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Factors of horocycle flows

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Dedicated to the memory of V. M. Alexeyev

Abstract. We classify up to an isomorphism all factors of the classical horocycle flow on the unit tangent bundle of a surface of constant negative curvature with finite volume.

Let $T=\{T_t, t\in R\}$ and $S=\{S_t, t\in R\}$ be two measure preserving (m.p.) flows on probability spaces (X,μ) and (Y,ν) respectively. We say that S is a factor of T if there is a measure preserving

$$\psi: X\to Y \quad \text{such that } \psi(T_t x)=S_t\psi(x)$$

for all $t\in R$ and μ-almost every (a.e.) $x\in X$. ψ is called a conjugacy between T and S. T and S are called isomorphic ($T\sim S$) if there is an invertible conjugacy between T and S, called an isomorphism. We write $(T,S)\sim(T',S')$ if $T\sim T'$ and $S\sim S'$. S is called trivial if there is $y\in Y$ such that $\nu\{y\}=1$. Henceforth the word 'factor' means non-trivial factor.

Let $\Phi(T)$ denote the set of all isomorphisms

$$\phi: X\to X \quad \text{such that } \phi(T_t x)=T_t\phi(x)$$

for all $t\in R$ and a.e. $x\in X$ and let $\Psi=\Psi(T,S)$ denote the set of all conjugacies between T and S. We say that $\psi_1\in\Psi$ and $\psi_2\in\Psi$ are equivalent ($\psi_1\sim\psi_2$) if there are $\phi_1\in\Phi(T)$ and $\phi_2\in\Phi(S)$ such that $\psi_2=\phi_2\circ\psi_1\circ\phi_1$ a.e.

Let $\pi(T,S)$ denote the set of equivalence classes in Ψ. It is clear that if $(T,S)\sim(T',S')$ then there is a natural one-to-one correspondence between $\pi(T,S)$ and $\pi(T',S')$. So $|\pi(T,S)|$ is an invariant of the isomorphism class of (T,S).

One would naturally raise the following problems: (1) classifying all possible factors of a given m.p. flow T up to an isomorphism; (2) describing $\pi(T,S)$ for a given factor S of T.

In this paper we shall solve these problems for the classical horocycle flow on the unit tangent bundle of a surface of constant negative curvature with finite volume.

Let G denote the group $\mathrm{SL}(2,R)$ equipped with a left invariant Riemannian metric and let $\mathcal{T}$ be the set of all discrete subgroups Γ of G such that the quotient space $M=\Gamma\backslash G=\{\Gamma g: g\in G\}$ has finite volume. M can be viewed as the unit tangent bundle of a surface of constant negative curvature with finite volume. Let F be an element of the Lie algebra $\mathcal{A}$ of G and let $F_t=\exp(tF)\in G$. The flow $f=\{f_t, t\in R\}$

on M defined by $f_t(\Gamma g) = \Gamma g \cdot F_t$, $g \in G$, $t \in R$ is called the algebraic flow, generated by F. f preserves the Riemannian volume v on M derived from the Haar measure on G. v is defined on the Borel σ-algebra B_M of M and we denote by $(\mathcal{B}, \mu)$ the normalized completion of (B_M, v), $\mu(M) = 1$.

The horocycle flow

$$h = \{h_t, t \in R\}$$

on M is the algebraic flow, generated by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, i.e.

$$h_t(\Gamma g) = \Gamma g N_t,$$

where

$$N_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t \in R, g \in G.$$

It is well known that h is ergodic and mixing on (M, μ), in fact mixing of all degrees [**1**].

Let $F \in \mathcal{A}$, $\Gamma_i \in \mathcal{T}$, $i = 1, 2$ and let $f^{(i)}$ be the algebraic flow on $M_i = \Gamma_i \backslash G$, generated by F, $i = 1, 2$. It is easy to see that if $\Gamma_1 \subset \Gamma_2$ then $f^{(2)}$ is a factor of $f^{(1)}$. Indeed, let

$$\psi : M_1 \to M_2$$

be defined by

$$\psi(\Gamma_1 g) = \Gamma_2 g, \quad g \in G.$$

Then ψ is measure preserving and

$$\psi f_t^{(1)}(\Gamma_1 g) = \psi(\Gamma_1 g F_t) = \Gamma_2 g F_t = f_t^{(2)}(\Gamma_2 g) = f_t^{(2)}(\psi(\Gamma_1 g)).$$

We shall call $f^{(2)}$ an algebraic factor of $f^{(1)}$.

The following theorem shows that every factor of the horocycle flow is algebraic.

THEOREM 1. *Let $\Gamma_1 \in \mathcal{T}$, $M_1 = \Gamma_1 \backslash G$ and let S be a factor of the horocycle $h^{(1)}$ on (M_1, μ_1). Then there is $\Gamma_2 \in \mathcal{T}$ such that $\Gamma_1 \subset \Gamma_2$ and S is isomorphic to $h^{(2)}$ on (M_2, μ_2).*

It has been proved in [**4**] that for $\Gamma_1, \Gamma_2 \in \mathcal{T}$ the horocycle flows $h^{(1)}$ and $h^{(2)}$ are isomorphic iff Γ_1 and Γ_2 are conjugate in G, i.e. $\Gamma_2 = C\Gamma_1 C^{-1}$ for some $C \in G$. For $\Gamma \in \mathcal{T}$ we denote

$$\alpha(\Gamma) = \{\tilde{\Gamma} \in \mathcal{T} : \Gamma \subset \tilde{\Gamma}\}.$$

It is well known [**6**] that $\alpha(\Gamma)$ is finite. Γ is called maximal if $\alpha(\Gamma) = \{\Gamma\}$. We get the following corollary.

COROLLARY 1. *The number of non-isomorphic factors of the horocycle flow h on $M = \Gamma \backslash G$, $\Gamma \in \mathcal{T}$ is finite and equals the number of conjugacy classes in $\alpha(\Gamma)$.*

It was proved in [**4**] that if $\Gamma_2 \in \alpha(\Gamma_1)$ and $\psi : M_1 \to M_2$ is a conjugacy between $h^{(1)}$ and $h^{(2)}$ then there is $C \in G$ such that

$$C\Gamma_1 C^{-1} \subset \Gamma_2 \quad \text{and} \quad \psi(\Gamma_1 g) = h_\sigma^{(2)} \psi_C(\Gamma_1 g)$$

for some $\sigma \in R$ and a.e. $\Gamma_1 g \in M_1$, $g \in G$, where $\psi_C(\Gamma_1 g) = \Gamma_2 C g$. This says that $\psi \sim \psi_C$.

For $\Gamma_2 \in \alpha(\Gamma_1)$ we denote

$$\mathscr{C}(\Gamma_1, \Gamma_2) = \{C \in G : C\Gamma_1 C^{-1} \subset \Gamma_2\} = \{C \in G : C^{-1}\Gamma_2 C \in \alpha(\Gamma_1)\}$$

and

$$\kappa(\Gamma_1, \Gamma_2) = \{\Gamma \in \alpha(\Gamma_1) : \Gamma = C^{-1}\Gamma_2 C \text{ for some } C \in G\}.$$

It follows from [**4**] that

$$\psi_{C_1} \sim \psi_{C_2}, \quad C_1, C_2 \in \mathscr{C}(\Gamma_1, \Gamma_2)$$

iff $C_2 = CC_1D$ for some $C \in \tilde{\Gamma}_2$ and some $D \in \tilde{\Gamma}_1$, where $\tilde{\Gamma}$ denotes the normalizer of Γ in G, i.e.

$$\tilde{\Gamma} = \{C \in G : C\Gamma C^{-1} = \Gamma\}.$$

In this case we write $C_2 \sim C_1$. $\sim$ is an equivalence relation in $\mathscr{C}(\Gamma_1, \Gamma_2)$. For $\Gamma', \Gamma'' \in \kappa(\Gamma_1, \Gamma_2)$ we write $\Gamma' \sim \Gamma''$ if $\Gamma'' = D^{-1}\Gamma' D$ for some $D \in \tilde{\Gamma}_1$. It is clear that $C_2 \sim C_1$ in $\mathscr{C}(\Gamma_1, \Gamma_2)$ iff $C_2^{-1}\Gamma_2 C_2 \sim C_1^{-1}\Gamma_2 C_1$ in $\kappa(\Gamma_1, \Gamma_2)$. We have just proved the following theorem.

THEOREM 2. *Let $\Gamma_1, \Gamma_2 \in \mathscr{T}$ and $\Gamma_1 \subset \Gamma_2$. Then*

$$\pi(h^{(1)}, h^{(2)}) = \{[\psi_C] : C \in \mathscr{C}(\Gamma_1, \Gamma_2)\},$$

where $[\psi]$ denotes the equivalence class of $\psi \in \Psi(h^{(1)}, h^{(2)})$. $\pi(h^{(1)}, h^{(2)})$ is finite and $|\pi(h^{(1)}, h^{(2)})|$ equals the number of equivalence classes in $\kappa(\Gamma_1, \Gamma_2)$.

COROLLARY 2. *If Γ is maximal and S is a factor of h on $\Gamma\backslash G$, then S is isomorphic to h and $|\pi(h, S)| = 1$.*

THEOREM 3. *Let S on (Y, ν) be a factor of h_1 (the time-one transformation of the horocycle flow) on $(M = \Gamma\backslash G, \mu)$, $\Gamma \in \mathscr{T}$ with a conjugacy $\psi : M \to Y$, $\psi h_1(x) = h_1\psi(x)$ a.e. $x \in M$. Then there exists a m.p. flow $\{S_t, t \in R\}$ on (Y, ν) such that $S = S_1$ and $\psi h_t(x) = S_t\psi(x)$ for all $t \in R$ and a.e. $x \in M$.*

COROLLARY 3. *If S is a factor of $h_1^{(1)}$ on $M_1 = \Gamma_1\backslash G$ then there is $\Gamma_2 \supset \Gamma_1$ such that S is isomorphic to $h_1^{(2)}$ on $M_2 = \Gamma_2\backslash G$. If Γ_1 is maximal then every factor of $h_1^{(1)}$ is isomorphic to $h_1^{(1)}$.*

The geodesic flow $g = \{g_t, t \in R\}$ on $M = \Gamma\backslash G$, $\Gamma \in \mathscr{T}$ is the algebraic flow, generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathscr{A}$, i.e.

$$g_t(\Gamma x) = \Gamma x \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(-t) \end{pmatrix}, \quad x \in G.$$

g and h satisfy the following commutation relation:

$$g_t \circ h_s = h_{s \exp(2t)} \circ g_t, \quad t, s \in R. \tag{$*$}$$

($*$) shows that h_α and h_β are isomorphic if $\alpha \cdot \beta > 0$ and that the entropy of h is zero.

