Factorization of Operator Valued Entire Functions

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1. Introduction. Let w be a complex valued entire function of exponential type τ that is not identically zero. Exponential type is defined as in Boas [2]. If w has non-negative values on the real axis and

$$(1) \qquad J(w) = \int_{-\infty}^{\infty} (1 + x^2)^{-1} \log^+ w(x)\, dx < \infty,$$

then ([2, p. 125], [3, p. 34], [6, p. 437])

$$w = f^*f$$

where f is an entire function of exponential type $\frac{1}{2}\tau$ that has no zeros in the upper half-plane $y > 0$. For any entire function f, f^* is defined by $f^*(z) = \bar{f}(\bar{z})$ for all complex z. In this paper we prove an analogous theorem for operator valued entire functions. We deduce as a corollary a factorization theorem for polynomials with operator coefficients which have non-negative values on the real axis.

A novel aspect of our work is the use of abstract shift and Toeplitz operator theory to prove an abstract Hilbert space theorem (Theorem 3) that is the key to the factorization theory. This is done in Section 3, and the main theorem is there proved under the assumption that the given function is bounded on the real axis. Section 2 is devoted to statements of our main results. In Section 4 we review the function theory that is needed elsewhere in the paper. In Section 5 we prove the main theorems. A strengthened form of the factorization theorem is derived in Section 6.

Our results were announced in [10]. For clarity of exposition we have chosen to limit our main discussion (Sections 2–5) to an essential case of the announced factorization theorem, leaving the general case to Section 6. Our treatment does not follow a direct path to the factorization theorem. Rather, we have intended to emphasize the algebraic structure underlying the proof.

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2. Definitions and main results. Throughout the paper $\mathcal{C}$ will denote a separable complex Hilbert space. We use I to denote the identity operator on $\mathcal{C}$ as well as on any other Hilbert space. If c is a vector in $\mathcal{C}$ and A is a bounded operator on $\mathcal{C}$, then we write $|c|$, $|A|$, and $\bar{A}$ for the norm of c, the norm of A, and the adjoint of A respectively. When we speak of a vector or operator valued function we mean a function whose values are vectors in $\mathcal{C}$ or bounded operators on $\mathcal{C}$ respectively. Definitions of measurability and analyticity are taken in the weak sense. For any operator valued function A on the complex plane, A^* is the function defined by $A^*(z) = \bar{A}(\bar{z})$ for all complex z. The argument of a function is usually suppressed, but not always. For example, if W and A are operator valued entire functions, we will write "$W = A^*A$" or "$W(z) = A^*(z)A(z)$", whichever is more convenient.

We write $L^2_{\mathcal{C}} = L^2_{\mathcal{C}}(-\infty, \infty)$ for the Hilbert space of measurable vector valued functions f on $(-\infty, \infty)$ such that

$$||f||^2 = \int_{-\infty}^{+\infty} |f(x)|^2 \, dx < \infty.$$

$L^2(a, b)$ is similarly defined for any finite or infinite subinterval (a, b) of $(-\infty, \infty)$. We indulge in the usual functional abuses regarding equivalence classes under the a.e. equivalence relation. There is a unique unitary transformation $\mathfrak{F}$ on $L^2_{\mathcal{C}}$ to $L^2_{\mathcal{C}}$ that assigns to each $\hat{f}$ in $L^2_{\mathcal{C}}$ an a.e. unique f in $L^2_{\mathcal{C}}$ in such a way that for every vector c in $\mathcal{C}$, $\langle f(x), c\rangle_{\mathcal{C}}$ is the Fourier-Plancherel transform of $\langle \hat{f}(x), c\rangle_{\mathcal{C}}$. When $\hat{f}$ and f are related in this way we write

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{ixt} \hat{f}(t) \, dt.$$

The Hardy space $H^2_{\mathcal{C}}$ is the set of functions f in $L^2_{\mathcal{C}}$ of the form

$$f(x) = (2\pi)^{-1/2} \int_{0}^{\infty} e^{ixt} \hat{f}(t) \, dt,$$

where $\hat{f} \in L^2_{\mathcal{C}}(0, \infty)$. So $H^2_{\mathcal{C}} = \mathfrak{F}L^2_{\mathcal{C}}(0, \infty)$.

If $\mathcal{B}$ is a closed subspace of $\mathcal{C}$ we write $L^2_{\mathcal{B}}$ and $H^2_{\mathcal{B}}$ for the closed subspaces of $L^2_{\mathcal{C}}$ and $H^2_{\mathcal{C}}$ of functions with values in $\mathcal{B}$. In case $\mathcal{C}$ is the one-dimensional space of complex numbers we write simply L^2 and H^2 for the above spaces.

A complex valued function $a(x)$ on $(-\infty, \infty)$ is *outer* if it is a.e. equal to the boundary value function of a function $a(z)$ defined for $y > 0$ by

$$a(z) = C \exp\left(\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{1 + tz}{t - z} \frac{\log k(t)}{1 + t^2} \, dt\right),$$

where C is a constant of absolute value one, $k(t) \geq 0$ on $(-\infty, \infty)$, and $(1 + t^2)^{-1} \log k(t) \in L^1(-\infty, \infty)$. In this case $|a(x)| = k(x)$ a.e. To motivate our extension of this concept to operator valued functions we recall two results about scalar valued outer functions. First, a bounded measurable scalar valued function a on $(-\infty, \infty)$ is outer if and only if multiplication by a in H^2 is an everywhere defined linear transformation with range dense in H^2. Second, a measurable scalar valued function a on $(-\infty, \infty)$ is outer if and only if there exists a bounded outer function f such that $f(x)a(x)$ is bounded and outer.

Now let A be a bounded measurable operator valued function on $(-\infty, \infty)$. A is said to be *outer* if left multiplication by A in $H^2_{\mathfrak{C}}$ is an everywhere defined linear transformation with range dense in a subspace of the form $H^2_{\mathfrak{B}}$, where $\mathfrak{B}$ is a closed subspace of $\mathfrak{C}$. An arbitrary measurable operator valued function A on $(-\infty, \infty)$ is *outer* if there exists a bounded complex valued outer function f such that $f(x)A(x)$ is bounded and outer in the sense just described. One easily checks that if g is a second bounded complex valued outer function such that $g(x)A(x)$ is bounded, then the closure of the range of multiplication by $g(x)A(x)$ in $H^2_{\mathfrak{C}}$ is again $H^2_{\mathfrak{B}}$. Thus $\mathfrak{B}$ is independent of the choice of f, and we say that A *acts in* $\mathfrak{B}$.

If A is a scalar or operator valued entire function, we say that A is *outer* if its restriction to the real axis is outer in the above sense.

The factorizations we will be considering, when viewed on the real axis, are of the form $W(x) = \tilde{A}(x)A(x)$ where A is outer. The following lemma shows that A is determined up to a constant partially isometric prefactor.

