid. Define the *co-cardinality* of $e \in 2^X$ to be the cardinality of its complement in X; then $F(\mathcal{I}_X)$ consists of the partial bijections whose domain and range have equal co-cardinality. This is all of the symmetric inverse monoid, if X is finite; the connection with Dedekind finiteness is explored further in [3]. Note that the restriction effect is achieved by the choice of K_e, which identifies all maps which act on e in the same way.

The orbit Ω_e consists of all subsets of X with the same cardinality and co-cardinality as e. Moreover $C(e) \cong \mathcal{S}_e \times \mathcal{S}_{X \setminus e}$ and $K_e \cong \mathcal{S}_{X \setminus e}$. If $|X| = n$ is finite, and $|e| = r$, we have from section 5.4 the well-known results that

$$|\Omega_e| = \binom{n}{r}, \qquad |H_e| = |\mathcal{S}_e| = r!,$$
$$|D_e| = \binom{n}{r}^2 r!, \text{ and } |\mathcal{I}_n| = \sum_{r=0}^{n} \binom{n}{r}^2 r!.$$

If X is infinite, let X' be a set disjoint from X and of equal cardinality, and set $Z = X \sqcup X'$. $\mathcal{I}_X$ is thereby included in $\mathcal{I}_Z$ in the obvious way, preserving the cardinality of the domain of every element. But the co-cardinality in Z of each domain is now $|X|$, hence $\mathcal{I}_X$ is actually included in $F(\mathcal{I}_Z)$. Combining this with the Wagner-Preston embedding of any inverse semigroup S in $\mathcal{I}_{|S|}$, we see that every inverse semigroup embeds in a factorizable inverse monoid [1].

In fact any embedding of an inverse semigroup S in a factorizable inverse monoid T' may be modified so as to be a *strict* embedding $\theta : S \longrightarrow T$ (that is, for all $s \in S$, there is $u \in U(T)$ such that $u \geq s\theta$) by taking $G = \{u \in U(T') \mid u \geq s\theta \text{ for some } s \in S\} = U(T') \cap (S\theta)^{\uparrow}$ and $T = GE(T')$. Recall from section 4 that $T = GE$ is the image of $G \ltimes E$ by ϕ. Then the diagram

$$S \overset{\theta}{\hookrightarrow} GE \overset{\substack{\text{ip.}-\text{sep.}\\ \phi}}{\twoheadleftarrow} G \ltimes E$$

may be completed to the commutative square

$$\begin{array}{ccc} P & \overset{\iota}{\hookrightarrow} & G \ltimes E \\ \downarrow \psi & & \downarrow \phi \\ S & \overset{\theta}{\hookrightarrow} & T = GE \end{array}$$

where ι is inclusion, P is E-unitary, and ψ is a *covering*, that is, surjective and idempotent-separating; moreover, $P/\sigma \cong G$. So we have McAlister's Covering Theorem [24] that every inverse semigroup has an E-unitary cover. On the converse side, any covering $\psi : P \to S$ such that P is E-unitary and $P/\sigma \cong G$ may be completed to essentially the same diagram, giving rise to a strict embedding of S into a factorizable inverse monoid GE [28]. There is a thorough exposition of this equivalence of strict factorizable embeddings with E-unitary covers in [18], especially sections 2.2 and 8.2.

The method for constructing a factorizable inverse monoid given in this section can be varied by using actions of related groups. For instance, the braid group is used in [5]. The alternating group gives the monoid of alternating charts [21]. For infinite X, the restricted symmetric group (generated by transpositions) yields the inverse monoid of partial permutations of finite shift, and has an alternating couterpart [22].

8.2 The factorizable part of the dual symmetric inverse monoid

The action of $\mathcal{S}_X$ on X may also be extended to an action on $\mathrm{Eq}(X)$, the (upper) semilattice of equivalence relations on X, as follows: for $u \in \mathcal{S}_X$ and

$e \in \mathrm{Eq}(X)$, define

$$\gamma : (u, e) \mapsto e^u = \{(x \cdot u, y \cdot u) \mid (x, y) \in e\}.$$

Sometimes it is more convenient to think of the equivalence e in terms of its corresponding partition of X into disjoint proper subsets X_i (called *blocks*), say $X = \sqcup_{i \in I} X_i$. Then γ maps this partition to the partition $X = \sqcup_{i \in I} (X_i \cdot u)$, where $X_i \cdot u = \{x \cdot u \mid x \in X_i\}$ (at notational variance from section 8.1 above). Note that X_i and $X_i \cdot u$ have the same cardinality for each $i \in I$. With this action and the choice

$$K_e = \{u \in U \mid (x, x \cdot u) \in e \text{ for each } x \in X\},$$

the linking conditions (eqns 4.3) are satisfied. (The proof of this is straightforward; one need only remember that $ef = e \vee f$, the intersection of all equivalences containing both e and f.) The resulting factorizable inverse monoid is the factorizable part $F(\mathcal{I}_X^*)$ of the dual symmetric inverse monoid ([10], Prop. 3.1).