It is well known that g is Bernoulli [**2**] and therefore g has uncountably many non-isomorphic factors. (*) shows that the entropy of g equals 2 for every $\Gamma \in \mathcal{T}$. This implies that $g^{(1)}$ is isomorphic to $g^{(2)}$ for any $\Gamma_1, \Gamma_2 \in \mathcal{T}$. One can show that $\pi(g^{(1)}, g^{(2)})$ is uncountable.

The proof of theorem 1 consists of three basic steps: (1) We show (§ 3) that if a flow S on (Y, ν) is a factor of the horocycle flow h on (M, μ) with a factor map $\psi: M \to Y$ then $\psi^{-1}\{y\}$ is finite for a.e. $y \in Y$. This uses the basic estimates on divergence of horocycles (§ 2) to show that ψ is locally 1–1; (2) using (1) we show that any factor map of the horocycle flow must be a factor map of the entire action of SL$(2, \boldsymbol{R})$ (§ 4); (3) using (2), we construct a discrete subgroup of SL$(2, \boldsymbol{R})$ for which the factor is a horocycle flow (the end of § 4).

Section 1 contains some measure-theoretical background and in § 5 we prove theorem 3.

I am grateful to Joe Wolf for valuable discussions.

1. *Factors and invariant partitions*

Henceforth all measure spaces are assumed to be separable and complete.

Let $S = \{S_t, t \in \boldsymbol{R}\}$ on (Y, ν) be a factor of $T = \{T_t, t \in \boldsymbol{R}\}$ on (X, μ) with a conjugacy $\psi: X \to Y$

$$\psi T_t(x) = S_t \psi(x) \quad \text{for all } t \in \boldsymbol{R} \text{ and a.e. } x \in X. \tag{1.1}$$

We can assume without loss of generality that (1.1) holds for all $x \in X$. ψ induces a measurable partition

$$\xi = \xi(\psi) = \{\psi^{-1}\{y\} : y \in Y\}$$

of X (see [**5**]), invariant under T, i.e. for every $t \in \boldsymbol{R}$

$$C \in \xi \quad \text{iff } T_t C \in \xi.$$

Let X/ξ be the quotient space, induced by ξ and let $\pi: X \to X/\xi$ be the projection $\pi(x) = C(x)$, where $C(x)$ denotes the atom of ξ, containing x. A set $A \subset X/\xi$ is called measurable in X/ξ if $\pi^{-1}(A)$ is measurable in X. We define a measure μ_ξ on X/ξ by $\mu_\xi(A) = \mu(\pi^{-1}(A))$. π is a conjugacy between T and the m.p. flow T^ξ on X/ξ defined by

$$T_t^\xi(C(x)) = C(T_t x), \quad x \in X, t \in \boldsymbol{R}.$$

It is clear, that T^ξ is isomorphic to S.

It is well known (see [**5**]) that for a.e. $C \in \xi$ there is a probability measure μ_C on C such that if $A \subset X$ is measurable in X then $A \cap C$ is measurable in C and

$$\mu(A) = \int_{X/\xi} \mu_C(A \cap C)\, d\mu_\xi(C). \tag{1.2}$$

Henceforth it will be clear from the context when $C \in \xi$ is considered as a subset of X and when it is considered as a point of X/ξ. The family of measures $\{\mu_C\}$ is unique in the following sense: a family $\{\mu'_C\}$ satisfies (1.2) iff $\mu'_C = \mu_C$ for a.e.

$C \in X/\xi$. This says that by possibly changing $\{\mu_C\}$ on a set of μ_ξ-measure zero we can get a set

$$\Omega \subset X/\xi, \quad T_t^\xi \Omega = \Omega, \quad t \in R, \quad \mu_\xi(\Omega) = 1$$

such that if $C \in \Omega$ then

$$\begin{aligned} &A \subset C \text{ is measurable in } C \text{ iff } T_t A \text{ is measurable in } T_t C \\ &\text{and } \mu_C(A) = \mu_{T_t C} T_t A \text{ for all } t \in R. \end{aligned} \tag{1.3}$$

We can assume without loss of generality that (1.3) holds for all $C \in X/\xi$, since T^ξ restricted on Ω is isomorphic to T^ξ on X/ξ.

We say that μ_C is atomic if there is $x \in C$ s.t. $\mu_C\{x\} > 0$.

PROPOSITION 1.1. *Suppose that T is ergodic and that there is $Z \subset X/\xi$, $\mu_\xi(Z) > 0$ such that μ_C is atomic for every $C \in Z$. Then there are*

$$U \subset X/\xi, \quad T_t^\xi U = U, \quad t \in R, \quad \mu_\xi(U) = 1,$$
$$D \subset X, \quad T_t D = D, \quad t \in R, \quad \mu(D) = 1$$

and an integer $n > 0$ such that for every $C \in U$, $D \cap C$ consists of exactly n points $x_1(C), \ldots, x_n(C)$ with

$$\mu_C\{x_i(C)\} = \frac{1}{n}, \quad i = 1, \ldots, n.$$

Proof. Let $m: X/\xi \to R$ be defined by

$$m(C) = \sup\{\mu_C\{x\} : x \in C\}.$$

m is measurable [**5**] and (1.3) shows that m is constant on orbits of T^ξ. Since T^ξ is ergodic, there is

$$U' \subset X/\xi, \quad T_t^\xi U' = U', \quad t \in R, \quad \mu_\xi(U') = 1$$

such that m equals a constant α on U'. Since

$$\mu_\xi(Z \cap U') > 0 \quad \text{and} \quad m(C) > 0$$

for every $C \in Z$, α must be positive.

Let

$$D = \{x \in X : C(x) \in U' \text{ and } \mu_C\{x\} = \alpha\}.$$

D is measurable [**5**] and (1.3) shows that D consists of orbits of T. It is clear, that $\mu(D) > 0$. Since T is ergodic, $\mu(D) = 1$.

Let

$$U = \{C \in U' : \mu_C(C \cap D) = 1\},$$
$$\mu_\xi(U) = 1, \quad T_t^\xi U = U, \quad t \in R.$$

If $x \in C \cap D$ then $\mu_C\{x\} = \alpha > 0$, $C \in U$. This says that $C \cap D, C \in U$ consists of finite many points $x_1(C), \ldots, x_n(C)$ and that $\alpha = 1/n$, since $\mu_C(C \cap D) = 1$, $C \in U$. This completes the proof. □

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It also follows from [5] that if a.e. $C \in X/\xi$ consists of n points of equal μ_C-measure, then there are a measurable

$$V \subset X/\xi, \quad \mu_\xi(V)=1, \quad \pi^{-1}(V)=\tilde{X}, \quad \mu(\tilde{X})=1$$

and pairwise disjoint measurable $X_i \subset X$, $i=1,\ldots,n$,

$$X=\bigcup_{i=1}^{n} X_i, \quad \mu(X_i)=\frac{1}{n}, \quad i=1,\ldots,n$$

such that if $C \in V$ then

$$C \cap X_i = \{x_i(C)\}$$

consists of exactly one point and the maps $\phi_i : \tilde{X}$ onto X_i defined by

$$\phi_i(x)=x_i(C(x))$$

are measurable, $i=1,\ldots,n$. The pair (X_i, ϕ_i) is called a measurable cross-section of ξ, $i=1,\ldots,n$.

2. *Properties of the covering horocycle flow in G*

Let $p: G \to M = \Gamma\backslash G$, $\Gamma \in \mathcal{T}$ be the covering projection $p(g)=\Gamma g$. Let

$$G_t g = g \cdot \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad H_t g = g \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad g \in G, t \in R$$

be the geodesic and the horocycle flows on G, covering $\{g_t\}$ and $\{h_t\}$ on M respectively. We shall also consider the flow

$$H_t^* g = g \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on G, covering the flow

$$h_t^*(\Gamma g) = \Gamma H_t^* g$$

on M.

We have

$$\begin{aligned} G_t \circ H_s &= H_{s \exp(2t)} \circ G_t \\ G_t \circ H_s^* &= H^*_{s \exp(-2t)} \circ G_t \end{aligned}, \quad t, s \in R. \tag{2.1}$$

We assume that G is equipped with a left invariant Riemannian metric, in which the length of the orbit intervals $[g, G_t g]$, $[g, H_t g]$ and $[g, H_t^* g]$ is t, $g \in G$. Let $d: G \times G \to R^+$ be the left invariant metric on G, induced by this Riemannian metric and let e denote the identity element of G.

Denote

$$\Delta(g) = \max\{|1-a|, |b|, |c|\} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

It is well known, that there is $A>1$ such that

$$A^{-1}\Delta(g) \le d(e, g) \le A\Delta(g) \quad \text{for all } g \in G \text{ with } d(g, e) \le 1. \tag{2.2}$$

For $x, y \in G$ we have

$$d(H_s x, H_s y) = d(e, N_{-s} \cdot g \cdot N_s)$$

where $g = x^{-1} \cdot y$ and $N_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$. It follows from (2.2) that if $d(H_s x, H_s y) \le 1$, then

$$A^{-1}\Delta(N_{-s} \cdot g \cdot N_s) \le d(H_s x, H_s y) \le A\Delta(N_{-s} \cdot g \cdot N_s)$$

where, (2.3)

$$\Delta(N_{-s} \cdot g \cdot N_s) = \max\{|1 - a - bs|, |b|, |bs^2 + s(a-d) - c|\}$$

and

$$g = x^{-1} \cdot y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $0 < \varepsilon \le 1$ be small and suppose that $d(x, y) < \varepsilon$. We shall now estimate the length of the time the horocycle orbits $H_t x$ and $H_t y$ stay within ε. (2.3) shows that $d(H_s x, H_s y)$ grows polynomially in s. We have

$$\{s \in R^+ : d(H_s x, H_s y) \le \varepsilon\} \subset \{s \in R^+ : \Delta(N_{-s} \cdot g \cdot N_s) \le A \cdot \varepsilon\} = E(g, \varepsilon) \quad (2.4)$$

where $g = x^{-1} \cdot y$ and $\Delta(N_{-s} \cdot g \cdot N_s)$ are as in (2.3).