Lemma 1. *For each $j = 1, 2$ let A_j be an operator valued outer function which acts in $\mathfrak{B}_j$. In order that*

$$\tilde{A}_1(x)A_1(x) = \tilde{A}_2(x)A_2(x)$$

a.e. it is necessary and sufficient that $A_2(x) = UA_1(x)$ a.e., where U is a partially isometric operator on $\mathfrak{C}$ with initial set $\mathfrak{B}_1$ and final set $\mathfrak{B}_2$.

The proof is given in Section 5.

Let w be a complex valued entire function of exponential type τ that does not vanish identically. We say that w is *factorable* if $w = a^*a$ where a is an outer entire function. Clearly a is determined to within a multiplicative constant of absolute value one.

It develops that this view of the scalar factorization problem is equivalent to the classical versions. See the references cited in the introduction. The following theorem is proved in Section 4.

Theorem 1. *Let w be a complex valued entire function of exponential type τ that does not vanish identically. Assume that w has non-negative values on the real axis. Then w is factorable if and only if $J(w) < \infty$. If $w = a^*a$ where a is an outer entire function, then* $\exp\left(-\tfrac{1}{2}i\tau z\right)a(z)$ *is of exponential type $\tfrac{1}{2}\tau$ and has no zeros in the upper half-plane.*

We proceed to the operator factorization problem. A vector valued entire function f and an operator valued entire function W are said to be of exponential type τ, $\tau \geqq 0$, if for each vector c in $\mathfrak{C}$ the scalar valued entire functions $\langle f(z), c\rangle_{\mathfrak{C}}$ and $\langle W(z)c, c\rangle_{\mathfrak{C}}$ are of exponential type τ. We note that a different definition of exponential type for operator valued entire functions was used in our announcement [10]. The difference is reconciled in Section 6.

An operator valued entire function W of exponential type τ is called *factorable* if $W = A^*A$ where A is an outer operator valued entire function. It follows from Lemma 1 that A is unique up to a constant partially isometric prefactor in this case.

Our main result can now be stated.

Theorem 2. *Suppose W is an operator valued entire function of exponential type τ, and w is a scalar valued entire function of exponential type, not identically zero, such that*

$$0 \leqq W(x) \leqq w(x)I$$

for all real x. If w is factorable, then so is W. If $W = A^*A$ where A is outer, then $\exp(-\frac{1}{2}i\tau z)A(z)$ *is of exponential type $\frac{1}{2}\tau$.*

In Theorem 2 "$\leqq$" refers to the partial ordering of self-adjoint operators on $\mathfrak{C}$.

We believe Theorem 2 is rather remarkable since under the assumption of a weak dominant from above one infers a factorization. This is in contradistinction to the general factorization theorems given, for example, in Douglas [4] and Helson [5, Lecture XI]. In the factorization theorems of Devinatz and Lowdenslager a necessary hypothesis is that the function is not "too close" to zero. Our situation is closer to that of the operator Fejér–Riesz Factorization Theorem as presented in [9], and the problem succumbs to a similar type of analysis.

Theorem 2 and its corollaries are proved in Sections 3 and 5.

Corollary 1. *Suppose $W = [w_{jk}]$ is an $n \times n$ matrix of entire functions of exponential type τ. Assume that $W(x) \geqq 0$ for all real x and*

$$\int_{-\infty}^{+\infty} (1 + x^2)^{-1} \log^+ (tr\, W(x))\, dx < \infty .$$

Then W has a factorization of the form $W = A^*A$ where $A = [a_{jk}]$ is an $n \times n$ matrix of entire functions such that $\exp(-\frac{1}{2}i\tau z)a_{jk}(z)$ *is of exponential type $\frac{1}{2}\tau$ for each $j, k = 1, \cdots, n$. If* $\det W(z)$ *is not identically zero the factorization may be chosen such that* $\det A(z)$ *has no zeros in the open upper half-plane.*

Corollary 2. *Let N be a non-negative integer, and let*

$$W(x) = \sum_{j=0}^{2N} C_j x^j$$

be a polynomial of degree $2N$ with operator coefficients that has non-negative values on $(-\infty, \infty)$. Then $W(x) = \bar{A}(x)A(x)$ where

$$A(x) = \sum_{j=0}^{N} A_j x^j$$

is a polynomial of degree N with operator coefficients such that A is outer.

In the case where $\dim \mathcal{C} < \infty$ Professors Kevin McCrimmon and Eugene Paige have shown that Corollary 2 may be deduced by algebraic techniques. The argument (which we do not reproduce) is based on Kronecker's reduction process.

3. An abstract factorization theorem. We consider now operators on a complex Hilbert space $\mathcal{H}$ of arbitrary non-finite dimension. By a shift operator S on $\mathcal{H}$ we mean an isometry such that $S^{*n} \to 0$ strongly as $n \to \infty$. Fix a shift S on $\mathcal{H}$. A bounded operator T on $\mathcal{H}$ is called *S-Toeplitz* if $S^*TS = T$, and *S-analytic* if $ST = TS$. If T is S-analytic and the closure of the range of T reduces S, then T is called *S-outer*. See Lemma 5 for the connection between S-outer operators on $\mathcal{H}$ and outer functions. Analogous definitions are made with respect to any shift operator.

We refer to B. Sz.-Nagy and C. Foiaş [11, pp. 130–142] for the fundamental properties of continuous semi-groups of contraction operators and, in particular, continuous semi-groups of shift operators. The shift S is the cogenerator of a continuous semi-group $\{V^t\}_{t \geq 0}$ of shift operators given by

$$V^t = \exp[-t(I + S)(I - S)^{-1}], \ t \geqq 0, \tag{2}$$

where, by definition, the right side of (2) is the strong limit as r increases to 1 of

$$\exp[-t(I + rS)(I - rS)^{-1}].$$

The limit exists by the Nagy–Foiaş functional calculus. We find by direct calculation that for each r, $0 < r < 1$,

$$rS = I - 2 \int_0^\infty e^{-t} \exp[-t(I + rS)(I - rS)^{-1}] \, dt,$$

where the integral is taken in the weak sense. Letting r increase to 1 we obtain

$$S = I - 2 \int_0^\infty e^{-t} V^t \, dt, \tag{3}$$

an identity also employed by Masani [8]. In fact, (3) is a special case of a more general formula for the cogenerator of a continuous semi-group of contraction operators. The integral in (3) can also be taken as $\int_0^\infty = \lim_{a \to \infty} \int_0^a$ in the uniform topology, where for each $a > 0$ the finite integral $\int_0^a$ is a norm limit of Riemann sums.

Lemma 2. Let T be a bounded operator on $\mathcal{H}$. Then T is S-Toeplitz, S-analytic, or S-outer if and only if for each $t > 0$, T is V^t-Toeplitz, V^t-analytic, or V^t-outer respectively.

Proof. For the analytic and outer cases the lemma follows directly from (2) and (3).