If $|X| = n$, finite, and e is a partition of type $1^{r_1} 2^{r_2} \ldots n^{r_n}$ (meaning that there are $r_i \geq 0$ blocks of size i) then

$$C(e) \cong \prod_i (\mathcal{S}_i)^{r_i} \times \mathcal{S}_{r_i},$$

since e is fixed by permutations of elements within the same block and also by permutations of whole blocks of the same size; and $K_e \cong \prod_i (\mathcal{S}_i)^{r_i}$, since e absorbs just those permutations of the first kind. So

$$\begin{aligned} |\Omega_e| &= n! \textstyle\prod_i \left\{(i!)^{r_i} r_i!\right\}^{-1}, \\ |H_e| &= \textstyle\prod_i r_i!, \qquad \text{and} \\ |D_e| &= n! n! \textstyle\prod_i \left\{(i!)^{2r_i} r_i!\right\}^{-1}, \text{ etc.} \end{aligned}$$

Again, there are alternating, braid, and 'finite shift' versions of the construction in this subsection.

8.3 Partial automorphisms of a vector space

The defining action of the general linear group $\mathcal{GL}(V)$ on a finite-dimensional vector space V (here written $x \mapsto x \cdot u$ for $x \in V$ and $u \in \mathcal{GL}(V)$) extends naturally to an action on the intersection semilattice $\mathrm{Sub}(V)$ of subspaces of V defined by

$$W^u = \{x \in V \mid x = y \cdot u \text{ for some } y \in W\},$$

where $W \in \mathrm{Sub}(V)$ and $u \in \mathcal{GL}(V)$. If we also define

$$K_W = \{u \in \mathcal{GL}(V) \mid x \cdot u = x \text{ for all } x \in W\},$$

then the linking conditions are satisfied. The resulting factorizable inverse monoid has multiplication

$$[u, W][v, Z] = [uv, W^v \cap Z]$$

in the notation of eqn. (4.2) and may be readily identified as the inverse monoid $\mathcal{PA}(V)$ of partial automorphisms of V mentioned as example 3.7 in [16]. (When V is infinite-dimensional, this construction results in only the factorizable part of $\mathcal{PA}(V)$.)

We may seek to dualise this construction in the same kind of way that section 8.2 dualises section 8.1. Namely, we seek "natural" partitions of V and arrange $\mathcal{GL}(V)$ to act on them. So consider, for W a subspace of V, the equivalence $\varepsilon_W = \varepsilon(W)$ defined by

$$(x, y) \in \varepsilon_W \iff x - y \in W;$$

ε_W partitions V into translates $W + x$ of the subspace W. Let us write $\mathbb{A}(V) = \{\varepsilon_W \mid W \in \mathrm{Sub}(V)\}$ and note that for subspaces W_1, W_2,

$$\varepsilon_{W_1 \cap W_2} = \varepsilon_{W_1} \cap \varepsilon_{W_2} \text{ and } \varepsilon_{W_1 + W_2} = \varepsilon_{W_1} \vee \varepsilon_{W_2},$$

so that $\mathrm{Sub}(V)$ and $\mathbb{A}(V)$ are isomorphic semilattices, and $\mathbb{A}(V)$ is closed under intersection and join. Now define an action by setting, for $\varepsilon \in \mathbb{A}(V)$ and $u \in \mathcal{GL}(V)$,

$$\varepsilon^u = \{(x \cdot u, y \cdot u) \mid (x, y) \in \varepsilon\}.$$

Clearly, $x - y \in W \iff x \cdot u - y \cdot u \in W^u$ so that $\varepsilon_W^u = \varepsilon_{W^u}$ and $\mathbb{A}(V)$ is also closed under this action. Next, for $\varepsilon \in \mathbb{A}(V)$, put

$$K_\varepsilon = \{u \in \mathcal{GL}(V) \mid (x, x \cdot u) \in \varepsilon \text{ for all } x \in V\},$$

and note that $\varepsilon_1 \subseteq \varepsilon_2$ implies $K_{\varepsilon_1} \leq K_{\varepsilon_2}$. For this to satisfy condition (v) of Lemma 4.1, the order-reversal property, we must employ the dual order in $\mathbb{A}(V)$, i.e. that inherited from the dual symmetric inverse monoid *via* the upper semilattice of equivalences on V and the corresponding join operation. With this choice, $\mathbb{A}(V)$ is dual-isomorphic with $\mathrm{Sub}(V)$, and it is simple technique to verify that these are representations that satisfy equations (4.3), the linking conditions.

