

# Failure and Rescue in an Interbank Network

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September 23, 2011

## Abstract

This paper is concerned with systemic risk in the interbank market. We model this market as a directed graph in which the banks represent the nodes and the liabilities between the banks represent the edges. Our work builds on the modelling paradigm of Eisenberg and Noe (2001), extending it by introducing default costs in the system. We provide a rigorous analysis of those situations in which banks have incentives to bailout distressed banks. Such incentives exist under very mild conditions. We illustrate our results with some simple examples, and go on to discuss possible measures of soundness of a financial system, together with possible policy implications for resolution of distress.

## 1 Introduction

Over the last ten or so years, there has been a growth of interest in the general phenomenon of the spread of bank failure through a network of interbank obligations. As the Asian banking crisis of the late 90s, and the more recent banking crisis of 2007-8 have shown, the banking system generally can be very vulnerable to deterioration of assets, loss of liquidity and loss of confidence. The modelling effort to date is largely restricted to static models, reflecting the urgent nature of the crises; traditional solutions such as the issuance of new equity or bonds cannot be achieved in the timescales available, typically just a few days. Among such studies of static failure models, there are two main types. The first, see for example Eisenberg and Noe (2001), Cifuentes et al. (2005), supposes that the network of interbank obligations is given to us, and then attempts to understand how it might fail, where it is weak, and what steps could be taken to strengthen it. The second, see for example Nier et al. (2007) and Beyeler et al. (2007), supposes some regular or symmetric structure<sup>1</sup> for the network of interbank obligations, and tries to derive general results for such a network. While the second approach lends itself better to elegant theory, we view its value as being largely indicative; in practice, we are *not* sampling from some distribution of possible bank networks, nor are we working with a symmetric network of interbank obligations - there is an *actual* network to be saved, and it will be a complicated ugly object.

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<sup>1</sup>The regular structure may be stochastic in nature, as in Nier et al. (2007)

Our contribution therefore will be of the first type, modelling the system as a directed graph with edge weights, representing the indebtedness of each bank to the others. Each bank is owed money by external borrowers, and initially all banks are solvent, but the value of the banks' loan portfolio is subject to variation, and it may come about that one bank's loan portfolio falls in value so that the bank is technically insolvent. We investigate the consequences of such a default on the system.

The model we present here is an extension of a model proposed by Eisenberg and Noe (2001). They assume that a defaulting bank calls in its loans, realising 100% of their face value. However, this must be treated with care. It is more likely, that in such a default situation the bank will only realise some fraction and not 100% of the face value. Allowing for this makes the model much more realistic as we will see. When a bank defaults it does not repay its obligations in full which may precipitate other collapses.

We are particularly interested in situations in which banks have incentives to form rescue consortia. We provide analytic results for the existence of these rescue consortia.

The outline of our paper is as follows. In Section 2 we describe the model for the interbank market which extends the model by Eisenberg and Noe (2001) by introducing default costs.

In Section 3 we study how the default of one bank affects the system and we show the existence of a clearing vector which represents a vector of payments to settle all liabilities in the system within a simultaneous clearing mechanism. Moreover we propose an algorithm which allows us to compute clearing vectors very efficiently.

Section 4 contains the main theoretical results of this paper on merger decisions in the interbank market. We show in Theorem 4.10 that, on the one hand, if there are no default costs, there is no incentive for rescue consortia to form. On the other hand, in Theorem 4.11, we prove that if there are strictly positive default costs and some mild regularity conditions are additionally satisfied, then rescue consortia do exist. This shows that there is almost always benefit to be derived from the solvent banks in a system rescuing the insolvent banks, and it is indeed in the interests of the solvent banks to do so. Why then does it seem so difficult in times of crisis for the system to act in its own interest and mend itself? There are of course many aspects to this question: How is a rescue to be coordinated? Would a rescue consortium be able to satisfy itself as to the risks involved in a rescue? Why would a bank join a rescue consortium and thereby take on losses if by waiting other banks might take the losses instead? In Section 6 we discuss some of these questions, and present some measures which can be used to assess the state of a financial system and particular its risk of contagion. In Section 5 we provide some examples and illustrate our theoretical results in these situations. Section 7 concludes. The Appendix contains all proofs of the theoretical results.

## **1.1 Related Literature**

Contagion in financial networks has been frequently studied in the past, see e.g. Diamond and Dybvig (1983), Rochet et al. (1996), Allen and Gale (2000), among other papers and the survey by DeBandt (2000).

Once bankruptcy of one or several financial institutions has occurred or is imminent, the natural question is whether these institutions should be bailed-out and if yes, by whom?

Aghion et al. (1999) discuss exactly such a bailout problem and show how and when a bailout should be done. Cordella and Yeyati (2003) show that bailing out banks under certain conditions can outweigh possible moral hazard effects. The main problem generally is that if a lender of last resort is present this might increase the risk appetite of financial institutions. If there is not such a lender of last resort the resulting contagion effects and loss of confidence could seriously affect the financial system. Perotti and Suarez (2002) consider the possibility of solvent banks taking over distressed banks and further show how this can stabilize the financial system. More recent approaches to optimal resolution of bank failures are provided in Acharya and Yorulmazer (2007) and Acharya and Yorulmazer (2008).

Acharya et al. (2011) focus on the related problem, namely the banks' choice of liquidity and how this depends on the resolution mechanism. They particularly provide evidence for the fact that providing liquidity support to failing banks can make them more likely to hold less liquidity in the first place.

Besides theoretical results on bank failure and contagion, there is also a wide range of empirical studies such as e.g. Iyer and Peydro (2011). Many authors have also considered stress testing in financial networks, e.g. Amini et al. (2011) or general simulation studies like Upper and Worms (2004), Upper (2007), Elsinger et al. (2006). Many of those are built on the modelling paradigm of Eisenberg and Noe (2001).

Our paper complements the literature on dealing with financial networks under stress and bank merger decisions. Its spirit is closest to work by Leitner (2005) who particularly investigated the role of linkage in interbank networks as a reason for private sector bailouts. Leitner (2005) considers also *optimal* networks in the sense that they provide an optimal tradeoff between risk sharing and the collapse of the system. We do not consider optimal designs but just take a network as given and provide condition when rescue consortia exist. Our analysis can therefore in principle be applied to all financial networks (assuming that the liabilities structure can be observed). We also show that banks in such a network very often have incentives to rescue other banks and hence a lender of last resort is not required for the bailout itself but for insuring an appropriate coordination mechanism. The role of the regulator in our paper is therefore very similar to the one considered by Leitner (2005).

## 2 Definition of the Financial System

We use a model similar to Eisenberg and Noe (2001), though we modify their notation slightly. We consider a market with  $n$  banks with indices in  $\mathcal{N} := \{1, \dots, n\}$  which we represent by nodes in a network. Each bank has liabilities to other banks in the system. We represent these liabilities in terms of a matrix.

**Definition 2.1** (Liabilities Matrix).

*The liabilities matrix is given by  $\mathbf{L} \in \mathbb{R}^{n \times n}$ , where the  $i_j^{\text{th}}$  entry  $L_{ij}$  represents the nominal liability of bank  $i$  to bank  $j$ . We assume that  $L_{ij} \geq 0 \forall i, j$  and  $L_{ii} = 0 \forall i$ .*

Those nominal liabilities that are strictly positive represent the directed edges of the network.

**Definition 2.2** (Obligations).

The total nominal obligations of bank  $i$  to all other banks in the system are given by  $\bar{L}_i = \sum_{j=1}^n L_{ij}$  and  $\bar{L}$  is the corresponding vector of the total nominal obligations.

**Definition 2.3** (Relative Liabilities Matrix).

Let  $\mathbf{L}$  be a liabilities matrix and  $\bar{L}$  the corresponding vector of total nominal obligations. The relative liabilities matrix  $\Pi \in \mathbb{R}^{n \times n}$  is defined by

$$\pi_{ij} := \begin{cases} L_{ij}/\bar{L}_i & \text{if } \bar{L}_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the rows of  $\Pi$  all sum up to 1 if  $\bar{L} > 0$ . If for some  $i \in \mathcal{N}$  the total liabilities  $\bar{L}_i = 0$ , then the corresponding row of  $\Pi$  sums up to 0.

**Definition 2.4** (Net Assets).

We denote by  $e_i \geq 0$  the net assets of bank  $i$  from sources outside the banking system<sup>2</sup>. The corresponding vector of net assets is denoted by  $e$ .

**Definition 2.5** (Financial System).

Let  $e$  be a vector of net external assets. We consider two constants  $\alpha, \beta \in (0, 1]$ , where the constant  $\alpha$  is the fraction of the face value of net external assets realized on liquidation, and the constant  $\beta$  is the fraction of the face value of interbank assets realized on liquidation. Then we define a financial system as a quadruple  $(\mathbf{L}, e, \alpha, \beta)$ , where  $\mathbf{L}$  is a liabilities matrix.

Once a financial system  $(\mathbf{L}, e, \alpha, \beta)$  is specified one can immediately derive the corresponding relative liabilities matrix  $\Pi$  and the vector of total obligations  $\bar{L}$ . Similarly, if we start with a vector of total obligations  $\bar{L}$  and a relative liabilities matrix  $\Pi$  we can derive the corresponding liabilities matrix  $\mathbf{L}$ . This leads to equivalent definitions of a the financial system which we will use interchangeably as it is convenient.

Next, we introduce the clearing vector. A clearing vector specifies payments between the banks in the system which are consistent with some rules. The rules considered are the three rules proposed by (Eisenberg and Noe, 2001, p. 239):

1. Limited liabilities: Each node never pays more than its available cash flow.
2. Priority of debt claims over equity: Paying off the liabilities  $L_{ij}$  has priority, even if the external assets  $e_i$  have to be used for that.
3. Proportionality: If default occurs the defaulting bank pays all claimant banks in proportion to the size of their nominal claims on the assets of the defaulting bank.

This leads to the following definition.

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<sup>2</sup>This would be equity plus loans less deposits.

**Definition 2.6** (Clearing Vector).

A clearing vector for the financial system  $(\mathbf{L}, e, \alpha, \beta)$  is a vector  $L^* \in [0, \bar{L}]$  such that

$$L^* = \Phi(L^*),$$

where  $\Phi$  is the function defined by

$$\Phi(L)_i \equiv \begin{cases} \bar{L}_i, & \text{if } \bar{L}_i \leq e_i + \sum_{j=1}^n L_j \pi_{ji}, \\ \alpha e_i + \beta \sum_{j=1}^n L_j \pi_{ji}, & \text{else.} \end{cases} \quad (1)$$

Simple but very important properties of the mapping  $\Phi$  are given by the following result.

**Lemma 2.7.** *The mapping  $\Phi$  has the following properties:*

- (i)  $\Phi$  is bounded above by  $\bar{L}$ : for any  $L$  we have  $\Phi(L) \leq \bar{L}$ ;
- (ii)  $\Phi$  is monotone: if  $\tilde{L} \leq L$  then  $\Phi(\tilde{L}) \leq \Phi(L)$ .

A proof of these results is provided in Appendix A.