Suppose T is S-Toeplitz and $T \geqq I$. By [9, p. 144], T has a factorization of the form $T = A^*A$ where A is an S-analytic operator on $\mathcal{H}$. But A is then V^t-

analytic, and so T is V^t-Toeplitz for every $t > 0$. Any selfadjoint S-Toeplitz operator is a multiple of the identity operator I plus an S-Toeplitz operator T_0 such that $T_0 \geqq I$, and therefore every selfadjoint S-Toeplitz operator is V^t-Toeplitz for every $t > 0$. Finally, any S-Toeplitz operator is a linear combination of two selfadjoint ones and hence is V^t-Toeplitz for each $t > 0$.

Conversely suppose T is V^t-Toeplitz for each $t > 0$. By (3),

$$S^*TS - T = 4\int_0^\infty \int_0^\infty e^{-s-t} V^{t*}TV^s \, ds \, dt - 2\int_0^\infty e^{-s}TV^s \, ds - 2\int_0^\infty e^{-t}V^{t*}T \, dt.$$

Since $V^{t*}TV^t = T, t > 0$, the first term on the right is

$$4\int_0^\infty \int_s^\infty e^{-s-t} V^{(t-s)*}T \, dt \, ds + 4\int_0^\infty \int_t^\infty e^{-s-t}TV^{s-t} \, ds \, dt$$
$$= 2\int_0^\infty e^{-t}V^{t*}T \, dt + 2\int_0^\infty e^{-s}TV^s \, ds.$$

Therefore $S^*TS = T$ and T is S-Toeplitz. The lemma follows.

Now let T be a bounded non-negative S-Toeplitz operator on $\mathcal{H}$. Let $\mathcal{R}$ be the closure of the range of T, considered as an inner product space with inner product

$$(f, g)_T = \langle Tf, g \rangle, \quad f, g \in \mathcal{R}.$$

Let $\mathcal{H}_T$ denote a completion of $\mathcal{R}$. Thus $\mathcal{H}_T$ is a Hilbert space with inner product $(\cdot\,, \cdot)_T$ which contains $\mathcal{R}$ as a dense linear subspace. The fact that T is S-Toeplitz implies [9, p. 142] that there exists a unique isometry $\mathbf{S}$ in $\mathcal{H}_T$ such that

$$(\mathbf{S}f, g)_T = \langle TSf, g \rangle$$

for each f, g in $\mathcal{R}$. But T is also V^t-Toeplitz for each $t > 0$ by Lemma 2. Therefore for each $t > 0$ there is a unique isometry $\mathbf{V}^t$ in $\mathcal{H}_T$ such that

$$(\mathbf{V}^t f, g)_T = \langle TV^t f, g \rangle$$

for f, g in $\mathcal{R}$. We define $\mathbf{V}^0$ to be the identity operator on $\mathcal{H}_T$.

Lemma 3. *The family* $\{\mathbf{V}^t\}_{t \geq 0}$ *is a continuous semi-group in* $\mathcal{H}_T$ *with cogenerator* $\mathbf{S}$. $\mathbf{S}$ *is a shift operator if and only if* $\mathbf{V}^{t_0}$ *is a shift operator for some (and hence any)* $t_0 > 0$.

Proof. Let Q be the projection operator in $\mathcal{H}$ with range $\mathcal{R}$. For any $t \geqq 0$, $\mathbf{V}^t \,|\mathcal{R} = QV^t|\,\mathcal{R}$. We have that $\ker T = \ker T^{1/2}$, and

$$\|T^{1/2}V^t f\|^2 = \langle V^{t*}TV^t f, f \rangle = \langle Tf, f \rangle = \|T^{1/2}f\|^2$$

for each f in $\mathcal{R}$. Hence $\ker T$ is invariant under V^t and $QV^t = QV^tQ$. Therefore if f is in $\mathcal{R}$, then

$$\mathbf{V}^t\mathbf{V}^s f = QV^tQV^s f = QV^{t+s} f = \mathbf{V}^{t+s} f.$$

Since $\mathfrak{R}$ is a dense subspace of $\mathcal{H}_T$, $\{\mathbf{V}^t\}_{t \geq 0}$ is a semi-group of isometries in $\mathcal{H}_T$.

If $f \in \mathfrak{R}$, then

$$\|(\mathbf{V}^t - I)f\|_T = \|T^{1/2}(V^t - I)f\| \leqq \|T^{1/2}\| \, \|(V^t - I)f\|.$$

Therefore $V^t f \to f$ in the metric of $\mathcal{H}_T$ as t decreases to 0 for f in a dense subspace of $\mathcal{H}_T$. It follows that the semi-group is continuous.

For any f, g in $\mathfrak{R}$,

$$(f, g)_T - 2 \int_0^\infty e^{-t}(\mathbf{V}^t f, g)_T \, dt = \langle f, Tg \rangle - 2 \int_0^\infty e^{-t} \langle V^t f, Tg \rangle \, dt,$$

which by (3) is $\langle TSf, g \rangle = (\mathbf{S}f, g)_T$. Therefore

$$\mathbf{S} = I - 2 \int_0^\infty e^{-t} \mathbf{V}^t \, dt,$$

and by a previous remark $\mathbf{S}$ is the cogenerator of $\{\mathbf{V}^t\}_{t \geq 0}$.

Finally, for any $t_0 > 0$ we have

$$\bigcap_{n=0}^\infty \mathbf{S}^n \mathcal{H}_T = \bigcap_{t \geq 0} \mathbf{V}^t \mathcal{H}_T = \bigcap_{n=0}^\infty \mathbf{V}^{nt_0} \mathcal{H}_T$$

by [11, p. 141] or [8, p. 626]. It follows from the Wold decomposition [11, p. 3] that S is a shift operator if and only if $\mathbf{V}^{t_0}$ is one.

In the spirit of [9] we say that a bounded operator T on $\mathcal{H}$ is *S-entire of type* (τ_1, τ_2) if T is S-Toeplitz and τ_1, τ_2 are non-negative numbers such that TV^{τ_1} and $T^*V^{\tau_2}$ are S-analytic. By Lemma 2, an S-Toeplitz operator T is S-entire of type (τ_1, τ_2) if and only if

$$TV^{t+\tau_1} = V^t TV^{\tau_1} \quad \text{and} \quad T^*V^{t+\tau_2} = V^t T^* V^{\tau_2} \tag{4}$$

for all $t > 0$. Justification for the terminology will come later (Theorem 4). The following lemma lists some immediate consequences of the definition.

Lemma 4. *(i) If T is S-entire of type (τ_1, τ_2) and F is S-entire of type $(0, \sigma)$, then TF and F^*T are S-entire of types $(\tau_1, \tau_2 + \sigma)$ and $(\tau_1 + \sigma, \tau_2)$ respectively.*

(ii) If T is S-entire of type $(0, \tau)$, then $V^{\tau/2*}T$ is S-entire of type $(\frac{1}{2}\tau, \frac{1}{2}\tau)$.