The resulting factorizable inverse monoid S (say) has multiplication

$$[u, \varepsilon][v, \eta] = [uv, \varepsilon^v \vee \eta]$$

(cf. eqn. (4.2)) and so is a submonoid of $F\left(\mathcal{I}^*_{|V|}\right)$. Moreover, for finite-dimensional V, it can be identified as the *dual partial automorphism monoid* of V; since the category of linear spaces and maps is self-dual, S is isomorphic with $\mathcal{PA}(V)$. The general set-up is described *passim* in sections 1 and 5 of [10], but it may be of interest to exhibit an explicit isomorphism. For this, first equip V with an inner product $\langle \_ \mid \_ \rangle$. Then note that $(W + Z)^\perp = W^\perp \cap Z^\perp$, and so the map $\psi : W \mapsto \varepsilon\left(W^\perp\right)$ is an isomorphism of $\mathrm{Sub}(V)$ to $\mathbb{A}(V)$. Also $x \in \left(W^\perp\right)^u$ if and only if $x = y \cdot u$ where $\langle y \mid w \rangle = 0$ for all $w \in W$; equivalently, $\langle x \cdot u^{-1} \mid w \rangle = 0 = \left\langle x \mid w \cdot \left(u^{-1}\right)^* \right\rangle$ where $(\_)^*$ is the adjoint map. So writing ϕ for the involutory automorphism $\phi : u \mapsto \left(u^{-1}\right)^*$ of $\mathcal{GL}(V)$, we have $\left(W^{u\phi}\right)^{\perp} = \left(W^{\perp}\right)^u$ and so

$$\varepsilon\left(\left(W^{u\phi}\right)^{\perp}\right) = \varepsilon\left(\left(W^{\perp}\right)^{u}\right), \text{ i.e., } \left(W^{u}\right)\psi = \left(W\psi\right)^{u\phi}.$$

Also

$$\begin{aligned} u\phi \in K_{W\psi} &\iff x \cdot u^* - x \in W^{\perp} \text{ for all } x \in V \\ &\iff \langle x \cdot u^* \mid w\rangle = \langle x \mid w\rangle \text{ for all } x \in V, w \in W \\ &\iff \langle x \mid w \cdot u - w\rangle = 0 \text{ for all } x \in V, w \in W \\ &\iff u \in K_W. \end{aligned}$$

So $(K_W)\phi = K_{W\psi}$ and, by Theorem 4.5, (ϕ, ψ) is an isomorphism of the category $\mathbf{Q}$; hence $[u, W] \mapsto [u\phi, W\psi] = \left[\left(u^{-1}\right)^*, \varepsilon\left(W^{\perp}\right)\right]$ is an isomorphism of S with $\mathcal{PA}(V)$.

8.4 Partial right translations of a group

Associated with any group G is its action on the underlying set $|G| = X$, say, defined by right multiplication, thus: $\gamma(x, g) = xg$ for $x \in |G|, g \in G$. This action extends to the intersection semilattice $E = 2^{|G|}$ by

$$e^g = \{xg \mid x \in e\}$$

and it is also natural to define (for $e \in E$)

$$K_e = \{g \in G \mid xg = x \text{ for all } x \in e\}.$$

This gives a (factorizable inverse) submonoid of $\mathcal{I}_{|G|}$ which we may call the inverse monoid of *partial right translations* of G. K_e is trivial for all $e \neq \varnothing$, so these are E^*-unitary examples.

8.5 The coset monoid of a group

Any group G acts by conjugation on $E = \mathrm{Sub}(G)$, the upper semilattice of its subgroups. The obvious choice for the pre-kernel map in this case is the identity map, $K_e = e$. (This is analogous with earlier examples, because

$$K_e = \{u \in G \mid x \cdot u \in e \text{ for all } x \in e\} = e$$

follows from $x \cdot u \in e \iff u \in x^{-1}e = e$.) Now $e \leq f \iff K_e \subseteq K_f$ and so for condition (v) of Lemma 4.1, the antitone property, to hold we must use the dual order again on $\mathrm{Sub}(G)$, so that the semilattice operation is join; we write ef for $e \vee f$. Then

