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RESEARCH ARTICLE

Factorization of matrices over elementary divisor domain

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Dedicated to the memory of my teacher P.S. Kazimirskij

Abstract. We propose constructive criteria of divisibility and associativity of matrices over commutative elementary divisor ring without zero divisors. On this base, the explicit form for all non-associated divisors which have prescribed canonical diagonal forms (c.d.f.) is indicated. A relation between c.d.f. for matrix and c.d.f. for its divisors is established. The uniqueness theorem is proved.

One of the most important problems in the matrix theory is the problem of factorization of a matrices over rings, classification and investigation of the structure of its divisors. Due to practical application, the main consideration was given to matrices over complex number field. In particular, the article of P. Kazimirskij [1,2] and its followers (V. Petrychkovych [3], V. Zelisko [4,5], V. Shchedryk [5,6] and many others), P. Lancaster, I.Gohberg, L. Rodman [7], A. Malyshev [8], Langer H. [9] deal with the given issue. The valuable problem contribution of these authors concern the existence and description of regular divisors of a matrix polynomials. Research activity in this area has been continued until now. The recent papers of T. Laffei [10,11], I. Krupnyk [12] and other deal primarily with the conditions of a matrix polynomials decomposition into the product of linear multipliers. Later, the investigation of a matrix polynomials, was extended to some other classes of rings, in particular, the polynomial

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ring in n variables [13], the ring of integer [14], principal ideal rings [15], dedekind rings [16].

1. The structure of matrices divisors

Let A be a matrix over a commutative elementary divisor ring R [17] without zero divisors. Matrices A and A_1 are called right associate (left associate, written $A \overset{l}{\sim} A_1$) if $A_1 = AU$ ($A_1 = UA$) for some invertible matrix U. Let B be a left divisor of the matrix A, i.e., $A = BC$. It is obvious that all matrices which are right associate to the matrix B are also left divisors of the matrix A. Hence, it is natural to describe the left divisors of the matrix A up to right associates. In this paper we have proposed the solution of this problem. As corollary we have obtained the description of monic divisors of matrix polynomials and solution to unilateral matrix equations over fields.

Let A, B be $n \times n$ matrices over R. There are invertible matrices P_A, P_B, Q_A, Q_B such that

$$P_A A Q_A = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots, 0) = \Psi,$$

$$P_B B Q_B = \operatorname{diag}(\varphi_1, \ldots, \varphi_t, 0, \ldots, 0) = \Phi,$$

where $\varepsilon_k \neq 0, \varphi_t \neq 0$, $\varepsilon_i | \varepsilon_{i+1}, \varphi_j | \varphi_{j+1}, i = 1, \ldots, k-1, j = 1, \ldots, t-1$. The matrices Ψ, Φ are called canonical diagonal forms (c.d.f.) of the matrices A and B, respectively. Consider the sets of matrices

$$\mathbf{G}_\Phi = \{H \in GL_n(R) | H\Phi = \Phi K \quad \text{for some} \quad K \in GL_n(R)\},$$

$$\mathbf{L}(\Psi, \Phi) = \{L \in GL_n(R) | L\Psi = \Phi S \quad \text{for some} \quad S \in M_n(R)\}.$$

A trivial verification shows that these sets have the following properties:

Proposition 1. *The set $\mathbf{G}_\Phi$ is a multiplicative group.* □

Proposition 2. $\mathbf{G}_\Phi \mathbf{L}(\Psi, \Phi) = \mathbf{L}(\Psi, \Phi)$. □

Proposition 3. *If $H \in \mathbf{G}_\Psi$ then $\mathbf{L}(\Psi, \Phi)H = \mathbf{L}(\Psi, \Phi)$.* □

The following results show that these sets play the main role in description of matrix divisors.

Theorem 1. *The matrix B is a left divisor of the matrix A , i.e., $A = BC$ if and only if $P_B P_A^{-1} \in \mathbf{L}(\Psi, \Phi)$.*

Proof. Necessity. Note that $A = P_A^{-1}\Psi Q_A^{-1}, B = P_B^{-1}\Phi Q_B^{-1}$ we have

$$P_B A = P_B(BC) = (P_B B)C = (\Phi Q_B^{-1})C = \Phi(Q_B^{-1}C).$$

On the other hand,

$$P_B A = (P_B P_A^{-1})(P_A A) = (P_B P_A^{-1})\Psi Q_A^{-1}.$$

Hence,

$$(P_B P_A^{-1})\Psi = \Phi S,$$

where $S = Q_B^{-1}CQ_A$. This means that $P_B P_A^{-1} \in \mathbf{L}(\Psi, \Phi)$.

Sufficiency. Since

$$P_B A = P_B(P_A^{-1}\Psi Q_A^{-1}) = (P_B P_A^{-1})\Psi Q_A^{-1} = \Phi S Q_A^{-1},$$

we have

$$A = P_B^{-1}\Phi S Q_A^{-1} = (P_B^{-1}\Phi Q_B^{-1})(Q_B S Q_A^{-1}) = BC,$$

where $C = Q_B S Q_A^{-1}$. □

Corollary 1. *All left divisors of the matrix $A = P_A^{-1}\Psi Q_A^{-1}$ with c.d.f. Φ have the form $(LP_A)^{-1}\Phi Q$, where $L \in \mathbf{L}(\Psi, \Phi), Q \in GL_n(R)$.* □

Corollary 2. *The matrices $A = P_A^{-1}\Phi Q_A^{-1}, B = P_B^{-1}\Phi Q_B^{-1}$ are right associate if and only if $P_B P_A^{-1} \in \mathbf{G}_\Phi$, i.e., $P_B = HP_A$, where $H \in \mathbf{G}_\Phi$.* □

Corollary 3. *If $P_A A Q_A = P'_A A Q'_A = \Psi$, then $P'_A = HP_A$, where $H \in \mathbf{G}_\Psi$.* □

Let us denote by $\mathbf{W}(\Psi, \Phi)$ the set of representatives of the left conjugate class of the set $\mathbf{L}(\Psi, \Phi)$ by the group $\mathbf{G}_\Phi$. Corollaries 1 and 2 can be summarized in the following statement:

Theorem 2. *The set $(\mathbf{W}(\Psi, \Phi)P_A)^{-1}\Phi$ consists of all left up to right associate divisors of the matrix A which have c.d.f. Φ.* □

Theorem 3. *Let $P_A A Q_A = P'_A A Q'_A = \Psi$ and $B \in (\mathbf{W}(\Psi, \Phi)P'_A)^{-1}\Phi$. Then the set $(\mathbf{W}(\Psi, \Phi)P_A)^{-1}\Phi$ contain a matrix which are right associate to the matrix B.*

Proof. Let $B = (WP'_A)^{-1}\Phi$, where $W \in \mathbf{W}(\Psi, \Phi)$. By Corollary 3 $P'_A = SP_A$, where $S \in \mathbf{G}_\Psi$. Therefore $B = (WSP_A)^{-1}\Phi$. According to Proposition 3 the matrix WS belong to the set $\mathbf{L}(\Psi, \Phi)$. Consequently, $WS \in \mathbf{G}_\Phi W_1$, where $W_1 \in \mathbf{W}(\Psi, \Phi)$. It follows that there exist matrix $H \in \mathbf{G}_\Phi$, such that $WS = HW_1$. Hence

$$B = (WP'_A)^{-1}\Phi = (WSP_A)^{-1}\Phi = (HW_1P_A)^{-1}\Phi = \\ = (W_1P_A)^{-1}H^{-1}\Phi = (W_1P_A)^{-1}\Phi K^{-1} = B_1K^{-1},$$

where $B_1 \in (\mathbf{W}(\Psi, \Phi)P_A)^{-1}\Phi, K^{-1} \in GL_n(R)$. □

Let $\sigma_{ij}(L)$ denote the greatest common divisor of matrix entries

$$\left\| \begin{matrix} l_{i1} & \dots & l_{ij} \\ \dots & \dots & \dots \\ l_{n1} & \dots & l_{nj} \end{matrix} \right\|,$$

which are submatrix of the matrix $L = \| \ l_{ij} \ \|_1^n$.

