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LU-FACTORIZATION OF OPERATORS ON l_1

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ABSTRACT. Necessary and sufficent conditions are obtained for LU-factorization of operators on l_1. In particular it is shown that uniform invertibility of the compressions of the operator is not sufficient to insure an LU-factorization of the operator, thus answering a question of de Boor, Jia, and Pinkus.

The question of when a bounded linear operator on l_p, $1 \leqslant p \leqslant \infty$, has an LU-factorization has been much studied recently. Barkar and Gohberg [2] have shown that if A is an operator on l_p which has an LU-factorization, then A and its compressions $A_n = P_n A P_n$ are uniformly invertible, i.e. $\sup_n\{\|A_n^{-1}\|, \|A^{-1}\|\} < \infty$. In the other direction, various classes of operators such as invertible, diagonally dominant operators on l_1 [7] and invertible, totally positive operators [3, 1] on l_p have been shown to have LU-factorizations. For these kinds of operators it is known [1] that their compressions satisfy a stronger condition than uniform invertibility; namely, that the inverses of the compressions are order bounded, i.e. $\|\sup_n|A_n^{-1}|\| < \infty$. Left open, then, is the possibility (first raised in [3] with a negative expectation) that uniform invertibility might be sufficient for a matrix operator on l_∞ to have an LU-factorization. In this paper an example is given that shows that uniform invertibility is *not* sufficient for factoring an operator on l_∞ (or l_1). However, we also show that uniform invertibility of the compressions is sufficient to ensure an LU-factorization when the operator has an inverse whose columns decay at a certain rate away from the diagonal. Among the operators with this property are the banded operators.

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We now fix some terminology and notation. If $x = (x_i)$ is an element of l_1 we denote its usual projection onto the span of the first n basis vectors by $P_n x$. A bounded linear operator A on l_1 is said to be upper (respectively lower) triangular if $P_n A P_n = A P_n$ (respectively $P_n A$) for all n. We say that A is unit upper (lower) triangular if it is upper (lower) triangular and its diagonal entries in the matrix representation for A relative to the usual basis e_i of l_1 are all ones. An operator A is said to have an LU-factorization (relative to the usual basis e_i of l_1) if there exist invertible operators L and U so that $A = LU$ and the operators L, L^{-1} are unit lower triangular while U, U^{-1} are upper triangular. An operator A is said to be

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banded if there exist integers m and l so that $A(i,j)=0$ if $j \notin [i-l, i-l+m]$. The absolute value of an operator $A=(a_{ij})$ is the operator $|A|=(|a_{ij}|)$. Finally, we let A_n^{-1} denote the operator on l_1 whose decomposition with respect to P_n and $I-P_n$ is given by

$$\begin{pmatrix} (P_nAP_n)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

EXAMPLE. For each m, let B_m be the operator on l_1^m given by $B_m e_j = e_1 - e_{j+1}$, $j=1,2,\ldots,m-1$, and $B_m e_m = e_1$. Then each B_m is invertible relative to l_1^m; in fact, $B_m^{-1}e_1 = e_m$ and $B_m^{-1}e_j = e_m - e_{j-1}$, $j=2,3,\ldots,m$. Since for each i, $P_iB_mP_i = B_i$, we have that the compressions of each B_m are invertible and so each B_m has an LU-factorization [**4**, p. 178]. In fact, $B_m = L_mU_m$ where $L_me_j = e_j - e_{j+1}$, $j=1,2,\ldots,m-1$, and $L_me_m = e_m$ and $U_me_j = \sum_{k=1}^{j} e_k$, $j=1,2,\ldots,m$. Note that $\|U_m\| = m$. If we now let $A = \oplus_{m=1}^{\infty} B_m$ then A and its compressions are uniformly invertible; in fact, $\sup_n\{\|A_n^{-1}\|, \|A^{-1}\|, \|A\|\} = 2$. But if $A = LU$ then $\|U\| \geqslant \sup_n \|P_nUP_n\| \geqslant \sup_n\|U_m\| = \infty$, so A does not have an LU-factorization. This fact can also be easily obtained using Theorem 2 of [**1**] since $B_m^{-1}e_1 = e_m$ implies that $(\sup_m|B_m^{-1}|)e_1 = \sum_m e_m$, i.e. $\|\sup_m|B_m^{-1}|\| = \infty$. Consequently, the block diagonal matrix A must also have $\|\sup_n|A_n^{-1}|\| = \infty$ and so does not have an LU-factorization. We remark that $A^*\colon l_\infty \to l_\infty$ does not have an LU-factorization either. For if $A^* = LU$, since L and U are operators on l_∞ representable as matrices, $A = U_*L_*$ is an LU-factorization for A where U_* and L_* are the preadjoints of U and L [**8**]. This fulfills the expectation raised in [**3**].

