Tôhoku Math. Journ.
27 (1975), 321-336.

# FACTORIZATION OF ENTIRE FUNCTIONS

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(Received April 2, 1974)

1. A meromorphic function $F(z) = f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is non-linear and meromorphic and $g(z)$ is non-linear and entire ($g$ may be meromorphic when $f(z)$ is rational). $F(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that $g(z)$ is linear (a polynomial) unless $f(z)$ is linear (rational). An entire function $F(z)$ is said to be $E$-prime if it is prime for entire $f$ and $g$.

Gross [7] posed an open problem whether there exist prime entire periodic functions. In this paper we shall prove the existence of an entire periodic function which is prime (Theorem 2). Our proof is very hard and needs a new idea. We make use of a regular function in $0 < |w| < \infty$. In [7] it was shown that the $E$-primeness does not imply the primeness. We shall give here another example showing this fact. Our example needs a slightly complicated consideration in its proof. However it seems to be interesting in its own right. Gross' proof is very simple. We shall give several related results.

2. We need several known results.

LEMMA 1. [4]. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{a_n\}_{n=1}^{\infty}$ such that all the roots of the equations $f(z) = a_n (n = 1, 2, \cdots)$ lie on a single straight line. Then $f(z)$ is a polynomial of degree at most two.*

This and the following lemma play an important role in the factorization theory.

LEMMA 2. [11]. *Let $F(z)$ be an entire function of finite order. Assume that $F(z) = f(g(z))$ holds with two transcendental entire functions $f$ and $g$. Then the order $\rho_f$ of $f$ is equal to zero and $\rho_g \leqq \rho_F$.*

This result holds for meromorphic $F$. Indeed Edrei and Fuchs [5] proved the following.

LEMMA 3. [5]. *Let $f$ be meromorphic of positive order, and let $g(z)$ be transcendental entire. Then $F(z) = f(g(z))$ is of infinite order.*

The following lemma was firstly stated in [3] and a complete proof of its general form was given in [8].

LEMMA 4. *Let* $a_i(z)$ *be entire and of finite order* $\rho$. *Let* $g_i(z)$ *also be entire, and let* $g_i(z) - g_j(z)$, $(i \neq j)$ *be a transcendental function or polynomial of degree greater than* $\rho$. *Then*

$$\sum_{i=1}^{n} a_i(z)e^{g_i(z)} = a_0(z)$$

*holds only when* $a_0(z) = a_1(z) = \cdots = a_n(z) = 0$.

We shall use the following notations: $\rho_f$, $\lambda_f$ and $\hat{\rho}_f$ are the order, the lower order and the hyper-order of $f$, which are defined by

$$\overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}, \quad \underline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}$$

and

$$\overline{\lim_{r\to\infty}} \frac{\log\log T(r, f)}{\log r}.$$

LEMMA 5. [9]. *Let* $f$ *be entire with* $\rho_f < \infty$. *Then*

$$\hat{\rho}_{f(g)} \leqq \rho_g .$$

LEMMA 6. [10]. *Let* $f$ *be entire with* $\lambda_f > 0$. *Then*

$$\hat{\rho}_{f(g)} \geqq \rho_g .$$

LEMMA 7. *Let* $f$ *be* $\exp L(z)$ *with transcendental entire* $L(z)$. *Then* $\lambda_f = \infty$. *If* $L(z)$ *is not a constant, then* $\lambda_f \geqq 1$.

This is very easy to prove by Pólya's method. The second part was stated already in [10].

Further we shall make use of several results on factorization.

LEMMA 8. [2]. *Let* $p(z)$ *be any non-constant polynomial. Then* $e^z + p(z)$ *is prime.*

LEMMA 9. [1], [6]. *If* $f$ *is any entire function of order less than* 1/2 *and* $g$ *is entire, then* $f(g)$ *is periodic if and only if* $g$ *is.*

LEMMA 10. [12]. *If* $f$ *is an arbitrary non-constant entire function and* $p$ *an arbitrary polynomial of degree* $\geqq 3$, *then* $f(p)$ *is not periodic.*

LEMMA 11. [12]. *If* $p$ *is a non-constant polynomial and* $g$ *an entire function, then the periodicity of* $p(g)$ *implies that of* $g$.

Lemma 11 is a special case of Lemma 9. We need another growth lemma.

LEMMA 12. *Let $h(w)$ be single-valued and regular in $0 < |w| < \infty$. If $h(e^z)$ is of finite order, then $h(w)$ is of order zero around $w = 0$ and $w = \infty$.*

PROOF. $h(w)$ can be represented as a sum $h(w) = I(w) + J(w)$, where $I(w)$, $J(w)$ are regular in $|w| < \infty$, $0 < |w|$ respectively. Suppose $\rho_J > 0$ or $\rho_I > 0$, then $\rho_{J(e^z)} = \infty$ or $\rho_{I(e^z)} = \infty$. Since $I(e^z)$ is bounded in $\mathscr{R}z \leqq \delta$ and $J(e^z)$ is bounded in $\mathscr{R}z \geqq -\delta$, $\delta > 0$,

$$M_{h(e^z)}(r) = \max(M_{I(e^z)}(r)\ ,\quad M_{J(e^z)}(r)) + O(1)\ .$$

Hence $\rho_{J(e^z)} = \infty$ or $\rho_{I(e^z)} = \infty$ or both imply that $\rho_{h(e^z)} = \infty$. Thus we have the desired result.