The abstract factorization theorem now follows quickly from previous work of the first author [9].

Theorem 3. *Let T be an S-entire operator on $\mathcal{H}$ of type (τ, τ). If $T \geqq 0$ then $T = F^*F$ where F is an S-outer operator on $\mathcal{H}$ that is S-entire of type $(0, \tau)$.*

Proof. We show that the operator $\mathbf{V}^1$ on $\mathcal{H}_T$ is a shift. It is sufficient [9, p. 143] to prove that

$$\limsup_{n \to \infty} \{ |\langle Tk, V^n f \rangle| : f \in \mathcal{H}, \langle Tf, f \rangle = 1 \} = 0 \tag{5}$$

for all k in the kernel of V^{1*}. But $V^{(1+\tau)*}T = V^{\tau*}TV^{1*}$, so the supremum is zero when $n \geqq 1 + \tau$. Therefore $\mathbf{V}^1$ is a shift, and by Lemma 2 so is $\mathbf{S}$. By Lowdenslager's Theorem as stated in [9, p. 144], $T = F^*F$ where F is a bounded S-outer operator on $\mathfrak{H}$.

For all $t \geqq 0$ we have $F^*FV^{t+\tau} = V^tF^*FV^\tau$, so $F^*V^{t+\tau} - V^tF^*V^\tau = 0$ on $F\mathfrak{H}$. But F is S-outer and hence V^t-outer, so ker F^* is a reducing subspace for V^τ and $V^{t+\tau}$. Hence $F^*V^{t+\tau} - V^tF^*V^\tau = 0$ on ker F^*, and therefore on all of $\mathfrak{H}$. It follows that F is S-entire of type $(0, \tau)$.

In the rest of this section we take $\mathfrak{H}$ to be the separable Hardy space $H^2_{\mathfrak{C}}$ and S to be the shift

$$S : f(x) \to (x - i)(x + i)^{-1}f(x), \quad f \in H^2_{\mathfrak{C}}. \tag{6}$$

The multiplicity of S is equal to the dimension of $\mathfrak{C}$. The semi-group $\{V^t\}_{t \geq 0}$ defined by (2) is given by

$$V^t : f(x) \to e^{ixt}f(x), \quad f \in H^2_{\mathfrak{C}}, \tag{7}$$

$t \geqq 0$. If $t > 0$, V^t is a shift of infinite multiplicity. The kernel of S^* is the set of functions in $H^2_{\mathfrak{C}}$ of the form

$$f(x) = c/(x + i), \quad c \in \mathfrak{C}.$$

For any $a > 0$, the kernel of V^{a*} is the set of functions in $H^2_{\mathfrak{C}}$ of the form

$$f(x) = (2\pi)^{-1/2} \int_0^a e^{ixt}g(t)\, dt, \qquad g \in L^2_{\mathfrak{C}}(0, a).$$

Lemma 5. *For each bounded measurable operator valued function W on $(-\infty, \infty)$ let T_W be the bounded operator on $H^2_{\mathfrak{C}}$ defined by*

$$T_W : f(x) \to P_+W(x)f(x), \quad f \in H^2_{\mathfrak{C}},$$

where P_+ is the projection mapping $L^2_{\mathfrak{C}}$ onto $H^2_{\mathfrak{C}}$. Then $W \to T_W$ establishes a one-to-one correspondence between the class of all bounded measurable operator valued functions W on $(-\infty, \infty)$ and the class of all bounded S-Toeplitz operators on $H^2_{\mathfrak{C}}$. The adjoint of T_W is $T_{\bar{W}}$. The operator T_W is

(*i*) *selfadjoint if and only if $W(x) = \bar{W}(x)$ a.e.,*
(*ii*) *non-negative if and only if $W(x) \geqq 0$ a.e.,*
(*iii*) *S-analytic if and only if $W(x)f(x) \in H^2_{\mathfrak{C}}$ whenever $f(x) \in H^2_{\mathfrak{C}}$,*
(*iv*) *S-outer if and only if W is an outer function.*

In Lemma 5, as usual, functions which are equal a.e. are considered identical. The proof of the lemma is straightforward and left to the reader.

The next theorem closes the gap between the abstract theory and its application to operator valued entire functions. An operator valued entire function W is said to be of class $\mathfrak{N}(\tau_1, \tau_2)$, $\tau_1 \geqq 0$, $\tau_2 \geqq 0$, if for every c in $\mathfrak{C}$ the scalar valued entire function $\langle W(z)c, c\rangle_{\mathfrak{C}}$ is of bounded type separately in the upper and lower half-planes, and

$$\limsup_{y \to \infty} y^{-1} \log |\langle W(iy)c, c\rangle_{\mathfrak{C}}| \leqq \tau_1 ,$$

$$\limsup_{y\to\infty} y^{-1} \log |\langle W(-iy)c, c\rangle_{\mathfrak{C}}| \leqq \tau_2 .$$

In this case W is of exponential type τ where $\tau = \max(\tau_1, \tau_2)$ (see Section 4).

Theorem 4. *Let W be a bounded measurable operator valued function on $(-\infty, \infty)$ and let τ_1, τ_2 be non-negative numbers. Then T_W is S-entire of type (τ_1, τ_2) if and only if $W(x) = \tilde{W}(x)$ a.e. where $\tilde{W}$ is an operator valued entire function of class $\mathfrak{N}(\tau_1, \tau_2)$.*

Proof of sufficiency. Assume $\tilde{W}$ exists, and for each $\epsilon > 0$ set $W_\epsilon(x) = \chi_\epsilon(x) W(x)$ and $\tilde{W}_\epsilon(x) = \chi_\epsilon(x)\tilde{W}(x)$, where

(8) $$\chi_\epsilon(x) = (e^{i\epsilon x} - 1)/(i\epsilon x).$$

Then $W_\epsilon(x) = \tilde{W}_\epsilon(x)$ a.e. where $\tilde{W}_\epsilon$ is an operator valued entire function of class $\mathfrak{N}(\tau_1, \tau_2 + \epsilon)$. For all c_1, c_2 in $\mathfrak{C}$,

(9) $$\int_{-\infty}^{+\infty} |\langle W_\epsilon(x)c_1, c_2\rangle_{\mathfrak{C}}|^2\, dx \leqq K_\epsilon\, |c_1|^2\, |c_2|^2,$$

where K_ϵ is a positive constant. By Lemma 8, Section 4,

$$\langle W_\epsilon(x)c_1, c_2\rangle_{\mathfrak{C}} = (2\pi)^{-1/2} \int_{-\tau_1}^{\tau_2+\epsilon} e^{ixt} \frac{d}{dt} \langle G_\epsilon(t)c_1, c_2\rangle_{\mathfrak{C}}\, dt$$

for all c_1, c_2 in $\mathfrak{C}$, where G_ϵ is some weakly absolutely continuous operator valued function on $(-\tau_1, \tau_2 + \epsilon)$. We use this to show that T_{W_ϵ} is S-entire of type $(\tau_1, \tau_2 + \epsilon)$. It is required to show that $T_{W_\epsilon}V^{\tau_1}$ and $T^*_{W_\epsilon}V^{\tau_2+\epsilon}$ are S-analytic. This follows from a simple calculation using the following fact: an S-Toeplitz operator T is S-analytic if and only if $\langle TS^j k_1, k_2\rangle = 0$ for all k_1, k_2 in $\ker S^*$ and all $j = 1, 2, 3, \cdots$. Thus T_{W_ϵ} is S-entire of type $(\tau_1, \tau_2 + \epsilon)$. But as ϵ decreases to zero, $T_{W_\epsilon} \to T_W$ strongly. It follows that T_W is S-entire of type (τ_1, τ_2).