$$C(e) = \{u \in G \mid eu = ue\},$$

the normalizer of e in G (so immediately, $K_e = e \trianglelefteq C(e)$.) Also trivially, $K_{e^u} = e^u = K_e^u$. So all the conditions (4.3) hold, and the resulting factorizable inverse monoid is $\mathbb{K}(G)$, the coset monoid of G, introduced after Proposition 6.2. We remind the reader that each element is of the form $eu = K_e u$, thus a coset in G, and the product in $\mathbb{K}(G)$ is given by

$$K_{eu} \circ K_{fv} = \left(K_e \vee uK_f u^{-1}\right) uv.$$

$\mathbb{K}(G)$ is exact by definition: $K_e \circ K_f = K_e \vee K_f = e \vee f = ef$, and K is injective. From section 5.4, $|\mathbb{K}(G)| = |G|^2 \sum \left(|H| \, |N_G(H)|^{-1}\right)$, the sum being taken over representative subgroups H of the conjugacy classes in G.

8.6 Galois inverse monoid of a field extension

A classic instance of a group act: let F be a field, $\mathbb{k}$ a subfield of F (called the *ground field*), and $[\mathbb{k}, F]$ the lower semilattice of intermediate fields F' such that $\mathbb{k} \subseteq F' \subseteq F$. The *Galois group* $G := \mathrm{Aut}(F : \mathbb{k})$ of automorphisms of F which fix $\mathbb{k}$ pointwise acts on $[\mathbb{k}, F]$ in the obvious way. For $\mathbb{k} \subseteq f \subseteq F$, define the Galois connexion $K_f = \{\alpha \in G \mid x\alpha = x \text{ for all } x \in f\}$. If $e, f \in [\mathbb{k}, F]$ with $e \leq f$, we have $K_f \subseteq K_e$, so $K : e \mapsto K_e$ is an order (anti-)representation of $[\mathbb{k}, F]$ in $\mathrm{Sub}(G)$. Again the linking conditions of 4.3 are satisfied. When the extension is separable and normal, K is an (anti-)isomorphism. The resulting factorizable inverse monoid S may be called the *Galois inverse monoid* of the extension, and is the submonoid of $\mathcal{I}_{|F|}$ consisting of those $\alpha \in \mathcal{I}_{|F|}$ such that $\mathrm{dom}\,\alpha$, $\mathrm{ran}\,\alpha$ are subfields of F and α fixes $\mathbb{k}$ pointwise and extends to an automorphism of F. (The extension property follows from the rest when F is finite-dimensional over $\mathbb{k}$.) S is also isomorphic with the coset monoid of the Galois group of the extension, cf. section 9.2 below.

8.7 Restriction monoids

This and several other foregoing examples suggest the following generalisation. Let a group G act unitarily (on the right) on a set X, writing $x \mapsto x \cdot g$. Let E be a subset of 2^X containing X, and closed under intersection and the action induced by G (i.e., for $e, f \in E$ and $u \in G$, both $e \cap f$ and e^g are in E, where $e^g = \{x \cdot g \mid x \in e\}$). Such an E is called a *system of subsets* for the action of G; one such system is 2^X itself, as in section 8.1. Also let $K_e = K_e^{\mathrm{rest}} = \{g \in G \mid x \cdot g = x \text{ for all } x \in e\}$. Since $x \cdot g = x$ if and only if $(x \cdot u)\, u^{-1} g u = x \cdot u$, $(K_e)^u = K_{e^u}$; $K_X = \{1\}$ by unitarity; and $e \leq f \implies K_f \leq K_e$. In short, we have a factorizable inverse monoid which we may call the *restriction monoid* of E with respect to the action of G; it is a submonoid of $\mathcal{I}_X$.

This too may be dualised. Again starting with G acting unitarily on X, and E a subset of $\mathrm{Eq}(X)$ containing the identity equivalence, and closed under join of equivalences and the induced action of G, $\varepsilon^g = \{(x \cdot g, y \cdot g) \mid (x, y) \in \varepsilon\}$, we define $K_\varepsilon = \{u \in G \mid (x, x \cdot u) \in \varepsilon\}$, and verify directly or from Theorem 4.5

that the resulting factorizable inverse monoid is a submonoid of $\mathcal{I}_X^*$, which we could call the *dual restriction monoid* of E with respect to the action.

As a class of examples of these restriction monoids, consider the cases where X is endowed with some structure, for example a set Ω of operations; thus $A = (X, \Omega)$ is a universal algebra. Let $G = \mathrm{Aut}\,(A)$ and let E be respectively the lower semilattice of subalgebras of A or the upper semilattice of congruences of A. The (primal and dual) restriction monoids are then the factorizable parts of the inverse monoids of partial automorphisms and of bicongruences on A. The partial automorphism monoid $\mathcal{PA}\,(A)$ on a universal algebra A is factorizable if and only if each partial automorphism extends to an automorphism of A. The bicongruence monoid $\mathrm{Bicon}\,(A)$ on a universal algebra A is factorizable if and only if each element (regarded as a subalgebra of the square A^2) contains (the graph of) an automorphism of A. This latter is certainly the case when A is weakly diagonal, and in the presence of additional hypotheses $\mathrm{Bicon}\,(A)$ characterises the variety generated by A [15].