It is easy to compute that:

(1) $E(g, \varepsilon)$ consists of at most two connected components $E_0 = E_0(g, \varepsilon) \ni 0$ and $E_1 = E_1(g, \varepsilon)$;

(2) If

$$l = l(g, \varepsilon) = \max\{l(E_0), l(E_1)\} \ge 1 \text{ ($l(I)$ denotes the length of I)},$$

then for every $s \in E(g, \varepsilon)$ we have

$$|1 - a_s| \le D(\varepsilon)/l, \quad |b_s| \le D(\varepsilon)/l^2, \quad |c_s| \le \varepsilon \quad (2.5)$$

where

$$\begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} = N_{-s} \cdot g \cdot N_s \quad \text{and} \quad \varepsilon \le D(\varepsilon) \to 0$$

when $\varepsilon \to 0$.

It follows from (2.3) and (2.5) that if $l \ge 1$ then

$$\Delta(N_{-s-u} \cdot g \cdot N_{s+u}) \le 3D(\varepsilon) \quad \text{for all } s \in E(g, \varepsilon) \text{ and all } 0 \le u \le l.$$

This implies that

$$d(H_{s+u} x, H_{s+u} y) \le 3AD(\varepsilon) \quad \text{for all } s \in E(g, \varepsilon) \text{ and all } 0 \le u \le l. \quad (2.6)$$

Henceforth $D(\varepsilon)$ will always mean a constant depending only on ε and converging to 0 when $\varepsilon \to 0$.

Let us observe that if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \Delta(g) < \varepsilon$$

and ε is sufficiently small then

$$g = H_q H_r^* G_p e \text{ where } p = \log a/(1 + bc), \quad r = b\, e^{-p}, \quad q = c\, e^p. \quad (2.7)$$

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For $g \in G$ and $\alpha, \beta, \gamma \geq 0$ we define

$$U(g; \alpha, \beta, \gamma) = \{\tilde{g} \in G: \tilde{g} = H_q H_r^* G_p g \text{ for some } |p| \leq \alpha, |r| \leq \beta, |q| \leq \gamma\}.$$

It follows from (2.1) that for every $t \in R$

$$G_t U(g; \alpha, \beta, \gamma) = U(G_t g; \alpha, \beta e^{-2t}, \gamma e^{2t}). \tag{2.8}$$

It follows from (2.4), (2.5) and (2.7) that if $s \in E(x^{-1} \cdot y, \varepsilon)$ and $l = l(x^{-1} \cdot y, \varepsilon) \geq 1$ then

$$H_s y \in U(H_s x, D(\varepsilon)/l, D(\varepsilon)/l^2, D(\varepsilon)) \tag{2.9}$$

where $D(\varepsilon) \to 0$ when $\varepsilon \to 0$.

We shall need the following:

LEMMA 2.1. *Given $0 < \delta < 1$ there are $\tilde{\delta} > 0$ and $\bar{\delta} > 0$ depending only on δ such that if $d(x, y) < \tilde{\delta}$, $x, y \in G$ then for every $s \in E(x^{-1} \cdot y, 1)$ and every $0 \leq u < \bar{\delta} l(x^{-1} \cdot y, 1)$ with $s + u \notin E(x^{-1} y, 1)$*

$$\text{either } d(H_{s+u} x, H_{s+u+1} y) < \delta \text{ or } d(H_{s+u} x, H_{s+u-1} y) < \delta. \tag{2.10}$$

Proof. It is enough to show that there are $\tilde{\delta} > 0$ and $\bar{\delta} > 0$ such that if $\Delta(g) < \tilde{\delta}$, $g \in G$ then for every $s \in E(g, 1)$ and every

$$0 \leq u \leq \bar{\delta} l(g, 1) \quad \text{and} \quad s + u \notin E(g, 1)$$

we have $|c_{s+u}| > 1$ and

$$\max\{|1 - a_{s+u}|, |b_{s+u}|, |c_{s+u} - \operatorname{sign} c_{s+u}|\} \leq \delta,$$

where

$$g_s = N_{-s} \cdot g \cdot N_s = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}$$

and $\operatorname{sign} c = c/|c|$ if $c \neq 0$.

Let $0 < \tilde{\delta} < \delta$ be so small that if $\Delta(g) < \tilde{\delta}$ then

$$\Delta(g_s) < 1 \quad \text{for all } 0 \leq s \leq 2D(1)/\delta.$$

(see (2.5) for the definition of $D(1)$). This says that

$$l = l(g, 1) \geq 2D(1)/\delta.$$

Let $\bar{\delta} = \delta/4D(1)$ and let $s \in E(g, 1)$, $0 \leq u \leq \bar{\delta} l$. We have using (2.3) and (2.5)

$$\begin{aligned} |b_{s+u}| &= |b_s| \leq D(1)/l^2 \leq \delta \\ |1 - a_{s+u}| &= |1 - a_s - b_s u| \leq D(1)/l + \bar{\delta} l \cdot D(1)/l^2 \leq \delta \end{aligned} \tag{2.11}$$

$$|c_{s+u} + c_s| = |b_s u^2 + u(a_s - d_s)| \leq \bar{\delta}^2 D(1) + 3\bar{\delta} D(1) \leq 4D(1)\bar{\delta} = \delta. \tag{2.12}$$

(2.11) shows that

$$|c_{s+u}| > 1 \quad \text{if } s + u \notin E(g, 1)$$

since $\Delta(g_{s+u}) > 1$ for $s + u \notin E(g, 1)$. Also $|c_s| \leq 1$ for $s \in E(g, 1)$. This and (2.12) imply that

$$|c_{s+u} - \operatorname{sign} c_{s+u}| \leq \delta \quad \text{if } s + u \notin E(g, 1).$$

This completes the proof. □

Denote

$$W_\varepsilon(g) = U(g; \varepsilon, \varepsilon, 0), g \in G.$$

We say that $x, y \in G$, $y \in W_\varepsilon(x)$ form an ε-strip of length $t \geq 0$ if for every $s \in [0, t]$ there is $q(s) \geq 0$, $q(0) = 0$ such that

$$H_{q(s)}y \in W_\varepsilon(H_s x). \tag{2.13}$$

$q(s) = q(s, x, y)$ is uniquely defined by (2.13) and is a smooth function of (s, x, y). It is easy to compute that

$$|q(s) - s| = D(\varepsilon)s, \tag{2.14}$$

where $D(\varepsilon) \to 0$ when $\varepsilon \to 0$. It follows from (2.1) that if x, y form an ε-strip of length t then $G_\tau x$, $G_\tau y$, $\tau \geq 0$ form an ε-strip of length $t e^{2\tau}$.

3. *h-invariant partitions*

Let $h = \{h_t, t \in R\}$ be the horocycle flow on $(M = \Gamma\backslash G, \mu)$ and let S on (Y, ν) be a factor of h_1 (the time-one transformation of the flow h_t) with a conjugacy $\psi: M \to Y$

$$\psi h_1(x) = S\psi(x) \quad \text{for a.e. } x \in M. \tag{3.1}$$

LEMMA 3.1. *Let ζ be the partition of M induced by ψ (see § 1). Then there exists $Z \subset M/\zeta$, $\mu_\zeta(Z) > 0$ such that μ_C is atomic for every $C \in Z$.*

Proof. We can assume without loss of generality that Y is a compact metric space and S is a homeomorphism of Y onto itself. Moreover, there exists $\varepsilon_Y > 0$ such that

$$d_Y(y, Sy) > \varepsilon_Y \quad \text{for every } y \in Y, \tag{3.2}$$

where d_Y denotes the metric in Y (see for instance [**3**]).

Let $0 < \theta < 0.01$ be fixed.

Since $\psi: M \to Y$ is measurable, there is $\Lambda \subset M$, $\mu(\Lambda) > 1 - \theta$ such that ψ is uniformly continuous on Λ (see lemma 3.1 in [**4**]).

Let $0 < \delta < 1$ be such that

$$\text{if } d(w_1, w_2) < \delta, \quad w_1, w_2 \in \Lambda \quad \text{then } d_Y(\psi w_1, \psi w_2) < \varepsilon_Y.$$

Let $\tilde{\delta} = \tilde{\delta}(\delta) > 0$ and $\bar{\delta} = \bar{\delta}(\delta) > 0$ be as in lemma 2.1. Since h_1 is ergodic, there are $V \subset M$, $\mu(V) > 1 - \bar{\delta}/100$ and an integer $n_0 > 0$ such that

$$\begin{aligned} &\text{if } n \geq n_0 \text{ and } x \in V \text{ then the relative frequency of} \\ &\Lambda \text{ on } \{x, h_1 x, \ldots, h_n x\} \text{ is at least } 1 - 2\theta. \end{aligned} \tag{3.3}$$

Let $\tilde{V} \subset M$, $\mu(\tilde{V}) > 1 - \theta$ and an integer $n_1 > n_0$ be such that

$$\begin{aligned} &\text{if } n \geq n_1 \text{ and } x \in \tilde{V} \text{ then the relative frequency of} \\ &V \text{ on } \{x, h_1 x, \ldots, h_n x\} \text{ is at least } 1 - \bar{\delta}/90. \end{aligned} \tag{3.4}$$

Let $0 < \delta_1 < \tilde{\delta}$ be so small that if $d(x, y) < \delta_1$, $x, y \in G$ then

$$d(H_s x, H_s y) < 1 \quad \text{for all } 0 \leq s \leq 2n_1/\bar{\delta}. \tag{3.5}$$

We claim that

$$d(u, v) \geq \delta_1 \tag{3.6}$$

for every $u, v \in C \cap \tilde{V}$, $u \neq v$ and every $C \in \zeta$.

Suppose on the contrary that there are $C_0 \in \zeta$ and $u_0, v_0 \in C_0 \cap \tilde{V}$, $u_0 \neq v_0$ such that $d(u_0, v_0) < \delta_1$.

Let $x_0 = p^{-1}(u_0)$, $y_0 = p^{-1}(v_0)$, $x_0, y_0 \in G$ be such that $d(x_0, y_0) = d(u_0, v_0)$ and let $E = E(x_0^{-1} \cdot y_0, 1) = E_0 \cup E_1$ be as in (2.5) (E_1 can be empty), $E_0 = [0, s_0]$, $E_1 = [s_1, s_2]$, $s_1 > s_0$.

(3.5) implies that

$$2n_1/\bar{\delta} \leq l(E_0) \leq \max\{l(E_0), l(E_1)\} = l.$$

Denote

$$F_0 = [s_0, s_0 + \bar{\delta}l/2], \quad F = [0, s_0] \cup F_0 \quad \text{if } s_1 - s_0 > \bar{\delta}l$$

and

$$F_0 = [s_2, s_2 + \bar{\delta}l/2], \quad F = [0, s_2] \cup F_0 \quad \text{if } s_1 - s_0 \leq \bar{\delta}l.$$

We have $F_0 \subset F - E$ and

$$|F| \geq n_1 \quad \text{and} \quad |F_0|/|F| \geq \bar{\delta}/20. \tag{3.7}$$

where $|F|$ denotes the number of integers in F.