Proof of necessity. Assume T_W is S-entire of type (τ_1, τ_2). Let $\epsilon > 0$ be given and define W_ϵ as in the proof of sufficiency. If we regard χ_ϵ as an operator valued function, then $T_{W_\epsilon} = T_W T_{\chi_\epsilon}$. Obviously T_{χ_ϵ} is S-entire of type $(0, \epsilon)$, and so by Lemma 4, T_{W_ϵ} is S-entire of type $(\tau_1, \tau_2 + \epsilon)$. We use this fact and (9) to deduce that $W_\epsilon(x) = \tilde{W}_\epsilon(x)$ a.e. where $\tilde{W}_\epsilon$ is an operator valued entire function of class $\mathfrak{N}(\tau_1, \tau_2 + \epsilon)$.

For arbitrary $\eta > 0$ and c in $\mathfrak{C}$, let $f(x) = \chi_\eta(x)c$ where χ_η is defined by (8) with ϵ replaced by η. Then $f \in \ker V^{\eta^*}$, and for any $t \geqq \eta$,

$$\langle T_{W_\epsilon} f, V^{t+\tau_2+\epsilon} f\rangle = \langle f, T^*_{W_\epsilon} V^{t+\tau_2+\epsilon} f\rangle = \langle f, V^t T^*_{W_\epsilon} V^{\tau_2+\epsilon} f\rangle = 0.$$

Similarly $\langle T_{W_\epsilon} V^{t+\tau_1} f, f\rangle = 0$ if $t \geqq \eta$. It follows from the definition of the inner product in $H^2_{\mathfrak{C}}$ that

$$\int_{-\infty}^{+\infty} \langle W_\epsilon(x)c, c\rangle\, |\chi_\eta(x)|^2\, e^{-ixy}\, dx = 0$$

for y outside $(-\tau_1 - \eta, \tau_2 + \epsilon + \eta)$. Therefore the L^2-transform

$$(2\pi)^{-1/2} \int_{-\infty}^{+\infty} \langle W_\epsilon(x)c, c\rangle e^{-ixy}\, dx$$

vanishes a.e. outside $(-\tau_1, \tau_2 + \epsilon)$. By Lemma 8, $W_\epsilon(x) = \tilde{W}_\epsilon(x)$ a.e. where $\tilde{W}_\epsilon$ is an operator valued entire function of class $\mathfrak{N}(\tau_1, \tau_2 + \epsilon)$.

By construction $W(x) = \tilde{W}(x)$ a.e. where $\tilde{W}(z) = \tilde{W}_\epsilon(z)/\chi_\epsilon(z)$ for all complex z. Since ϵ is arbitrary, $\tilde{W}$ is entire. It follows from Lemma 6, Section 4, that $\tilde{W}$ is of class $\mathfrak{N}(\tau_1, \tau_2 + 2\epsilon)$. But again ϵ is arbitrary, so actually $\tilde{W}$ is of class $\mathfrak{N}(\tau_1, \tau_2)$. This completes the proof.

A special case of our main result now follows.

Proof of Theorem 2 when W is bounded on the real axis. In Theorem 3 take $\mathfrak{H} = H^2_{\mathfrak{C}}$ and S to be the shift (6), so V^t is given by (7). Let W be an operator valued entire function of exponential type τ that is bounded and non-negative on the real axis (w is taken to be a constant function, and hence it is trivially factorable). Let $T = T_W$ be the S-Toeplitz operator induced by W restricted to the real axis. Clearly T is bounded and non-negative. By Theorem 4, T is S-entire of type (τ, τ). Hence by Theorem 3, $T = F^*F$ where F is S-outer and S-entire of type $(0, \tau)$. By Lemma 4 (ii), $V^{1/2\tau^*}F$ is S-entire of type $(\frac{1}{2}\tau, \frac{1}{2}\tau)$. Thus by Theorem 4, $F = T_A$ where $A(z)$ is an operator valued entire function such that $\exp(-\frac{1}{2}i\tau z)A(z)$ is of exponential type $\frac{1}{2}\tau$. The identity $W = A^*A$ holds by construction, and Theorem 2 follows in case W is bounded on the real axis.

4. Function theory. In this section we collect the function theory that is needed for the main development in the paper. Except when otherwise stated the functions which appear here are scalar valued.

If f is an entire function of exponential type, its exact type is

$$\alpha = \limsup_{|z|\to\infty} |z|^{-1} \log |f(z)|,$$

so $-\infty \leqq \alpha < \infty$. The functional J is defined by (1).

An entire function g is said to be of class $\mathfrak{N}$ if the restrictions of g and g^* to the upper half-plane $y > 0$ are of bounded type in the half-plane. By a theorem of M. G. Kreĭn ([7], [3, pp. 30, 38]), this is the case if and only if $J(g) < \infty$ and g is of exponential type, and moreover, then the exact type of g is $\alpha = \max(\alpha_+, \alpha_-)$, where

$$\alpha_\pm = \limsup_{y\to\infty} y^{-1} \log |g(\pm iy)|.$$

Following L. de Branges [3] we call α_+ and α_- the *mean types* of g and g^* respectively (with respect to the upper half-plane). If g is of class $\mathfrak{N}$ and not identically zero, then α_+ and α_- are finite numbers, $\alpha_+ + \alpha_- \geqq 0$, and so $|\alpha_\pm| \leqq \alpha$ where α is the exact type of g [2, p. 69].

An entire function g is said to be of class $\mathfrak{N}(\tau_1, \tau_2)$, $\tau_1 \geqq 0$, $\tau_2 \geqq 0$, if g is of class $\mathfrak{N}$ and $\alpha_+ \leqq \tau_1$, $\alpha_- \leqq \tau_2$, where α_+ and α_- are the mean types of g and g^* respectively.