8.8 Reflection monoids and Renner monoids

A *root system* Φ in a vector space V is defined as a finite set of vectors spanning V and closed under reflection in the subspaces $w^{\perp}$ defined by $w \in \Phi$. These subspaces $w^{\perp}$ have codimension 1 and so are *hyperplanes*: they constitute the *hyperplane arrangement* $\mathcal{H}_\Phi$ associated with Φ. Write σ_w for the reflection in $w^{\perp}$. The subgroup $\langle \sigma_w \mid w \in \Phi \rangle$ of $\mathcal{GL}\,(V)$ is a reflection group, the *Weyl group* $\mathcal{W}_\Phi$ associated with Φ, and it has an action on V inherited from $\mathcal{GL}\,(V)$. A *system of subspaces* for this action is a set $\mathcal{H}$ of subspaces of V closed under intersection and the action of $\mathcal{W} = \mathcal{W}_\Phi$; clearly the lower semilattice $\langle \mathcal{H}_\Phi \rangle$ generated by $\mathcal{H}_\Phi$ is one such (but not the only one). The restriction monoid of $\mathcal{H}$ with respect to the action of $\mathcal{W}$ is called a *reflection monoid* [8].

A related situation arises from a connected regular linear algebraic monoid; that is, loosely, a monoid M which is simultaneously an affine variety in the sense of algebraic geometry. The group U of units of M is a linear algebraic group, which has a maximal torus T with normaliser $N = N_U\,(T)$. It turns out that the Zariski closures of N and T, denoted $\overline{N}$ and $\overline{T}$, are submonoids of M, and that $\overline{N} = NE\left(\overline{T}\right)$ is a factorizable inverse submonoid of M. Indeed N acts by conjugation on $E\left(\overline{T}\right)$, but not unitarily. The *Renner monoid* R of M is the quotient $\overline{N}/\mu$ and is a factorizable inverse monoid with group of units $W = N_U\,(T)\,/T$, the Weyl group of U [30](Ch. 11), [32]. The action of W on $E\left(\overline{T}\right)$ is unitary, and $R \cong W \ltimes E\left(\overline{T}\right)/\mu$. Cf. section 9.1 below. Comparisons of reflection and Renner monoids are made in [8].

8.9 Dynamical systems

By (one) definition, a *dynamical system* is a group G acting (continuously and unitarily) on a topological space (X, Ω); that is, there is a representation $\gamma : g \mapsto \gamma_g$ of G in $\mathrm{Aut}\,(X, \Omega)$, the group of homeomorphisms $X \to X$. The topology Ω may be regarded as a lower semilattice, and the action extends to Ω by defining, for $A \in \Omega$, $A^g = \{x \cdot g \mid x \in A\} \in \Omega$ (clearly, $A \subseteq B$ if and only if $A^g \subseteq B^g$). The action also extends to an action on the complete semilattice of closed sets, $(X \setminus A)^g = X \setminus A^g$. In the restriction spirit, it is most natural to take

$$K_A = \{g \in G \mid x \cdot g = x \text{ for all } x \in A\},$$

so that if $A \subseteq B$ then $K_B \leq K_A$. The resulting factorizable inverse monoid is the inverse monoid of partial automorphisms of X formed by restriction of elements of G.

8.10 Groups acting on graphs

Let Γ be an undirected simple graph, with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Regarding $\mathcal{V}$ as a set of singleton sets, and $\mathcal{E}$ as 2-element sets, $\Gamma^0 = \mathcal{V} \cup \mathcal{E} \cup \{\varnothing\}$ thus forms a lower semilattice (a subset of $2^{\mathcal{V}}$). Automorphisms of the semilattice Γ^0 effectively coincide with automorphisms of the graph Γ , so $G = \mathrm{Aut}\,(\Gamma)$ has a natural action on Γ^0. Put $K_\varnothing = G$ and $K_\gamma = \{g \in G \mid v \cdot g = v \text{ for all } v \in \gamma\}$ for $\gamma \in \mathcal{V} \cup \mathcal{E}$; this is a restriction-style definition which satisfies the linking conditions and leads to a factorizable inverse monoid.