3. We shall start from proving the following proposition.

PROPOSITION 1. *Let $F(z)$ be*

$$(e^z - 1)\exp(e^z - 2z)\ .$$

*Then $F(z)$ is E-prime.*

PROOF. Suppose that $F(z) = f(g(z))$ with two non-linear entire functions $f$ and $g$. Evidently $F(z) = 0$ has roots on a single straight line, that is, the imaginary axis. Assume that $f(w) = 0$ has an infinite number of roots $\{w_j\}$. Then $g(z) = w_j$ must have its roots on the imaginary axis for all $j$. By Lemma 1 $g(z)$ must be a polynomial of degree at most two. We may assume that $g(z) = az^2 + bz + c$, $a \neq 0$. Evidently $\hat{\rho}_F = 1$, $\rho_F = \infty$. Let $f_1(w)$ be the canonical product formed by the zeros of $f(w)$. Although $\rho_f = \infty$, $\rho_{N(r,0,F)} = 1$ and hence $\rho_{N(r,0,f)} = 1/2$ imply that $f_1(w)$ is well defined and $\rho_{f_1} = 1/2$. Let $\exp L(w)$ be $f(w)/f_1(w)$. Then $\rho_{\exp L} = \infty$. Hence $L(w)$ is not a polynomial. Since $F(z)$ has two expressions

$$\begin{aligned} F(z) &= (e^z - 1)\exp(e^z - 2z) \\ &= f_1(az^2 + bz + c)e^{L(az^2+bz+c)} \end{aligned}$$

we have

$$e^z - 1 = f_1(az^2 + bz + c)e^{X(z)}\ ,$$

$$e^z - 2z = L(az^2 + bz + c) - X(z) + d\ ,\quad d = 2p\pi i\ .$$

Evidently $\rho_{\exp(X)} \leqq 1$. Hence $X(z)$ has the form $\alpha z + \beta$. Thus

$$e^z - (2 - \alpha)z + \beta - d = L(az^2 + bz + c)\ .$$

If $\alpha \neq 2$, then we have a contradiction by Lemma 8. Hence $\alpha = 2$. Then

$$e^z + \beta - d = L(az^2 + bz + c)$$

and

$$e^z - 1 = Af_1(az^2 + bz + c)e^{2z}\,, \quad A \neq 0\,.$$

By cancelling out the exponential term $e^z$ and then putting $w = az^2 + bz + c$, we have

$$Af_1(w) = \frac{L(w) + d - \beta - 1}{(L(w) + d - \beta)^2}\,.$$

Since $L(az^2 + bz + c) = e^z + \beta - d$, $\rho_{L(w)} = 1/2$. Hence $L(w) + d - \beta$ has zeros which do not coincide with the zeros of $L(w) + d - \beta - 1$. Thus $f_1(w)$ is not entire and hence $f(w) = f_1(w)e^{L(w)}$ is not entire, which is a contradiction.

Assume that $f(w) = 0$ has only a finite number of roots. Then $f(w) = Q(w)\exp M(w)$ with a polynomial $Q$ and an entire function $M$. Assume that $M$ is not a constant. In this case $\lambda_f \geqq 1$. Hence by Lemma 6, $\hat{\rho}_F \geqq \rho_g$. Since $\hat{\rho}_F = 1$, we have $\rho_g \leqq 1$. By the two expressions of $F(z)$ we have

$$Q(g(z)) = (e^z - 1)e^{X(z)}\,,$$
$$X(z) + M(g(z)) = e^z - 2z + d\,.$$

Here $X(z)$ is entire. Thus $\rho_{Q(g)} = \rho_g \leqq 1$ implies $X(z) = \alpha z + \beta$ and then $\rho_g = 1$. If $\alpha \neq -2$, then the identity

$$e^z - (2 + \alpha)z + d - \beta = M(g(z))$$

implies that $M$ should be linear by Lemma 8. Thus putting $M(w) = Aw + B$ we have

$$Ag(z) + B = e^z - (2 + \alpha)z + d - \beta\,,$$
$$Q(g(z)) = (e^z - 1)e^{\alpha z + \beta}\,.$$

Cancelling out $g(z)$ we have

$$A_0 + A_1e^{\alpha_1 z} + \cdots + A_m e^{\alpha_m z} = 0$$

with polynomials $A_j$ ($j = 0, 1, \cdots, m$), which are not zero, and non-zero constants $\alpha_j$ such that $\alpha_k \neq \alpha_j$ for $k \neq j$. This gives a contradiction by Lemma 4. If $\alpha = -2$,

$$e^z + d - \beta = M(g(z))$$

and Lemma 1 imply that $g(z)$ is a polynomial of degree at most two, when $M(w) = 0$ has an infinite number of roots. This is untenable, since $\rho_g = 1$. Hence $M(w)$ has only a finite number of zeros. By Lemma 2 $\rho_M = 0$. Hence $M(w)$ is not transcendental, that is, $M(w)$ is a polynomial. In this case we have

$$Q(g(z)) = D_1(e^z - 1)e^{-2z} ,$$
$$M(g(z)) = e^z + D_2 .$$

By cancelling out $e^z$ and then putting $w = g(z)$

$$Q(w) = \frac{D_1(M(w) - D_2 - 1)}{(M(w) - D_2)^2} .$$

Then $Q(w)$ is not entire, which is a contradiction.