The question remains as to whether there are any easily recognized situations in which uniform invertibility of the compressions is sufficient to insure an LU-factorization of the operator. In order to give an example of such a situation we find it convenient to give a characterization of when an operator on l_1 has an LU-factorization. This characterization is similar to that presented in Theorem 2 of [**1**] where the finiteness of $\|\sum|A_{n+1}^{-1} - A_n^{-1}|\|$ is replaced by the finiteness of $\|\sup_n|A_n^{-1}|\|$. As further motivation we recall that if an operator A and its compressions are uniformly invertible, then $A_n^{-1}e_i \to A^{-1}e_i$ for all i. Our first result shows that for A to have an LU-factorization this convergence must be of a telescoping variety.

THEOREM 1. *A bounded linear operator A on l_1 has an LU-factorization if and only if, for each n, $A_n = P_nAP_n$ is invertible and*

$$\sup_i \sum_{n=1}^{\infty} \left\|\left(A_{n+1}^{-1} - A_n^{-1}\right)e_i\right\| = \left\|\left(\sum_{n=1}^{\infty} \left|A_{n+1}^{-1} - A_n^{-1}\right|\right)\right\| < \infty.$$

PROOF. If $A = LU$ then $A_n = P_nLP_nUP_n$ and hence $A_n^{-1} = P_nU^{-1}P_nL^{-1}P_n = U^{-1}P_nL^{-1}$ since U^{-1} is upper triangular and L^{-1} is lower triangular. Consequently, $(A_{n+1}^{-1} - A_n^{-1})(e_i) = U^{-1}(P_{n+1} - P_n)L^{-1}e_i$, so

$$\sup_i \sum_{n=1}^{\infty} \left\|\left(A_{n+1}^{-1} - A_n^{-1}\right)(e_i)\right\| \leqslant \sup_i \|U^{-1}\| \sum_{n=1}^{\infty} \left\|\left(P_{n+1} - P_n\right)L^{-1}e_i\right\|$$

$$\leqslant \|U^{-1}\| \sup_i \left\|L^{-1}e_i\right\| = \|U^{-1}\|\|L^{-1}\| < \infty.$$

For the converse, note that the hypothesis implies that

$$Be_i \equiv A_1^{-1}e_i + \sum_{n=1}^{\infty} \left(A_{n+1}^{-1} - A_n^{-1}\right)e_i$$

exists for each i and $\sup_i \|Be_i\| < \infty$. Hence B extends to a bounded linear operator on l_1 and since $Be_i = \lim_n A_n^{-1}e_i$ it follows quickly that $B = A^{-1}$. Now for each N,

$$A^{-1}e_i = A_N^{-1}e_i + \sum_{n=N}^{\infty} \left(A_{n+1}^{-1} - A_n^{-1}\right)(e_i)$$

and so

$$A_N^{-1} = A^{-1} + \sum_{n=N}^{\infty} \left(A_{n+1}^{-1} - A_n^{-1}\right)$$

pointwise. Hence

$$\sup_N |A_N^{-1}| \leqslant |A^{-1}| + \sum_n |A_{n+1}^{-1} - A_n^{-1}|$$

pointwise and, consequently,

$$\left\|\sup_N |A_N^{-1}|\right\| \leqslant \|A^{-1}\| + \left\|\sum_n |A_{n+1}^{-1} - A_n^{-1}|\right\| < \infty.$$