Let

$$\tilde{J} = \{m \in F: m \text{ is an integer and } h_m u_0 \in V, h_m v_0 \in V\}.$$

It follows from (3.4) that

$$|\tilde{J}|/|F| \geq 1 - \bar{\delta}/40$$

since $u_0, v_0 \in \tilde{V}$ and $|F| > n_1$. This and (3.7) imply that there is an integer m_0 such that

$$m_0 \in F_0 \cap \tilde{J}.$$

Denote

$$J = \{m \in [m_0, m_0 + \bar{\delta}l/2]: m \text{ is an integer and } h_m u_0 \in \Lambda, h_{m-1} v_0 \in \Lambda, h_{m+1} v_0 \in \Lambda\}.$$

It follows from (3.3) that

$$|J|/|[m_0, m_0 + \bar{\delta}l/2]| \geq 1 - 6\theta,$$

since

$$h_{m_0} u_0, h_{m_0} v_0 \in V \quad \text{and} \quad \bar{\delta}l/2 > n_1 > n_0.$$

This implies that there is

$$m_1 \in [m_0, m_0 + \bar{\delta}l/2] \subset [s_0, s_0 + \bar{\delta}l/2] \cup [s_2, s_2 + \bar{\delta}l/2]$$

such that

$$h_{m_1} u_0 \in \Lambda, \quad h_{m_1 - 1} v_0 \in \Lambda \quad \text{and} \quad h_{m_1 + 1} v_0 \in \Lambda. \tag{3.8}$$

It follows from lemma 2.1 that

$$\text{either } d(h_{m_1} u_0, h_{m_1+1} v_0) < \delta \quad \text{or} \quad d(h_{m_1} u_0, h_{m_1-1} v_0) < \delta \tag{3.9}$$

since $d(u_0, v_0) < \delta_1 < \tilde{\delta}(\delta)$.

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Assume for simplicity that the first condition of (3.9) holds. We have by (3.8) and our choice of δ

$$d_Y(\psi h_{m_1}u_0, \psi h_{m_1+1}v_0) < \varepsilon_Y. \tag{3.10}$$

(3.1) implies that

$$\psi(h_{m_1+1}v_0) = S\psi(h_{m_1}v_0).$$

Also

$$\psi(h_{m_1}u_0) = \psi(h_{m_1}v_0) = y$$

since $u_0, v_0 \in C_0 \in \zeta$. (3.10) implies then that

$$d_Y(y, Sy) < \varepsilon_Y$$

which contradicts (3.2). So we have proved (3.6).

Since $\mu(\tilde{V}) > 0$ there is $Z \subset M/\zeta$, $\mu_\zeta(Z) > 0$ such that

$$\mu_C(C \cap \tilde{V}) > 0 \quad \text{for every } C \in Z. \tag{3.11}$$

(3.6) implies that $C \cap \tilde{V}$ is at most countable. This implies via (3.11) that μ_C is atomic for every $C \in Z$. This completes the proof. □

Note 3.1. It follows from the proof of lemma 3.1 that given $0 < \theta < 0.01$ there are a compact $K \subset M$, $\mu(K) > 1 - \theta$ and $\delta_1 > 0$ such that

$$d(u, v) \geq \delta_1 \quad \text{for every } u, v \in C \cap K, u \neq v \text{ and every } C \in \zeta.$$

4. *Algebraicity of* ξ

From now on our discussion will be similar to [**4**].

Let $S = \{S_t, t \in R\}$ on (Y, ν) be a factor of $h = \{h_t, t \in R\}$ on (M, μ) with a conjugacy $\psi: M \to Y$

$$\psi h_t(x) = S_t\psi(x) \quad \text{for all } t \in R \text{ and a.e. } x \in M,$$

and let ξ be the h-invariant partition of M, induced by ψ. It follows from proposition 1.1 and lemma 3.1, that there are $D \subset M$, $h_tD = D$, $t \in R$, $\mu(D) = 1$, $U \subset M/\xi$, $h_t^\xi U = U$, $t \in R$, $\mu_\xi(U) = 1$ and an integer $n > 0$ such that for every $C \in U$ the intersection $D \cap C$ consists of exactly n points with μ_C-measure $1/n$.

We assume without loss of generality that $D = M$ and $U = M/\xi$. Thus each $C \in \xi$ consists of n distinct points of μ_C-measure $1/n$.

Let $0 < \theta < 0.01$ be given. Using the discreteness of $\Gamma \in \mathcal{T}$, $M = \Gamma \backslash G$ and note 3.1, we can get a compact $K \subset M$, $\mu(K) > 1 - \theta^2/n^2$ and $\rho > 0$ such that

(1) if $x \in p^{-1}(K)$ then the projection $p: G \to M$, $p(g) = \Gamma g$ is an isometry on the ball of radius ρ centered at x. (4.1)

(2) $d(u, v) \geq \rho$ for every $u, v \in C \cap K$, $u \neq v$, $C \in \xi$.

Let

$$K' = \pi^{-1}\left\{C \in M/\xi : \mu_C(C \cap K) > 1 - \frac{\theta}{n}\right\},$$

where $\pi: M \to M/\xi$ is the projection $\pi(x) = \xi(x)$, $x \in M$. K' consists of atoms of ξ. We have

$$\mu(K') > 1 - \theta/n \quad \text{and} \quad K' \subset K, \tag{4.2}$$

since $\mu(K)>1-\theta^2/n^2$ and every $C\in\xi$ consists of n points of μ_C-measure $1/n$.

Let $0<\varepsilon<\rho/2$ be so small that

$$\varepsilon<1 \text{ (see (2.2))} \quad \text{and} \quad 3AD(\varepsilon)<\rho/2 \text{ in (2.6)}. \tag{4.3}$$

Let $0<\delta_0<\varepsilon$ be so small that if $d(x, y)<\delta_0$, $x, y\in G$ then

$$d(H_s x, H_s y)<\varepsilon \quad \text{for all } 0\le s\le 1. \tag{4.4}$$

Let $u\in K$, $v\in M$ and $d(u, v)<\delta<\delta_0$. Let $x, y\in G$ be such that $p(x)=u$, $p(y)=v$ and $d(x, y)<\delta$. Denote

$$E(u, v, \varepsilon)=E_0(x^{-1}\cdot y, \varepsilon)$$

where $E_0(x^{-1}\cdot y, \varepsilon)$ is defined in (2.5). $E(u, v, \varepsilon)$ is well defined and does not depend on the choice of $x\in p^{-1}(u)$, $y\in p^{-1}(v)$, since $u\in K$ and $\delta<\rho$. It follows from (4.4) that $l(E(u, v, \varepsilon))\ge 1$. Henceforth $\xi(v)$ denotes the atom of ξ, containing v.

LEMMA 4.1. *Let* $0<\delta<\delta_0$, $u, v\in M$ *and* $A_t=A_t(u, v, \delta)=\{s\in[0, t]$: *there exists* $v(s)\in\xi(v)$ *such that* $h_s v(s)\in K'$ *and* $d(h_s u, h_s v(s))<\delta\}$, $t\ge 1$. *If* $l(A_t)>0.9t$ *then there is* $s\in A_t$ *such that* $l(E(h_s u, h_s v(s), \delta))\ge 0.2t$.

Proof. The proof is similar to that of lemma 2.1 in [**4**]. Let

$$E_s=s+E(h_s u, h_s v(s), \delta), \quad s\in A_t.$$

We claim that

$$\text{if } s_1\in A_t \quad \text{and} \quad v(s_1)\ne v(s) \quad \text{then } s_1\notin E_s. \tag{4.5}$$

Indeed, suppose on the contrary that $s_1\in E_s$. Then

$$d(h_{s_1}u, h_{s_1}v(s))<3AD(\varepsilon)<\rho/2$$

by (2.6) and (4.3). Also we have

$$d(h_{s_1}v(s_1), h_{s_1}u)<\delta<\rho/2,$$

since $s_1\in A_t$. This implies that

$$d(h_{s_1}v(s_1), h_{s_1}v(s))<\rho. \tag{4.6}$$

We have

$$h_{s_1}v(s)\in\xi(h_{s_1}v(s_1)),$$

since $v(s), v(s_1)\in\xi(v)$. Also

$$h_{s_1}v(s_1)\in K',$$

since $s_1\in A_t$ and therefore

$$h_{s_1}v(s)\in K',$$

since K' consists of atoms of ξ. This and (4.6) imply that

$$h_{s_1}v(s)=h_{s_1}v(s_1)$$

which contradicts $v(s)\ne v(s_1)$ in (4.5).

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Let $\beta=\{E_1,\ldots,E_m\}$ be the collection of pairwise disjoint intervals $E_i=[s_i,\tau_i]\subset[0,t]$, $s_j>\tau_i$, $j>i$, such that $E_i=E_s$ for some $s\in A_t$, $i=1,\ldots,m$ and $A_t\subset\bigcup_{i=1}^m E_i$ and let $d(E_i,E_j)=s_j-\tau_i$.