Lemma 1. $\det L = \sigma_{ii}(L)l_i, l_i \in R, i = 1, \dots, n.$

Proof. Since $\sigma_{ii}(L)$ is a divisor of all minors of maximal order of the matrix

$$\left\| \begin{matrix} l_{i1} & \dots & l_{in} \\ \dots & \dots & \dots \\ l_{n1} & \dots & l_{nn} \end{matrix} \right\|,$$

$i = 1, \dots, n$, the proof is immediate. □

Corollary 4. *If L is an invertible matrix then $\sigma_{ii}(L) \in U(R), i = 1, \dots, n$.* □

Now, let us establish the relation between c.d.f. of the matrices A and B. It is well known, that Φ divides Ψ provided R is a principal ideal ring [18] or an adequate ring [19]. The following theorem asserts that this statement is still true provided R is an elementary divisor ring.

Theorem 4. *The matrix $A = P_A^{-1}\Psi Q_A^{-1}$, where $\Psi = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_k, 0, \dots, 0)$, has a divisor B with c.d.f. $\Phi = \operatorname{diag}(\varphi_1, \dots, \varphi_t, 0, \dots, 0)$, if and only if Φ divide Ψ.*

Proof. Necessity. Let $A = BC$. By Theorem 1, $L\Psi = \Phi S$, where $L = P_B P_A^{-1} = \| \ l_{ij} \ \|_1^n$, $S = \| \ s_{ij} \ \|_1^n$. Therefore

$$\left\| \begin{array}{cccccc} \varepsilon_1 l_{11} & \dots & \varepsilon_k l_{1k} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \varepsilon_1 l_{t1} & \dots & \varepsilon_k l_{tk} & 0 & \dots & 0 \\ \varepsilon_1 l_{t+1.1} & \dots & \varepsilon_k l_{t+1.k} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \varepsilon_1 l_{n1} & \dots & \varepsilon_k l_{nk} & 0 & \dots & 0 \end{array} \right\| = \left\| \begin{array}{ccc} \varphi_1 s_{11} & \dots & \varphi_1 s_{1n} \\ \vdots & & \vdots \\ \varphi_t s_{t1} & \dots & \varphi_t s_{tn} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{array} \right\|. \quad (1)$$

It follows that

$$\left\| \begin{array}{ccc} l_{t+1.1} & \dots & l_{t+1.k} \\ \dots & \dots & \dots \\ l_{n1} & \dots & l_{nk} \end{array} \right\| = \mathbf{0}.$$

Applying Corollary 4 we conclude that the element $l_{t+1.k}$ lies below the main diagonal, i.e., $t+1 > k$. Hence, $t \geq k$. From (1) we conclude that $\varphi_i | \varepsilon_j l_{ij}, i = 1, \dots, t, j = 1, \dots, k$. Consequently, $\frac{\varphi_i}{(\varphi_i, \varepsilon_j)} | \frac{\varepsilon_j}{(\varphi_i, \varepsilon_j)} l_{ij}$. It follows that $\frac{\varphi_i}{(\varphi_i, \varepsilon_j)} | l_{ij}$, i.e., $l_{ij} = f_{ij} l'_{ij}$, where $f_{ij} = \frac{\varphi_i}{(\varphi_i, \varepsilon_j)}$. Therefore the matrix L has the form

$$L = \left\| \begin{array}{cccccc} \frac{\varphi_1}{(\varphi_1, \varepsilon_1)} l'_{11} & \dots & \frac{\varphi_1}{(\varphi_1, \varepsilon_k)} l'_{1k} & l_{1.k+1} & \dots & l_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\varphi_t}{(\varphi_t, \varepsilon_1)} l'_{t1} & \dots & \frac{\varphi_t}{(\varphi_t, \varepsilon_k)} l'_{tk} & l_{t.k+1} & \dots & l_{tn} \\ 0 & \dots & 0 & l_{t+1.k+1} & \dots & l_{t+1.n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & l_{n.k+1} & \dots & l_{nn} \end{array} \right\|. \quad (2)$$

According to

$$f_{i+r.j-l} = f_{ij} \frac{\left(\varphi_{i+r}, \frac{\varphi_{i+r}}{\varphi_i} \varepsilon_j\right)}{(\varphi_{i+r}, \varepsilon_{j-l})}, \quad l < j,$$

we have $f_{ij} | \sigma_{ij}(L)$. By Corollary 4, $f_{ii} = \frac{\varphi_i}{(\varphi_i, \varepsilon_i)} \in U(R)$, $i = 1, \dots, k$. It follows that $\varphi_i | \varepsilon_i, i = 1, \dots, k$, hence, $\Phi | \Psi$.

Sufficiency is obvious. □

Therefore decomposition procedure of the matrix A falls naturally into two steps. At first we decompose the c.d.f. of the matrix A into two factors: $\Psi = \Phi\Delta$, where $\Phi = \operatorname{diag}(\varphi_1, \dots, \varphi_t, 0 \dots, 0), \varphi_i | \varphi_{i+1}$, $i = 1, \dots, t-1$. Secondly, we seek divisors of the matrix A with prescribed c.d.f. Φ.

Corollary 5. *The set* $\mathbf{L}(\Psi,\Phi)$ *consists of all invertible matrices of the form*

$$L = \left\| \begin{matrix} L_1 & * \\ L_2 & * \\ \mathbf{0} & * \end{matrix} \right\|, \qquad (3)$$

where

$$L_1 = \left\| \begin{matrix} l_{11} & l_{12} & \dots & l_{1.k-1} & l_{1k} \\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)} l_{21} & l_{22} & \dots & l_{2.k-1} & l_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_k}{(\varphi_k,\varepsilon_1)} l_{k1} & \frac{\varphi_k}{(\varphi_k,\varepsilon_2)} l_{k2} & \dots & \frac{\varphi_k}{(\varphi_k,\varepsilon_{k-1})} l_{k.k-1} & l_{kk} \end{matrix} \right\|, \qquad (4)$$

$$L_2 = \left\| \begin{matrix} \frac{\varphi_{k+1}}{(\varphi_{k+1},\varepsilon_1)} l_{k+1.1} & \dots & \frac{\varphi_{k+1}}{(\varphi_{k+1},\varepsilon_k)} l_{k+1.k} \\ \dots & \dots & \dots \\ \frac{\varphi_t}{(\varphi_t,\varepsilon_1)} l_{t1} & \dots & \frac{\varphi_t}{(\varphi_t,\varepsilon_k)} l_{tk} \end{matrix} \right\|,$$

$l_{ij} \in R$.