The interpretation of the clearing vector  $L^*$  is that  $L_i^*$  represents the cash which bank  $i$  has available to pay out to other banks. The value of the assets (external and interbank) available to bank  $i$  will be the sum  $e_i + \sum_j L_j^* \pi_{ji}$ , and if this is at least  $\bar{L}_i$  then bank  $i$  is able to meet its obligations. If this inequality does not hold, then bank  $i$  is in default, and must call in its assets; it recovers only a fraction  $\alpha$  of the external assets, and a fraction  $\beta$  of the interbank assets. The two fractions may conceivably be different; we would typically expect that  $\alpha$  would be low, because the bank would be having to sell off its loan portfolio, probably at a knock-down price. On the other hand,  $\beta$  might be much closer to 1, since an obligation from a solvent bank would probably be paid in full (though perhaps with some negotiated discount to compensate for the inconvenience of early repayment). An obligation from a liquidated bank would also probably be paid in full, since  $L_j^*$  represents realized cash, but there might also be some deduction for the costs of receivership.

Thus on default of bank  $i$ , there are actual losses of at least

$$(1 - \alpha)e_i + (1 - \beta) \sum_{j=1}^n L_j^* \pi_{ji}.$$

For  $0 < \alpha, \beta < 1$ , this is a positive amount - bank failure really costs something. This assumption is new and was not considered in Eisenberg and Noe (2001) who in effect have assumed  $\alpha = \beta = 1$  throughout. We will soon see how this extension leads to more realistic behaviour, and more interesting properties.

**Remark 2.8.** This model can be easily extended by using different fractions  $\alpha_i, \beta_i$  corresponding to the different banks  $1 \leq i \leq n$  in the model.

We conclude this section by defining the value of a bank.

**Definition 2.9** (Value of Banks).

The value  $\mathcal{V}$  of the banks corresponding to a clearing vector  $L$  in a financial system  $(\mathbf{L}, e, \alpha, \beta)$  is defined as

$$\mathcal{V}(L, e)_i := (\Pi^\top L + e - L)_i \mathbb{I}_{\{L_i \geq \bar{L}_i\}}. \quad (2)$$

**Remark 2.10.** From Definition 2.9 it is clear that as soon as a bank's clearing payment is strictly less than its total liabilities, its value is zero. Moreover,  $\Pi^\top L + e$  can be interpreted as cash-flow into the bank and  $L$  as cash-flow out of the bank. The value of the bank is then the net cash position after clearing and this is assumed to be 0 if the bank defaults. A similar concept was considered in (Eisenberg and Noe, 2001, p. 239).

### 3 Existence and Construction of Clearing Vectors

In the following we show that a clearing vector exists for all  $0 < \alpha, \beta \leq 1$ . We also propose an algorithm to compute a clearing vector. Eisenberg and Noe (2001) only considered the case  $\alpha = \beta = 1$ .

#### 3.1 Existence of Clearing Vectors

**Theorem 3.1** (Existence of Clearing Vectors).

For every financial system  $(\mathbf{L}, e, \alpha, \beta)$  there exist clearing vectors  $L^*$  and  $L_*$  such that for any clearing vector  $L$ , we have

$$L_* \leq L \leq L^* \quad (3)$$

*Proof of Theorem 3.1.* Consider a sequence of vectors  $L^{(n)}$ ,  $n = 0, 1, \dots$ , defined recursively by  $L^{(0)} := \bar{L}$ ,

$$L^{(n+1)} = \Phi(L^{(n)}) \quad (4)$$

for  $n = 0, 1, \dots$ , where  $\Phi$  is as at (1). From Lemma 2.7 we have  $L^{(1)} \leq L^{(0)} = \bar{L}$ , and hence for all  $n$ ,

$$L^{(n+1)} \leq L^{(n)}$$

by induction. Since all the  $L^{(n)}$  are non-negative, there is a monotone limit  $L^* := \downarrow \lim_{n \rightarrow \infty} L^{(n)}$ . Notice that the set  $A_n := \{i : L_i^{(n)} < \bar{L}_i\}$  is non-decreasing in  $n$ , and therefore is eventually constant. Note that  $\Phi$  is continuous from above. Hence it is clear that  $L^*$  satisfies

$$L^* = \Phi(L^*), \quad (5)$$

that is,  $L^*$  is a clearing vector.

We may similarly start the recursion from the zero vector  $L^{(0)} := 0$ , in which case we obtain an increasing sequence of vectors, with limit  $\hat{L}$ . In contrast to the first situation the limit  $\hat{L}$  does not have to be a clearing vector unless  $\alpha = \beta = 1$  (see Example 3.3). This is due to the fact that the function  $\Phi$  is not continuous from below. If a bank just becomes solvent with the limit payment vector  $\hat{L}$ , one needs to restart the iteration from  $\hat{L}$  and continue until one reaches the next limit (monotone convergence). Since

there are at most  $n$  banks where that could happen, one has to restart the iteration at most  $n - 1$  times. We denote the vector which we obtain from this iteration by  $L_*$ . This vector is then the least clearing vector and a fixed point of  $\Phi$ .

Since  $L^*$  is non-negative, and a fixed point of  $\Phi$ , monotonicity of  $\Phi$  implies that  $L^* \geq L_*$ . Moreover, since any clearing vector is bounded below by 0 and above by  $\bar{L}$ , we have that any clearing vector must be bounded between  $L_*$  and  $L^*$ .  $\square$

**Remark 3.2.** • It is clear from (3) that the greatest and least clearing vectors are unique.

- The proof of Theorem 3.1 can be used directly to compute a clearing vector. However, it is not guaranteed that it will terminate after a finite number of steps. We will therefore present an algorithm in Definition 3.6 which returns a clearing vector after at most  $n = |\mathcal{N}|$  steps.
- An alternative proof of the existence results uses Tarski's fixpoint theorem, see Tarski (1955), and proceeds then similarly as in Eisenberg and Noe (2001).

**Example 3.3.** • We consider the financial system

$$(\mathbf{L}, e, \alpha, \beta) = \left( \begin{pmatrix} 0 & K_1 \\ K_2 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha, \beta \right)$$

with  $\alpha = \beta = \frac{1}{2}$ ,  $K_1 = \frac{1}{1-\alpha} = 2$ ,  $K_2 = 2.2$ . If we start from  $L_i^{(0)} := 0$ ,  $i \in \{1, 2\}$ , one can show by induction that  $L_i^{(n)} = \frac{2^n - 1}{2^n}$  and therefore  $\lim_{n \rightarrow \infty} L_i^{(n)} = 1$  for  $i = 1, 2$ . But  $(1, 1)^\top$  is not a clearing vector since

$$\Phi \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

However, if we start the iteration again with  $L^{(0),new} := \lim_{n \rightarrow \infty} L^{(n)} = (1, 1)^\top$ ,  $L^{(n+1),new} = \Phi(L^{(n),new})$ , then we obtain

$$\lim_{n \rightarrow \infty} L^{(n),new} = \begin{pmatrix} 2 \\ 2.2 \end{pmatrix}.$$

Therefore we have found the clearing vector  $(2, 2.2)^\top$ .

- If we change the financial system by setting  $K_1 = 2.2$ . Then  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 2.2 \\ 2.2 \end{pmatrix}$  are both clearing vectors. If additionally,  $\alpha = \beta = 1$ , then only  $\begin{pmatrix} 2.2 \\ 2.2 \end{pmatrix}$  is a clearing vector.

**Remark 3.4.** The construction of the upper solution  $L^*$  is achieved by assuming initially that all banks are sound, then knocking out banks which are insolvent, or which become insolvent as the result of the failures of others. Thus we are building a solution which *spreads insolvency across the network*. The lower solution  $L_*$  is achieved by supposing initially that banks can only rely on their own assets  $e_i$ ; any which can cover their interbank obligations with their own assets are certain to be secure, and thus their

obligations will be paid in full. This in turn may make other insolvent banks secure; this solution *spreads solvency across the network*. From a practical point of view, it is likely to be the upper solution  $L^*$  which most concerns us; we would envisage a situation where initially all banks were (or were thought to be) solvent, and insolvency is seen spreading across the banking network.

**Remark 3.5.** Notice that in the case  $\alpha = \beta = 1$  the function  $\Phi$  defined in (1) takes the simpler form

$$\Phi(L)_i = \bar{L}_i \wedge \left( e_i + \sum_{j=1}^n L_j \pi_{ji} \right), \quad (6)$$

from which it is obvious that  $\Phi$  is continuous and increasing.

It was shown in Eisenberg and Noe (2001) that if the banking network satisfied a technical condition (called *regularity*), and if  $\alpha = \beta = 1$ , then there is only one clearing vector:  $L^* = L_*$ .

### 3.2 Construction of Clearing Vectors

Theorem 3.1 gives a recursive method for calculating approximations to the greatest/least clearing vector which in practice converges very rapidly and stably. However, this algorithm is not guaranteed to converge in any fixed finite number of steps, and this is certainly a drawback, at least at the theoretical level.

In this subsection, we propose an algorithm which is a modification of the *fictitious default algorithm* introduced by Eisenberg and Noe (2001). This algorithm by contrast will find the greatest clearing vector in at most  $n$  steps; moreover, on the way we find an interesting understanding of the way default spreads through the network which we return to later.

**Definition 3.6** (Greatest Clearing Vector Algorithm, GA).

For a financial system  $(\mathbf{L}, e, \alpha, \beta)$  the Greatest Clearing Vector Algorithm (GA) constructs a sequence  $(\Lambda^{(\nu)})$  as follows. Again  $\Pi = (\pi_{ij})$  and  $\bar{L}$  are defined as before.

1. Set  $\nu = 0$ ,  $\Lambda^{(0)} := \bar{L}$  and  $\mathcal{I}_{-1} := \emptyset$ .
2. For all nodes  $i$  compute  $v_i^{(\nu)} := \sum_{j=1}^n \Lambda_j^{(\nu)} \pi_{ji} + e_i - \bar{L}_i$ .
3. Define

$$\mathcal{I}_\nu := \{1 \leq i \leq n : v_i^{(\nu)} < 0\},$$

the set containing all indices of insolvent banks, and

$$\mathcal{S}_\nu := \{1 \leq i \leq n : v_i^{(\nu)} \geq 0\}$$

the set containing all indices of solvent banks.

4. If  $\mathcal{I}_\nu \equiv \mathcal{I}_{\nu-1}$  terminate the algorithm.



5. Otherwise set

$$\Lambda_j^{(\nu+1)} := \bar{L}_j \quad \forall j \in \mathcal{S}_\nu$$

and determine the remaining clearing payments by finding the maximal solution to the system of linear equations

$$x_i = \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\nu} x_j \pi_{ji} \right\} \quad \forall i \in \mathcal{I}_\nu \quad (7)$$

which is bounded above by  $\Lambda^{(\nu)}$ , and setting  $\Lambda_i^{(\nu+1)} := x_i$  for  $i \in \mathcal{I}_\nu$ .

6. Set  $\nu \rightarrow \nu + 1$  and go back to 2.

When the algorithm has terminated, the vector  $\Lambda^{(\nu)}$  is a clearing vector.