Let $x\in G$ be such that $p(x)=u$, $x_i=H_{s_i}x$, $p(x_i)=h_{s_i}u=u_i$ and let $y_i\in G$ be such that $d(x_i,y_i)<\delta$ and $p(y_i)=h_{s_i}v(s_i)=v_i$. We have

$$E_i=s_i+E_0(x_i^{-1}\cdot y_i,\delta)\subset s_i+E(x_i^{-1}\cdot y_i,\delta)$$

and

$$l(E_i)\le l(x_i^{-1}\cdot y_i,\delta)=l_i$$

(see (2.5)). Suppose that $s_j-s_i=q$ and $v(s_i)=v(s_j)$. We have

$$(h_{s_j}u,h_{s_j}v(s_j))=(u_j,v_j)=(h_qu_i,h_qv_i).$$

Though $d(x_i,y_i)<\delta$, $p(x_i)=u_i$, $p(y_i)=v_i$ and $d(u_j,v_j)<\delta$, it is not necessarily true that

$$d(H_qx_i,H_qy_i)<\delta,$$

but there is a unique $\mathscr{D}\in\Gamma$ such that

$$d(H_qx_i,\mathscr{D}\cdot H_qy_i)<\delta. \tag{4.7}$$

We write $E_i\overset{\Gamma}{\sim}E_j$ if $v(s_i)=v(s_j)$ and $\mathscr{D}\neq e$ in (4.7), $E_i\overset{e}{\sim}E_j$ if $v(s_i)=v(s_j)$ and $\mathscr{D}=e$ in (4.7) and $E_i\overset{\xi}{\sim}E_j$ if $v(s_i)\neq v(s_j)$. It follows from (2.6) and (4.3) that

$$d(H_{q_i+s}x_i,H_{q_i+s}y_i)\le 3AD(\varepsilon)<\rho/2 \tag{4.8}$$

for all $0\le s\le l_i$, where $q_i=\tau_i-s_i$, $i=1,\ldots,m$. This implies via (4.1) that

$$s_j-\tau_i=d(E_i,E_j)\ge l_i \text{ if } E_i\overset{\Gamma}{\sim}E_j \tag{4.9}$$

since $y_j\in p^{-1}(K)$. (4.8) also shows that

$$d(h_{\tau_i+s}u,h_{\tau_i+s}v(s_i))=d(h_{q_i+s}u_i,h_{q_i+s}v_i)<\rho/2$$

for all $0\le s\le l_i$. This implies that

$$s_j-\tau_i=d(E_i,E_j)\ge l_i \quad \text{if } E_i\overset{\xi}{\sim}E_j, \tag{4.10}$$

since otherwise we would have

$$d(h_{s_j}v(s_i),h_{s_j}v(s_j))<\rho$$

which contradicts (4.1), since $v(s_i)\neq v(s_j)$, $h_{s_j}v(s_j)\in K'$ and $h_{s_j}v(s_i)\in\xi(h_{s_j}v(s_j))\subset K'$.

Let us now define a new collection $\bar\beta=\{\bar E_1,\ldots,\bar E_{\bar m}\}$ by the following procedure. We set $\bar E_1=E_1$ unless $E_1\overset{e}{\sim}E_2$ and $d(E_1,E_2)\le l(E_1)$. In this last case we set $\bar E_1=[s_1,\tau_2]\supset E_1\cup E_2$. Suppose $\bar E_k$, $k=1,\ldots,p$ have been defined. To define $\bar E_{p+1}$ we apply the same construction to the first $E\in\beta$, which has not been included in any $\bar E_k$, $k=1,\ldots,p$.

It follows from the construction of $\bar\beta$ that

$$d(\bar E_k,\bar E_{k+1})\ge l(\bar E_k) \quad \text{if } \bar E_k\overset{e}{\sim}\bar E_{k+1} \tag{4.11}$$

and for each $\bar{E}_k \in \bar{\beta}$ there is $E_{i_k} \in \beta$ such that

$$\text{either } \bar{E}_k = E_{i_k} \quad \text{or} \quad \bar{E}_k \supset (E_{i_k} \cup E_{i_k+1}) \quad \text{and} \quad l(\bar{E}_k) \le 3l_{i_k}. \tag{4.12}$$

This, (4.9) and (4.10) imply

$$d(\bar{E}_k, \bar{E}_{k+1}) \ge l_{i_k} \ge l(\bar{E}_k)/3$$

if $\bar{E}_k \overset{\Gamma}{\sim} \bar{E}_{k+1}$ or $\bar{E}_k \overset{\xi}{\sim} \bar{E}_{k+1}$. This and (4.11) give

$$d(\bar{E}_k, \bar{E}_{k+1}) \ge l(\bar{E}_k)/3 \quad \text{for all } k = 1, \ldots, \bar{m}-1. \tag{4.13}$$

Denote

$$l(\bar{\beta}) = \sum_{k=1}^{\bar{m}} l(\bar{E}_k).$$

We have

$$l(\bar{\beta}) > 0.9t,$$

since $A_t \subset \bigcup_{k=1}^{m} \bar{E}_k$.

This and (4.13) imply that there is $\bar{E} \in \bar{\beta}$ such that

$$l(\bar{E}) \ge 0.6t.$$

This implies via (4.12) that there is $E \in \beta$ such that $l(E) \ge 0.2t$. This completes the proof. □

Corollary 4.1. *Let $u, v \in M$ and let $l(A_t) > 0.9t$ for all $t \ge t_0 > 1$, where $A_t = A_t(u, v, \delta)$ as in lemma 4.1. Then there is $\tilde{v} \in \xi(v)$ such that $\tilde{v} = h_q u$ for some $q = q(u, v, \delta)$, $|q| < \delta$.*

Proof. It follows from the proof of lemma 4.1 that there is $s \ge 0$ such that

$$l(E(h_s u, h_s v(s), \delta)) \ge 0.2t \quad \text{for all } t \ge t_0.$$

(2.5) shows that this may happen only if $h_s v(s) = h_q h_s u$ for some $|q| < \delta$. We get $\tilde{v} = v(s) = h_q u$, $\tilde{v} \in \xi(v)$. □

For $A \subset M$ we shall write $A < \xi$ if A consists of atoms of ξ.

According to § 1 there are $X < \xi$, $\mu(X) = 1$ and pairwise disjoint measurable sets

$$X_i \subset X, i = 1, \ldots n, \bigcup_{i=1}^{n} X_i = X, \mu(X_i) = \frac{1}{n}$$

such that for every $x \in X$ the intersection

$$\xi(x) \cap X_i = \{x_i(x)\}$$

consists of exactly one point and the map $\phi_i: X$ onto X_i defined by $\phi_i(x) = x_i(x)$ is measurable, $i = 1, \ldots, n$.

Let K' be the set defined in (4.2) and let

$$\tilde{K} = K' \cap X, \quad \mu(\tilde{K}) = \mu(\tilde{K}') > 1 - \frac{\theta}{n}, \quad \tilde{K} < \xi.$$

Since $\phi_i: X \to X_i$ is measurable, $i = 1, \ldots, n$ there is $\Lambda \subset X$, $\mu(\Lambda) > 1 - \theta$ such that $\Lambda < \xi$ and each ϕ_i, $i = 1, \ldots, n$ is uniformly continuous on Λ (see lemma 3.1 in [**4**]).

Let

$$Q = \Lambda \cap \tilde{K}, \quad \mu(Q) > 1 - 2\theta, \quad Q < \xi$$

and let Ω be the generic set of Q for h,

$$h_t\Omega = \Omega, \quad t \in R, \quad \mu(\Omega) = 1, \quad \Omega < \xi.$$

LEMMA 4.2. *For every $0 < \delta < \delta_0$ there is $\omega = \omega(\delta) > 0$ such that if $u_1, v_1 \in \Omega$, $v_1 = g_p u_1$ for some $|p| < \omega$, then for every $u_2 \in \xi(u_1)$ there is $v_2 \in \xi(v_1)$ such that $v_2 = h_b g_p u_2$ for some $b = b(u_1, u_2, p)$, $|b| < \delta$ and $b(h_t u_1, h_t u_2, p) = b(u_1, u_2, p)$ for all $t \in R$.*

Proof. Since ϕ_i, $i = 1, \ldots, n$ are uniformly continuous on Λ there is $0 < \omega < \delta/2$ such that

$$\text{if } d(w_1, w_2) < \omega,\ w_1, w_2 \in \Lambda \text{ then } d(\phi_i \cdot w_1, \phi_i \cdot w_2) < \delta/2,\ i = 1, \ldots, n. \tag{4.14}$$

Let $u_1, v_1 \in \Omega$, $v_1 = g_p u_1$ for some $|p| < \omega$. Let $\lambda_0 > 0$ be such that

$$\text{if } \lambda \geq \lambda_0 \text{ then the relative length measure of } Q \text{ on } [u_1, h_\lambda u_1] \text{ and on } [v_1, h_\lambda v_1] \text{ is at least } 1 - 3\theta. \tag{4.15}$$

Let $x, y \in G$, $y = G_p x$ be such that $p(x) = u_1$, $p(y) = v_1$. x and y form an ω-strip of length λ for every $\lambda > 0$. We have

$$H_{q(s)} y = G_p H_s \quad \text{(see (2.13)) and} \quad h_{q(s)} v_1 = g_p h_s u_1 \quad \text{for all } s \geq 0.$$

Denote

$$F_\lambda = \{s \in [0, \lambda]: h_s u_1 \in Q,\ h_{q(s)} v_1 \in Q\}.$$

It follows from (4.15) that

$$l(F_\lambda) > (1 - 7\theta)\lambda \tag{4.16}$$

if $\omega > 0$ is sufficiently small and $\lambda \geq \lambda_\delta$, $q(\lambda) \geq \lambda_0$ (see (2.14)).

Let $u_2 \in \xi(u_1)$. We write $j(t) = i \in \{1, \ldots, n\}$ if $h_t u_2 \in X_i$.

We have

$$\phi_{j(s)}(h_s u_1) = h_s u_2 \in X_{j(s)}$$

$$\phi_{j(s)}(h_{q(s)} v_1) \in \xi(h_{q(s)} v_1) = h_{q(s)} \xi(v_1)$$

or

$$\phi_{j(s)}(h_{q(s)} v_1) = h_{q(s)} v_1(q(s)),$$

where $v_1(q(s)) \in \xi(v_1)$ and if $s \in F_\lambda$ then

$$h_s u_2 \in K', \quad h_{q(s)} v_1(q(s)) \in K'$$

and

$$d(h_s u_2, h_{q(s)} v_1(q(s))) < \delta/2 \tag{4.17}$$

by (4.14). Let $w = g_p u_2$. We have

$$h_{q(s)} w = g_p h_s u_2$$

and therefore

$$d(h_s u_2, h_{q(s)} w) < \omega.$$

This and (4.17) imply that

$$d(h_{q(s)}w, h_{q(s)}v_1(q(s))) < \omega + \delta/2 \le \delta \tag{4.18}$$

for all $s \in F_\lambda$ and all $\lambda \ge \lambda_0$, $q(\lambda) \ge \lambda_0$.

Let $A_t = A_t(w, v_1, \delta)$ be as in lemma 4.1. (4.16) and (4.18) show that there is $t_0 > 1$ such that

$$l(A_t) > 0.9t \quad \text{for all } t \ge t_0.$$

It follows then from corollary 4.1 that there is $v_2 \in \xi(v_1)$ such that $v_2 = h_b w = h_b q_p u_2$ for some $b = b(u_1, u_2, p)$, $|b| < \delta$. It is clear, that $b(h_t u_1, h_t u_2, p) = b(u_1, u_2, p)$ for all $t \in R$, $|p| < \omega$. □

It follows from lemma 4.2 that there exists $\omega_0 > 0$ such that

$$g_p w \in \Omega \text{ iff } g_p u \in \Omega$$

for every $u \in \Omega$, $w \in \xi(u)$, $|p| < \omega_0$, since Ω is h-invariant and $\Omega < \xi$.