Proof. On account of the proof of Theorem 4 any matrix L from $\mathbf{L}(\Psi,\Phi)$ has form (2). Since $\frac{\varphi_i}{(\varphi_i,\varepsilon_{i+j})} \in U(R)$, $j = 0,\dots,k-i$, we see that the elements $l_{i.i+j}$ have no restrictions. Hence, the matrix L has form (3).

Conversely, suppose that the matrix L has form (3). An easy computation shows that $L\Psi = \Phi S$, where

$$S = \left\| \begin{matrix} M_1 & \mathbf{0} \\ M_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{matrix} \right\|,$$

$$M_1 = \left\| \begin{matrix} \frac{\varepsilon_1}{\varphi_1} l_{11} & \frac{\varepsilon_2}{\varphi_1} l_{12} & \dots & \frac{\varepsilon_{k-1}}{\varphi_1} l_{1.k-1} & \frac{\varepsilon_k}{\varphi_1} l_{1k} \\ \frac{\varepsilon_1}{(\varphi_2,\varepsilon_1)} l_{21} & \frac{\varepsilon_2}{\varphi_2} l_{22} & \dots & \frac{\varepsilon_{k-1}}{\varphi_2} l_{2.k-1} & \frac{\varepsilon_k}{\varphi_2} l_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\varepsilon_1}{(\varphi_k,\varepsilon_1)} l_{k1} & \dots & \dots & \frac{\varepsilon_{k-1}}{(\varphi_k,\varepsilon_{k-1})} l_{k.k-1} & \frac{\varepsilon_k}{\varphi_k} l_{kk} \end{matrix} \right\|,$$

$$M_2 = \left\| \begin{matrix} \frac{\varepsilon_1}{(\varphi_{k+1},\varepsilon_1)} l_{k+1.1} & \dots & \frac{\varepsilon_k}{(\varphi_{k+1},\varepsilon_k)} l_{k+1.k} \\ \dots & \dots & \dots \\ \frac{\varepsilon_1}{(\varphi_t,\varepsilon_1)} l_{t1} & \dots & \frac{\varepsilon_k}{(\varphi_t,\varepsilon_k)} l_{tk} \end{matrix} \right\|.$$

□

Corollary 6. *The group* $\mathbf{G}_\Phi$ *consists of all invertible matrices of the form*

$$H = \left\| \begin{matrix} H_1 & * \\ \mathbf{0} & N \end{matrix} \right\|,$$

where

$$H_1 = \left\| \begin{array}{ccccc} h_{11} & h_{12} & \ldots & h_{1.t-1} & h_{1t} \\ \frac{\varphi_2}{\varphi_1}h_{21} & h_{22} & \ldots & h_{2.t-1} & h_{2t} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\varphi_t}{\varphi_1}h_{t1} & \frac{\varphi_t}{\varphi_2}h_{t2} & \ldots & \frac{\varphi_t}{\varphi_{t-1}}h_{t.t-1} & h_{tt} \end{array} \right\|,$$

$N \in GL_{n-t}(R)$. □

We can now rephrase Theorem 1 as follows.

Theorem 5. *The matrix $B = P_B^{-1}\Phi Q_B^{-1}$ is the left divisor of the matrix $A = P_A^{-1}\Psi Q_A^{-1}$ if and only if the matrix $P_B P_A^{-1}$ has form* (3). □

Theorem 6. *The matrix A has a unique up to associate divisor with c.d.f.*
$\Phi = \operatorname{diag}(\varphi_1, \ldots, \varphi_t, 0, \ldots, 0)$ if and only if one of following three cases holds:

1) $k = t = n \Longrightarrow (\varphi_n, \varepsilon_j) = \varphi_j$, $j = 1, \ldots, n-1$;

2) $k < n, t = n \Longrightarrow \varphi_{k+1} = \varphi_{k+2} = \ldots = \varphi_n$, and $(\varphi_n, \varepsilon_j) = \varphi_j$, $j = 1, \ldots, k$;

3) $k, t < n \Longrightarrow k = t$, and $(\varphi_k, \varepsilon_j) = \varphi_j$, $j = 1, \ldots, k-1$.

Proof. We follow the notations of Corollaries 5 and 6. According to Theorem 2 the matrix A has a unique up to associate divisor with c.d.f. Φ if and only if $\mathbf{W}(\Psi, \Phi) = \{E\}$, i.e., $\mathbf{L}(\Psi, \Phi) = \mathbf{G}_\Phi$.

Let $k = t = n$. The equality of these sets is equivalent to $L_1 = H_1$. Therefore $(\varphi_i, \varepsilon_j) = \varphi_j$, $i = 2, \ldots, n$, $j = 1, \ldots, n-1, i > j$. Specifically, $(\varphi_n, \varepsilon_j) = \varphi_j$, $j = 1, \ldots, n-1$.

Conversely, if $\left(\frac{\varphi_n}{\varphi_j}, \frac{\varepsilon_j}{\varphi_j}\right) = 1$, $j = 1, \ldots, n-1$, we have $\left(\frac{\varphi_i}{\varphi_j}, \frac{\varepsilon_j}{\varphi_j}\right) = 1$, $i = j+1, \ldots, n$, so that

$$(\varphi_i, \varepsilon_j) = \varphi_j \left(\frac{\varphi_i}{\varphi_j}, \frac{\varepsilon_j}{\varphi_j}\right) = \varphi_j.$$

Case 2. The equality of the sets $\mathbf{L}(\Psi, \Phi)$ and $\mathbf{G}_\Phi$ is equivalent to

$$H_1 = \left\| \begin{array}{cc} L_1 & * \\ L_2 & * \end{array} \right\|.$$

This is equivalent to

$$\frac{\varphi_i}{\varphi_j} = 1, i = k+2, k+3, \ldots, n, j = k+1, k+2, \ldots, n-1, i > j. \quad (5)$$

and

$$(\varphi_i, \varepsilon_j) = \varphi_j, i = 2, \ldots, n, j = 1, \ldots, k, i > j. \quad (6)$$

Specifically,
$$\frac{\varphi_{k+2}}{\varphi_{k+1}} = \frac{\varphi_{k+3}}{\varphi_{k+2}} = \cdots = \frac{\varphi_n}{\varphi_{n-1}} = 1.$$
Hence,
$$\varphi_{k+1} = \varphi_{k+2} = \ldots = \varphi_n. \tag{7}$$
Having noticed
$$\frac{\varphi_p}{\varphi_q} = \frac{\varphi_p}{\varphi_{p-1}} \frac{\varphi_{p-1}}{\varphi_{p-2}} \cdots \frac{\varphi_{q+1}}{\varphi_q},$$
where $p > q$, we conclude that equalities (5) and (7) are equivalent. In the same manner as above we can see that *ii*) and (6) are equivalent.