**Remark 3.7.** The maximal solution will be unique if  $(\beta P)^n \rightarrow 0$ , where  $P$  is the restriction of  $\Pi$  to  $\mathcal{I}_\nu \times \mathcal{I}_\nu$ . This condition will always be satisfied if  $\beta < 1$ .

**Theorem 3.8.** *The greatest clearing vector algorithm (GA) stated in Definition 3.6 produces a sequence of vectors  $(\Lambda^{(\nu)})$  decreasing in at most  $n$  iterations to the greatest clearing payment vector.*

Hence we have shown how we can find the greatest clearing payment vector. The proof is given in Appendix B.

There is an analogue of the algorithm (GA) which starts from  $\Lambda^{(0)} = 0$ ,  $\mathcal{I}_{-1} = \{1, \dots, n\}$ , and proceeds exactly as (GA), with the sole exception that the solution to (7) we take is the minimal non-negative solution. This algorithm produces a sequence  $(\Lambda^{(\nu)})$  of vectors increasing to  $L_*$  in at most  $n$  steps. The method of proof is a direct translation of the proof of Theorem 3.8; since this result is of less interest, we omit it completely.

The construction of Theorem 3.8 leads to an increasing sequence  $\mathcal{I}_\nu$  of insolvency sets which have an important and natural interpretation.

**Definition 3.9.** *We call the set  $\mathcal{I}_\nu$  the level- $\nu$  insolvency set.*

Notice that the level-0 insolvency set is the set of those banks which would default *even if all other banks paid their obligations in full*. The level- $\nu$  insolvency set is the set of all those banks which would not be able to meet their obligations *if all the level- $(\nu - 1)$  insolvent banks were to default*. Thus the insolvency sets  $\mathcal{I}_\nu$  trace the spread of default through the financial system. We shall comment further on this later in the context of possible rescue schemes and policy for limiting the damage of bank failures.

## 4 Merged Banks as Rescue Consortium

The model we have introduced already allows us to study how default of one bank affects other banks in the system. However, it also allows us to study situations where a bank failure might be avoided by some solvent bank(s) stepping forward to rescue the failing bank. By so doing, the rescuer takes on the loss

incurred by the distressed bank, but this may work out cheaper than the loss suffered by the rescuer once the losses of the defaulting bank have been spread and *amplified* by the network of interbank obligations. To model this amplification effect, the inclusion of default costs is essential.

Recall that the banks are indexed by  $\mathcal{N} = \{1, \dots, n\}$ . We define a bank merger in the following.

**Definition 4.1** (Merger).

*In the financial system  $(\mathbf{L}, e, \alpha, \beta)$  consisting of  $n$  banks, the merger of all banks in  $B \subset \mathcal{N}$  is a new financial system  $(\tilde{\mathbf{L}}, \tilde{e}, \alpha, \beta)$  indexed by  $\tilde{\mathcal{N}} := \{0\} \cup B^c$ . It is defined by*

$$\tilde{e} := Me,$$

and

$$\tilde{\mathbf{L}}_{ij} := \begin{cases} 0 & \text{if } i=j=0, \\ \mathbf{L}'_{ij} & \text{otherwise,} \end{cases}$$

where

$$\mathbf{L}' := M\mathbf{L}M^\top,$$

and  $M$  is the  $|\tilde{\mathcal{N}}| \times |\mathcal{N}|$  matrix,

$$M_{0i} = \begin{cases} 1 & \text{if } i \in B, \\ 0 & \text{else,} \end{cases}$$

$$M_{ij} = \delta_{ij} \quad (i, j \notin B).$$

From this definition we see that for those banks which merge the corresponding liabilities to the other (non merging) banks in the network are just added up. The liabilities to those banks which merge are just cancelled. Also the external assets of the merged bank are just the sum of the external assets of the banks which merged. Hence, this is a very natural definition of a merger.

There is the possibility that a rescue consortium is formed which just provides the necessary assets such that no bank defaults. This works as follows. Suppose the system  $(\mathbf{L}, e, \alpha, \beta)$  is subjected to stress, by reducing the banks' initial external asset vector  $e$  to  $\tilde{e}$ . Here we assume that initially (i.e. with external assets  $e$ ) all banks are sound but under reasonable model assumptions there comes a point where one or more banks become insolvent if we reduce the external assets to  $\tilde{e}$ .

**Definition 4.2** (Bailout costs).

*Given a financial system  $(\mathbf{L}, \tilde{e}, \alpha, \beta)$ . We define*

$$\delta := \max \left\{ 0, - \left( \Pi^\top \bar{L} + \tilde{e} - \bar{L} \right) \right\}$$

and refer to  $\sum_{j=1}^n \delta_j = \sum_{j \in \mathcal{I}_0} \delta_j$  as the bailout costs.

In the following we define a rescue consortium assuming that the level-0 insolvency set is non-empty.

**Definition 4.3** (Rescue consortium).

Let  $(\mathbf{L}, \tilde{e}, \alpha, \beta)$  be a financial system where the level-0 insolvency set  $\mathcal{I}_0$  is non-empty. The corresponding upper clearing vector is denoted by  $L^*$ . We define

$$\tilde{V} := \max\{0, \Pi^\top \bar{L} + \tilde{e} - \bar{L}\}, \quad (8)$$

$$\Delta V := \tilde{V} - \mathcal{V}(L^*, \tilde{e}). \quad (9)$$

A rescue consortium is a set  $A \subseteq \mathcal{N} \setminus \mathcal{I}_0$  such that the following two conditions hold:

1. *Rescue incentive:*

$$\sum_{i \in A} \Delta V_i > \sum_{j \in \mathcal{I}_0} \delta_j, \quad (10)$$

2. *Rescue ability:*

$$\sum_{i \in A} \tilde{V}_i > \sum_{j \in \mathcal{I}_0} \delta_j. \quad (11)$$

Conditions (10) and (11) distinguish between situations in which banks *want* to rescue other banks and those in which they *can* rescue. We assume that both conditions need to be satisfied. We establish a relationship between these two conditions:

**Theorem 4.4.** 1. *Every rescue consortium which has an incentive to rescue the failing banks also has the ability to rescue the failing banks.*

2. *Suppose that the set of banks at risk of contagious default  $\mathcal{R} := \cup_{\nu} \mathcal{I}_\nu \setminus \mathcal{I}_0$  is non-empty, and suppose, further that some subset  $A \subseteq \mathcal{R}$  is able to rescue the failing banks. Then  $A$  also has an incentive to rescue the failing banks.*

A proof is provided in Appendix C.

**Remark 4.5.** We assume that a rescue consortium will always try to rescue all banks which are technically insolvent at once. One could change this definition and consider individual rescue consortia for different defaulting banks as well.

In the following we describe the new financial system once a rescue consortium was formed.

**Definition 4.6** (Rescued Financial System).

Let  $(\mathbf{L}, \tilde{e}, \alpha, \beta)$  be a financial system in which the level-0 insolvency set  $\mathcal{I}_0$  is non-empty, and suppose that a rescue consortium defined by a set of indices  $A$  exists. Then the rescued financial system is the financial system obtained by a merger of all banks in  $\mathcal{I}_0 \cup A$ .

From this definition we immediately get the following results using the notation of Definition 4.6:

**Lemma 4.7.** *In the situation of Definition 4.6, the value of the new bank obtained as a merger of all banks in  $\mathcal{I}_0 \cup A$  is  $\sum_{i \in A} \tilde{V}_i - \sum_{j \in \mathcal{I}_0} \delta_j > 0$ .*

The following result will be very useful to establish more general results about the existence of a rescue consortium.

**Lemma 4.8.** *Consider a financial system  $(\mathbf{L}, e, \alpha, \beta)$  in which all banks are initially solvent. Suppose that the assets  $e$  are reduced to  $\tilde{e}$ , with  $\tilde{e}_i \leq e_i \forall i$  such that some banks have become insolvent:  $\mathcal{I}_0 \neq \emptyset$ . Let  $L^*$  be the greatest clearing payment vector in  $(\mathbf{L}, \tilde{e}, \alpha, \beta)$ . Then*

$$0 \leq \sum_{i=1}^n (\mathcal{V}(\bar{L}, e)_i - \mathcal{V}(L^*, \tilde{e})_i) = \sum_{i=1}^n (e_i - \tilde{e}_i) + \sum_{i=1}^n \left( (1 - \alpha)\tilde{e}_i + (1 - \beta) \sum_{j=1}^n L_j^* \pi_{ji} \right) \mathbb{I}_{\{L_i^* < \bar{L}_i\}}.$$

A proof of Lemma 4.8 is provided in Appendix C.

**Remark 4.9.** Lemma 4.8 shows how much money is lost in the financial system when the external assets are reduced from  $e$  to  $\tilde{e}$ . We see that for  $\alpha, \beta < 1$ , this loss is usually larger than  $\sum_{i=1}^n (e_i - \tilde{e}_i)$ .

As we will see in the following accounting for default costs is necessary for the existence of a rescue consortium. Indeed as we prove in the following, for  $\alpha = \beta = 1$  banks have no incentive to form rescue consortia.

**Theorem 4.10.** *[Absence of Rescue Consortium] Consider a financial system  $(\mathbf{L}, e, \alpha, \beta)$  in which all banks are initially solvent. Suppose the assets  $e$  are reduced to  $\tilde{e}$ , with  $\tilde{e}_i \leq e_i \forall i$  with the result that at least one bank becomes level-0 insolvent. Let  $L^*$  be the greatest clearing payment vector in  $(\mathbf{L}, \tilde{e}, \alpha, \beta)$ . Suppose that  $\alpha = \beta = 1$ . Then no bank in the network has an incentive to rescue the insolvent bank(s).*

Again, a proof can be found in Appendix C. Therefore we see that for  $\alpha = \beta = 1$  it is never beneficial for a solvent bank to take over an insolvent bank. But as we will see in the following for  $\alpha, \beta < 1$  it can be beneficial for some bank(s) to take over an insolvent bank and in many realistic situations it will be beneficial.

**Theorem 4.11.** *[Presence of Rescue Consortium] Let  $(\mathbf{L}, \tilde{e}, \alpha, \beta)$  be a financial system with  $\alpha, \beta \in [0, 1)$ . Suppose that  $\mathcal{I}_0$  is a proper subset of  $\mathcal{N}$ :  $\emptyset \subsetneq \mathcal{I}_0 \subsetneq \mathcal{N}$ . Let  $L^*$  be the corresponding greatest clearing vector and suppose that there exists a bank  $k$  such that  $L_k^* < \bar{L}_k$  which satisfies at least one of the following two conditions:*

1.  $\tilde{e}_k > 0$ ,
2. there exists  $j \neq k$  such that  $L_j^* \pi_{jk} > 0$ .

*Then there exists a rescue consortium.*

A proof is given in Appendix C.

**Remark 4.12.** • We therefore see that in the presence of default costs a rescue consortium exists under very mild regularity conditions. The two conditions stated in Theorem 4.11 have a clear interpretation. The first condition requires the existence of one defaulting bank with strictly positive external assets. This is a very natural assumption and very often satisfied. The second condition requires the existence of at least one defaulting bank and another bank in the network which after clearing still has strictly positive liabilities to the defaulting bank. This condition is also often satisfied.