Next assume that $f(w) = Q(w)$ is a polynomial. Then

$$F(z) = Q(g(z)) = (e^z - 1)\exp(e^z - 2z) .$$

Assume that $Q(w)$ has at least two different zeros $w_1, w_2$. Then $g(z) - w_j$ has zeros whose counting function is of order at most one. By forming the canonical product by the zeros of $g(z) - w_j$ and denoting it by $G_j(z)$ we have

$$g(z) - w_j = G_j(z)e^{H_j(z)} .$$

Evidently $\rho_{G_j} \leqq 1$. Further $\hat{\rho}_g = \hat{\rho}_{Q(g)} = \hat{\rho}_F = 1$. Hence $\hat{\rho}_{\exp H_j} = 1$. By Lemma 7 $\lambda_{\exp w} = 1$ and then by Lemma 6 $\rho_{H_j} \leqq \hat{\rho}_{\exp H_j} = 1$. On the other hand by Lemma 5 $\hat{\rho}_{\exp H_j} \leqq \rho_{H_j}$. Hence $\rho_{H_j} = 1$. Let us consider

$$w_2 - w_1 = G_2e^{H_2} - G_1e^{H_1} .$$

By Lemma 4 $H_2 - H_1$ should be a linear polynomial, since $\rho_{G_j} \leqq 1$. Then

$$w_2 - w_1 = (AG_2e^{\alpha z} - G_1)e^{H_1} .$$

If $AG_2\exp(\alpha z) - G_1 \not\equiv 0$, the right hand side has $\infty$ as its order. However $w_2 - w_1$ is a constant. By $w_2 \neq w_1 AG_2\exp(\alpha z) - G_1 \not\equiv 0$. Thus we arrive at a contradiction. Therefore $Q(w) = A(w - w_1)^n$. Then $AG_1(z)^n = (e^z - 1)e^X$ has only simple zeros. Thus $n = 1$, that is, $Q(w)$ is linear. Thus we have the desired result.

Evidently $F(z)$ admits $F(z) = f(g(z))$,

$$f(w) = \frac{w-1}{w^2}e^w , \quad g(z) = e^z .$$

Hence we have

THEOREM 1. *The E-primeness does not imply the primeness.*

4. In this section we shall prove the following.

THEOREM 2. *There is an entire periodic function, which is prime.*

PROOF. Let us consider

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{e^z}{\exp e^n}\right)\prod_{n=1}^{\infty}\left(1 - \frac{e^{-z}}{\exp\exp e^n}\right).$$

Put

$$\Pi_1(w) = \prod_{n=1}^{\infty}\left(1 - \frac{w}{\exp e^n}\right).$$

Then by a simple consideration we have

$$\log M(r, \Pi_1) \leqq \log r \log\log r\ ,$$

where $M(r, f)$ is the maximum modulus of $f$ on $|z| = r$. Hence

$$\begin{aligned}\log M(r, \Pi_1(e^z)) &\leqq \log M(M(r, e^z), \Pi_1)\\ &\leqq \log M(r, e^z)\log\log M(r, e^z)\\ &= r\log r\ ,\end{aligned}$$

which shows that $\rho_{\Pi_1(e^z)} = 1$. Similarly we have $\rho_{\Pi_2(e^{-z})} = 1$ for

$$\Pi_2(w) = \prod_{n=1}^{\infty}\left(1 - \frac{w}{\exp\exp e^n}\right).$$

Thus $\rho_F \leqq 1$. $\rho_F \geqq 1$ is evident. Hence $\rho_F = 1$.

(a) Suppose that $F(z) = f(g(z))$ with two transcendental entire functions $f$ and $g$. Then $\rho_F = 1$ implies $\rho_f = 0$ and $\rho_g \leqq \rho_F = 1$. By Lemma 9 $g(z)$ is periodic. Its period should be $n_0 2\pi i$ with a positive integer $n_0$. Let $g(e^n + 2j\pi i)$ and $g(-\exp e^n + 2j\pi i)$ be denoted by $X_{nj}$ and $Y_{nj}$, respectively. Then $X_{nj} = X_{nk}$, $Y_{nj} = Y_{nk}$ for $j - k = pn_0$. Consider the equations $g(z) = X_{nj}$, $g(z) = Y_{nj}$, $j = 0, 1, \cdots, n_0 - 1$. These have solutions $e^n + 2l\pi i$, $-\exp e^n + 2m\pi i$, respectively. Hence $f(X_{nj}) = 0$, $f(Y_{nj}) = 0$. There is no other zero of $f(x)$. Hence by $\rho_f = 0$

$$f(x) = A\prod_{n=1}^{\infty}\prod_{j=0}^{n_0-1}\left(1 - \frac{x}{X_{nj}}\right)\prod_{n=1}^{\infty}\prod_{j=0}^{n_0-1}\left(1 - \frac{x}{Y_{nj}}\right),$$

$$F(z) = A\prod\prod\left(1 - \frac{g(z)}{X_{nj}}\right)\prod\prod\left(1 - \frac{g(z)}{Y_{nj}}\right).$$