Now consider Case 3. Thus we get
$$\left\| \begin{matrix} L_1 & * \\ L_2 & * \\ \mathbf{0} & * \end{matrix} \right\| = \left\| \begin{matrix} H_1 & * \\ \mathbf{0} & N \end{matrix} \right\|.$$

The sizes of zero submatrices are $(n-t) \times k$ and $(n-t) \times t$ hence, $k = t$. It follows that the matrix L_2 is empty. Furthermore the analysis similar to above shows that $(\varphi_k, \varepsilon_j) = \varphi_j$, $j = 1, \ldots, k-1$. □

This result generalizes the known results of Z. Borevich [15], V. Zelisko [4] and V. Petrychkovych [3].

2. Finding the set $\mathbf{W}(\Psi, \Phi)$

This part of our paper is devoted to the study of set $\mathbf{W}(\Psi, \Phi)$, where A is a nonsingular matrix.

Let us denote by $K(f)$ the set of representatives of the conjugate classes of R/Rf, $f \in R$. Let $\mathbf{V}(\Psi, \Phi)$ denote the set of lower unitriangular matrices of the form
$$\left\| \begin{matrix} 1 & 0 & \ldots & 0 & 0 \\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)} k_{21} & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\varphi_n}{(\varphi_n,\varepsilon_1)} k_{n1} & \frac{\varphi_n}{(\varphi_n,\varepsilon_2)} k_{n2} & \ldots & \frac{\varphi_n}{(\varphi_n,\varepsilon_{n-1})} k_{n.n-1} & 1 \end{matrix} \right\|,$$
where $k_{ij} \in K\left(\frac{(\varphi_i,\varepsilon_j)}{\varphi_j}\right)$, $i = 2, \ldots, n$, $j = 1, \ldots, n-1$, $i > j$. For the first time in the case $R = \mathbb{C}[x]$ these matrices were introduced by Kazimirskij P.S. [1,2].

Proposition 4. $\mathbf{V}(\Psi, \Phi) \subseteq \mathbf{W}(\Psi, \Phi)$.

Proof. Let V, V_1 be matrices from $\mathbf{V}(\Psi, \Phi)$ with the elements $f_{ij}v_{ij}$, $f_{ij}u_{ij}$, respectively, where $f_{ij} = \frac{\varphi_i}{(\varphi_i, \varepsilon_j)}, i > j$, and let $HV = V_1$, where $H \in \mathbf{G}_\Phi$. The assertion follows if we prove that $V = V_1$. It is obvious that the matrix H is also a lower unitriangular matrix with the elements $\frac{\varphi_i}{\varphi_j}h_{ij}, i > j$. We have $H = V_1V^{-1}$. Putting $n = 2$, we get

$$f_{21}u_{21} - f_{21}v_{21} = \frac{\varphi_2}{\varphi_1}h_{21}.$$

Notice that

$$\frac{\varphi_2}{\varphi_1} = f_{21}\frac{(\varphi_2, \varepsilon_1)}{\varphi_1},$$

thus

$$u_{21} - v_{21} = \frac{(\varphi_2, \varepsilon_1)}{\varphi_1}h_{21}.$$

Hence,

$$u_{21} \equiv v_{21} \left(\operatorname{mod} \frac{(\varphi_2, \varepsilon_1)}{\varphi_1} \right).$$

This means that $u_{21} = v_{21}$, so $V = V_1$.

Suppose that the assumption holds for the matrices of order $n - 1$, we will prove it for n. The equality $HV = V_1$ implies that the equalities

$$H'V' = \left\| \begin{matrix} 1 & & & 0 \\ \frac{\varphi_2}{\varphi_1}h_{21} & 1 & & \\ \dots & \dots & & \\ \frac{\varphi_{n-1}}{\varphi_1}h_{n-1.1} & \frac{\varphi_{n-1}}{\varphi_2}h_{n-1.2} & \dots \quad \frac{\varphi_{n-1}}{\varphi_{n-2}}h_{n-1.n-2} & 1 \end{matrix} \right\| \times$$

$$\times \left\| \begin{matrix} 1 & & & 0 \\ f_{21}v_{21} & 1 & & \\ \dots & \dots & & \\ f_{n-1.1}v_{n-1.1} & f_{n-1.2}v_{n-1.2} & \dots \quad f_{n-1.n-2}v_{n-1.n-2} & 1 \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} 1 & 0 & & 0 \\ f_{21}u_{21} & 1 & & \\ \dots & \dots & & \\ f_{n-1.1}u_{n-1.1} & f_{n-1.2}u_{n-1.2} & \dots \quad f_{n-1.n-2}u_{n-1.n-2} & 1 \end{matrix} \right\| = V_1',$$

and

$$H''V'' = \left\| \begin{matrix} 1 & & & 0 \\ \frac{\varphi_3}{\varphi_2}h_{21} & 1 & & \\ \dots & \dots & & \\ \frac{\varphi_n}{\varphi_2}h_{n2} & \frac{\varphi_n}{\varphi_3}h_{n3} & \dots \quad \frac{\varphi_n}{\varphi_{n-1}}h_{n.n-1} & 1 \end{matrix} \right\| \times$$

$$\times \left\| \begin{matrix} 1 & & & & 0 \\ f_{32}v_{32} & 1 & & & \\ \dots & \dots & & & \\ f_{n2}v_{n2} & f_{n3}v_{n3} & \dots & f_{n.n-1}v_{n.n-1} & 1 \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} 1 & & & & 0 \\ f_{32}u_{32} & 1 & & & \\ \dots & \dots & & & \\ f_{n2}u_{n2} & f_{n3}u_{n3} & \dots & f_{n.n-1}u_{n.n-1} & 1 \end{matrix} \right\| = V_1''$$

hold. Since $H' \in \mathbf{G}_{\mathrm{diag}(\varphi_1,\dots,\varphi_{n-1})}$ and $V', V_1' \in \mathbf{V}(\mathrm{diag}(\varepsilon_1,\dots,\varepsilon_{n-1}), \mathrm{diag}(\varphi_1,\dots,\varphi_{n-1}))$, by the induction hypothesis, $V' = V_1'$. Analogously, $H'' \in \mathbf{G}_{\mathrm{diag}(\varphi_2,\dots,\varphi_n)}$ and $V'', V_1'' \in \mathbf{V}(\mathrm{diag}(\varepsilon_2,\dots,\varepsilon_n), \mathrm{diag}(\varphi_2,\dots,\varphi_n))$ implies $V'' = V_1''$. It follows that the matrices V, V_1 differ from each other by the entry $(n,1)$ at most. Hence,

$$V_1V^{-1} = \left\| \begin{matrix} 1 & & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & \\ s_{n1} & 0 & \dots & 0 & 1 \end{matrix} \right\|,$$

where $s_{n1} = f_{n1}(u_{n1} - v_{n1})$. Thus,

$$u_{n1} \equiv v_{n1} \left(\mathrm{mod}\, \frac{(\varphi_n, \varepsilon_1)}{\varphi_1} \right).$$

This means that $u_{n1} = v_{n1}$. Consequently, $V = V_1$ and the proof is complete. $\square$

Let $2 \le j_1 < j_2 \dots < j_g \le n$ the set of all indices such that $\varphi_i \ne \varphi_{i-1}$, $i = j_1, j_2, \dots, j_g$.

Theorem 7. *The sets* $\mathbf{V}(\Psi, \Phi)$ *and* $\mathbf{W}(\Psi, \Phi)$ *coincide if and only if any divisor of the element* $\frac{\varphi_i}{\varphi_{i-1}}$ *has a common divisor with the element* $\frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})}$, $i = j_1, j_2, \dots, j_g$.