- The statement of Theorem 4.11 can also be motivated by considering a *mega-merger* in which all banks in the financial network merge into one bank. The value of such a bank is  $\sum_{i=1}^n e_i$  and hence strictly positive if at least one bank in the system has strictly positive external assets. Moreover, by definition such a mega-bank cannot go bankrupt because it does not have any liabilities to satisfy.

Key in all considerations regarding rescue consortia is that we assume that once the external assets are reduced, banks have time to actually form a rescue consortium. Hence it is assumed that the default of a bank does not automatically trigger the clearing procedure, but banks may merge before clearing takes place. This is an important assumption. In a different set-up, Leitner (2005), discusses the importance of a coordinating device, once default or possible contagion is unavoidable. We suppose that this coordination mechanism is available here too; provided the conditions of Theorem 4.11 are satisfied, a rescue consortium will be able to form.

We have seen so far that the existence of default costs is necessary for the existence of rescue consortia. One could think this further and assume that the parameters  $\alpha, \beta$  could be chosen to some extent by regulators if they confiscate or tax assets of failing banks. Then there would be a much higher chance that the system sorts out its problem on its own without relying on bailouts using external money.

## 5 Examples

In the following we consider two symmetric examples and one asymmetric example to illustrate our results.

### 5.1 Circular Network

**Definition 5.1.** Let  $n = 2N$ ,  $N \in \mathbb{N}$ , the (even) number of banks in the network. We refer to a financial system  $(\mathbf{L}, e, \alpha, \beta)$  as circular if the external assets are  $e = (1 - \varepsilon, 1 + \varepsilon, \dots)^\top \in \mathbb{R}^n$  and the liabilities matrix  $\mathbf{L} \in \mathbb{R}^{n \times n}$  is given by

$$\mathbf{L} = \begin{pmatrix} 0 & a & \dots & 0 & 0 \\ 0 & 0 & a + \varepsilon & \dots & 0 \\ \dots & & & & \\ 0 & 0 & \dots & 0 & a \\ a + \varepsilon & 0 & 0 & \dots & 0 \end{pmatrix},$$

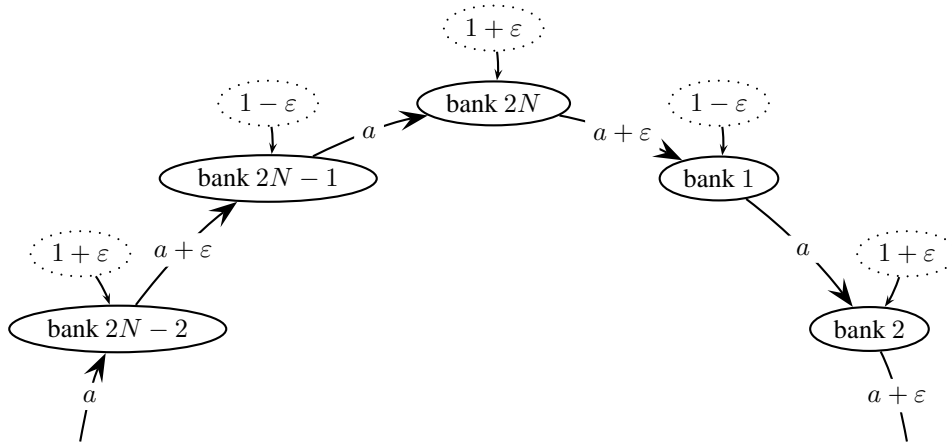


Figure 1: Circular arrangement.

where  $a > 0$ ,  $0 < \varepsilon \leq 1$ .

Figure 1 shows such a circular financial system. We see that the value of all banks is 1 and there are two different types of banks. We can therefore simplify our analysis by just considering a network consisting of bank 1 and bank 2. We investigate the effects of changing the external asset vector  $e$  to  $\gamma e$  where  $0 < \gamma \leq 1$ . By construction all banks are solvent if  $\gamma = 1$ . The minimum amount of external assets required for the two banks is

$$\xi = \max\{0, -(\Pi^\top \bar{L} - \bar{L})\} = (0, \varepsilon)^\top.$$

Hence, we see that only bank 2 can be level-0 insolvent when the external assets are reduced, in particular, this is the case if and only if

$$\gamma(1 + \varepsilon) < \varepsilon \Leftrightarrow \gamma < \frac{\varepsilon}{1 + \varepsilon}.$$

In the following we will always assume that

$$0 < \gamma < \frac{\varepsilon}{1 + \varepsilon} \tag{12}$$

is satisfied and hence default is triggered. We need to distinguish two situations: bank 1 suffers a contagious default or bank 1 does not default.

**Lemma 5.2.** *Suppose a circular financial system with two banks is given and suppose (12) is satisfied.*

1. *Bank 1 suffers a contagious default if and only if*

$$\gamma < \frac{a(1 - \beta)}{\alpha(1 + \varepsilon) + (1 - \varepsilon)}. \tag{13}$$

2. Suppose (13) is not true. Then the greatest clearing vector is

$$L^* = (a, a + \alpha\gamma(1 + \varepsilon))^\top. \quad (14)$$

3. Suppose (13) is true. Then the greatest clearing vector is

$$L^* = \frac{\alpha\gamma}{1 - \beta^2} (\beta(1 + \varepsilon) + 1 - \varepsilon, 1 + \varepsilon + \beta(1 - \varepsilon))^\top. \quad (15)$$

A proof is provided in Appendix D.

**Theorem 5.3.** *We consider a circular financial system with two banks. Suppose (12) holds, and suppose  $\alpha \in [0, 1)$  and  $\beta \in [0, 1]$ . Then bank 2 will always be rescued from default by a rescue consortium consisting of bank 1.*

Again, the proof can be found in Appendix D.

**Remark 5.4.** • From Lemma 5.2 we see that for  $\beta = 1$  a contagious default of bank 1 will never happen.

• Theorem 5.3 guarantees the existence of a rescue consortium by only assuming that  $\alpha < 1$ .

## 5.2 Star Network

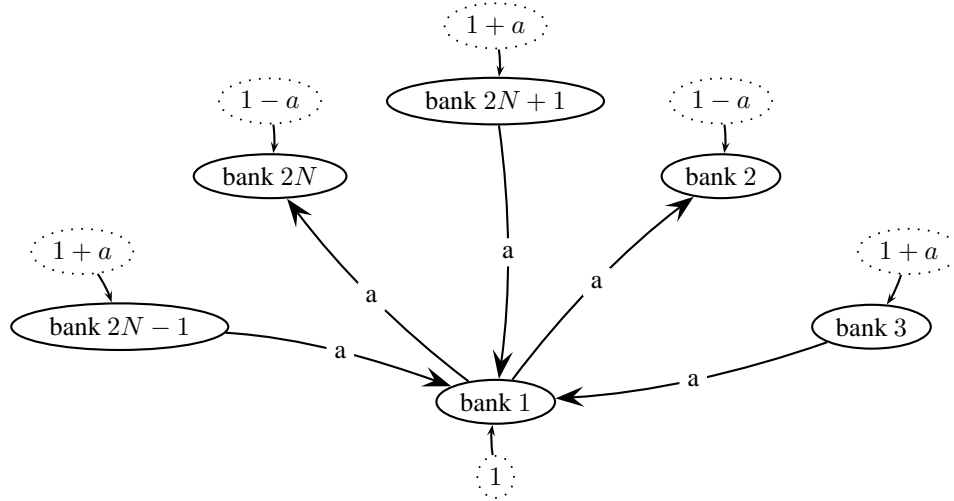


Figure 2: Star arrangement.

We now consider a star arrangement as shown in Figure 2.

**Definition 5.5.** *Let  $n = 2N + 1$ ,  $N \in \mathbb{N}$ , the (odd) number of banks in the network. We refer to a network  $(\mathbf{L}, e, \alpha, \beta)$  as star shaped if the external assets are  $e = (1, 1 - a, 1 + a, 1 - a, 1 + a, \dots)^\top$*

and the liabilities matrix  $\mathbf{L}$  is given by

$$\mathbf{L} = \begin{pmatrix} 0 & a & 0 & a & \dots & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ a & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ a & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ a & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where  $0 < a < 1$ .

Again we investigate the effects of changing the external asset vector  $e$  to  $\gamma e$  where  $0 < \gamma \leq 1$ .

The value of all banks is 1 and we see that there are three different types of banks. In order to simplify the presentation we consider a network consisting of bank 1, 2, 3 in the following. The results obtained for bank 2 carry over for all banks with an even number and the results obtained for bank 3 carry over to all banks with an odd number strictly greater than 1.

We compute the minimum amount of external assets required for the three banks to avoid them being level-0 insolvent:

$$\xi = \max\{0, -(\Pi^\top \bar{L} - \bar{L})\} = (0, 0, a)^\top.$$

Hence the only bank which can become level-0 insolvent is bank 3 and this happens if and only if  $\gamma(1+a) < a \Leftrightarrow \gamma < \frac{a}{1+a}$ . Therefore we will assume in the following that

$$0 < \gamma < \frac{a}{1+a} \tag{16}$$

is satisfied.

If we now account for possible default costs modelled by  $\alpha \in (0, 1]$  and  $\beta \in (0, 1]$ , we investigate in which situation bank 3 is the only defaulting bank in the network and what the corresponding clearing vectors are.

**Lemma 5.6.** *Suppose a star shaped financial system  $(\mathbf{L}, e, \alpha, \beta)$  is given and suppose that (16) is satisfied.*

1. *Bank 3 is the only defaulting bank in the network if and only if*

$$\alpha \geq \frac{aN - \gamma}{N\gamma(1+a)}. \tag{17}$$

2. *Suppose (17) holds. Then the greatest clearing vector is given by*

$$L^* = (a, 0, \alpha\gamma(1+a))^\top. \tag{18}$$



3. Suppose that (16) holds and suppose that

$$\alpha < \frac{aN - \gamma}{N\gamma(1 + a)}. \quad (19)$$

Then, both bank 1 and bank 3 default and the corresponding upper clearing vector is

$$L^* = (N\beta\alpha\gamma(1 + a) + \alpha\gamma, 0, \alpha\gamma(1 + a))^T. \quad (20)$$

A proof can be found in Appendix D. Note, since  $\alpha \leq 1$  we see that for small  $\gamma$  condition (17) will not be satisfied and then more banks will default.

The following theorem guarantees the existence of a rescue consortium for bank 3 if default costs exist.

**Theorem 5.7.** *Suppose a star shaped financial system  $(\mathbf{L}, e, \alpha, \beta)$  is given and suppose that (16) is satisfied.*

1. *If (17) holds true and  $\alpha \in [0, 1)$ , then bank 3 will be rescued from default by a rescue consortium consisting of bank 1.*
2. *If (19) is satisfied and if  $\alpha, \beta \in [0, 1)$ , then there exists a rescue consortium which rescues bank 3 from default.*

These results are proved in Appendix D. Note that the first result is completely independent of the choice of  $\beta$ .