From the first factor all the zeros of $F(z)$ lying in the right half-plane, that is, $e^n + 2l\pi i$ appear and the second factor carries all the zeros of $F(z)$ lying in the left half-plane. Hence by

$$\rho_{\Pi_1(e^z)} = \rho_{\Pi\Pi(1-g/X_{nj})} = 1$$

$$\Pi_1(e^z) = A_1\prod\prod\left(1 - \frac{g(z)}{X_{nj}}\right)e^{\varepsilon\alpha z}\ ,$$

$$\Pi_2(e^{-z}) = A_2\prod\prod\left(1 - \frac{g(z)}{Y_{nj}}\right)e^{-\varepsilon\alpha z}\ ,\quad A_1A_2 = A\ ,$$

where $\varepsilon = 0$ or $\pm 1$. Firstly we shall consider the case $\varepsilon = 0$. $g(z)$ is representable as $h_1(w) \circ \exp(z/n_0)$ with a regular function $h_1(w)$ in $0 < |w| < \infty$. Let $w$ be $\exp(z/n_0)$. Then

$$\Pi_1(w^{n_0}) = A_1 \prod \prod \left(1 - \frac{h_1(w)}{X_{nj}}\right),$$

$$\Pi_2(w^{-n_0}) = A_2 \prod \prod \left(1 - \frac{h_1(w)}{Y_{nj}}\right).$$

Assume that $h_1(w)$ is regular at $w = 0$. Then $\Pi_2(w^{-n_0})$ should be regular at $w = 0$ but it has an essential singularity at $w = 0$. This is impossible. Hence $h_1(w)$ has a pole or an essential singularity at $w = 0$. Then $\Pi_1(w^{n_0})$ should have an essential singularity at $w = 0$ but this is not the case. This is again impossible. We next consider the case $\varepsilon = \pm 1$. The case $\varepsilon = -1$ is quite similar as in the case $\varepsilon = 1$. It is very easy to prove $\alpha = p/n_0$ with a positive integer $p$. ($p$ may be negative, but it does not have any effect in the following discussion.) Let $w$ be $\exp(z/n_0)$. Then

$$\Pi_1(w^{n_0}) = A_1 \prod \prod \left(1 - \frac{h_1(w)}{X_{nj}}\right) w^p ,$$

$$\Pi_2(w^{-n_0}) = A_2 \prod \prod \left(1 - \frac{h_1(w)}{Y_{nj}}\right) w^{-p} .$$

The same process as in the case $\varepsilon = 0$ does work in this case and leads us to a contradiction.

(b) Suppose that $f$ is transcendental entire and $g$ is a polynomial of degree at least two. Then by Lemma 10 $g$ is a quadratic polynomial. We put

$$g(z) = \alpha(z - a)^2 + \beta .$$

Assume that two points $z_1$ and $z_2$ satisfy $g(z_1) = g(z_2)$. Then $2a = z_1 + z_2$. Therefore all the zeros of $F(z)$ are distributed symmetrically with respect to $a$. Consider the asymptotic distribution of the zeros of $F(z)$. They are distributed more densely in the right half-plane than in the left. So there is no centre point of symmetry of them. This is a contradiction.

(c) Suppose that $f$ is a polynomial of degree at least two and $g$ is transcendental entire. By Lemma 11 $g$ is periodic. Let $f$ be $A(w - w_1) \cdots (w - w_p)$. There is at most one $w_j$, say $w_1$, such that $g(z) - w_1$ has no zero. Then $g(z) - w_1 = Be^{\alpha z}$, since $\rho_g = 1$. Here $B \neq 0, \alpha \neq 0$. In this case

$$F(z) = ABe^{\alpha z}(w_1 - w_2 + Be^{\alpha z}) \cdots (w_1 - w_p + Be^{\alpha z}) .$$

Let $n_0 2\pi i$ be the period of $g(z)$. Then $\alpha = s/n_0$ with an integer $s$. Of course $n_0$ is a positive integer. Let $x$ be $\exp(z/n_0)$. Then

$$\Pi_1(x^{n_0})\Pi_2(x^{-n_0}) = ABx^s(w_1 - w_2 + Bx^s) \cdots (w_1 - w_p + Bx^s) .$$