9 Constructing pre-kernels

The examples in section 8 have shown many more or less natural ways in which a given group action can be matched up with a pre-kernel map. In this section we set out two further fairly general ways. The first, dubbed the fundamental way, constructs a pre-kernel map directly from a group acting on a semilattice. The second, named the Galois closure way, constructs an exact restriction of any pre-kernel map which participates in a Galois connection. It can thus compare a complete factorizable inverse algebra with the coset algebra of its unit group.

9.1 The fundamental way

Given any action of U on E, we may construct pre-kernels by defining $K = K^{\mathrm{fund}}$ by

$$K_e^{\mathrm{fund}} = \bigcap\nolimits_{i \leq e} C\,(i)\,.$$

This satisfies the linking conditions (4.3): $K_e \leq C\,(e)$ by construction, and $e \leq f$ implies $e^{\downarrow} \subseteq f^{\downarrow}$, so that $\bigcap_{i \leq e} C\,(i) \geq \bigcap_{i \leq f} C\,(i)\,$; and moreover

$$K_{e^u} = \bigcap\nolimits_{i \leq e} C\,(i^u) = \left(\bigcap\nolimits_{i \leq e} C\,(i)\right)^u = K_e^u.$$

By Proposition 6.2, this choice for K_e defines the maximum idempotent-separating congruence μ on the semidirect product $U \ltimes E$. So the resulting factorizable inverse monoid is fundamental.

9.2 The Galois closure way

This subsection is inspired by section 5 of [20], although that deals especially with inverse algebras arising from the restriction set-up of section 8.7. Let $S = GE$ be a factorizable inverse monoid, and suppose the associated pre-kernel map $K : E \longrightarrow \mathrm{Sub}\,(G)$ is part of a Galois connection, that is, there exists an antitone map $f : E \longrightarrow \mathrm{Sub}\,(G)$ such that, for all $e \in E$ and $H \leq G$,

$$e \leq f\,(H) \iff K\,(e) \geq H. \tag{9.1}$$

Before proceeding, we note that if this connection exists then $S = GE$ is actually an inverse algebra, because for any $u \in G$ we may take $H = \mathrm{gp}\,\langle u \rangle$, so that $e \leq f\,(\mathrm{gp}\,\langle u \rangle) \iff K\,(e) \geq \mathrm{gp}\,\langle u \rangle \iff e \leq u$, and $f\,(\mathrm{gp}\,\langle u \rangle)$ serves as the fixed-point idempotent $f\,[u]$ (cf. Remark 5.3).

We use the Galois connection (9.1) to construct a new factorizable inverse monoid. It follows as usual that $f \circ K \circ f = f$ and $K \circ f \circ K = K$, and that $K \circ f$ and $f \circ K$ are closure operators (monotone, increasing and idempotent). Thus $e \in E$ is called *closed* if $e = f \circ K\,(e)$ or, equivalently, $e = f\,(H)$ for some $H \in \mathrm{Sub}\,(G)$; similarly, $H \in \mathrm{Sub}\,(G)$ is *closed* if $H = K \circ f\,(H)$. If e_1 and e_2 are closed idempotents, then the following chain of equivalent statements (for any $i \in E$) shows that $e_1 e_2 = f\,(K\,(e_1) \vee K\,(e_2))$:

$$\begin{array}{lll} i \leq e_1 e_2; & i \leq e_1, e_2; & i \leq f \circ K\,(e_1)\,, f \circ K\,(e_2)\,; \\ K\,(i) \geq K\,(e_1)\,, K\,(e_2)\,; & K\,(i) \geq K\,(e_1) \vee K\,(e_2)\,; & i \leq f\,(K\,(e_1) \vee K\,(e_2))\,. \end{array}$$

Similarly, if $H_1, H_2 \in \mathrm{Sub}\,(G)$ are closed, then $H_1 \vee H_2 = K\,(f(H_1) \cdot f\,(H_2))$.Thus the closed idempotents form a subsemilattice of E, denoted $\overline{E}$, isomorphic with the subsemilattice $\overline{\mathrm{Sub}}\,(G)$ of closed subgroups of G. Moreover, restriction of K and f to the closed idempotents and subgroups gives a pair of mutually inverse maps, which we denote K^{Gal} and f^{Gal}, and which are semilattice homomorphisms with respect to the operations inherited from E and $\mathrm{Sub}\,(G)$. Clearly, for $g \in G$, e is closed if and only if e^g is closed, and H is closed if and only if H^g is closed, so by section 5.1 we have a factorizable inverse submonoid $S^{\mathrm{Gal}} = G\overline{E}$ of the original S. We shall call it the *Galois-closed inverse monoid* associated with S. It is exact, and in fact also embeds in the coset monoid $\mathbb{K}\,(G)$, via $K^{\mathrm{Gal}} : \overline{E} \rightarrow \mathrm{Sub}\,(G)$ and the identity on G. In a sense, S^{Gal} provides a qualitative measure of the nearness of $S = GE$ to the corresponding coset monoid $\mathbb{K}\,(G)$.