In order to prove this theorem we establish a series of facts which present interest in their own right.

Lemma 2. *Let S be an $n \times m$ matrix and* $\Phi_i = \mathrm{diag}\left(\frac{\varphi_i}{\varphi_1}, \dots, \frac{\varphi_i}{\varphi_{i-1}}, \underbrace{1, \dots, 1}_{n-i+1} \right)$, $i = 2, \dots, n$. *If* $H \in \mathbf{G}_\Phi$ *then* $\Phi_i HS \overset{l}{\sim} \Phi_i S$, $i = 2, \dots, n$.

Proof. Since the j-th column of the matrix H has the form

$$h_j = \left\| h_{1j} \quad \cdots \quad h_{jj} \quad \frac{\varphi_{j+1}}{\varphi_j} h_{j+1.j} \quad \cdots \quad \frac{\varphi_n}{\varphi_j} h_{nj} \right\|^T, \quad j = 1, \dots, n-1,$$

we obtain

$$\Phi_i h_j = \left\| \frac{\varphi_i}{\varphi_1} h_{1j} \quad \cdots \quad \frac{\varphi_i}{\varphi_{j-1}} h_{j-1.j} \quad \frac{\varphi_i}{\varphi_j} h_{jj} \quad \frac{\varphi_i}{\varphi_j} h_{j+1.j} \quad \cdots \right.$$
$$\left. \cdots \quad \frac{\varphi_i}{\varphi_j} h_{ij} \quad \frac{\varphi_{i+1}}{\varphi_j} h_{i+1.j} \quad \cdots \quad \frac{\varphi_n}{\varphi_j} h_{nj} \right\|^T =$$

$$= \frac{\varphi_i}{\varphi_j} \left\| \frac{\varphi_j}{\varphi_1} h_{1j} \quad \cdots \quad \frac{\varphi_j}{\varphi_{j-1}} h_{j-1.j} \quad h_{jj} \quad \cdots \right.$$
$$\left. \cdots \quad h_{ij} \quad \frac{\varphi_{i+1}}{\varphi_i} h_{i+1.j} \quad \cdots \quad \frac{\varphi_n}{\varphi_i} h_{nj} \right\|^T,$$

$i = 2, \dots, n,\ i > j$. It follows that

$$\Phi_i H = K_i \Phi_i, \tag{8}$$

where the matrix K_i is the quotient of dividing $\Phi_i H$ by Φ_i. Since $\det \Phi_i \neq 0$ and $H \in GL_n(R)$, (8) shows that $K_i \in GL_n(R)$, therefore $\Phi_i HS = K_i \Phi_i S$. Consequently, one has $\Phi_i S \overset{l}{\sim} \Phi_i HS$, $i = 2, \dots, n$. □

Lemma 3. *Let L be an $n \times n$ invertible matrix of form* (4)*. Then*

$$\left(\frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})}, l_{ij}, l_{i+1.j}, \dots, l_{nj} \right) = 1, \quad i = 2, \dots, n, j = i, i+1, \dots, n.$$

Proof. Suppose, contrary to our claim, that

$$\left(\frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})}, l_{ij}, l_{i+1.j}, \dots, l_{nj} \right) = \delta_{ij} \neq 1.$$

Let us consider the submatrix

$$L_{ij} = \left\| \begin{matrix} \frac{\varphi_i}{(\varphi_i, \varepsilon_1)} l_{i1} & \cdots & \frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})} l_{i.i-1} & l_{ij} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\varphi_n}{(\varphi_n, \varepsilon_1)} l_{n1} & \cdots & \frac{\varphi_n}{(\varphi_n, \varepsilon_{i-1})} l_{n.i-1} & l_{nj} \end{matrix} \right\|$$

of the matrix L. Since $\frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})} \Big| \frac{\varphi_k}{(\varphi_k, \varepsilon_s)}$, $k = i, i+1, \dots, n$, $s = 1, \dots, i-1$, we have $\delta_{ij} | L_{ij}$. By Lemma 1, $\delta_{ij} | \det L$. This contradicts to the fact that L is an invertible matrix. □

Lemma 4. *Let S be a lower unitriangular matrix from $\mathbf{L}(\Psi,\Phi)$. Then there is a matrix $H \in \mathbf{G}_\Phi$ such that $HS \in \mathbf{V}(\Psi,\Phi)$.*

Proof. Let S, H_0 be lower unitriangular matrices with elements $\frac{\varphi_i}{(\varphi_i,\varepsilon_j)}s_{ij}$, $\frac{\varphi_i}{\varphi_j}h_{ij}$, respectively, where h_{ij} are parameters, $i=2,\dots,n$, $j=1,\dots,n-1$, $i>j$. If $n=2$, then

$$H_0S=\begin{Vmatrix} 1 & 0 \\ \frac{\varphi_2}{\varphi_1}h_{21} & 1 \end{Vmatrix}\begin{Vmatrix} 1 & 0 \\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)}s_{21} & 1 \end{Vmatrix} = \\ = \begin{Vmatrix} 1 & 0 \\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)}\left(\frac{(\varphi_2,\varepsilon_1)}{\varphi_1}h_{21}+s_{21}\right) & 1 \end{Vmatrix} = S_1.$$

Let $s_{21} \equiv k_{21}\left(\operatorname{mod}\frac{(\varphi_2,\varepsilon_1)}{\varphi_1}\right)$, where $k_{21} \in K\left(\frac{(\varphi_2,\varepsilon_1)}{\varphi_1}\right)$. It follows that $k_{21}=s_{21}+\frac{(\varphi_2,\varepsilon_1)}{\varphi_1}r_{21}$ for some $r_{21}\in R$. Setting $h_{21}=r_{21}$, we obtain $S_1 \in \mathbf{V}(\Psi,\Phi)$.

Suppose that the assumption holds for the matrices of the order $n-1$, we will prove it for n. The matrix H_0S is also a lower unitriangular matrix with the elements d_{ij}, $i>j$. We have

$$d_{nj}=\begin{Vmatrix} \frac{\varphi_n}{\varphi_1}h_{n1} \cdots \frac{\varphi_n}{\varphi_{n-1}}h_{n.n-1} & 1 \end{Vmatrix} \times \\ \times \begin{Vmatrix} \underbrace{0\dots0}_{j-1} & 1 & \frac{\varphi_{j+1}}{(\varphi_{j+1},\varepsilon_j)}s_{j+1.j} \cdots \frac{\varphi_n}{(\varphi_n,\varepsilon_j)}s_{nj} \end{Vmatrix}^T =$$

$$= \frac{\varphi_n}{\varphi_j}h_{nj}+\frac{\varphi_n}{(\varphi_{j+1},\varepsilon_j)}h_{n.j+1}s_{j+1.j}+\cdots \\ +\frac{\varphi_n}{(\varphi_{n-1},\varepsilon_j)}h_{n.n-1}s_{n-1.j}+\frac{\varphi_n}{(\varphi_n,\varepsilon_j)}s_{nj}=$$

$$=\frac{\varphi_n}{(\varphi_n,\varepsilon_j)}\left(\frac{(\varphi_n,\varepsilon_j)}{\varphi_j}h_{nj}+\frac{(\varphi_n,\varepsilon_j)}{(\varphi_{j+1},\varepsilon_j)}h_{n.j+1}s_{j+1.j}+\cdots \right. \\ \left. +\frac{(\varphi_n,\varepsilon_j)}{(\varphi_{n-1},\varepsilon_j)}h_{n.n-1}s_{n-1.j}+s_{nj}\right),$$