Let

$$\Omega_p = \{u \in \Omega : g_p u \in \Omega\}, |p| < \omega_0.$$

Ω_p is h-invariant, $\mu(\Omega_p) = 1$ and $\Omega_p < \xi$.

LEMMA 4.3. *There is an h-invariant* $\Omega'_p \subset \Omega_p$, $\Omega'_p < \xi$, $\mu(\Omega'_p) = 1$ *such that* $b(u, w, p) = 0$ *for all* $u \in \Omega'_p$, $w \in \xi(u)$, $|p| < \omega_0$.

Proof. It follows from the definition of $b(u, w, p)$ that it is measurable and

$$b(u, w, p) = -b(w, u, p)$$
$$b(x, w, p) = b(u, w, p) - b(u, x, p), x, w \in \xi(u), \tag{4.19}$$
$$u \in \Omega_p, |p| < \omega_0.$$

Define $\bar{f}_p : \Omega_p \to R$ and $\tilde{f}_p : \Omega_p \to R$ by

$$\bar{f}_p(u) = \max\{b(u, w, p) : w \in \xi(u)\}$$
$$\tilde{f}_p(u) = \min\{b(u, w, p) : w \in \xi(u)\}.$$

The functions $\bar{f}_p$ and $\tilde{f}_p$ are measurable and constant on orbits of h. Since h is ergodic, there are $\Omega'_p \subset \Omega_p$, $\mu(\Omega'_p) = 1$, $\Omega'_p < \xi$ and constants $\bar{\sigma}, \tilde{\sigma}$ such that $\bar{f}_p = \bar{\sigma}$ and $\tilde{f}_p = \tilde{\sigma}$ on Ω'_p.

We claim that $\bar{\sigma} = \tilde{\sigma} = 0$. Indeed, suppose on the contrary that $\bar{\sigma} > 0$. Let $u \in \Omega'_p$ and $w \in \xi(u)$ be such that

$$b(u, w, p) = \bar{\sigma}.$$

Then

$$b(w, u, p) = -\bar{\sigma} < 0$$

and therefore $\tilde{\sigma} < 0$.

Let $x \in \xi(u)$ be such that

$$b(u, x, p) = \tilde{\sigma}.$$

Then

$$b(x, w, p) = \bar{\sigma} - \tilde{\sigma} > \bar{\sigma}$$

by (4.19) which contradicts the fact that $\bar\sigma = \max\{b(x, w, p): w \in \xi(x)\}$. Therefore $\bar\sigma = \tilde\sigma = 0$. This completes the proof. □

Let

$$\tilde\Omega = \bigcap_{\substack{p \text{ is rational} \\ |p|<\omega_0}} \Omega'_p.$$

$\tilde\Omega$ is h-invariant, $\mu(\tilde\Omega) = 1$ and $\tilde\Omega < \xi$. We have

$$g_p(\xi(u)) = \xi(g_p u)$$

for all $u \in \tilde\Omega$ and all rational $|p| < \omega_0$.

Let $\bar\Omega = \{u \in M: \tilde\Omega$ is dense on the geodesic orbit of $u\}$. $\bar\Omega$ is h-invariant, $\mu(\bar\Omega) = 1$ and $\bar\Omega \cap \tilde\Omega < \xi$. Lemma 4.2 shows that $b(u, w, p)$ is continuous in p. This implies that

$$g_p(\xi(u)) = \xi(g_p u) \tag{4.20}$$

for all $u \in \bar\Omega \cap \tilde\Omega$ and all $p \in R$ with $g_p u \in \Omega$.

Let $g_p u \in M - \Omega$ for some $u \in \bar\Omega \cap \tilde\Omega$, $p \in R$. We have

$$\xi(g_p u) \subset M - \Omega, \quad \text{since } \Omega < \xi;$$
$$g_p(\xi(u)) \subset M - \Omega \quad \text{by (4.20)}.$$

Let us define a partition $\bar\xi$ on $\bar\Omega$ by

$$\bar\xi(g_p u) = \xi(g_p u) \quad \text{if } u \in \bar\Omega \cap \tilde\Omega,\ g_p u \in \Omega$$
$$\bar\xi(g_p u) = g_p(\xi(u)) \quad \text{if } u \in \bar\Omega \cap \tilde\Omega,\ g_p u \notin \Omega.$$

We have

$$\bar\xi = \xi \text{ on } \bar\Omega \cap \Omega < \xi \quad h_t\bar\xi(u) = \bar\xi(h_t u) \quad g_t\bar\xi(u) = \bar\xi(g_t u) \tag{4.21}$$

for all $u \in \bar\Omega$ and all $t \in R$.

Let $Q \subset M$, $\mu(Q) > 1 - 2\theta$, $Q < \xi$ be as in lemma 4.2. Since h is ergodic, there are $Z \subset \Omega$, $Z < \xi$, $\mu(Z) > 1 - \theta$ and $\bar t > 0$ such that

$$\begin{array}{l}\text{if } z \in Z,\ t > \bar t \text{ then the relative length measure of } Q \\ \text{on } [z, h_t z] \text{ is at least } 1 - 3\theta.\end{array} \tag{4.22}$$

Let $\bar Z \subset \bar\Omega$ be the generic set of Z for the geodesic flow g, $\bar Z < \bar\xi$, $\mu(\bar Z) = 1$.

LEMMA 4.4. *There exists $\gamma > 0$ such that if $u, v \in \bar Z$ and $v = h_r^* u$ for some $|r| < \gamma$ then*

$$\bar\xi(v) = h_r^* \bar\xi(u).$$

Proof. The proof is similar to that of lemma 4.2. Since ϕ_i, $i = 1, \ldots, n$ are uniformly continuous on Q, given $0 < \delta < \delta_0$ there is $0 < \omega = \omega(\delta) < \delta/2$ such that

$$\text{if } d(w_1, w_2) < \omega, \quad w_1, w_2 \in Q \text{ then } d(\phi_i w_1, \phi_i w_2) < \delta/2 \quad \text{for all } i = 1, \ldots, n. \tag{4.23}$$

Let $0 < \gamma < \omega$ be such that if $x, y \in G$, $y \in W_\gamma(x)$ then x, y form an ω-strip of length 1 (see (2.13)). Let

$$u, v \in \bar Z,\ v = h_r^* u \quad \text{for some } |r| < \gamma.$$

We shall show that

$$h_r^* u_1 \in \bar{\xi}(v) \quad \text{for every } u_1 \in \bar{\xi}(u).$$

Let $x, y \in G$, $p(x) = u$, $p(y) = v$, $y = H_r^* x$. x and y form an ω-strip of length 1. Since $u, v \in \bar{Z}$, there is a sequence $0 < \tau_k \to \infty$, $k \to \infty$ such that $\exp(2\tau_k) > \bar{t}$ and

$$u^{(k)} = g_{\tau_k} u \in Z, \quad v^{(k)} = g_{\tau_k} v \in Z, k = 1, 2, \ldots.$$

Let $x^{(k)} = G_{\tau_k} x$, $y^{(k)} = G_{\tau_k} y$. We have $p(x^{(k)}) = u^{(k)}$, $p(y^{(k)}) = v^{(k)}$ and $x^{(k)}, y^{(k)}$ form an ω-strip of length $t_k = \exp(2\tau_k) > \bar{t}$. This means (see (2.13)) that

$$H_{q(s)} y^{(k)} \in W_\omega(H_s x^{(k)}) \quad \text{for all } s \in [0, t_k]$$

or

$$h_{q(s)} v^{(k)} \in W_\omega(h_s u^{(k)}), \quad s \in [0, t_k].$$

Let

$$B_k = \{s \in [0, t_k]: h_s u^{(k)} \in Q, h_{q(s)} v^{(k)} \in Q\}.$$

$k = 1, 2, \ldots$ (4.22) implies that

$$l(B_k) > (1 - 7\theta) t_k, \quad k = 1, 2, \ldots \tag{4.24}$$

if ω is sufficiently small, $t_k > \bar{t}$, $q(t_k) > \bar{t}$.

Let $u_1 \in \bar{\xi}(u)$. Then

$$u_1^{(k)} = g_{\tau_k} u_1 \in \bar{\xi}(u^{(k)})$$

by (4.21). We write $j_k(s) = i \in \{1, \ldots, n\}$ if $h_s u_1^{(k)} \in X_i$. We have that if $s \in B_k$ then

$$h_s u_1^{(k)} = \phi_{j_k(s)} h_s u^{(k)}$$

$$\phi_{j_k(s)} h_{q(s)} v^{(k)} \in \xi(h_{q(s)} v^{(k)}) = h_{q(s)}(\xi(v^{(k)}))$$

or

$$\phi_{j_k(s)} h_{q(s)} v^{(k)} = h_{q(s)} v^{(k)}(q(s))$$

for some

$$v^{(k)}(q(s)) \in \xi(v^{(k)}) = \bar{\xi}(v^{(k)}),$$

since $v^{(k)} \in Z \subset \Omega$, and

$$d(h_s u_1^{(k)}, h_{q(s)} v^{(k)}(q(s))) < \delta/2 \quad k = 1, 2, \ldots \tag{4.25}$$

by (4.23). Let

$$w = h_r^* u_1 \quad \text{and} \quad w^{(k)} = g_{\tau_k} w, \quad k = 1, 2, \ldots$$

We have

$$d(h_s u_1^{(k)}, h_{q(s)} w^{(k)}) < \omega, \quad s \in [0, t_k], \quad k = 1, 2, \ldots$$

This and (4.25) imply that

$$d(h_{q(s)} w^{(k)}, h_{q(s)} v^{(k)}(q(s))) < \omega + \delta/2 < \delta.$$

Also

$$h_{q(s)} v^{(k)}(q(s)) \in K' \quad \text{if } s \in B_k. \tag{4.26}$$

Let

$$A_k = A_{q(t_k)}(w^{(k)}, v^{(k)}, \delta) \subset [0, q(t_k)]$$

be as in lemma 4.1. We have

$$l(A_k) \geq 0.9 q(t_k), \quad k = 1, 2, \ldots$$

by (4.24) and (4.26), if ω is sufficiently small. This implies via lemma 4.1 that there is $s_k \in [0, q(t_k)]$ such that

$$E(h_{s_k} w^{(k)}, h_{s_k} v^{(k)}(s_k), \delta) \geq 0.2 q(t_k), \quad k = 1, 2, \ldots.$$