The left hand side has an essential singularity at $x = \infty$ but the right hand side does not. This is impossible. Hence $g(z) - w_j$ has a zero and hence in virtue of its periodicity it has infinitely many zeros. Further there is no multiple zero of $f$, since $F(z)$ has no multiple zero. We should remark that Rényi's proof of Lemma 11 shows that $1 \leqq n_0 \leqq p =$ the degree of $f(w)$. Let $g(z)$ be represented as $h_1(e^{z/n_0})$ with a regular function $h_1(x)$ in $0 < |x| < \infty$. Put $x = \exp(z/n_0)$. Then

$$\Pi_1(x^{n_0})\Pi_2(x^{-n_0}) = f(h_1(x)) .$$

Hence $x = 0$ and $x = \infty$ are singularities of $h_1(x)$. Since $h_1(e^{z/n_0})$ is of the first order, $\rho_{h_1} = 0$ around $x = 0$ and $x = \infty$. This is due to Lemma 12. Let $z_{1l}$ be the zeros of $g(z) - w_1$. Consider the set $\{\mathscr{R} z_{1l}\}$. Its elements have the form $n_0 \log |x_j|$ with $h_1(x_j) - w_1 = 0$, $z_{1l} = n_0 \log x_j + 2n_0 p\pi i$. There are only a finite number of different $x_j$ with the same modulus. Suppose that the set $\{\mathscr{R} z_{1l}\}$ is a finite set. Then $h_1(x) - w_1$ has only a finite number of zeros. This implies that $h_1(x) - w_1$ is a rational function of $x$. Thus $\Pi_1(x^{n_0})\Pi_2(x^{-n_0}) = A(h_1(x) - w_1) \cdots (h_1(x) - w_p)$ has only a finite number of zeros but the left hand side does have an infinite number of zeros. This is untenable. Hence all the equations $g(z) = w_j$, $j = 1, \cdots, p$ have an infinite number of roots having different real parts. Let $W_j$ be the set of zeros of $g(z) - w_j$. Each $W_j$ contains points $z_{jl}$ such that $\mathscr{R} z_{jl_\mu} \to +\infty$, $\mathscr{R} z_{jl_\nu} \to -\infty$ along suitable subsequences $\{l_\mu\}$, $\{l_\nu\}$ of $\{l\}$. All the zeros of $F(z)$ are divided into $p$ different (disjoint) groups $W_j$, $j = 1, \cdots, p$. We transfer these by $\exp(z_{jl}/n_0)$. Then $h_1(x) - w_j$ has roots $\exp(z_{jl}/n_0)$. If $z_{jl} = z_{jk} + 2sn_0\pi i$, then $\exp(z_{jl}/n_0) = \exp(z_{jk}/n_0)$. We denote $x_{jl}$ the different $\exp(z_{jl}/n_0)$. Then, if $p = n_0$,

$$x_{jn} = \varepsilon(j, n) \exp(e^n/n_0)$$

and

$$x_{j,-n} = \varepsilon(j, -n) \exp(-(\exp e^n)/n_0) ,$$

where $\varepsilon(j, \pm n)$ is an $n_0$-th root of unity which depends on $j$ and $\pm n$. In what follows it is sufficient to consider the case

$$h_1(x) = w_j + A_j \prod_{s=1}^{\infty}\left(1 - \frac{x}{x_{j,s}}\right) \prod_{t=1}^{\infty}\left(1 - \frac{x_{j,-t}}{x}\right) R_j(x)$$
$$\equiv w_j + \pi_j(x) R_j(x) .$$

Here $R_j(x)$ is a rational function of $x$. This representation is possible. In fact

$$(h_1(x) - w_j)/\pi_j(x) \equiv M(x)$$

has no zero in $0 < |x| < \infty$ and hence it has the form

$$R(x)e^{S(x)}$$

with $R(x) = x^q$ and rational $S(x)$, where $q$ is an integer. However $M(x)$ is of order zero around $x = 0$ and $x = \infty$. Hence $S(x)$ should be a constant. Hence $M(x) = R_j(x) \neq 0$ in $0 < |x| < \infty$. Let us consider $\pi_j(x)$. Put $x = x_{1m}$. Then

$$\pi_j(x_{1m}) = A_j X_1 X_2 ,$$
$$X_1 = \prod_{n=1}^{\infty}\left(1 - \frac{x_{1m}}{x_{j,n}}\right), \quad X_2 = \prod_{n=1}^{\infty}\left(1 - \frac{x_{j,-n}}{x_{1m}}\right).$$

Evidently $X_2 \to 1$ if $x_{1m} \to \infty$. Let us consider $|X_1|$. For $|X_1|$ we have

$$|X_1| \geqq \prod_{n=1}^{m-1}(\exp((e^m - e^n)/n_0) - 1)$$
$$\times \left|1 - \frac{\varepsilon(1, m)}{\varepsilon(j, m)}\right| \prod_{n=m+1}^{\infty}\left(1 - \frac{1}{\exp((e^m - e^n)/n_0)}\right).$$

For the first factor

$$\prod_{1}^{m-1}\left(\exp\frac{e^m - e^n}{n_0} - 1\right)$$
$$= \exp\frac{1}{n_0}\sum_{1}^{m-1}(e^m - e^n)\prod_{1}^{m-1}\left(1 - \frac{1}{\exp((e^m - e^n)/n_0)}\right)$$
$$\geqq (1 - \varepsilon)\exp\left(\frac{1}{n_0}(m - 1)(1 - \varepsilon')e^m\right).$$
$$\varepsilon' \to 0, \varepsilon \to 0 \text{ as } m \to \infty .$$