$j=1,\dots,n-1$. Let $j=n-1$ and $s_{n.n-1}\equiv k_{n.n-1}\left(\operatorname{mod}\frac{(\varphi_n,\varepsilon_{n-1})}{\varphi_{n-1}}\right)$, where $k_{n.n-1}\in K\left(\frac{(\varphi_n,\varepsilon_{n-1})}{\varphi_{n-1}}\right)$. This gives $k_{n.n-1}=s_{n.n-1}+\frac{(\varphi_n,\varepsilon_{n-1})}{\varphi_{n-1}}r_{n.n-1}$ for some $r_{n.n-1} \in R$. We put $h_{n.n-1} = r_{n.n-1}$. Set $j = n-2$ and

$$s_{n.n-2} + \frac{(\varphi_n, \varepsilon_{n-2})}{(\varphi_{n-1}, \varepsilon_{n-2})} r_{n.n-1} s_{n-1.n-2} \equiv k_{n.n-2} \left(\operatorname{mod} \frac{(\varphi_n, \varepsilon_{n-2})}{\varphi_{n-2}} \right),$$

for some $k_{n.n-2} \in K\left(\frac{(\varphi_n, \varepsilon_{n-2})}{\varphi_{n-2}}\right)$. Then there exists $r_{n.n-2} \in R$ such that

$$k_{n.n-2} = s_{n.n-2} + \frac{(\varphi_n, \varepsilon_{n-2})}{(\varphi_{n-1}, \varepsilon_{n-2})} r_{n.n-1} s_{n-1.n-2} + \frac{(\varphi_n, \varepsilon_{n-2})}{\varphi_{n-2}} r_{n.n-2}.$$

We set $h_{n.n-2} = r_{n.n-2}$. We continue in this fashion obtaining the lower unitriangular matrix

$$H_1 = \left\| \begin{matrix} E_{n-1} & \mathbf{0} \\ h & 1 \end{matrix} \right\|,$$

where E_{n-1} is the identity $(n-1) \times (n-1)$ matrix,

$$h = \left\| \frac{\varphi_n}{\varphi_1} r_{n1} \quad \dots \quad \frac{\varphi_n}{\varphi_{n-1}} r_{n.n-1} \right\|,$$

such that

$$H_1 S = \left\| \begin{matrix} S' & \mathbf{0} \\ g & 1 \end{matrix} \right\|,$$

where S' is a lower unitriangular matrix from $\mathbf{L}(\operatorname{diag}(\varepsilon_1, \dots, \varepsilon_{n-1}), \operatorname{diag}(\varphi_1, \dots, \varphi_{n-1}))$,

$$g = \left\| \frac{\varphi_n}{(\varphi_n, \varepsilon_1)} k_{n1} \quad \dots \quad \frac{\varphi_n}{(\varphi_n, \varepsilon_{n-1})} k_{n.n-1} \right\|,$$

$k_{nj} \in K\left(\frac{(\varphi_n, \varepsilon_j)}{\varphi_j}\right)$, $j = 1, \dots, n-1$. Thus, by the induction hypothesis, there exists $H' \in \mathbf{G}_{\operatorname{diag}(\varphi_1, \dots, \varphi_{n-1})}$ such that $H'S' \in \mathbf{V}(\operatorname{diag}(\varepsilon_1, \dots, \varepsilon_{n-1}), \operatorname{diag}(\varphi_1, \dots, \varphi_{n-1}))$. Hence, $(H' \oplus 1) H_1 S \in \mathbf{V}(\Psi, \Phi)$, and this is precisely the assertion of the Lemma. □

We proceed to the proof of Theorem 7.

Proof. Necessity. Let δ_i be a non-trivial divisor of $\frac{\varphi_i}{\varphi_{i-1}}$, $i_1 \le i \le i_g$. Suppose, contrary to our claim, that $\left(\frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})}, \delta_i\right) = 1$. Then there exist $u, v \in R$ such that $u \frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})} + v\delta_i = 1$. Consider the matrix

$$L_i = E_{i-2} \oplus \left\| \begin{matrix} v & -u \\ \frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})} & \delta_i \end{matrix} \right\| \oplus E_{n-i}.$$

It is obvious that $L_i \in \mathbf{L}(\Psi,\Phi)$. Denote by S_i the matrix consisting of last $n-i+1$ columns of the matrix L_i. It is easy to check that

$$\operatorname{diag}\left(\frac{\varphi_i}{\varphi_1},\cdots,\frac{\varphi_i}{\varphi_{i-1}},1,\ldots,1\right)S_i = \Phi_i S_i \overset{l}{\sim} \left\|\begin{matrix}\mathbf{0}\\ \delta_i\oplus E_{n-i}\end{matrix}\right\|.$$

On the other hand, if $M \in \mathbf{V}(\Psi,\Phi)$, then

$$\Phi_i M_i \overset{l}{\sim} \left\|\begin{matrix}\mathbf{0}\\ E_{n-i+1}\end{matrix}\right\|,$$

where M_i is a matrix, which consists of the last $n-i+1$ columns of a matrix M. We conclude from Lemma 2 that $\mathbf{V}(\Psi,\Phi)$ does not contain the representative of the conjugate class $\mathbf{G}_\Phi L_i$. This contradicts the fact that $\mathbf{V}(\Psi,\Phi)=\mathbf{W}(\Psi,\Phi)$.

Sufficiency. Let $\left(\frac{\varphi_i}{(\varphi_i,\varepsilon_{i-1})}b,d\right)=1$ and $\left(\frac{\varphi_i}{\varphi_{i-1}}b,d\right)=\alpha_i$ for same $b,d\in R$, $2\le i\le n$. Since $\alpha_i|\frac{\varphi_i}{\varphi_{i-1}}$ and $\alpha_i|d$, by the assertion assumption of the theorem, $\left(\frac{\varphi_i}{(\varphi_i,\varepsilon_{i-1})}b,d\right)\neq 1$, a contradiction. Hence, the equality $\left(\frac{\varphi_i}{(\varphi_i,\varepsilon_{i-1})}b,d\right)=1$ implies that $\left(\frac{\varphi_i}{\varphi_{i-1}}b,d\right)=1$, $i=2,\ldots,n$.

Let L be an invertible matrix of form (4). In order to prove this statement we only need to show that there exist a matrix $H\in\mathbf{G}_\Phi$ such that $HL\in\mathbf{V}(\Psi,\Phi)$. The proof will be divided into 2 steps. At the first step we will find a matrix $H_1\in\mathbf{G}_\Phi$ such that H_1L has a lower unitriangular form.