Proof of Theorem 1. Suppose w is a scalar entire function of exponential type and exact type τ. If w is factorable, then the definition of an outer function forces w to satisfy (1). Conversely if (1) is satisfied, the classical result quoted in the introduction applies, and so $w = f^*f$ where f is an entire function of exponential type and exact type $\frac{1}{2}\tau$ with no zeros in the open upper half-plane. By Nevanlinna's factorization ([3, p. 22])

$$f(z) = Ce^{-ihz} \exp\left(\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{1 + tz}{t - z}\frac{\log|f(t)|}{1 + t^2}\, dt\right)$$

for $y > 0$, where h is the mean type of f. Necessarily $h \leqq \frac{1}{2}\tau$. We have $f(z) = \exp(-ihz)a(z)$ where a is an outer entire function such that $w = a^*a$. We complete the proof by showing that $b(z) = \exp(-\frac{1}{2}i\tau z)a(z)$ is of exponential type $\frac{1}{2}\tau$. The mean type of $b(z) = \exp(-i(\frac{1}{2}\tau - h)z)\, f(z)$ is $\frac{1}{2}\tau - h + h = \frac{1}{2}\tau$, and the mean type of b^* is $\leqq -\frac{1}{2}\tau + h + \frac{1}{2}\tau = \frac{1}{2}\tau$. It follows that b has exact type $\frac{1}{2}\tau$, and the proof is complete.

Lemma 6. *Suppose h is an entire function of the form $h = f/g$ where f and g are entire functions, g not identically zero, such that f is of class $\mathfrak{N}(\tau_1, \tau_2)$ and g is of class $\mathfrak{N}(\sigma_1, \sigma_2)$. Then h is of class $\mathfrak{N}(\tau_1 + \sigma_1, \tau_2 + \sigma_2)$.*

Proof. This follows directly from the definitions.

Lemma 7. *Let f be an entire function such that $J(f) < \infty$, and let τ, ϵ be positive numbers. Assume that fgg^* is of exponential type $\tau + \epsilon$ for every entire function g of the form*

$$g(z) = \int_0^\epsilon e^{izt}\phi(t)\, dt,$$

where $\phi \in L^2(0, \epsilon)$. Then f is of exponential type τ.

Proof. Choose $g(z) = (e^{i\epsilon z} - 1)/(i\epsilon z)$. The hypotheses imply that fgg^* and f^*gg^* are of bounded type and mean type $\leqq \tau + \epsilon$ in the upper half-plane. Since gg^* is of bounded type and mean type ϵ for $y > 0$, f and f^* are of bounded type and mean type $\leqq \tau$ for $y > 0$. The result now follows from Kreĭn's Theorem.

We also use a theorem of Beurling and Malliavin [1] (see also [3, p. 254]). Let f be an entire function of exponential type such that $J(f) < \infty$. Then for every $\epsilon > 0$ there exists an entire function g, not identically zero, of exponential type ϵ such that g and fg are bounded on the real axis. We show that g can be chosen to be outer and zero free in any preassigned bounded region G. First choose g_1 of exponential type ϵ such that fg_1 and g_1 are bounded on the real axis and g_1 is not identically zero. We may assume that g_1 and g_1^* are zero free in G because we can always divide out a finite number of zeros. We next apply Theorem 1 to $g_1^*g_1$. This gives $g_1^*g_1 = g^*g$ where g is an outer entire function. Clearly g has the required properties.

We turn our attention now to operator valued functions. An operator valued entire function W is said to be of class $\mathfrak{N}(\tau_1, \tau_2)$, $\tau_1 \geqq 0$, $\tau_2 \geqq 0$, if for every vector c

in $\mathfrak{C}$ the scalar valued entire function $\langle W(z)c, c\rangle_{\mathfrak{C}}$ is of class $\mathfrak{N}(\tau_1, \tau_2)$. This is equivalent to the definition given in Section 3.

Lemma 8. *Let W be a measurable operator valued function on $(-\infty, \infty)$ such that*

$$\int_{-\infty}^{+\infty} |\langle W(x)c_1, c_2\rangle_{\mathfrak{C}}|^2 \, dx \leqq K |c_1|^2 |c_2|^2,$$

for some positive constant K and all c_1, c_2 in $\mathfrak{C}$. Let τ_1, τ_2 be non-negative numbers. Then the following are equivalent:

(i) *$W(x) = \tilde{W}(x)$ a.e. where $\tilde{W}$ is an operator valued entire function of class $\mathfrak{N}(\tau_1, \tau_2)$,*

(ii) *The L^2-transform*

$$(2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-ixt} \langle W(t)c, c\rangle_{\mathfrak{C}} \, dt$$

vanishes a.e. outside $(-\tau_1, \tau_2)$ for each c in $\mathfrak{C}$,

(iii) *there exists a weakly absolutely continuous operator valued function G on $(-\tau_1, \tau_2)$ such that*

$$\langle W(x)c, c\rangle_{\mathfrak{C}} = (2\pi)^{-1/2} \int_{-\tau_1}^{\tau_2} e^{ixt} \frac{d}{dt} \langle G(t)c, c\rangle_{\mathfrak{C}} \, dt$$

for all c in $\mathfrak{C}$.

Proof. (i) $\Rightarrow$ (ii). This follows from the classical Paley-Wiener Theorem ([2, p. 103], [3, p. 46]).

(ii) $\Rightarrow$ (iii). There exists a unique (locally) weakly absolutely continuous operator valued function G on $(-\infty, \infty)$ such that

$$\langle G(x)c, c\rangle_{\mathfrak{C}} = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \frac{e^{-ixt} - 1}{-it} \langle W(t)c, c\rangle_{\mathfrak{C}} \, dt$$

for all c in $\mathfrak{C}$ and all real x. We have then

$$\frac{d}{dx} \langle G(x)c, c\rangle_{\mathfrak{C}} = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-ixt} \langle W(t)c, c\rangle_{\mathfrak{C}} \, dt$$

a.e., and so

$$\langle W(x)c, c\rangle_{\mathfrak{C}} = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{ixt} \frac{d}{dt} \langle G(t)c, c\rangle_{\mathfrak{C}} \, dt$$

a.e.. The last two integrals are taken in the mean square sense. By (ii), $(d/dx) \langle G(x)c, c\rangle_{\mathfrak{C}} = 0$ a.e. outside $(-\tau_1, \tau_2)$. This gives (iii).

(iii) $\Rightarrow$ (i). This is routine.