This implies via (2.9) that

$$h_{s_k} v^{(k)}(s_k) \in U(h_{s_k} w^{(k)}, D(\varepsilon)/t_k, D(\varepsilon)/t_k^2, D(\varepsilon))$$

and therefore

$$\begin{aligned} g_{-\tau_k} h_{s_k} v^{(k)}(s_k) &= h_{s_k \exp(-2\tau_k)} g_{-\tau_k} v^{(k)}(s_k) \\ &= h_{s_k \exp(-2\tau_k)} \tilde{v}(k) \in U(h_{s_k \exp(-2\tau_k)} w, D(\varepsilon)/t_k, D(\varepsilon)/t_k, D(\varepsilon)/t_k), \\ & \qquad k = 1, 2, \ldots \quad (4.27) \end{aligned}$$

where $\tilde{v}(k) = g_{-\tau_k} v^{(k)}(s_k) \in \bar{\xi}(v)$ by (4.21). (4.27) may happen only if

$$w = h_r^* u_1 \in \bar{\xi}(v)$$

since $s_k \exp(-2\tau_k) \in [0, q(1)]$, $k = 1, 2, \ldots$, and $\bar{\xi}(v)$ is finite. This completes the proof. □

For $w \in M$ we denote

$$W^{(u)}(w) = \{w' \in M: w' = h_r g_p w \text{ for some } p, r \in R\}.$$

$W^{(u)}(w)$, $w \in M$ form the unstable foliation $W^{(u)}$ for the geodesic flow g. The set $\bar{\Omega}$ consists of leaves of $W^{(u)}$. It follows from (4.21) that if $w_k \in W^{(u)}(w)$, $w \in \bar{\Omega}$ and $w_k \to w$ in the topology of $W^{(u)}(w)$, then

$$\bar{\xi}(w_k) \to \bar{\xi}(w), \quad k \to \infty.$$

Let

$$\tilde{Z} = \{w \in \bar{\Omega}: \bar{Z} \text{ is dense on the } h^*\text{-orbit of } w\},$$

$\mu(\tilde{Z}) = 1$ and let

$$\bar{W} = \{w \in \bar{\Omega}: \bar{Z} \cap \tilde{Z} \text{ is dense in } W^{(u)}(w)\}, \; \mu(\bar{W}) = 1.$$

It follows from lemma 4.4 and (4.21) that $\bar{W} < \bar{\xi}$ and

$$\text{if } u, v \in \bar{W}, v = h_q h_r^* g_p u \quad \text{for some } p, q, r \in R \text{ then } \bar{\xi}(v) = h_q h_r^* q_p \bar{\xi}(u). \tag{4.28}$$

This implies that if

$$w_k \in \bar{W}, \quad w'_k \in \bar{W}, \quad w_k \to w \in M, \quad w'_k \to w$$

when $k \to \infty$ then

$$\lim_{k \to \infty} \bar{\xi}(w_k) = \lim_{k \to \infty} \bar{\xi}(w'_k)$$

and this limit equals $\bar{\xi}(w)$, if $w \in \bar{W}$. This implies that

$$\text{if } w \in M - \bar{W}, \quad w_k \to w, \quad w_k \in \bar{W} \quad \text{then } \lim_{k \to \infty} \bar{\xi}(w_k) \subset M - \bar{W}.$$

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Let us define a partition $\tilde{\xi}$ on M by

$$\tilde{\xi}(u)=\bar{\xi}(u) \quad \text{if } u\in\bar{W} \text{ and}$$
$$\tilde{\xi}(u)=\lim_{k\to\infty}\bar{\xi}(u_k),\ u_k\in\bar{W},\ u_k\to u,\ k\to\infty.$$

$\tilde{\xi}$ is well defined and

$$\begin{aligned}&\tilde{\xi}=\xi \text{ on } \bar{W}\cap\bar{\Omega}\cap\Omega \text{ by (4.21) and if } v=h_q h_r^* g_p u,\ u\in M\\ &\text{then } \tilde{\xi}(v)=h_q h_r^* g_p\tilde{\xi}(u) \text{ by (4.28).}\end{aligned} \tag{4.29}$$

(4.29) shows that h^{ξ} on M/ξ and $h^{\tilde{\xi}}$ on $M/\tilde{\xi}$ are isomorphic, since $\bar{W}\cap\bar{\Omega}\cap\Omega$ is h-invariant and $\mu(\bar{W}\cap\bar{\Omega}\cap\Omega)=1$.

Proof of theorem 1. Denote

$$\tilde{\Gamma}(u)=p^{-1}(\tilde{\xi}(u)),\ u\in M \quad \text{and} \quad \tilde{\Gamma}=\tilde{\Gamma}(u_o),$$

where $u_0=p(e)$. We shall show that $\tilde{\Gamma}$ is a subgroup of G.

We say that $J\in G$ is a chain in G if $J=J_1\cdots J_k$ where

$$J_i=H_{q_i}H^*_{r_i}G_{p_i}e=e\cdot\begin{pmatrix}\exp(p_i) & \\ & \exp(-p_i)\end{pmatrix}\cdot\begin{pmatrix}1 & r_i\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ q_i & 1\end{pmatrix}$$

for some $p_i, q_i, r_i\in R$, $i=1,\cdots,k$. It is clear, that for any $g_1, g_2\in G$ there is a chain $J\in G$ such that $g_2=g_1\cdot J$.

Let $g, \tilde{g}\in\tilde{\Gamma}$ and let

$$g=e\cdot J,\quad \tilde{g}=e\cdot\tilde{J}$$

for some chains

$$J=J_1\cdots J_k,\quad J_i=H_{q_i}H^*_{r_i}G_{p_i}e,\qquad i=1,\ldots,k$$

and

$$\tilde{J}=\tilde{J}_1\cdots\tilde{J}_{\tilde{k}},\quad \tilde{J}_i=H_{\tilde{q}_i}H^*_{\tilde{r}_i}G_{\tilde{p}_i}e,\qquad i=1,\ldots,\tilde{k}.$$

We write

$$p(J_i)=h_{q_i}h^*_{r_i}g_{p_i}p(e)=(hh^*g)_i(u_0),\quad i=1,\ldots,k.$$

We have

$$p(g)=(hh^*g)_k\cdots(hh^*g)_1(u_0)\in\tilde{\xi}(u_0)$$
$$p(\tilde{g})=(\widetilde{hh^*g})_{\tilde{k}}\cdots(\widetilde{hh^*g})_1(u_0)\in\tilde{\xi}(u_0)$$

since $g, \tilde{g}\in\tilde{\Gamma}$. This implies by (4.29) that

$$\begin{aligned}&(hh^*g)_k\cdots(hh^*g)_1(\tilde{\xi}(u_0))=\tilde{\xi}(u_0)\\ &\text{and}\\ &(\widetilde{hh^*g})_{\tilde{k}}\cdots(\widetilde{hh^*g})_1(\tilde{\xi}(u_0))=\tilde{\xi}(u_0).\end{aligned} \tag{4.30}$$

We have

$$g\cdot\tilde{g}=e\cdot J\cdot\tilde{J}$$

and

$$p(g\cdot\tilde{g})=(\widetilde{hh^*g})_{\tilde{k}}\cdots(\widetilde{hh^*g})_1(hh^*g)_k\cdots(hh^*g)_1(u_0)\in\tilde{\xi}(u_0)$$

by (4.30).

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This implies that $g \cdot \tilde{g} \in \tilde{\Gamma}$ and that $\tilde{\Gamma}$ is a subgroup of G. It is clear that $\tilde{\Gamma}$ is discrete and $\Gamma \subset \tilde{\Gamma}$.

Let $g \in \tilde{\Gamma}(u)$, $u \in M$ and let $g = e \cdot J$ for some chain $J \in G$. (4.29) shows that then

$$\tilde{\Gamma}(u) = \tilde{\Gamma} \cdot J = \tilde{\Gamma} g.$$

Define $\tilde{\psi}: \tilde{\Gamma}/G$ onto $M/\tilde{\xi}$ by

$$\tilde{\psi}(\tilde{\Gamma} g) = \tilde{\xi}(p(g)).$$

It is clear that $\tilde{\psi}$ is measure preserving and

$$\tilde{\psi}\tilde{h}_t(\tilde{\Gamma} g) = \tilde{\psi}(\tilde{\Gamma} g \cdot N_t) = \tilde{\xi}(p(g \cdot N_t)) = \tilde{\xi}(h_t g) = h_t^{\tilde{\xi}} \tilde{\xi}(g).$$

This shows that $\tilde{\psi}$ is an isomorphism between $\tilde{h}$ and $\tilde{\Gamma}/G$ and $h^{\tilde{\xi}}$ on $M/\tilde{\xi}$. This implies via (4.29) that $\tilde{h}$ is isomorphic to h^{ξ} on M/ξ. □

5. *Proof of theorem 3*

Let S on (Y, ν) be a factor of h_1 on $(M = \Gamma\backslash G, \mu)$ with a conjugacy $\psi: M \to Y$

$$\psi h_1(x) = S\psi(x) \quad \text{for a.e. } x \in M,$$

and let ζ be the h_1-invariant partition of M, induced by ψ. It follows from proposition 1.1 and lemma 3.1 that there are $D \subset M$, $h_1 D = D$, $\mu(D) = 1$, $U \subset M/\zeta$, $h_1^{\zeta} U = U$, $\mu_{\zeta}(U) = 1$ and an integer $n > 0$ such that for every $C \in U$ the intersection $C \cap D$ consists of exactly n points each of μ_C-measure $1/n$.

We assume without loss of generality that $D = M$ and $U = M/\zeta$. So each $C \in \zeta$ consists of n distinct points of μ_C-measure $1/n$.

Let θ, K, ρ, K', ε and δ_0 be as in § 4 for ζ.

We omit the proof of the following lemma, since it is fully analogous to the proof of lemma 4.1 and corollary 4.1.

LEMMA 5.1. *Let* $0 < \delta < \delta_0$, $u, v \in M$ *and let*

$$A_k = \{m \in \{0, 1, \ldots, k\}\text{: there exists } v(m) \in \zeta(v) \text{ such that } h_m v(m) \in K' \text{ and } d(h_m u, h_m v(m)) < \delta\}.$$

If $|A_k|/k > 0.9$ *for all integers* $k > k_0 > 0$ *then there is* $\tilde{v} \in \zeta(v)$ *such that*

$$\tilde{v} = h_q u \quad \text{for some } q = q(u, v, \delta),\ |q| < \delta.$$

Let $X < \zeta$, $\mu(X) = 1$ and $X_i \subset X$, $i = 1, \ldots, n$.

$$X_i \cap X_j = \varnothing, \quad i \neq j,$$

$$\bigcup_{i=1}^{n} X_i = X, \quad \mu(X_i) = \frac{1}{n}, \quad i = 1, \ldots, n$$

be such that for every $x \in X$ the intersection $\zeta(x) \cap X_i$ consists of a single point $x_i(x)$ and the map $\phi_i: X$ onto X_i defined by $\phi(x) = x_i(x)$, is measurable.