In the case $n=2$ the invertibility of the matrix

$$L=\left\|\begin{matrix} l_{11} & l_{12}\\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)}l_{21} & l_{22}\end{matrix}\right\|$$

implies that

$$\left(\frac{\varphi_2}{(\varphi_2,\varepsilon_1)}l_{12},l_{22}\right)=1.$$

By the above reasoning

$$\left(\frac{\varphi_2}{\varphi_1}l_{12},l_{22}\right)=1.$$

Therefore,

$$\frac{\varphi_2}{\varphi_1}l_{12}u_1+l_{22}u_2=1$$

for some $u_1,u_2\in R$. Clearly, we have

$$\left\|\begin{matrix} l_{22} & -l_{12}\\ \frac{\varphi_2}{\varphi_1}u_1 & u_2\end{matrix}\right\|\left\|\begin{matrix} l_{11} & l_{12}\\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)}l_{21} & l_{22}\end{matrix}\right\|=\left\|\begin{matrix} e & 0\\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)}s_{21} & 1\end{matrix}\right\|,$$

where $e \in U(R)$. Hence

$$H_1 = \left\| \begin{matrix} e^{-1}l_{22} & -e^{-1}l_{21} \\ \frac{\varphi_2}{\varphi_1}u_1 & u_2 \end{matrix} \right\|$$

is the desired matrix.

Suppose that the assumption holds for the matrices of order $n-1$, we will prove it for n. Step by step, using Lemma 3 and the result which has been proved above, we obtain

$$\begin{aligned} &\left(\frac{\varphi_n}{\varphi_1}l_{1n}, \cdots, \frac{\varphi_n}{\varphi_{n-1}}l_{n-1.n}, l_{nn}\right) = \\ &= \left(\frac{\varphi_n}{\varphi_{n-1}}\left(\frac{\varphi_{n-1}}{\varphi_1}l_{1n}, \cdots, \frac{\varphi_{n-1}}{\varphi_{n-2}}l_{n-2.n}, l_{n-1.n}\right), l_{nn}\right) = \\ &= \left(\frac{\varphi_{n-1}}{\varphi_1}l_{1n}, \cdots, \frac{\varphi_{n-1}}{\varphi_{n-2}}l_{n-2.n}, l_{n-1.n}, l_{nn}\right) = \\ &= \left(\frac{\varphi_{n-1}}{\varphi_{n-2}}\left(\frac{\varphi_{n-2}}{\varphi_1}l_{1n}, \cdots, l_{n-2.n}\right), (l_{n-1.n},\ l_{nn})\right) = \\ &= \ldots = \left(\frac{\varphi_2}{\varphi_1}l_{1n}, (l_{2n}, \ldots, l_{nn})\right) = (l_{1n}, \ldots, l_{nn}) = 1. \end{aligned}$$

Then there exist $u_1, \ldots, u_n$ such that

$$\frac{\varphi_n}{\varphi_1}l_{1n}u_1 + \ldots + \frac{\varphi_n}{\varphi_{n-1}}l_{n-1.n}u_{n-1} + l_{nn}u_n = 1.$$

This implies that

$$\left(\frac{\varphi_n}{\varphi_1}u_1, \cdots, \frac{\varphi_n}{\varphi_{n-1}}u_{n-1}, u_n\right) = 1.$$

It is known (see, for example [18] p.13) that there exists an invertible matrix of the form

$$H_0 = \left\| \begin{matrix} u_{11} & u_{12} & \ldots & u_{1.n-1} & u_{1n} \\ 0 & u_{22} & \ldots & u_{2.n-1} & u_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & u_{n-1.n-1} & u_{n-1.n} \\ \frac{\varphi_n}{\varphi_1}u_1 & \frac{\varphi_n}{\varphi_2}u_2 & \ldots & \frac{\varphi_n}{\varphi_{n-1}}u_{n-1} & u_n \end{matrix} \right\| \in \mathbf{G}_\Phi.$$

Let $H_0L = \| \ t_{ij} \ \|_1^n$. Then $t_{nn} = 1$. So

$$\left\| \begin{matrix} 1 & 0 & \ldots & 0 & -t_{1n} \\ & \ddots & & & \vdots \\ 0 & & & 1 & -t_{n-1.n} \\ 0 & 0 & \ldots & 0 & 1 \end{matrix} \right\| H_0L = H_0'H_0L = \left\| \begin{matrix} L' & \mathbf{0} \\ g & 1 \end{matrix} \right\|.$$

Since $H_0' H_0 \in \mathbf{G}_\Phi$, the above equality shows that $H_0' H_0 L \in \mathbf{L}(\Psi, \Phi)$, by Property 2. Therefore
$L' \in \mathbf{L}(\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{n-1}), \operatorname{diag}(\varphi_1, \ldots, \varphi_{n-1}))$. By the induction hypothesis, there exists a matrix $H' \in \mathbf{G}_{\operatorname{diag}(\varphi_1,\ldots,\varphi_{n-1})}$ such that the matrix $H'L'$ has lower unitriangular form. Hence the matrix $H_1 = (H' \oplus 1) H_0' H_0$ is the desired matrix, i.e., $H_1 L \in \mathbf{L}(\Psi, \Phi)$ and has a lower unitriangular form.

By Lemma 4, $H_2 H_1 L \in \mathbf{V}(\Psi, \Phi)$ for some matrix $H_2 \in \mathbf{G}_\Phi$. The proof is complete. □

Combining Theorem 2 and 7 we obtain.

Theorem 8. *The set $(\mathbf{V}(\Psi, \Phi) P_A)^{-1} \Phi$ consists of all left up to right associate divisors of the matrix A which have c.d.f. Φ if and only if any divisor of element $\frac{\varphi_i}{\varphi_{i-1}}$ has common divisor with the element $\frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})}$, $i = j_1, j_2, \ldots, j_g$.* □

Let now R be a principal ideal ring and $A = P_A^{-1} \Psi Q_A^{-1}$, where $\Psi = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, is a nonsingular matrix over R. Let $\Phi = \operatorname{diag}(\varphi_1, \ldots, \varphi_n) | \Psi$, $\varphi_i | \varphi_{i+1}$, $i = 1, \ldots, n-1$. Decompose the elements $\frac{\varphi_i}{\varphi_{i-1}}$, $\frac{\varepsilon_{i-1}}{\varphi_{i-1}}$, $i = j_1, j_2, \ldots, j_g$, into product of the irreducible factors:

$$\frac{\varphi_i}{\varphi_{i-1}} = g_{i1}^{k_{i1}} \cdots g_{il}^{k_{il}}, \; \frac{\varepsilon_{i-1}}{\varphi_{i-1}} = g_{i1}^{q_{i1}} \cdots g_{il}^{q_{il}} h_{i1}^{p_{i1}} \cdots h_{ir}^{p_{ir}}. \tag{9}$$

Theorem 9. *The set $(\mathbf{V}(\Psi, \Phi) P_A)^{-1} \Phi$ consists of all left up to right associate divisors of the matrix A which have c.d.f. Φ if and only if $k_{ij} > q_{ij}$, $i = j_1, j_2, \ldots, j_g$, $j = 1, \ldots, l$.*