The above assumes the existence of a Galois connection. We note some cases where existence is assured. In the motivating example of Galois field extensions (section 8.6), separability and normality properties of the field extensions ensure that all intermediate extensions and subgroups are closed, and finite index ensures that all partial automorphisms of intermediate fields fixing the ground field $\Bbbk$ are restrictions of field automorphisms of F. Again, if E is complete and K sends arbitrary joins to intersections, the Galois connection exists (by the adjoint functor theorem applied to $K : E^{\mathrm{opp}} \rightarrow \mathrm{Sub}\,(G)$) and is given by $f\,(H) = \vee \{i \in E \mid K_i \geq H\}$.

In the next subsection, we consider concrete examples of the fundamental and Galois closure ways.

9.3 Further examples

As noted in section 8.1, the natural action of $G = \mathcal{S}_X$ on $E = 2^X$ leads to $C(i) = \{u \in G \mid x \in i \iff x \cdot u \in i\}$. It follows that $K^{\text{fund}} = K^{\text{rest}}$ (in the notation of section 8.7), and hence $F(\mathcal{I}_X) \cong (\mathcal{S}_X \ltimes 2^X)/\mu$. Now $K^{\text{rest}}_{\cup e_j} = \cap K^{\text{rest}}_{e_j}$ (for any subset $\{e_j\} \in E$) and thus, power sets being complete, the Galois adjoint f exists and is given, for $H \leq G$, by $f(H) = \{x \in X \mid x \cdot g = x \text{ for all } g \in H\}$, the set of points fixed by the subgroup H. (Similar remarks hold for lattices of subspaces as occur in some examples in section 8.3.) Observe that $Y \subseteq X$ is closed if and only if Y is not of the form $Y = X \setminus \{x\}$ for some $x \in X$. [Proof: $f \circ K(X \setminus \{x\}) = X = f \circ K(X)$, but if there exist distinct $x_0, x_1 \in X \setminus Y$ then there is $u \in K(Y)$ such that $x_0 \cdot u = x_1$ and $x_1 \cdot u = x_0$. So $y \in f \circ K(Y)$ implies $y \in Y$.] So S^{Gal} consists of those $\alpha \in F(\mathcal{I}_X)$ such that $\operatorname{dom}\alpha$ and $\ker\alpha$ are not of co-cardinality 1.

Note if $X = Y \cup \{x_0\}$ (for $x_0 \notin Y$) then the map given (for $\alpha \in F(\mathcal{I}_Y)$) by

$$\alpha \mapsto \begin{cases} \alpha & \text{if } \alpha \notin \mathcal{S}_Y \\ \alpha \cup \{(x_0, x_0)\} & \text{if } \alpha \in \mathcal{S}_Y \end{cases}$$

embeds $F(\mathcal{I}_Y)$ in S^{Gal} and hence in $\mathbb{K}(\mathcal{S}_X)$ (McAlister, [26]).

Again consider the action of $\mathcal{S}_X$ on $\operatorname{Eq}(X)$ discussed in section 8.2, and proceed on the fundamental way. If $\varepsilon \in \operatorname{Eq}(X)$ contains at least three distinct blocks, X_1, X_2, and X_3 say, and $x \in X_1$ while $x \cdot u \in X_2$, then $u : X_1 \to X_2$ bijectively, and so $u \notin C(\pi)$ where π has blocks X_1, $X_2 \cup X_3$ and so is an equivalence below ε. Hence $K^{\text{fund}}_\varepsilon = K^{\text{rest}}_\varepsilon = \{u \in U \mid (x, x \cdot u) \in \varepsilon \text{ for each } x \in X\}$. But if ε has rank 2, with blocks X_1 and X_2, then $K^{\text{fund}}_\varepsilon = K^{\text{rest}}_\varepsilon$ if and only if $|X_1| \neq |X_2|$ (if $|X_1| = |X_2|$, $K^{\text{fund}}_\varepsilon = C(\varepsilon)$). In particular, $F(\mathcal{I}^*_X)$ is fundamental if and only if $|X|$ is odd finite.