### 5.3 Asymmetric Network

Finally, we consider an asymmetric network consisting of six banks. Figure 3 shows the liabilities structure in this network and in (21) the liabilities matrix  $\mathbf{L}$  and external assets  $e$  are given.

$$L = \begin{pmatrix} 0 & 4.94 & 2.47 & 5.59 & 0 & 0 \\ 6 & 0 & 0 & 2 & 0 & 0 \\ 0 & 13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 12 & 0 & 0 & 0 & 0 & 0 \\ 2.79 & 6.21 & 0 & 0 & 0 & 0 \end{pmatrix}, e = \begin{pmatrix} 1 \\ 1 \\ 11.51 \\ 1.4 \\ 12.5 \\ 2 \end{pmatrix}. \quad (21)$$

To avoid level-0 insolvencies the external assets have to exceed

$$\xi = \max\{0, -(\Pi^T \bar{L} - \bar{L})\} = (0, 0, 10.53, 0.41, 12, 1)^T.$$

From this we see immediately that banks 3, 4, 5, 6 can become level-0 insolvent. We see that if the external assets  $e$  are reduced to  $\gamma e$  with  $0 < \gamma < 1$ , the first condition of Theorem 4.11 is still satisfied

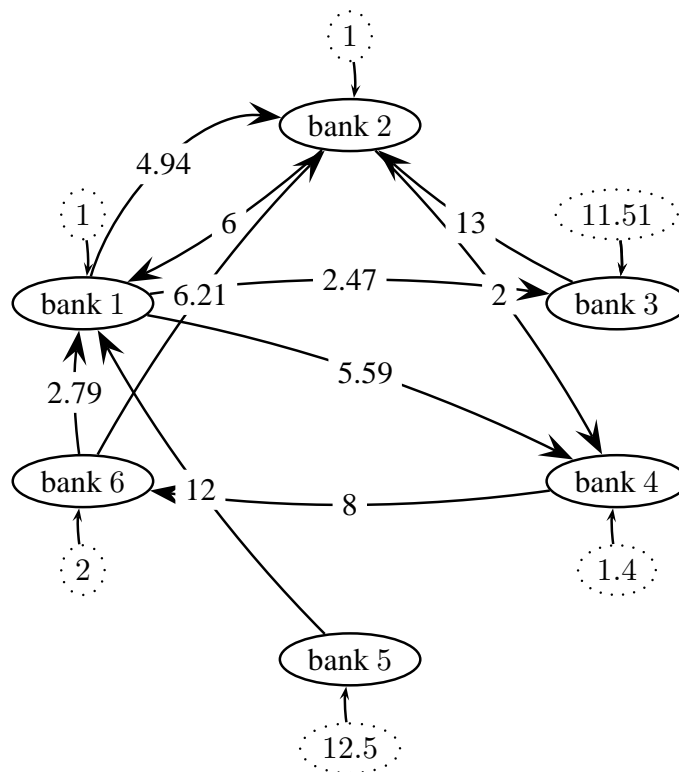


Figure 3: Asymmetric arrangement.

and therefore a rescue consortium does also exist in this asymmetric example.

## 6 Assessing and controlling contagion risks.

We have seen how default may spread through a financial system. A financial system which is initially solvent may get into difficulties because the external assets  $e$  fall to (or are revalued to) some lower values  $\tilde{e}$ , whereupon the set  $\mathcal{I}_0$  of level-0 insolvent banks becomes non-empty; one or more banks are in difficulties. Section 3 explains what happens if the problems are left unchecked: without intervention, the amounts recoverable of the interbank loans fall from the nominal  $\bar{L}$  to the maximal clearing vector  $L^*$ , and insolvency spreads from the initial level-0 set to level-1 insolvencies, level-2 insolvencies, ... Section 4 explains what *can* happen if a group of solvent banks get together and rescue the level-0 insolvent banks, and the circumstances in which there can be a rescue consortium with both the resources and the incentive to save the level-0 insolvents. However, what *can* happen is not necessarily what *does* happen, and in general a member of a rescue consortium would prefer *not* to take part in the rescue because doing so will incur costs for the rescuer; far better to let someone else stand up and take the bullets! It may be that a potential rescuer may see benefits in the rescue which make the cost worthwhile; for example, the rescued banks may have strengths in geographical or business areas where the rescuer is weak, and the combination would strengthen the rescuer. But if no such synergy exists, the regulator must either have power to compel the banks which it regulates to participate in a rescue for the good of the system as a whole, or else must have available sufficient cash to bail out the failing banks, which is the solution applied in the crisis of 2007-8. This solution has generated huge resentment in the democracies which resorted to the emergency bailouts, being seen quite correctly as the state being left with an enormous bill, while the banks are able to continue much as before. The alternative, which involves regulators being equipped with powers to compel banks to take part in rescues, would prove much more palatable to the electorates, though quite how it could be made to function in a landscape of overlapping regulatory responsibilities and transnational banks is very far from clear. Nevertheless, we shall attempt in this Section to make some suggestions in this direction.

But before we do this, we offer some possible diagnostics for the fragility of a financial system. We propose to base these on a thought experiment in which the external assets  $e$  of the individual banks are reduced by a common fraction<sup>3</sup>; as the losses mount, the pressure on the system increases until the set  $\mathcal{I}_0$  of level-0 insolvent banks becomes non-empty. So we consider the situation where the external assets  $e$  have been reduced to  $\tilde{e} = \gamma e$  for some  $0 < \gamma < 1$ , and the set  $\mathcal{I}_0$  is non-empty. Firstly we calculate the value  $\tilde{V}$  of the banks with external assets  $\tilde{e}$  on the assumption that all interbank liabilities are paid in full (8), and then we calculate the losses  $\Delta V$  which are suffered if defaults occur, as given by (9). We then attempt to assemble a rescue consortium, starting with all banks in  $\mathcal{I}_1$ , recruiting banks in decreasing order of their potential losses  $\Delta V_i$ ; then moving on to  $\mathcal{I}_2$  again adjoining banks in decreasing order of  $\Delta V_i$ ; continuing in this fashion through  $\mathcal{I}_3, \mathcal{I}_4 \dots$  until if necessary we add in banks which are in  $\mathcal{N} \setminus \cup_{\nu} \mathcal{I}_{\nu}$  which would in any case survive the wave of bankruptcies, but nonetheless might suffer losses

<sup>3</sup>This could represent a global decline in the value of assets used as collateral, or the effects of a recession. The assumption is probably an oversimplification, but is the simplest story that could be told.

in the process. If at some stage we achieve a rescue consortium, then this rescue happens, otherwise no rescue is possible, and the failure spreads through the network to bring down all the banks in  $\cup_{\nu} \mathcal{I}_{\nu}$ ; either way, we record the overall remaining value  $v(\gamma)$  of the financial system. If a rescue consortium  $A$  exists, then we obtain from Lemma 4.7, that

$$v(\gamma) = \sum_{i \in A} \tilde{V}_i - \sum_{j \in \mathcal{I}_0} \delta_j + \sum_{i \in \mathcal{N} \setminus (A \cup \mathcal{I}_0)} \tilde{V}_i = \sum_{i \in \mathcal{N}} \left( \Pi^\top \bar{L} + \gamma e - \bar{L} \right)_i = \gamma \bar{e},$$

where  $\bar{e} := \sum_{i \in \mathcal{N}} e_i$  is the initial value of the financial system before it was subjected to stress. We used the notation  $L^*(\gamma)$  to indicate that the greatest clearing vector does depend on  $\gamma$ . If a rescue consortium does not exist, then we obtain from Lemma 4.8, that

$$v(\gamma) = \sum_{i \in \mathcal{N}} \mathcal{V}(L^*(\gamma), \gamma e)_i = \gamma \bar{e} - \sum_{i=1}^n \left( (1-\alpha)\gamma e_i + (1-\beta) \sum_{j=1}^n L_j^*(\gamma) \pi_{ji} \right) \mathbb{I}_{\{L_i^*(\gamma) < \bar{L}_i\}}.$$

We then plot the function

$$\begin{aligned} \gamma \mapsto \lambda(\gamma) &:= \frac{\bar{e} - v(\gamma)}{\bar{e}} \\ &= \begin{cases} 1 - \gamma + \frac{\sum_{i=1}^n ((1-\alpha)\gamma e_i + (1-\beta) \sum_{j=1}^n L_j^*(\gamma) \pi_{ji}) \mathbb{I}_{\{L_i^*(\gamma) < \bar{L}_i\}}}{\bar{e}} & \text{without,} \\ 1 - \gamma & \text{with rescuing.} \end{cases} \end{aligned}$$

The function  $\lambda$  measures the relative losses due to default. It measures the difference between the initial value of the system and the value of the stressed system divided by the initial value of the system.

Note that the conditions in Theorem 4.11 on the existence of a rescue consortium are either satisfied for all  $0 < \gamma < 1$  or not at all.

Another informative plot we can present will display the function

$$\gamma \mapsto \eta(\gamma) := \frac{|\{i \in \mathcal{N} \mid L_i^*(\gamma) < \bar{L}_i\}|}{|\mathcal{N}|}, \quad (22)$$

where  $L^*(\gamma)$  is the greatest clearing vector if the external assets are given by  $\tilde{e} = \gamma e$ . This shows how the proportion of banks defaulting grows as  $\gamma$  decreases.

In the following we present these plots for our examples from Section 5 assuming that no rescue consortium forms. This is for illustration only as we know already from Section 5 that rescue consortia exist in all three examples. We immediately see a very different default behaviour in the three networks.

We have chosen the parameters in the symmetric networks such that the initial default conditions (12), (16) are satisfied at the same time. In all three examples we observe an increase in default costs and the proportion of defaulting banks for decreasing values of  $\gamma$  and hence decreasing external assets. We also find that higher values of  $\alpha$  can trigger earlier defaults of additional banks. This is particularly striking in the asymmetric example. For small values of  $\alpha$  we can observe almost individual defaults if we decrease  $\gamma$ . For larger values of  $\alpha$  we see many banks defaulting at once. In our examples, the star

shaped network is the only example where some banks will never default no matter how we reduce  $\gamma$  or  $\alpha$ . This is due to the fact that their liabilities are 0 and hence they can always satisfy them.

These diagnostics give a readily-understood picture of how fragile the given banking network is, and are more informative than any attempt to define a single scalar ‘fragility index’ for a financial system. They could be used to warn of looming dangers in the financial system, allowing time for corrective measures to be put in place.

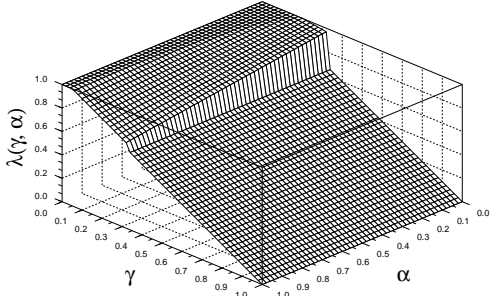
As we have noted, left to their own devices, banks would be unwilling to step forward to take on the losses of the failing banks, and without the possibility of compulsion and some kind of coordination, a chaotic collapse would be the likely outcome. Here there is a rôle for a bank regulator. We shall suppose that the regulator is able to observe the entire financial system; in particular, he knows whether or not the banks in the level-1 insolvency set  $\mathcal{I}_1$  are capable of rescuing the level-0 insolvent banks. If this is the case, then we propose that the rescue consortium will be made up of level-1 banks, which are after all those most perilously exposed to the level-0 banks. In the interests of containing the spread of contagion, we propose that the regulator should act so far as possible to leave banks distant from the level-0 banks completely unimpaired. If the banks in  $\mathcal{I}_1$  cannot mount a rescue, then the regulator will widen the net and try to assemble a rescue from the banks in  $\mathcal{I}_1 \cup \mathcal{I}_2$ , and so on. For the sake of discussion<sup>4</sup>, let us suppose that the banks in  $\mathcal{I}_1$  are capable of mounting a rescue. Then the regulator should be empowered to compel that group of banks to rescue the banks in  $\mathcal{I}_0$ . Different possible mechanisms could be proposed; here are a few that might be considered.