For the generalization in Section 6 we require

Lemma 9. *Let $W(x)$ be a bounded measurable operator valued function on $(-\infty, \infty)$. Let $\mathcal{C}_0$ be a dense subspace of $\mathcal{C}$. Assume that for each c in $\mathcal{C}_0$ there is a number $\tau(c) \geqq 0$ such that for each b in $\mathcal{C}$, $\langle W(x)c, b\rangle_{\mathcal{C}}$ is the a.e. restriction to the real axis of an entire function f_{cb} of exponential type $\tau(c)$. Then $W(x) = \tilde{W}(x)$ a.e. where W is an operator valued entire function such that for each c in $\mathcal{C}_0$, $W(z)c$ is of exponential type $\tau(c)$.*

Proof. For fixed z, $f_{cb}(z)$ is linear in c, $c \in \mathcal{C}_0$, and conjugate linear in b, $b \in \mathcal{C}$. For fixed c, b, $f_{cb}(z)$ is of bounded type and mean type $\leqq \tau(c)$ for $y > 0$. If the function does not vanish identically, then [3, p. 29]

$$\begin{aligned}\log |f_{cb}(z)| &\leqq \tau(c)y + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\log |f_{cb}(t)|}{(t-x)^2 + y^2}\,dt \\ &\leqq \tau(c)y + \log\left(\frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|f_{cb}(t)|}{(t-x)^2+y^2}\,dt\right) \\ &\leqq \tau(c)y + \log (K\,|b|\,|c|),\end{aligned}$$

i.e.

$$|f_{cb}(z)| \leqq Ke^{\tau(c)y}\,|b|\,|c|$$

for $y > 0$. Here K is chosen such that $|W(x)| \leqq K$ for all real x. The last inequality holds trivially if $f_{cb}(z) = 0$. Moreover the estimate extends to the real axis by continuity, and a similar estimate holds for $y < 0$. It follows that

$$f_{cb}(z) = \langle \tilde{W}(z)c, b\rangle, \quad c \in \mathcal{C}_0, \quad b \in \mathcal{C},$$

for a unique operator valued function $\tilde{W}$ defined in the complex plane. Clearly $\tilde{W}$ has the required properties.

5. Proof of the main results. We prepare the way for Lemma 1 with the following.

Lemma 10. *Suppose S is an arbitrary shift on a Hilbert space $\mathcal{H}$ and T_1 and T_2 are S-outer operators on $\mathcal{H}$. Then $T_1^*T_1 = T_2^*T_2$ if and only if there is a partial isometry V on $\mathcal{H}$ such that V commutes with S and S^*, V has initial set $\overline{T_1\mathcal{H}}$ and final set $\overline{T_2\mathcal{H}}$ and $T_2 = VT_1$.*

Proof. If such a partial isometry exists, then clearly $T_2^*T_2 = T_1^*V^*VT_1 = T_1^*T_1$.

Suppose $T_1^*T_1 = T_2^*T_2$, where T_1 and T_2 are S-outer. Then by [9, p. 141–142] there exist inner (S-analytic partially isometric) operators G_1 and G_2 such that $T_2 = G_1T_1$, $T_1 = G_2T_2$, $G_1^*G_1 = P_1$, $G_2^*G_2 = P_2$, where P_1 and P_2 are the projections on $\overline{T_1\mathcal{H}}$ and $\overline{T_2\mathcal{H}}$ respectively. Then $T_1 = G_2G_1T_1$, $T_2 = G_1G_2T_2$, $G_2^*T_1 = T_2 = G_1T_1$, and $G_1^*T_2 = T_1 = G_2T_2$, so $G_2G_1 = P_1$, $G_1G_2 = P_2$, $(G_2^* - G_1)G_1^* = 0$, and $(G_1^* - G_2)G_2^* = 0$. It follows that $P_2 = G_1G_1^*$ and $P_1 = G_2G_2^*$, so $G_1 = G_2^*$. Thus $V = G_1 = G_2^*$ is a partial isometry with the requisite properties.

Proof of Lemma 1. Choose a bounded scalar valued outer function f such that $B_j = fA_j$ is bounded for $j = 1, 2$, and hence outer.

Suppose first that $\tilde{A}_1(x)A_1(x) = \tilde{A}_2(x)A_2(x)$. Then $\tilde{B}_1(x)B_1(x) = \tilde{B}_2(x)B_2(x)$. If T_j is left multiplication by B_j in $H^2_{\mathfrak{C}}$, $j = 1, 2$, then $T_1^*T_1 = T_2^*T_2$ and the range of T_j is dense in $H^2_{\mathfrak{B}_j}$, where $\mathfrak{B}_j$ is a closed subspace of $\mathfrak{C}$, $j = 1, 2$. As in Section 2 we let S be the shift (6), so T_1 and T_2 are S-outer. By Lemma 10 there is a partial isometry V with initial set $H^2_{\mathfrak{B}_1}$ and final set $H^2_{\mathfrak{B}_2}$ such that $T_2 = VT_1$ and V commutes with S and S^*. Thus there exists a partial isometry U on $\mathfrak{C}$ with initial set $\mathfrak{B}_1$ and final set $\mathfrak{B}_2$ such that $B_2(x) = UB_1(x)$, and hence $A_2(x) = UA_1(x)$.

Suppose, conversely, that $A_2(x) = UA_1(x)$ with U as in the statement of the lemma. Then $B_2(x) = UB_1(x)$, so $\tilde{B}_2(x)B_2(x) = \tilde{B}_1(x)B_1(x)$ and $\tilde{A}_2(x)A_2(x) = \tilde{A}_1(x)A_1(x)$, as required.

Completion of the proof of Theorem 2. We have already proved the theorem in the case where W is bounded on the real axis. We now use the Beurling-Malliavin Theorem (see Section 4) to reduce the general case to this case.

Let $M > 0$ and $\epsilon > 0$ be given. If w is factorable, then it satisfies the hypotheses of the Beurling-Malliavin Theorem. Choose a scalar valued outer entire function g of exponential type ϵ such that g and wg are bounded on the real axis, and $g(z)$ is zero-free for $|z| < M$. The operator entire function $W_1 = g^*gW$ is non-negative and bounded on the real axis, and it is entire of exponential type $\tau + 2\epsilon$. By the special case of the theorem already proved $W_1 = A_1^*A_1$, where A_1 is an outer operator valued entire function such that $e^{-(1/2)i(\tau+2\epsilon)z}A_1(z)$ is of exponential type $\frac{1}{2}\tau + \epsilon$.

We show that $A = A_1/g$ is entire. Clearly $A(z)$ is analytic for $|z| < M$ and elsewhere in the complex plane except for isolated poles. Repeat the above construction with M replaced by a number $\tilde{M} > M$. We get a second factorization $\tilde{g}^*\tilde{g}W = \tilde{A}_1^*\tilde{A}_1$ as above. We have then

$$\begin{aligned}(g^*\tilde{A}_1^*)(g\tilde{A}_1) &= g^*g\tilde{g}^*\tilde{g}W \\ &= (\tilde{g}^*A_1^*)(\tilde{g}A_1),\end{aligned}$$

where $g\tilde{A}_1$ and $\tilde{g}A_1$ are outer functions that are bounded on the real axis. By Lemma 1, $\tilde{g}(x)A_1(x) = Ug(x)\tilde{A}_1(x)$, where U is a partially isometric operator on $\mathfrak{C}$. Since $\tilde{g}(z)$ is zero-free for $|z| < M$, $A = A_1/g = U\tilde{A}_1/\tilde{g}$ is analytic for $|z| < \tilde{M}$. Since $\tilde{M}$ is arbitrary, A is entire as claimed.