The last factor tends to 1 if $m \to \infty$. Further

$$\varepsilon(j, m) \neq \varepsilon(1, m)$$

for $j \neq 1$ and hence

$$|\varepsilon(j, m) - \varepsilon(1, m)|$$
$$\geqq |1 - \exp(2\pi i/n_0)| \geqq \delta > 0 .$$

Hence as $m \to \infty$

$$|X_1| \geqq \delta(1 - \varepsilon'')\exp\left(\frac{m - 1}{n_0}e^m\right).$$

On the other hand

$$|R(x_{1m})| \leqq A \exp\left(\frac{q}{n_0} e^m\right).$$

Hence as $m \to \infty$

$$\pi_j(x_{1m}) R_j(x_{1m}) \to \infty .$$

Now let us consider $w_1 + \pi_1(x)R_1(x) = w_j + \pi_j(x)R_j(x)$. Along $\{x_{1m}\}$ we have $w_1 - w_j = \pi_j(x)R_j(x) \to \infty$ by $\pi_1(x_{1m}) = 0$. This is evidently a contradiction. Several variants of the above case may occur. However the same consideration does work in every case. If $p > n_0 \geqq 1$, then we make $h_1(x) = w_j + \pi_j^*(x)$ and

$$w_2 - w_1 = \pi_1^*(x) - \pi_2^*(x) , \quad \pi_j^* = \pi_j \cdot R_j .$$

Let $x_{1n}$ be zeros of $\pi_1^*(x)$. Its subset tending to $\infty$ is denoted by $\{x_{1n}\}$ again. This is a subset of $\{\varepsilon(1, n) \exp(e^n/n_0)\}$. Let $x$ be $x_{1n}$. Then $\pi_1^*(x_{1n}) = 0$ and $\pi_2^*(x_{1n}) \to \infty$. This part is quite similar as in the above case. Then we arrive at a contradictory relation $w_2 - w_1 = \infty$. Hence $f$ should be a linear polynomial.

(d) In order to go further we need the following

PROPOSITION 2. *Let $F(z)$ be an entire function which admits a factorization $f(g(z))$ with transcendental meromorphic (not entire) $f$ and transcendental entire $g$. Then*

$$f(w) = f^*(w)/(w - w_1)^n , \quad g(z) = w_1 + e^{M(z)} ,$$
$$f^*(w_1) \neq 0 ,$$

*where $f^*$ is transcendental entire, $M$ a non-constant entire function and $n$ a positive integer.*

PROOF. Assume that $f(w) = \infty$ has two different roots $w_1$ and $w_2$. Then one of two equations $g(z) = w_1$, $g(z) = w_2$ admits an infinite number of roots, which must be poles of $F(z)$. This is a contradiction. Hence $f(w) = \infty$ has only one root $w_1$ and then $g(z) = w_1$ has no root. Hence we have the desired result.

Suppose that $f$ is transcendental meromorphic (not entire) and $g$ is transcendental entire. By Lemma 3 we have $\rho_f = 0$. Let $\{w_j^*\}$ be the set of zeros of $f^*(w)$. $\rho_{f^*} = \rho_f = 0$ implies that $\{w_j^*\}$ is an infinite set. Then by the second fundamental theorem

$$\begin{aligned} T(r, F) &\geqq N(r, 0, F) \geqq \sum_1^{K+1} N(r, w_j^*, g) \\ &\geqq Km(r, g) - O(\log rm(r, g)) \\ &\geqq K'm(r, g) . \end{aligned}$$

Hence

$$\rho_g \leqq \rho_F = 1 .$$

However by its form $\rho_g \geqq 1$. Hence $\rho_g = 1$. Therefore $M(z) = \alpha z + \beta$. Then the set of solutions of

$$w_j^* = w_1 + Be^{\alpha z}$$

coincides with the set of roots of $F(z) = 0$. They are

$$z = \frac{1}{\alpha} \log \frac{w_j^* - w_1}{B} + \frac{2p\pi i}{\alpha} .$$

By the periodicity of $F$ with period $2\pi i$ we have $\alpha = 1$ or $-1$. If $\alpha = 1$, then

$$F(z) = Ae^{-nz} f^*(w_1 + Be^z)$$

and

$$\Pi_1(w)\Pi_2(w^{-1}) = \frac{A}{w^n} f^*(w_1 + Bw) .$$

This is a contradiction, since the left hand side has $w = 0$ as an essential singularity but the right hand side has $w = 0$ as a pole. If $\alpha = -1$, then

$$F(z) = Ae^{nz} f^*(w_1 + Be^{-z})$$

and

$$\Pi_1(w^{-1})\Pi_2(w) = \frac{A}{w^n} f^*(w_1 + Bw) .$$

This is again a contradiction by the similar reasoning.

(e) Suppose that $f$ is transcendental meromorphic (not entire) and $g$ is a non-linear polynomial. In this case $f$ has at least one pole $w_0$ and $g(z) = w_0$ has at least one root. Hence $F(z)$ has at least one pole, which is clearly untenable.