As in § 4 we denote

$$\tilde{K} = K' \cap X, \quad \tilde{K} < \zeta, \quad \mu(\tilde{K}) = \mu(K') > 1 - \theta/n,$$

pick

$$\Lambda \subset X, \quad \Lambda < \zeta, \quad \mu(\Lambda) > 1 - \theta$$

such that each ϕ_i, $i=1,\ldots,$ is uniformly continuous on Λ and take

$$Q=\Lambda\cap\tilde{K},\quad \mu(Q)>1-2\theta,\quad Q<\zeta.$$

Let $F\subset M$ be the generic set of Q for h_1. We have

$$h_1F=F,\quad F<\zeta\quad\text{and}\quad \mu(F)=1.$$

LEMMA 5.2. *For every* $0<\delta<\delta_0$ *there is* $\beta=\beta(\delta)$ *such that if* $u_1, v_1\in F$, $v_1=h_tu_1$ *for some* $|t|<\beta$ *then for every* $u_2\in\zeta(u_1)$ *there is* $v_2\in\zeta(v_1)$ *such that* $v_2=h_au_2$ *for some* $a=a(u_1,u_2,t)$, $|a|<\delta$ *and* $a(h_1u_1,h_1u_2,t)=a(u_1,u_2,t)$.

Proof. The proof is similar to that of lemma 4.2. Let $\beta>0$ be such that

$$\text{if } d(w_1,w_2)<\beta,\ w_1,w_2\in\Lambda \text{ then}$$
$$d(\phi_iw_1,\phi_iw_2)<\delta,\quad i=1,\ldots,n. \tag{5.1}$$

Let

$$u_1,v_1\in F\quad\text{and}\quad v_1=h_tu_1\quad\text{for some } |t|<\beta.$$

Since $u_1, v_1\in F$ there is $k_0>0$ such that if $k\ge k_0$ and

$$B_k=\{m\in\{0,1,\ldots,k\}: h_mu_1\in Q, h_mv_1\in Q\}$$

then

$$|B_k|/k>1-7\theta \tag{5.2}$$

where $|B|$ denotes the number of points in B.

Let $u_2\in\zeta(u_1)$. We write $j(m)=i\in\{1,\ldots,n\}$ if $h_mu_2\in X_i$, $m=1,2,\ldots$. We have

$$\phi_{j(m)}(h_mu_1)=h_mu_2\in X_{j(m)}$$
$$\phi_{j(m)}(h_mv_1)\in\zeta(h_mv_1)=h_m\zeta(v_1)$$

or

$$\phi_{j(m)}(h_mv_1)=h_mv_1(m)$$

for some $v_1(m)\in\zeta(v_1)$ and if $m\in B_k$ then

$$h_mu_2\in K',\quad h_mv_1(m)\in K'$$

and

$$d(h_mu_2,h_mv_1(m))<\delta$$

by (5.1). This and (5.2) imply via lemma 5.1 that there is $v_2\in\zeta(v_1)$ such that

$$v_2=h_au_2$$

for some $a=a(u_1,u_2t)$, $|a|<\delta$. It is clear that

$$a(h_1u_1,h_1u_2,t)=a(u_1,u_2,t). \qquad\square$$

Let $T(x)$ denote the h_t-orbit of $x\in M$ and let

$$\bar{F}=\{x\in M: F\cap T(x) \text{ is dense in } T(x)\}.$$

$\bar{F}$ is h_t-invariant, $t\in R$ and $\mu(\bar{F})=1$.

It follows from lemma 5.2 that if $x\in\bar{F}$, $x_i\in T(x)\cap F$, $i=1,2,\ldots$ and $x_i\to x$, $i\to\infty$ in the topology of $T(x)$ then the $\lim_{i\to\infty}\zeta(x_i)$ exists and does not depend on the sequence $x_i\in T(x)\cap F$, $x_i\to x$, $i\to\infty$. If $x\in\bar{F}\cap F$ then this limit equals to $\zeta(x)$.

We define $\bar{\zeta}$ on $\bar{F}$ by

$$\bar{\zeta}(x)=\zeta(x) \quad \text{if } x\in\bar{F}\cap F$$

and

$$\bar{\zeta}(x)=\lim_{i\to\infty}\zeta(x_i) \quad \text{if } x\in\bar{F}-F$$

where $x_i\in T(x)\cap F$, $i=1,2,\ldots$ and $x_i\to x$, $i\to\infty$ in $T(x)$.

$\bar{\zeta}$ is well defined and

$$\bar{\zeta}(x)=\zeta(x) \quad \text{for a.e. } x\in M.$$

Proof of theorem 3. In order to prove the theorem it is enough to show that there exists an h_t-invariant set

$$F'\subset\bar{F},\quad \mu(F')=1,\quad F'<\zeta$$

such that

$$h_t(\bar{\zeta}(x))=\bar{\zeta}(h_tx) \quad \text{for all } x\in F' \text{ and all } t\in R.$$

It follows from lemma 5.2 that for every $x\in\bar{F}$, $\tilde{x}\in\bar{\zeta}(x)$ and $t\in R$ there is $a=a(x,\tilde{x},t)\in R$ such that

$$\begin{gathered} h_a\tilde{x}\in\bar{\zeta}(h_tx)\\ a(h_1x,h_1\tilde{x},t)=a(x,\tilde{x},t)\\ a(x,x,t)=t,\quad a(x,\tilde{x},0)=0,\quad a(x,\tilde{x},1)=1. \end{gathered} \tag{5.3}$$

The function $a(x,\tilde{x},t)$ is uniformly continuous in t for every $x\in\bar{F}$, $\tilde{x}\in\bar{\zeta}(x)$.

Denote

$$\begin{aligned} r^-(x,t)&=\min\{a(x,\tilde{x},t):\tilde{x}\in\bar{\zeta}(x)\}\\ r^+(x,t)&=\max\{a(x,\tilde{x},t):\tilde{x}\in\bar{\zeta}(x)\},\ x\in\bar{F},\ t\in R. \end{aligned}$$

$r^-(x,t)$ and $r^+(x,t)$ are continuous in t and are constant on the h_1-orbit of x. Since h_1 is ergodic, there is $F_t\subset\bar{F}$, $F_t<\bar{\zeta}$, $h_1F_t=F_t$, $\mu(F_t)=1$ such that $r^+(x,t)$ and $r^-(x,t)$ equal constants $r^+(t)$ and $r^-(t)$ respectively on F_t.

Let

$$\tilde{F}=\bigcap_{t\text{ is rational}}F_t,\quad \mu(\tilde{F})=1,\quad h_1\tilde{F}=\tilde{F},\quad \tilde{F}<\bar{\zeta}.$$

We have

$$\begin{aligned} r^-(x,t)&=r^-(t)\\ r^+(x,t)&=r^+(t) \end{aligned} \tag{5.4}$$

for every $x\in\tilde{F}$ and every rational t. Since $r^+(x,t)$ and $r^-(x,t)$ are continuous in t, (5.4) holds for all $t\in R$.

Let

$$F'=\{x\in\bar{F}:\tilde{F}\cap T(x)\text{ is dense in } T(x)\},$$

$h_tF'=F'$, $t\in R$, $F'<\bar{\zeta}$ and $\mu(F')=1$. (5.4) implies that

$$r^-(x,t)=r^-(t),\quad r^+(x,t)=r^+(t)$$

for all $x \in F'$ and all $t \in R$, since

$$r^+(x,t) = \lim_{i\to\infty} r^+(x_i,t),\ r^-(x,t) = \lim_{i\to\infty} r^-(x_i,t)$$

if $x_i \in T(x) \cap \tilde{F}$ and $x_i \to x$ in $T(x)$.

Take $x \in F'$ and let $\tilde{x} \in \bar{\zeta}(x)$ be such that

$$h_{r^-(t)}\tilde{x} \in \bar{\zeta}(h_t x).$$

We have

$$a(x, \bar{x}, t) \geq r^-(t) = a(x, \tilde{x}, t) \quad \text{for every } \bar{x} \in \bar{\zeta}(x).$$

This implies that

$$a(\tilde{x}, \bar{x}, r^-(t)) \geq r^-(t) \quad \text{for all } \bar{x} \in \bar{\zeta}(x)$$

and therefore

$$r^-(r^-(t)) = r^-(t) \quad \text{for all } t \in R. \tag{5.5}$$

We claim that

$$r^-(t) = r^+(t) = t \quad \text{for all } t \in R. \tag{5.6}$$

Indeed, it follows from (5.3) and the definition of r^+ and r^- that

$$r^-(0) = r^+(0) = 0$$
$$r^-(1) = r^+(1) = 1 \tag{5.7}$$

and

$$r^-(t) + r^+(1-t) = 1.$$

Let us first show that

$$r^-(\tfrac{1}{2}) = r^+(\tfrac{1}{2}) = \tfrac{1}{2}.$$

Since $r^-(t)$ is continuous, there is $t_0 \in (0, 1)$ such that

$$r^-(t_0) = \tfrac{1}{2}.$$

This and (5.5) imply that

$$r^-(\tfrac{1}{2}) = \tfrac{1}{2}$$

and therefore

$$r^+(\tfrac{1}{2}) = \tfrac{1}{2}$$

by (5.7). We have shown that if $x \in F'$ then

$$h_{1/2}\bar{\zeta}(x) = \bar{\zeta}(h_{1/2}x).$$

This implies that

$$r^-(t) + r^+(\tfrac{1}{2} - t) = \tfrac{1}{2} \quad \text{for all } t \in R.$$

Arguing as above we get that (5.6) holds for $t = \frac{1}{4}$ and $t = \frac{3}{4}$. Proceeding by induction, we get that (5.6) holds for all $t \in R$ of the form $k/2^n$, $k, n = 1, 2, \ldots$. Since r^- and r^+ are continuous, (5.6) holds for all $t \in R$. (5.6) implies that

$$h_t\bar{\zeta}(x) = \bar{\zeta}(h_t x) \quad \text{for all } x \in F' \text{ and all } t \in R.$$

This completes the proof. □

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