We have already obtained the factorization $W = A^*A$, where A is an outer operator valued entire function. Since $e^{-(1/2)i\tau z}A_1(z)$ is of exponential type $\frac{1}{2}\tau + 2\epsilon$ and g is of exponential type ϵ, it follows from Lemma 7 that

$$e^{-(1/2)i\tau z}A(z) = e^{-(1/2)i\tau z}A_1(z)/g(z)$$

is of exponential type $\frac{1}{2}\tau + 3\epsilon$. But ϵ is an arbitrary positive number, so $e^{-(1/2)i\tau z}A(z)$ is of exponential type $\frac{1}{2}\tau$.

Proof of Corollary 1. We choose $w(x) = \operatorname{tr} W(x)$ in Theorem 2. The last statement is a consequence of a lemma in [5, p. 125].

Proof of Corollary 2. In this case W admits a scalar polynomial as a dominant. In fact, since

$$|W(x)| \leqq \sum_{j=0}^{2N} |C_j|\, |x^j| \leqq \sum_{j=0}^{2N} \tfrac{1}{2}\, |C_j|\, (x^2+1)^j$$

for all real x, we may choose

$$w(z) = \sum_{j=0}^{2N} |C_j|\, (z^2+1)^j.$$

Now W is of exponential type 0, so by Theorem 2, $W = A^*A$, where A is an outer operator valued entire function of exponential type 0. Moreover, for x real,

$$|A(x)|^2 = |\bar{A}(x)A(x)| = |W(x)| = O(x^{2N}),$$

as $|x| \to \infty$. From the classical theory of entire functions (see [2, p. 83]) it follows that for any vectors c_1, c_2 in the scalar entire function $\langle A(z)c_1, c_2\rangle$ is a polynomial of degree $\leqq N$. Therefore A is a polynomial of degree $\leqq N$.

6. Strengthened form of the factorization theorem. We change the definition of exponential type for operator valued entire functions. Exponential type for vector valued entire functions is defined as before. However, we now call an operator valued entire function W of exponential type if there exists a dense subspace $\mathfrak{C}_0$ of $\mathfrak{C}$ such that for each c in $\mathfrak{C}_0$, $W(z)c$ is of exponential type $\tau(c)$ for some number $\tau(c) \geqq 0$.

Assume W is an operator valued entire function of exponential type, and let $\mathfrak{C}_0$ and $\tau(c)$, $c \in \mathfrak{C}_0$, be as above. Suppose there exists a scalar valued entire function w of exponential type and not identically zero such that

$$0 \leqq W(x) \leqq w(x)I$$

for all real x. If w is factorable, then

$$W = A^*A,$$

where A is an outer entire operator valued function such that for each c in $\mathfrak{C}_0$, $\exp\,(-\tfrac{1}{2}i\tau(c)z)A(z)c$ *is of exponential type $\tfrac{1}{2}\tau(c)$.*

We sketch the proof.

Assume first that W is bounded on the real axis. Let S and $\{V^t\}_{t\geq 0}$ be defined in $H^2_{\mathfrak{C}}$ by (6) and (7). Let T be the S-Toeplitz operator induced in $H^2_{\mathfrak{C}}$ by the restriction of W to the real axis, and construct $\mathfrak{K}_T$, $\mathbf{S}$, and $\{\mathbf{V}^t\}_{t\geq 0}$ for T as in Section 3.

We show that $\mathbf{V}^1$ is a shift. As in the proof of Theorem 3 it is enough to show that (5) holds for all k in $\ker V^{1*}$. By [9, p. 143] we need only take k in a dense subspace of $\ker V^{1*}$, and in fact it is enough to take k of the form $k(x) = g(x)c$ where $c \in \mathfrak{C}_0$ and

$$g(x) = \int_0^1 e^{ixt}\phi(t)\, dt \tag{10}$$

for some $\phi \in L^2(0, 1)$. We check (5) in this case by showing that $Tk \in \ker V^{n^*}$ as soon as $n \geqq \tau(c) + 1$. The vector valued function $W(x)k(x)$ has an entire extension which is of exponential type $\tau(c) + 1$. By the Paley-Wiener Theorem (in its classical scalar form) and our discussion of the Fourier transformation $\mathfrak{F}$ in L^2, it follows that

$$W(x)k(x) = \int_{-\tau(c)-1}^{\tau(c)+1} e^{ixt} f(t)\, dt$$

where $f \in L^2_{\mathfrak{C}}(-\tau(c) - 1, \tau(c) + 1)$. Hence

$$Tk(x) = P_+ W(x)k(x) = \int_0^{\tau(c)+1} e^{ixt} f(t)\, dt.$$

By its form this function belongs to $\ker V^{1^*}$. Therefore (5) holds, and so $\mathbf{V}^1$ is a shift. By Lemma 3, S is also a shift.

As in the proof of Theorem 3 it follows from [9, p. 144] that

$$T = F^*F,$$

where F is an S-outer operator on $H^2_{\mathfrak{C}}$. By Lemma 5, $F = T_A$ is induced by an outer operator valued function A on $(-\infty, \infty)$, and

$$W(x) = \tilde{A}(x)A(x)$$

a.e. By a lemma in [9, p. 145], if k is as above, *i.e.* $k(x) = g(x)c$, $c \in \mathfrak{C}_0$, and $g(x)$ is of the form (10), then

$$Fk \in \ker V^{(\tau(c)+1)^*} \tag{11}$$

(In [9, p. 145] take the underlying Hilbert space to be $H^2_{\mathfrak{C}}$, the shift to be $V^{\tau(c)+1}$, and $K = V^{\tau(c)+1}$.)

We use (11) to discuss the analyticity properties of A. We have from (11) that

$$A(x)g(x)c = \int_0^{\tau(c)+1} e^{ixt} f(t)\, dt,$$

where $f \in L^2_{\mathfrak{C}}(0, \tau(c) + 1)$. Therefore

$$e^{-(1/2)i\tau(c)x} A(x)\ |g(x)|^2\, c = \int_{-(1/2)\tau(c)-1}^{(1/2)\tau(c)+1} e^{ixt} f_1(t)\, dt,$$

where $f_1 \in L^2(-\frac{1}{2}\tau(c) - 1, \frac{1}{2}\tau(c) + 1)$. By this identity and Lemma 9 the function

$$A_1(x) = \exp\,(-\tfrac{1}{2}i\tau(c)x)A(x)\ |g(x)|^2,$$

has an entire extension A_1 such that for each c in $\mathfrak{C}_0$, $A_1(z)c$ is of exponential type $\frac{1}{2}\tau(c) + 1$. Since g is arbitrary, $A(x)$ has an entire extension A. By Lemma 7, $\exp\,(-\frac{1}{2}i\tau(c)z)A(z)c$ is of exponential type $\frac{1}{2}\tau(c)$. Thus

$$W = A^*A,$$

is the required factorization when W is bounded on the real axis.

In the general case we make a reduction to the case already considered as in Section 5. The modifications needed for this are left to the reader.

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