Now since A_n is invertible for all n, we have that $A_n = L_n U_n$. We shall show that the operators L_n^{-1} and U_n^{-1} are bounded and so deduce that A has an LU-factorization. (This part of the argument has already appeared in **[1]** but we include it here for the sake of completeness.) Now for each n,

$$L_n^{-1}(i,j) = -\sum_{k=1}^{i-1} A_{i-1}^{-1}(k,j)A(i,k) \quad \text{for } i > j$$

and

$$U_n^{-1}(i,j) = A_j^{-1}(i,j) \quad \text{for } i < j$$

[**1**, **2**]. It follows that

$$\sup_n |L_n^{-1}(i,j)| \leqslant \sum_{k=1}^{\infty} \sup_i |A_{i-1}^{-1}(k,j)||A(i,k)| \quad \text{for } i > j$$

and so

$$\sup_n \|L_n^{-1}\| \leqslant \left\|\sup_n |L_n^{-1}|\right\| \leqslant \left\|\sup_i |A_{i-1}^{-1}|\right\| \|A\| + 1 < \infty.$$

Similarly,

$$\sup_i \|U_n^{-1}\| \leqslant \left\|\sup_n |U_n^{-1}|\right\| \leqslant \left\|\sup_n |A_n^{-1}|\right\| < \infty.$$

Since $L_n = P_n L_{n+1} P_n$ and $U_n = P_n U_{n+1} P_n$ we have that $L_n^{-1} = P_n L_{n+1}^{-1} P_n$ and $U_n^{-1} = P_n U_{n+1}^{-1} P_n$. Consequently, for each x in l_1, the limits $\lim_n L_n x = Lx$, $\lim_n L_n^{-1}x = Vx$, $\lim_n U_n x = Ux$, and $\lim_n U_n^{-1}x = Wx$ exist and define bounded triangular operators on l_1. Now since

$$LVx = \lim_n L_n L_n^{-1}x = \lim_n I_n x = x = \lim_n I_n x = \lim_n L_n^{-1} L_n x = VLx$$

we have that $V = L^{-1}$. Similarly, $W = U^{-1}$. Finally, for each x in l_1, we have that $LUx = \lim_n L_n U_n x = \lim_n A_n x = Ax$ so A has the promised factorization.

We remark that Theorem 1 can be easily applied to the example preceding the theorem. In this case

$$A_2^{-1} - A_1^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3^{-1} - A_2^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$A_4^{-1} - A_3^{-1} = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad A_5^{-1} - A_4^{-1} = \begin{pmatrix} 0 & & 0 & \\ & & -1 & -1 \\ 0 & & 1 & 1 \end{pmatrix},$$

$$A_6^{-1} - A_5^{-1} = \begin{pmatrix} 0 & & 0 & \\ & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ & 1 & 1 & 1 \end{pmatrix}.$$

(Here we have displayed only the upper left hand, nonzero portion of each operator.) Hence

$$\sum_{n=1}^{\infty} \left\| \left(A_{n+1}^{-1} - A_n^{-1} \right)(e_2) \right\| = 3, \qquad \sum_{n=1}^{\infty} \left\| \left(A_{n+1}^{-1} - A_n^{-1} \right)(e_4) \right\| = 5$$

and, in general, by a routine but tedious induction argument,

$$\sum_{n=1}^{\infty} \left\| \left(A_{n+1}^{-1} - A_n^{-1} \right)\left(e_{k(k+1)/2+1} \right) \right\| = 2k + 1$$

so by Theorem 1 A does not have an LU-factorization.

THEOREM 2. *Let A be a bounded linear operator on l_1. If A and its compressions are uniformly invertible and, in addition,*

$$\sup_i \sum_{k=1}^{\infty} k \left| A^{-1}(i + k, i) \right| < \infty,$$

then A has an LU-factorization.