- (i) The simplest possibility would be that each bank in  $\mathcal{I}_1$  contributes to the bailout costs in proportion to the losses<sup>5</sup>  $\Delta V$  that they would experience if default were to occur, and receives shares in the rescued banks in proportion to their contribution.
- (ii) An alternative would be for the regulator to receive from each bank  $i$  in  $\mathcal{I}_1$  a sealed bid for the fraction  $\alpha_i$  of the defaulting banks which it was willing to take on, which would of course imply that bank  $i$  would assume responsibility for a fraction  $\alpha_i$  of the bailout costs. This would allow banks to bid higher if they thought that taking over the failing banks might be advantageous to their own business. If  $\sum_{i \in \mathcal{I}_1} \alpha_i \geq 1$ , then the bids received are sufficient to cover the bailout costs, and banks in  $\mathcal{I}_1$  are allocated fractions of the defaulting banks assets and liabilities proportional to their bids  $\alpha_i$ . If the total fractions bid fall short of 1, then each bank in  $\mathcal{I}_1$  contributes to the bailout proportionally to its potential losses  $\Delta V$  as in mechanism (i) above, but receives a fraction of the defaulting banks proportional to its bid. Thus a bank which bid zero would pay nothing towards the bailout if the total fraction bid by the other banks was at least 1, but it would then run the risk that the total was less than 1, in which case it would end up contributing to the rescue, but not receiving any part of the assets of the rescued bank. This threat would hopefully induce banks to make a realistic offer towards the bailout.

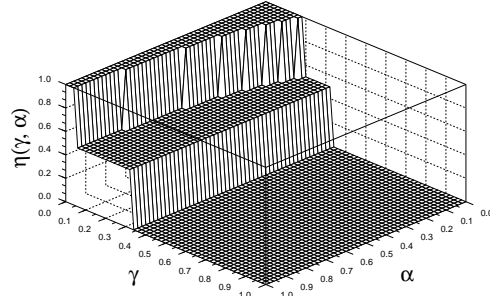
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<sup>4</sup>There is no loss of generality here: it is just notationally more compact.

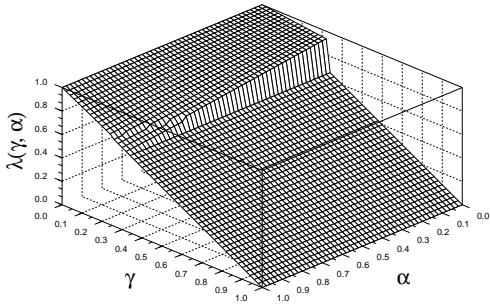
<sup>5</sup>See (9).



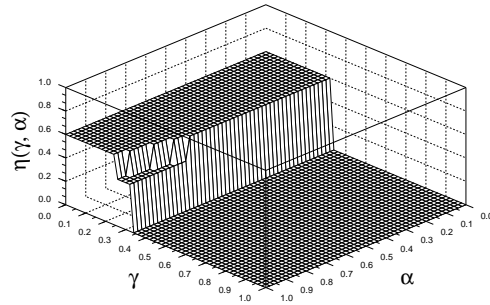
(a) Plot of  $\lambda$  for the circular network with 6 banks,  $\beta = 0.9$ ,  $a = 1$ ,  $\epsilon = 0.5$ ,  $N = 3$ .



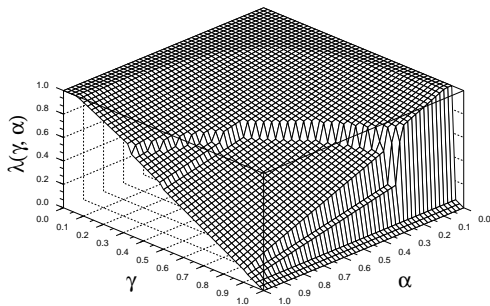
(b) Plot of  $\eta$  for the circular network.



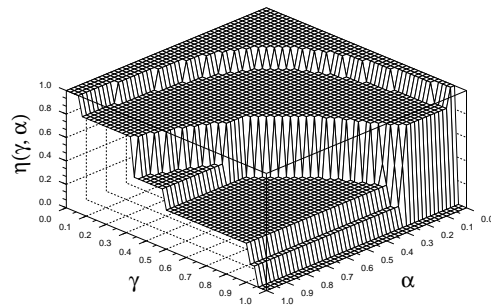
(c) Plot of  $\lambda$  for the star network with 5 banks,  $\beta = 0.9$ ,  $a = 0.5$ ,  $N = 2$ .



(d) Plot of  $\eta$  for the star network.



(e) Plot of  $\lambda$  for the asymmetric network with 6 banks,  $\beta = 0.9$ .



(f) Plot of  $\eta$  for the asymmetric network .

Figure 4: Plot of the functions  $\lambda$  and  $\eta$  considered as a function not just in  $\gamma$  but also in  $\alpha$ .

- (iii) Another somewhat riskier resolution mechanism which could be proposed is to allow the regulator to seize the assets of any failing bank, which would then pay out nothing to any bank to which it owed money. Those assets could now be used to compensate depositors, with any not used in this way being held by the government; but the main reason for allowing the regulator to seize the assets of a failing bank would be to give other banks in the system a very strong incentive to mount a rescue. This mechanism would also give failing banks a stronger position when bargaining with other more solvent banks. However, this mechanism would probably not be preferred because it does not as it stands guarantee that a catastrophic meltdown would be avoided, even were that possible.

The mechanisms proposed here all have the property that the regulatory authority is not required to inject any cash to rescue the failing banks, which would presumably be preferred by a rational democratic government, and the first two mechanisms would imply that the *total* cost to the financial system would be just the bailout costs, which is as low as it could possibly be.

Notice that while these proposals can arguably be effective in containing the damage caused by failure in an interconnected financial network, they would not work in the similar context of a failing sovereign nation, which could not be liquidated and shared among a consortium of rescuers as a bank could.

## 7 Conclusion

This paper has extended the modelling framework of Eisenberg and Noe (2001) to allow for the fact that when a bank defaults and has to call in its loans, it never realizes the face value of those loans but instead suffers a real loss. We used this to show that without intervention a failure of one or more banks could spread through the financial system, destroying value and taking down more banks as a result. For a given network, we are able to find what the network would look like when the spread of default is finished; we can work out which banks though initially solvent will fail if the banks initially in difficulties fail - and then we can work out what other banks will fail as a consequence. Dominoes of default sweep through the network until eventually only solvent banks remain.

We are then able to analyze how failing banks might be rescued by consortia of other banks, establishing the important results that any consortium which has an incentive to rescue the failing banks also has the means; and that any consortium of banks which would fail if default was allowed to spread will have an incentive to rescue if it has the means. These are hopeful conclusions, but not enough to ensure that failing banks will be rescued. We could have a situation where a group of banks might have the means to effect a rescue, but no incentive, as could happen if the banking network was in two geographical locations with weak linkages between them. Moreover, it does not deal with the moral hazard issue; a bank would prefer to let another bank do the rescuing, and would indeed have to have a good reason to act apparently against shareholder interests by bailing out a failing bank. We therefore see no alternative to some regulatory backstop compulsion if the banking sector is to collectively make good the losses of some of its members which would threaten further losses and overall stability. Otherwise, governments

will again be the unconscious underwriters of risky but profitable banking activities. Any framework of effective regulatory legislation runs the risk of driving financial services into the least-regulated jurisdiction, so some effective international coordination would be needed here. We offer some thoughts on how a rescue mechanism might look, but do so tentatively, well aware of the complexities lurking around these issues. Maybe all we have been able to do here is to present some ways of thinking about financial networks, some tools to analyze the default risks inherent in them, and some simple diagnostic methods that may help us to assess financial fragility; even this we believe has value.

## Acknowledgment

Financial support by the Cambridge Endowment for Research in Finance (CERF) is gratefully acknowledged.

## A Proofs of Section 2

*Proof of Lemma 2.7.* (i) The first property follows immediately from the definition of  $\Phi$  and the fact that  $\alpha, \beta \in (0, 1]$ .

(ii) Let  $\mathcal{I} = \{i : \bar{L}_i > e_i + \sum_{j=1}^n L_j \pi_{ji}\}$ , and  $\tilde{\mathcal{I}} = \{i : \bar{L}_i > e_i + \sum_{j=1}^n \tilde{L}_j \pi_{ji}\}$ . Clearly  $\mathcal{I} \subseteq \tilde{\mathcal{I}}$ . For any  $i \in \mathcal{I}$  the second alternative of the definition (1) obtains for both  $L$  and  $\tilde{L}$ , so it follows that  $\Phi(\tilde{L})_i \leq \Phi(L)_i$  for such  $i$ . For any  $i \in \tilde{\mathcal{I}} \setminus \mathcal{I}$ , since  $\alpha, \beta \in (0, 1]$ , we have

$$\Phi(\tilde{L})_i = \alpha e_i + \beta \sum_{j=1}^n \tilde{L}_j \pi_{ji} \leq e_i + \sum_{j=1}^n \tilde{L}_j \pi_{ji} < \bar{L}_i = \Phi(L)_i.$$

□

## B Proofs of Section 3

*Proof of Theorem 3.8.* We shall firstly prove that

$$\Lambda^{(\nu+1)} \leq \Lambda^{(\nu)} \quad \forall \nu = 0, \dots, n.$$

The proof proceeds by induction. To start the induction, we prove that  $\Lambda^{(1)} \leq \Lambda^{(0)} = \bar{L}$ . According to step 5 of the GA algorithm, we have  $\Lambda_i^{(1)} = \bar{L}_i = \Lambda_i^{(0)}$  for  $i \in \mathcal{S}_0$ , so now we just have to identify  $\Lambda^{(1)}$  on the insolvency set  $\mathcal{I}_0$ , as specified by (7). We construct the maximal solution  $x$  to (7) by a recursive method, starting from  $x^{(0)} = \Lambda^{(0)}$ , and defining recursively the  $x^{(k)}$  by

$$x_i^{(k+1)} = \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_0} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_0} x_j^{(k)} \pi_{ji} \right\}, \quad i \in \mathcal{I}_0. \quad (23)$$



Now for  $i \in \mathcal{I}_0$  we have

$$x_i^{(1)} = \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_0} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_0} \Lambda_j^{(0)} \pi_{ji} \right\} \leq e_i + \sum_{j=1}^n \Lambda_j^{(0)} \pi_{ji} < \bar{L}_i = \Lambda_i^{(0)} = x_i^{(0)} \quad (24)$$

where the first inequality holds because  $\alpha, \beta \in (0, 1]$  and  $\Lambda^{(0)} = \bar{L}$  on  $\mathcal{S}_0$ , and the second inequality holds because of the definition of the insolvency set  $\mathcal{I}_0$ . Thus we see that  $x^{(1)} \leq x^{(0)}$ , so the sequence  $x^{(k)}$  decreases to begin with, and hence due to the recursive definition decreases thereafter. The limit  $x := \downarrow \lim_{k \rightarrow \infty} x^{(k)}$  solves (7). If  $\tilde{x}$  was any other solution to (7) bounded above by  $\Lambda^{(0)}$ , then it is easy to see from the recursive recipe (23) that  $\tilde{x} \leq x$ , since  $\tilde{x} \leq \Lambda^{(0)} = x^{(0)}$ ; therefore the constructed  $x$  is the maximal solution to (7) bounded above by  $\Lambda^{(0)}$ .