$\operatorname{Eq}(X)$ is complete, but K^{rest} preserves joins (is exact) if and only if X is finite [6], [4]. So if X is infinite, there is no Galois adjoint. If X is finite, the Galois adjoint is $f(H) = \{(x, x \cdot u) \mid x \in X, u \in \mathcal{S}_X\}$, corresponding to the partition into orbits. All partitions are closed, and closed subgroups have the form $\prod \mathcal{S}_{X_i}$ for corresponding partitions $\varepsilon = \bigsqcup X_i$ (cf. section 8.2). So then $K^{\text{Gal}} = K^{\text{rest}}$ and we have all of $F(\mathcal{I}^*_X)$ as the Galois-closed monoid, which embeds in $\mathbb{K}(\mathcal{S}_X)$.

Curiously, $K = K^{\text{rest}}$ preserves meets: $K(\cap \varepsilon_j) = \cap K(\varepsilon_j)$ for any subset $\{\varepsilon_j\} \in E$, so has a Galois adjoint. But when we take $\operatorname{Eq}(X)$ as a lower semilattice, K fails the condition of Lemma 4.1(v), so we cannot construct a factorizable inverse monoid in this way. Instead, we may take the fundamental way with the natural action of section 8.2 applied to the lower semilattice $\operatorname{Eq}(X)$. Let $\varepsilon \in \operatorname{Eq}(X)$ and $u \in \mathcal{S}_X$. Then

$$C(\varepsilon) = \{u \in \mathcal{S}_X \mid (x, y) \in \varepsilon \iff (x \cdot u, y \cdot u) \in \varepsilon\}.$$

If x and $x \cdot u$ are in different non-unit blocks of ε, or if $x, x \cdot u$ and y are distinct elements of the same block of ε, then there is $\pi \in \mathrm{Eq}\,(X)$ with $\pi \subset \varepsilon$ such that $u \notin C\,(\pi)$, and so $u \notin K_\varepsilon^{\mathrm{fund}} = \cap \{C\,(\pi) \mid \pi \subseteq \varepsilon\}$. Conversely, if $u \in C\,(\varepsilon)$ fixes all elements in blocks of size > 2, then it stabilises any π which refines ε. So

$$K_\varepsilon^{\mathrm{fund}} = \{u \in \mathcal{S}_X \mid x \cdot u = u \text{ for all } x \text{ in blocks of size } > 2\}.$$

10 Some other classes of factorizable inverse monoids

We conclude this survey by noting a couple of interesting classes of factorizable inverse monoids which arise naturally from the recipe given in Theorem 4.2. Recall that, in a coset monoid $\mathbb{K}\,(U)$, the pre-kernel map $K_e = e$ is exact and bijective. In fact from Theorem 4.5, the class of coset monoids are those $S = EU$ such that the pre-kernel map is a semilattice isomorphism of E with $\mathrm{Sub}\,(U)$, as characterised in [26]. In this case E is complete, and the definition $f\,[u] = \bigwedge \{e \in E \mid e \leq u\} = \bigwedge \{e \in E \mid u \in K_e\} = \mathrm{gp}\,\langle u \rangle$ serves as a fixed-point idempotent for u; thus (by Remark 5.3) $\mathbb{K}\,(U) \cong S$ is an inverse algebra.

If we weaken the condition to K being exact and injective, then there is an embedding $eu \mapsto K_e u$ of S in $\mathbb{K}\,(U)$ as a cofull submonoid, and conversely this implies K is exact and injective. For example, we noted in section 9.3 that $F\,(\mathcal{I}_X^*)$ embeds in $\mathbb{K}\,(\mathcal{S}_X)$ when X is finite.

We turn to mere exactness or injectivity. From Lemma 6.3, the class of exact factorizable inverse monoids consists precisely of those which may be represented in the coset monoid of their group of units. Such a representation is not necessarily faithful, nor cofull. But each exact factorizable inverse monoid with group of units U has a minimum homomorphic image (and corresponding maximal congruence θ) which embeds, as a cofull factorizable inverse submonoid, in $\mathbb{K}\,(U)$; θ is also the maximum congruence which has trivial intersection with Green's relation $\mathcal{L}$ [6].

Finally, the factorizable inverse monoids such that K is injective are named *generalised coset monoids* in [4] and characterised as those factorizable inverse monoids which embed in some $\mathbb{K}\,(G)$. Such a G contains an isomorphic copy of U as a subgroup, but it may be a proper subgroup.

11 Acknowledgements

The author extends sincere thanks to Jonathan Leech, David Easdown and James East for collaborations in which he learnt about factorizable inverse monoids. He tenders his gratitude to John Fountain for detecting and correcting some errors, and patiently sharing his knowledge of reflection and Renner monoids in both Hobart and York. To two anonymous referees he offers profound thanks for intercepting yet other errors, remedying significant omissions and lax type-setting, and prompting sundry other improvements.

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