(f) Suppose that $f$ is non-linear rational (not a polynomial) and $g$ is meromorphic. Let $a_1$ be a pole of $f$. Then $g(z) - a_1 \neq 0$. Let $g_1(z)$ be $1/(g(z) - a_1)$. Then $F(z) = R(g_1(z))$ with rational $R$ and entire $g_1$. $R$ has only one pole $b$. Hence $g_1(z) = b + Ae^{\alpha(z)}$. Since $\rho_{g_1} = \rho_g = 1$, $\alpha(z)$ must be $\alpha z$. Thus

$$F(z) = P(b + Ae^{\alpha z})e^{-m\alpha z}$$

with a polynomial $P$ and a positive integer $m$. $\alpha$ should be $\pm 1$ as in (d). Then we can apply the same reasoning as in (d) and we arrive at a contradiction.

Summing up the above results we have the primeness of

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{e^z}{\exp e^n}\right)\prod_{n=1}^{\infty}\left(1 - \frac{e^{-z}}{\exp\exp e^n}\right),$$

which is clearly entire and periodic. Therefore we have the desired result.

It is very easy to show that $F(z) = h(e^z)$ with regular $h(w)$ in $0 < |w| < \infty$ is not pseudo-prime, when either $w = 0$ or $w = \infty$ is not an essential singularity of $h(w)$ but one of them is. Even if both two points are essential singularities of $h(w)$, $F(z)$ is not always prime. This is shown by two typical examples:

$$h(w) = A_0 + \sum_{j=1}^{\infty} A_j(w^j + w^{-j}) ,$$
$$F(z) = h(e^z)$$
$$= \left\{A_0 + \sum_{j=1}^{\infty} A_j(e^{j\sqrt{w}} + e^{-j\sqrt{w}})\right\} \circ z^2$$

and

$$h(w) = \sum_{j=0}^{\infty} A_j R(w)^j ,$$
$$F(z) = h(e^z) = \left\{\sum_{j=0}^{\infty} A_j W^j\right\} \circ R(e^z) ,$$

where $R$ is a rational function of $w$. We do not know whether these two are all possible types of factorization of our $F$.

5. Let us denote $\varepsilon m$ the class of entire functions satisfying the condition in Proposition 2.

THEOREM 3. *Let $F(z)$ belong to the class $\varepsilon m$. Assume that $F(z) = A$ has only finitely many roots for an $A \neq \infty$. Then $F(z)$ is not pseudo-prime in entire sense.*

PROOF. $F(z)$ is representable as $F(z) = f(g(z))$

$$f(w) = f^*(w)/(w - w_1)^n , \quad f^*(w_1) \neq 0 ,$$
$$g(z) = w_1 + Be^{M(z)} .$$

Consider $f(w) = A$ and $g(z) = w$. $g(z)$ has already two lacunary values $w_1$ and $\infty$. Hence $g(z) = w$ has infinitely many roots, if $w \neq w_1, \infty$. Since $A \neq \infty$, $w$ satisfying $f(w) = A$ does not coincide with $w_1, \infty$. Then $F(z) = A$ has an infinite number of roots, if $f(w) = A$ has at least one root. This is untenable. Hence $f(w) = A$ has no root and hence

$$f^*(w) = A(w - w_1)^n + e^{L(w)}$$

with a non-constant entire function $L(w)$. Therefore

$$F(z) = A + \frac{1}{B} e^{-nM(z)} e^{L(w_1 + Be^{M(z)})} .$$

We put

$$f_1(w) = A + \frac{1}{B} e^w ,$$
$$g_1(z) = -nM(z) + L(w_1 + Be^{M(z)}) .$$

Then $F(z) = f_1(g_1(z))$ .

Theorem 3 gives only a sufficient condition. This is shown by

$$F(z) = \{1 - \exp(-e^z)\} \exp \exp e^z .$$

It admits two factorizations

$$\{(1 - e^{-w}) \exp e^w\} \circ e^z , \quad \left\{\frac{w-1}{w} e^w\right\} \circ \exp e^z .$$

Another remark is the following: If $F(z) \in \varepsilon m$, then $\lambda_F \geqq 1$. A method of proof was shown in the (d) step of proving Theorem 2.

THEOREM 4. *Let $F(z) \in \varepsilon m$. If $\rho_F = 1$, then $F(z)$ is pseudo-prime in entire sense. If $1 < \rho_F$, $F(z)$ admits a factorization $f(g(M(z)))$, where $f$ or $g$ is a polynomial and $M(z)$ is entire.*

PROOF. Consider the case $\rho_F = 1$. By Proposition 2 we have

$$F(z) = Ae^{-n\alpha z} f^*(w_1 + Be^{\alpha z}) ,$$
$$A = e^{-n\beta} , \quad B = e^{\beta} , \quad f^*(w_1) \neq 0 .$$