To carry the inductive proof from  $\nu$  to  $\nu + 1$ , we observe that  $\mathcal{S}_\nu \subseteq \mathcal{S}_{\nu-1}$  by the inductive hypothesis, and so for all  $j \in \mathcal{S}_\nu$  we have  $\Lambda_j^{(\nu+1)} = \Lambda_j^{(\nu)} = \bar{L}_j$ . Now we again construct the maximal solution  $x$  to (7) by the obvious modification of the recursive recipe (23):

$$x_i^{(k+1)} = \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\nu} x_j^{(k)} \pi_{ji} \right\}, \quad i \in \mathcal{I}_\nu, \quad (25)$$

this time starting with  $x_i^{(0)} = \Lambda_i^{(\nu)}$  for  $i \in \mathcal{I}_\nu$ . The inequality (24) evolves to

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\nu} \Lambda_j^{(\nu)} \pi_{ji} \right\} \\ &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{S}_{\nu-1} \setminus \mathcal{S}_\nu} \Lambda_j^{(\nu)} \pi_{ji} + \sum_{j \in \mathcal{I}_{\nu-1}} \Lambda_j^{(\nu)} \pi_{ji} \right\} \\ &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{S}_{\nu-1} \setminus \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\nu-1}} \Lambda_j^{(\nu)} \pi_{ji} \right\} \\ &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_{\nu-1}} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\nu-1}} \Lambda_j^{(\nu)} \pi_{ji} \right\} \end{aligned} \quad (26)$$

since  $\Lambda^{(\nu)} = \bar{L}$  on  $\mathcal{S}_{\nu-1}$ . Now from (26) we see that for  $i \in \mathcal{I}_{\nu-1}$  we have  $x_i^{(1)} = \Lambda_i^{(\nu)} = x_i^{(0)}$ , and for  $i \in \mathcal{I}_\nu \setminus \mathcal{I}_{\nu-1}$  the expression (26) is

$$\begin{aligned} \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_{\nu-1}} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\nu-1}} \Lambda_j^{(\nu)} \pi_{ji} \right\} &\leq e_i + \sum_{j \in \mathcal{S}_{\nu-1}} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_{\nu-1}} \Lambda_j^{(\nu)} \pi_{ji} \\ &= e_i + \sum_{j=1}^n \Lambda_j^{(\nu)} \pi_{ji} \\ &< \bar{L}_i = \Lambda_i^{(\nu)} = x_i^{(0)}. \end{aligned}$$

This ensures that the sequence  $x^{(k)}$  starts decreasing, and therefore will always be decreasing. Whatever the limit is, it cannot be bigger than  $x^{(0)} = \Lambda^{(\nu)}$ , so we learn that  $\Lambda^{(\nu+1)} \leq \Lambda^{(\nu)}$ , as required.

The next task is to prove that  $\Lambda^{(\nu)} \geq L^*$  for all  $\nu$ , again by induction. It is clearly true when  $\nu = 0$ , so we now assume true up to  $\nu$  and try to extend to  $\nu + 1$ . If we make the natural notation

$$\mathcal{I}_* = \{i : e_i + \sum_j L_j^* \pi_{ji} < \bar{L}_i\}$$

then by the inductive hypothesis we have  $\mathcal{I}_\nu \subseteq \mathcal{I}_*$ . Observe that for  $i \in \mathcal{S}_\nu$  we have  $\Lambda_i^{(\nu+1)} = \bar{L}_i \geq L_i^*$ , so we just have to confirm that  $\Lambda^{(\nu+1)} \geq L^*$  also on  $\mathcal{I}_\nu$ . For this, we return to the recursive construction (25) and notice that when we start with  $x_i^{(0)} = \Lambda_i^{(\nu)}$  on  $\mathcal{I}_\nu$  we find

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\nu} \Lambda_j^{(\nu)} \pi_{ji} \right\} \\ &\geq \alpha e_i + \beta \left\{ \sum_{j \in \mathcal{S}_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\nu} L_j^* \pi_{ji} \right\} \\ &\geq \alpha e_i + \beta \sum_j L_j^* \pi_{ji} = L_i^*, \end{aligned}$$

exploiting the inductive hypothesis for the first inequality, and the fact that  $\bar{L} \geq L^*$  for the second, and finally using the defining property of  $L^*$  for the last equality, bearing in mind that  $i \in \mathcal{I}_\nu \subseteq \mathcal{I}_*$ . Once we have that  $x^{(1)} \geq L^*$  on  $\mathcal{I}_\nu$ , the recursive recipe (25) guarantees that  $x^{(k)} \geq L^*$  on  $\mathcal{I}_\nu$  for all  $k$ , and hence that the limit  $\Lambda^{(\nu+1)}$  is at least  $L^*$  on  $\mathcal{I}_\nu$ .

The proof is finished by the observation that when the algorithm terminates, as it must, the final vector  $\Lambda^{(\nu)}$  is a clearing vector. But we also know that  $\Lambda^{(\nu)} \geq L^*$ , as has just been proved; since  $L^*$  is the maximal clearing vector (Theorem 3.1), the only possibility is  $L^* = \Lambda^{(\nu)}$ , as required.  $\square$

## C Proofs of Section 4

*Proof of Theorem 4.4.* 1. This result follows from (10) and (11) and the fact that  $\mathcal{V}(L^*, \bar{e}) \geq 0$ .

2. For  $i \in \mathcal{R}$  we have  $\mathcal{V}(L^*, \bar{e}) = 0$  and hence  $\Delta V_i = \tilde{V}_i$ .  $\square$

*Proof of Lemma 4.8.* Note that in the original financial system  $(\mathbf{L}, e, \alpha, \beta)$  the vector  $\bar{L}$  is a clearing vector, since  $\mathcal{I}_0 = \emptyset$ . Hence,

$$\mathcal{V}(\bar{L}, e)_i = (\Pi^\top \bar{L} + e - \bar{L})_i = \sum_{j=1}^n \bar{L}_j \pi_{ji} + e_i - \bar{L}_i \geq 0 \quad \forall i \in \mathcal{N}.$$

Summing on  $i$  and using the fact that  $\Pi$  is a stochastic matrix<sup>6</sup> leads to the conclusion

$$\sum_{i=1}^n \mathcal{V}(\bar{L}, e)_i = \sum_{i=1}^n e_i. \quad (27)$$

In the distressed financial system, for a bank  $i$  with  $L_i^* < \bar{L}_i$  we have

$$L_i^* = \alpha \tilde{e}_i + \beta \sum_{j=1}^n L_j^* \pi_{ji}$$

and hence

$$\sum_{j=1}^n L_j^* \pi_{ji} + \tilde{e}_i - L_i^* = \sum_{j=1}^n L_j^* \pi_{ji} + \tilde{e}_i - \left( \alpha \tilde{e}_i + \beta \sum_{j=1}^n L_j^* \pi_{ji} \right) = (1 - \alpha) \tilde{e}_i + (1 - \beta) \sum_{j=1}^n L_j^* \pi_{ji}.$$

Thus we can rewrite the value of a bank:

$$\begin{aligned} \mathcal{V}(L^*, \tilde{e})_i &= (\Pi^\top L^* + \tilde{e} - L^*)_i \mathbb{I}_{\{L_i^* \geq \bar{L}_i\}} \\ &= \sum_{j=1}^n L_j^* \pi_{ji} + \tilde{e}_i - L_i^* - \left( (1 - \beta) \sum_{j=1}^n L_j^* \pi_{ji} + (1 - \alpha) \tilde{e}_i \right) \mathbb{I}_{\{L_i^* < \bar{L}_i\}}. \end{aligned}$$

Again summing on  $i$  and using the fact that  $\Pi$  is stochastic gives us

$$\sum_{i=1}^n \mathcal{V}(L^*, \tilde{e})_i = \sum_{i=1}^n \tilde{e}_i - \sum_{i=1}^n \left( (1 - \beta) \sum_{j=1}^n L_j^* \pi_{ji} + (1 - \alpha) \tilde{e}_i \right) \mathbb{I}_{\{L_i^* < \bar{L}_i\}} \quad (28)$$

Taking the difference of (27) and (28) proves the Lemma.  $\square$

*Proof of Theorem 4.10.* For  $i \in \mathcal{N}$ ,  $\mathcal{V}(\bar{L}, e)_i \geq 0$  by assumption. We shall suppose that  $\mathcal{V}(\bar{L}, e)_i = 0$  for all  $i \in \mathcal{I}_0$ , so that these banks are just on the brink of insolvency. Thus the bailout costs will be exactly

$$\sum_{i \in \mathcal{I}_0} (e_i - \tilde{e}_i).$$

Similarly, we shall suppose that  $e_j = \tilde{e}_j$  for  $j \in \mathcal{N} \setminus \mathcal{I}_0$ , so that any fall in the value of a solvent bank is due entirely to the defaults of the insolvent banks. From Lemma 4.8 and the fact that  $\alpha = \beta = 1$ , we see that

$$0 \leq \sum_{i=1}^n (\mathcal{V}(\bar{L}, e)_i - \mathcal{V}(L^*, \tilde{e})_i) = \sum_{i \in \mathcal{I}_0} (e_i - \tilde{e}_i).$$

Hence we see that the total loss in value of all banks when the insolvency spreads until we arrive at

<sup>6</sup>In fact, there may be rows of  $\Pi$  which are identically zero, but these are rows corresponding to banks which owe nothing to any other bank, and these contribute nothing to the sum.

the maximal clearing vector is equal to the total initial loss  $\sum_{i \in \mathcal{I}_0} (\tilde{e}_i - e_i)$ . Hence it can never be advantageous to any group of banks to rescue the failing banks, because in so doing they will have to assume liability for the losses  $\sum_{i \in \mathcal{I}_0} (\tilde{e}_i - e_i)$  of the failing banks, and this will be at least the loss that would be suffered if they did not intervene.  $\square$

*Proof of Theorem 4.11.* Let  $e$  be a vector such that  $e_i \geq \tilde{e}_i$  for all  $i \in \mathcal{N}$  and such that the financial system  $(\mathbf{L}, e, \alpha, \beta)$  does not contain a level-0 insolvent bank. (Note that such a vector always exists since we could choose  $e = \tilde{e} + \delta$ , where  $\delta$  is given below.)