Proof. Observing that

$$\frac{\varphi_i}{(\varphi_i, \varepsilon_{i-1})} = \frac{\varphi_i/\varphi_{i-1}}{\left(\varphi_i/\varphi_{i-1}, \varepsilon_{i-1}/\varphi_{i-1}\right)}, \quad i = j_1, j_2, \ldots, j_g,$$

and having Theorem 8 we prove this assertion. □

3. Application

Let us apply these results to factorization of a matrix polynomials and solution of unilateral matrix equations. Consider a nonsingular matrix polynomial

$$A(x) = A_s x^s + A_{s-1} x^{s-1} + \cdots + A_0,$$

$A_i \in M_n(P)$, $i = 0, \dots, s$, P is a field. We recall that a matrix polynomial $A(x)$ is monic if $A_s = E$. A matrix polynomial $A(x)$ is right-regularizable if there exists invertible matrix $U(x)$ such that $A(x)U(x)$ is a monic polynomial. Necessary and sufficient conditions for right-regularization of the matrix polynomial $A(x)$ are proposed in [2,5,20]. Since $A(x)$ is the matrix over elementary divisor ring $R = P[x]$, there exist $P_A(x), Q_A(x)$ such that

$$P_A(x)A(x)Q_A(x) = \operatorname{diag}(\varepsilon_1(x), \dots, \varepsilon_n(x)) = \Psi(x).$$

Let $\Phi(x) = \operatorname{diag}(\varphi_1(x), \dots, \varphi_n(x)), \varphi_i(x)|\varphi_{i+1}(x), i = 1, \dots, n-1$, be a divisor of the matrix $\Psi(x)$. Write the polynomials $\frac{\varphi_i(x)}{\varphi_{i-1}(x)}$, $\frac{\varepsilon_{i-1}(x)}{\varphi_{i-1}(x)}$, $i = 2, \dots, n$, in form (9). In order to describe divisors of the matrix $A(x)$ with c.d.f. $\Phi(x)$ we can use Theorem 9. To describe monic divisors of the matrix polynomial $A(x)$ we employ the following result.

Theorem 10. *Let deg* $\det \Phi(x) = nr$. *All left monic divisors of degree r of the matrix polynomial $A(x)$ with c.d.f. $\Phi(x)$ can be obtained by right-regularization of matrices from $(\mathbf{V}(\Psi(x), \Phi(x))P_A(x))^{-1}\Phi(x)$ if and only if $k_{ij} > q_{ij}$, $i = j_1, j_2, \dots, j_g$, $j = 1, \dots, l$.* □

Consider the matrix polynomial equation

$$X^s A_s + X^{s-1}A_{s-1} + \dots + A_0 = \mathbf{0}. \tag{10}$$

It is well known that the matrix B is the root of equation (10) if and only if the matrix polynomial $Ex - B$ is the left divisor of the corresponding matrix polynomial $A(x) = A_s x^s + A_{s-1}x^{s-1} + \dots + A_0$. Therefore, we can apply Theorem 10 to the solutions of unilateral matrix equation (10).

Consider some examples that illustrate the presented factorization theory.

Example 1. *Let* $R = \{a + b_1 x + b_2 x^2 + \dots \mid \quad a \in \mathbb{Z}, \quad b_i \in \mathbb{Q}, \quad i \in \mathbb{N}\}$ [21]. *Let us find all left up to right associate divisors of the matrix* $A = \operatorname{diag}(5, 5x^3) = \Psi$ *which have c.d.f.* $\Phi = \operatorname{diag}(1, 5x)$.

It is easy to check that any divisor of the element $\frac{\varphi_2}{\varphi_1} = 5x$ has common divisor with the element $\frac{\varphi_2}{(\varphi_2, \varepsilon_1)} = \frac{5x}{(5x,5)} = x$. Thus the set of desire divisors has the form

$$\left\{ \left\| \begin{matrix} 1 & 0 \\ xk & 1 \end{matrix} \right\|^{-1} \Phi \right\} = \left\{ \left\| \begin{matrix} 1 & 0 \\ -xk & 5x \end{matrix} \right\| \right\},$$

where $k \in K\left(\frac{(\varphi_2, \varepsilon_1)}{\varphi_1}\right) = K(5) = \{0, 1, 2, 3, 4\}$.

Example 2. *Let us solve the equation*

$$X^2 = \mathbf{0}, \tag{11}$$

where X is a 3×3 matrix.

The matrix polynomial $Ex^2=\Psi(x)$ corresponds to this matrix equation. The possible c.d.f. of the left monic divisors of Ex^2 are the matrices

$$\Phi_1(x)=Ex,\Phi_2(x)=\mathrm{diag}(1,x,x^2).$$

By Theorem 6 the matrix $\Phi_1(x)=Ex$ determines the unique solution $X_1=\mathbf{0}$. Consider the matrix

$$V(x)=\begin{Vmatrix} 1 & 0 & 0\\ a & 1 & 0\\ bx+c & d & 1\end{Vmatrix}\in\mathbf{V}(\Psi(x),\Phi_2(x)),$$

where $a,b,c,d\in P$. By Proposition 4 and Theorem 2 we conclude that $\mathbf{V}(\Psi(x),\Phi_2(x))^{-1}\Phi_2(x)$ is the set of left divisors of the matrix Ex^2. Using the results of papers [2,5,20] we can find the invertible matrix $U(x)$ such that

$$\mathbf{V}(\Psi(x),\Phi_2(x))^{-1}\Phi_2(x)U(x)=$$
$$=Ex-b^{-1}\begin{Vmatrix} c & d & 1\\ -ac & -ad & -a\\ (da-c)c & (da-c)d & da-c\end{Vmatrix}.$$

It follows that

$$\mathbf{X_2}=\left\{f\begin{Vmatrix} c & d & 1\\ -ac & -ad & -a\\ (da-c)c & (da-c)d & da-c\end{Vmatrix}\right\},$$

where $a,c,d\in P, f\in P^*$, is the set of solutions of equation (11). Since

$$\frac{\varphi_3(x)}{\varphi_2(x)}=x,\frac{\varepsilon_2(x)}{\varphi_2(x)}=x,$$

the conditions of Theorem 10 do not hold. This means that the set $\mathbf{V}(\Psi(x),\Phi_2(x))^{-1}\Phi_2(x)$ does not contain all left up to right associate divisors of the matrix Ex^2 with c.d.f. $\Phi_2(x)$.

Using the results of paper [22] we get that the set of all solutions of equation (11) consists of

$$X_1,\mathbf{X_2},\mathbf{X_3}=\left\{f\begin{Vmatrix} -c & -1 & 0\\ c^2 & c & 0\\ ac & a & 0\end{Vmatrix}\right\},\mathbf{X_4}=\left\{f\begin{Vmatrix} 0 & 0 & 0\\ -d & -c & -1\\ cd & c^2 & c\end{Vmatrix}\right\},$$

$$\mathbf{X_5}=\left\{\begin{Vmatrix} 0 & 0 & 0\\ g & 0 & 0\\ a & 0 & 0\end{Vmatrix}\right\},\mathbf{X_6}=\left\{\begin{Vmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ s & t & 0\end{Vmatrix}\right\},$$

where $a, c, d \in P, f, g \in P^*, s, t$ are not equal to zero simultaneously.

Remark. Let P be a commutative ring. It is easy to check that the matrices from $X_1, \mathbf{X_2}, \ldots, \mathbf{X_6}$ are also solutions of equation (11). Moreover, if f is the element of the center of a non commutative ring P, then the matrices from these sets are solutions of equation (11) as well.

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