Suppose that $F(z) = f_1(g_1(z))$ with two transcendental entire $f_1$ and $g_1$. Then $\rho_{f_1} = 0$ and hence by Lemma 9 $g_1(z)$ is periodic. Its period should be equal to $n_0 2\pi i/\alpha$. Therefore $g_1(z) = h_1(e^{\alpha z/n_0})$ with a regular function $h_1(x)$ in $0 < |x| < \infty$. Hence putting $w = \exp(\alpha z/n_0)$

$$Aw^{-nn_0} f^*(w_1 + Bw^{n_0}) = f_1(h_1(w)) .$$

If $h_1(w)$ has an essential singularity or a pole at $w = 0$, then $f_1(h_1(w))$ has $w = 0$ as an essential singularity. However $w = 0$ is only a pole of the left hand side. This is impossible. Hence $h_1(w)$ is entire. This is again a contradiction, since the left hand side has really a pole at $w = 0$. Hence we have the desired result, if $\rho_F = 1$. Next we shall consider the case $1 < \rho_F$. Then

$$F(z) = F_1(w) \circ M(z) ,$$
$$F_1(w) = e^{-nw} f^*(w_1 + Be^w)$$

with entire $M(z)$. Since $F_1(z)$ is pseudo-prime in entire sense, $F_1(z) = f_1(g_1(z))$. Here $f_1$ is a polynomial unless $g_1$ is so. Hence

$$F(z) = f_1(g_1(M(z))) .$$

This gives the desired result.

There remain two problems: When is an $F \in \varepsilon m$, $\rho_F = 1$, $E$-prime? Is every $F \in \varepsilon m$, $\rho_F < \infty$ pseudo-prime in entire sense?

THEOREM 5. *Let $F(z)$ belong to $\varepsilon m$. Assume that the lower order of $N(r, 0, F)$ is less than* 1. *Then $F(z)$ has* 0 *as a lacunary value.*

PROOF. Suppose firstly that $f^*(w) = 0$ has at least two roots. Here

$$F(z) = f(g(z)) , \quad f(w) = f^*(w)/(w - w_1)^n ,$$
$$g(z) = w_1 + e^{M(z)} , \quad f^*(w_1) \neq 0 .$$

Then by the second fundamental theorem

$$\lambda_{N(r,0,F)} \geqq \lambda_g \geqq 1 ,$$

since

$$N(r, 0, F) \geqq N(r, w_2, g) + N(r, w_3, g)$$
$$\geqq m(r, g) + O(\log rm(r, g)) .$$

This is impossible. Assume that $f^*(w) = 0$ has only one root. Then $N(r, 0, F) = N(r, w_2, g)$. Since $N(r, \infty, g) = N(r, w_1, g) = 0$, $\lambda_{N(r,0,F)} = \lambda_g \geqq 1$ by the second fundamental theorem. Thus we have again a contradiction. If $f^*(w)$ does not vanish, then $F(z) = 0$ has no root. Thus we have the desired result.

THEOREM 6. *Let $F(z)$ belong to $\varepsilon m$. Assume that the lower order of $N(r, 0, F)$ is less than* 1. *Then $\hat{\rho}_F \geqq 1$.*

PROOF. By the above proof of Theorem 5 to be considered is a case that $f^*(w) = 0$ has no root. Then $f^*(w) = \exp L(w)$ with entire $L$. Then

$$F(z) = e^{-nM(z)} \exp (L(w_1 + e^{M(z)})) .$$

Since $\lambda_{f^*} \geqq 1$, $\hat{\rho}_F \geqq \rho_g \geqq 1$. This is the desired result.

**6.** In this section we shall be concerned with a remark on Lemma 10, since it seems to belong to our range of idea of using the function $h(e^z)$. Let $F(z)$ be represented as $f(P(z))$ with entire $f$ and a polynomial $P(z)$. Suppose that $F$ is entire periodic. Then $P(z)$ is of degree at most two by Lemma 10. What can be said about $f$ when $P$ is quadratic? The following, which may or may not be new, gives the definite answer.

THEOREM 7. *Let $F(z)$ be entire periodic with period $2\pi i$. Assume that $F(z)$ is represented as $f(P(z))$ with entire $f$ and a polynomial $P(z)$ of degree $\geqq 2$. Then*

$$f(w) = g\left\{\cosh \sqrt{\frac{w-c}{B}}\right\}$$

*with entire $g$ and constants $c$ and $B$.*

PROOF. We may put $P(z) = B(z-\alpha)^2 + c$. Then

$$F(z) = f(P(z)) = h(e^z)$$

with regular $h(x)$ in $0 < |x| < \infty$. Let $z - \alpha$ be $x$. Then

$$f(Bx^2 + c) = h(Ae^x)\ , \quad A = e^\alpha$$

is an even function of $x$. Hence $h(Ae^x) = h(Ae^{-x})$. Let $h(w)$ be

$$\sum_{n=-\infty}^{\infty} a_n w^n\ .$$

Then $a_n A^n = a_{-n} A^{-n}$ for any $n$. Put $L_n = a_n A^n$. Then

$$h(Ae^x) = a_0 + \sum_{j=1}^{\infty} L_j(e^{jx} + e^{-jx})$$
$$= \sum_{n=0}^{\infty} M_n\left(\frac{e^x + e^{-x}}{2}\right)^n .$$

Let $g(X)$ be

$$\sum_{n=0}^{\infty} M_n X^n\ .$$

Then

$$f(Bx^2 + c) = g\,\{\cosh x\}\ .$$

Hence

$$f(w) = g\left\{\cosh \sqrt{\frac{w-c}{B}}\right\} .$$

COROLLARY. *Besides the assumptions in Theorem 7 we assume that $\rho_F > 1$. Then $f(w)$ is not pseudo-prime.*

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