Using the notation from Definitions 4.2 and 4.3, we see that

$$\begin{aligned}\tilde{V} &= \max\{0, \Pi^\top \bar{L} + \tilde{e} - \bar{L}\}, \\ \delta &= \max\{0, -(\Pi^\top \bar{L} + \tilde{e} - \bar{L})\}\end{aligned}$$

and hence

$$\begin{aligned}\tilde{V} - \delta &= \Pi^\top \bar{L} + \tilde{e} - \bar{L} = \Pi^\top \bar{L} + e - \bar{L} + (\tilde{e} - e) = \mathcal{V}(\bar{L}, e) + (\tilde{e} - e), \\ \Delta V &= \tilde{V} - \mathcal{V}(L^*, \tilde{e}) = \mathcal{V}(\bar{L}, e) - \mathcal{V}(L^*, \tilde{e}) + (\tilde{e} - e) + \delta.\end{aligned}$$

By summing over all components of  $\Delta V$  and applying Lemma 4.8 we obtain

$$\begin{aligned}\sum_{i=1}^n \Delta V_i &= \sum_{i=1}^n (\mathcal{V}(\bar{L}, e)_i - \mathcal{V}(L^*, \tilde{e})_i) + \sum_{i=1}^n (\tilde{e}_i - e_i) + \sum_{i=1}^n \delta_i \\ &= \sum_{i=1}^n (e_i - \tilde{e}_i) + \sum_{i=1}^n \left( (1 - \alpha)\tilde{e}_i + (1 - \beta) \sum_{j=1}^n L_j^* \pi_{ji} \right) \mathbb{I}_{\{L_i^* < \bar{L}_i\}} - \sum_{i=1}^n (e_i - \tilde{e}_i) + \sum_{i=1}^n \delta_i \\ &= \sum_{i=1}^n \left( (1 - \alpha)\tilde{e}_i + (1 - \beta) \sum_{j=1}^n L_j^* \pi_{ji} \right) \mathbb{I}_{\{L_i^* < \bar{L}_i\}} + \sum_{i=1}^n \delta_i.\end{aligned}$$

Hence we find that

$$\sum_{i=1}^n \Delta V_i > \sum_{i=1}^n \delta_i \Leftrightarrow \sum_{i=1}^n \left( (1 - \alpha)\tilde{e}_i + (1 - \beta) \sum_{j=1}^n \pi_{ji} L_j^* \right) \mathbb{I}_{\{L_i^* < \bar{L}_i\}} > 0$$

and the second inequality is true by our assumption that there exists a bank  $k$  which satisfies at least one of two properties and we see that each property guarantees the strict positivity of the expression.  $\square$

## D Proofs of Section 5

*Proof of Lemma 5.2.* 1. Bank 1 does suffer a contagious default if and only if

$$\beta a + \alpha\gamma(1 + \varepsilon) + \gamma(1 - \varepsilon) < a \Leftrightarrow \gamma < \frac{a(1 - \beta)}{\alpha(1 + \varepsilon) + (1 - \varepsilon)}.$$

2. + 3. These results follow from the definition of a clearing vector and the fact that bank 2 always defaults and bank 1 only suffers a contagious default if (13) is true.  $\square$

*Proof of Theorem 5.3.* Note that in any case, bank 2 is the only level-0 insolvent bank, but bank 2 can suffer a contagious default depending on the model parameters. The following bailout costs are always the same and are given by

$$\delta_2 := -(a + \gamma(1 + \varepsilon) - (a + \varepsilon)) = \varepsilon - \gamma(1 + \varepsilon).$$

Moreover,

$$\tilde{V} := \max\{0, \Pi^\top \bar{L} + \gamma e - \bar{L}\} = (\varepsilon + \gamma(1 - \varepsilon), 0)^\top.$$

1. Suppose we are in the situation in which the clearing vector is given by (14). Then

$$\begin{aligned} \mathcal{V}(L^*, \tilde{e})_1 &= a + \alpha\gamma(1 + \varepsilon) - a + \gamma(1 - \varepsilon) = \gamma(\alpha(1 + \varepsilon) + 1 - \varepsilon), \\ \Delta V_1 &= \tilde{V}_1 - \mathcal{V}(L^*, \tilde{e})_1 = \varepsilon - \alpha\gamma(1 + \varepsilon), \\ \Delta V_1 > \delta_2 &\Leftrightarrow \alpha < 1. \end{aligned}$$

2. Suppose we are in the situation in which the clearing vector is given by (15). Then

$$\begin{aligned} \mathcal{V}(L^*, \tilde{e})_1 &= \gamma(1 - \varepsilon) + \frac{\alpha\gamma}{1 - \beta^2}(1 + \varepsilon + \beta(1 - \varepsilon) - \beta(1 + \varepsilon) + \varepsilon - 1), \\ \Delta V_1 &= \tilde{V}_1 - \mathcal{V}(L^*, \tilde{e})_1 = \varepsilon + \varepsilon \frac{2\alpha\gamma}{1 - \beta^2}(-1 + \beta). \end{aligned}$$

From there we immediately obtain that

$$\begin{aligned} \Delta V_1 > \delta_2 &\Leftrightarrow \varepsilon + \varepsilon \frac{2\alpha}{1 - \beta^2}(-1 + \beta) > -(1 + \varepsilon) \\ &\Leftrightarrow \varepsilon \left(1 + \frac{2\alpha}{(1 - \beta)(1 + \beta)}(-1)(1 - \beta)\right) + (1 + \varepsilon) > 0 \\ &\Leftrightarrow 2\varepsilon \left(1 - \frac{\alpha}{1 + \beta}\right) + 1 > 0 \end{aligned}$$

and the last inequality is always true, since in this situation  $\alpha < 1 + \beta$  and  $\varepsilon > 0$ .

Hence we see that bank 1 always has an incentive to rescue bank 2 and from Lemma 4.4 we know that it

also has enough assets. □

*Proof of Lemma 5.6.* 1. We need to check when bank 1 defaults as well. Bank 1 does not default if and only if

$$N\alpha\gamma(1+a) + \gamma \geq Na \Leftrightarrow \alpha \geq \frac{aN-\gamma}{N\gamma(1+a)}.$$

2. + 3 This follows immediately from the first part and the definition of a clearing vector. □

*Proof of Theorem 5.7.* We prove the result for  $N = 1$ .

1. We compute the bailout costs:

$$\delta_3 := -(\gamma(1+a) - a) = a - \gamma(1+a).$$

Moreover,

$$\tilde{V} := \max\{0, \Pi^\top \bar{L} + \gamma e - \bar{L}\} = (\gamma, a + \gamma(1-a), 0)^\top,$$

since

$$\Pi^\top \bar{L} + \gamma e - \bar{L} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} + \begin{pmatrix} \gamma \\ \gamma(1-a) \\ \gamma(1+a) \end{pmatrix} - \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} = \begin{pmatrix} \gamma \\ a + \gamma(1-a) \\ \gamma(1+a) - a \end{pmatrix}.$$

Since (16) and (17) hold we are in the situation in which only bank 3 defaults. Then the clearing vector is given in (18) and

$$\begin{aligned} & \Pi^\top L^* + \gamma e - L^* \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ \alpha\gamma(1+a) \end{pmatrix} + \begin{pmatrix} \gamma \\ \gamma(1-a) \\ \gamma(1+a) \end{pmatrix} - \begin{pmatrix} a \\ 0 \\ \alpha\gamma(1+a) \end{pmatrix} = \begin{pmatrix} \alpha\gamma(1+a) + \gamma - a \\ a + \gamma(1-a) \\ \gamma(1+a)(1-\alpha) \end{pmatrix}. \end{aligned}$$

Hence, we find

$$\mathcal{V}(L^*, \tilde{e}) = (\alpha\gamma(1+a) + \gamma - a, a + \gamma(1-a), 0)^\top.$$

(Note that the first component is greater or equal to zero since (17) holds.) Then

$$\Delta V_1 = \gamma - (\alpha\gamma(1+a) + \gamma - a) = a - \alpha\gamma(1+a),$$

$$\Delta V_2 = a + \gamma(1-a) - (a + \gamma(1-a)) = 0.$$

Hence we see that  $\Delta V_1 \geq \Delta V_2$  and we therefore check whether a rescue consortium consisting only of bank 1 is sufficient for rescuing bank 3.

Note that  $\Delta V_1 > \delta_3$  for all  $0 \leq \alpha < 1$  and for all  $0 < \gamma < 1$ . (Hence we see that the strict inequality is not satisfied for  $\alpha = 1$ .) Hence condition (10) is satisfied for a rescuing consortium consisting of bank 1.

From Lemma 4.4, we know that condition (11) is also satisfied. Hence we see, that under the current parameter restrictions bank 1 will always rescue bank 3.

2. As before, the bail out costs are

$$\delta_3 := -(\gamma(1+a) - a) = a - \gamma(1+a)$$

and also  $\tilde{V}$  stays the same and is given by

$$\tilde{V} := \max\{0, \Pi^\top \bar{L} + \gamma e - \bar{L}\} = (\gamma, a + \gamma(1-a), 0)^\top.$$

Since the clearing vector is now given by (20) we obtain

$$\begin{aligned} & \max\{0, \Pi^\top L^* + \gamma e - L^*\} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta\alpha\gamma(1+a) + \alpha\gamma \\ 0 \\ \alpha\gamma(1+a) \end{pmatrix} + \begin{pmatrix} \gamma \\ \gamma(1-a) \\ \gamma(1+a) \end{pmatrix} - \begin{pmatrix} \beta\alpha\gamma(1+a) + \alpha\gamma \\ 0 \\ \alpha\gamma(1+a) \end{pmatrix} \\ &= \begin{pmatrix} \alpha\gamma(1+a)(1-\beta) + \gamma(1-\alpha) \\ \beta\alpha\gamma(1+a) + \alpha\gamma + \gamma(1-a) \\ \gamma(1+a)(1-\alpha) \end{pmatrix}. \end{aligned}$$

Since  $L_1^* < \bar{L}_1$  and  $L_3^* < \bar{L}_3$ , we obtain

$$\mathcal{V}(L^*, \bar{e}) = (0, \beta\alpha\gamma(1+a) + \alpha\gamma + \gamma(1-a), 0)^\top$$

and

$$\begin{aligned} \Delta V_1 &:= \gamma, \\ \Delta V_2 &:= a - \beta\alpha\gamma(1+a) - \alpha\gamma. \end{aligned}$$

Then

$$\begin{aligned} \Delta V_1 + \Delta V_2 &= \gamma + a - \beta\alpha\gamma(1+a) - \alpha\gamma > \delta_3 = a - \gamma(1+a), \\ &\Leftrightarrow 1 - \alpha\beta(1+a) - \alpha + 1 + a > 0 \\ &\Leftrightarrow (1+a)(1-\alpha\beta) + (1-\alpha) > 0 \end{aligned}$$

for  $\alpha, \beta \in [0, 1)$ .

Using Theorem 4.4, we therefore see that a rescue consortium consisting of bank 1 and 2 always

exists. For many realistic parameter combination one can also show that a rescue consortium consisting only of bank 2 exists.

□

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