PROOF. We start with $AA^{-1}e_i = e_i$. Hence

$$AP_n A^{-1} e_i + A(I - P_n) A^{-1} e_i = e_i \quad \text{for all } n$$

so

$$P_n A P_n A^{-1} e_i + P_n (I - P_n) A^{-1} e_i = P_n e_i = e_i \quad \text{for } i \leqslant n.$$

Hence

$$P_n A^{-1} e_i - A_n^{-1} e_i = A_n^{-1} P_n A (P_n - 1) A^{-1} e_i \quad \text{for } i \leqslant n$$

and thus

$$\left\| P_n A^{-1} e_i - A_n^{-1} e_i \right\| \leqslant M^2 \left\| (P_n - I) A^{-1} e_i \right\| \quad \text{for } i \leqslant n$$

where $M = \sup_n \{ \|A_n^{-1}\|, \|A\| \}$. Now

$$\begin{aligned} \left\| A_n^{-1} e_i - A^{-1} e_i \right\| &\leqslant \left\| P_n A^{-1} e_i - A_n^{-1} e_i \right\| + \left\| (I - P_n) A^{-1} e_i \right\| \\ &\leqslant (M^2 + 1) \left\| (I - P_n) A^{-1} e_i \right\| \quad \text{for } i \leqslant n. \end{aligned}$$

Hence

$$\sum_{n=i}^{\infty} \|A_{n+1}^{-1}e_i - A_n^{-1}e_i\| \leqslant 2(M^2+1)\sum_{n=i}^{\infty} \|(I-P_n)A^{-1}e_i\|$$
$$\leqslant 2(M^2+1)\sum_{n=i}^{\infty}\sum_{j=n+1}^{\infty} \left|\left\langle A^{-1}e_i, e_j\right\rangle\right|$$
$$\leqslant 2(M^2+1)\sum_{k=1}^{\infty} k\left|\left\langle A^{-1}e_i, e_{i+k}\right\rangle\right|.$$

Consequently,

$$\sup_i \sum_{n=1}^{\infty} \|(A_{n+1}^{-1} - A_n^{-1})e_i\| \leqslant \sup_i \|A_i^{-1}e_i\| + \sup_i \sum_{n=i}^{\infty} \|(A_{n+1}^{-1} - A_n^{-1})(e_i)\| < \infty.$$

So by Theorem 1 A has an LU-factorization.

It is not surprising that the additional condition imposed in Theorem 2 is far from necessary. For example, choose a so that $0 < a < 1$ and $\sum_n (a/n^2) < 1$ and let

$$B(i,j) = \begin{cases} a/(i-1)^2, & j = 1,\ i \neq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|B\| < 1$, and $B(i,i) = 0$ for all i so $A = I - B$ is invertible and strictly (column) diagonally dominant. Consequently, A has an LU-factorization [7]. But since $B^2 = 0$, $A^{-1} = I + B$ and so

$$\sum_{k=1}^{\infty} k|A^{-1}(k+1,1)| = \sum_{k=1}^{\infty} k\frac{a}{k^2} = \infty.$$

The problem here is the slowness of the decay rate of the entries of A^{-1} away from the main diagonal; however, for banded operators this poses no difficulty.

Corollary 3. *Let A be a banded operator on l_1. Then A has an LU-factorization if and only if A and its compressions are uniformly invertible.*

Proof. One direction is clear; for the other we recall from [5] that if A is banded and invertible then there are positive constants C and λ with $\lambda < 1$ so that $|A^{-1}(i,j)| \leqslant C\lambda^{|i-j|}$ for all i, j. Consequently,

$$\sup_i \sum_{k=1}^{\infty} k|A^{-1}(i+k,i)| \leqslant C\sum_{k=1}^{\infty} k\lambda^k < \infty.$$

So Theorem 2 shows that A has an LU-factorization.

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