

Fair Apportionment in the View of the Venice Commission's Recommendation

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Abstract

The Venice Commission in its Code of Good Practice in Electoral Matters specifies that (single-seat) constituencies should be drawn so that the size difference of a constituency's size from the average should not exceed a fixed limit while its borders must not cross the borders of administrative regions, such as states or counties.

Assuming that constituencies are of equal size within each of the administrative regions, the problem is equivalent to the apportionment problem, that is, the proportional allocation of voting districts among the administrative regions. We show that the principle of maximum admissible departure is incompatible with common apportionment properties, such as monotonicity and Hare-quota.

When multiple apportionments satisfy the smallest maximum admissible departure property we find a unique apportionment by the repeated application of the property. The allotment such that the differences from the average district size are lexicographically minimized can be found using an efficient algorithm. This apportionment rule is a well-defined allocation mechanism compatible with and derived from the recommendation of the Venice Commission. Finally, we compare this apportionment rule with mainstream mechanisms using data from

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1. Introduction

Fair representation is the cornerstone of representative democracies. The idea that each congressmen should represent the same number of citizens is as old as the United States. We study the so-called apportionment problem (for
5 recent texts see Pukelsheim, 2014 or Chapter 1 of Kubiak, 2009): sharing a given number of seats among a given set of regions with known populations in a fair way and look for mechanisms that provide a unique solution to each problem. Such a unique solution leaves no room for political bargaining and manipulation in designing or updating the legal framework for elections.

10 The stakes at the elections are very high and therefore the codification of any electoral law should be done with great care. Seemingly an easy problem, establishing electoral districts¹ with equal numbers of voters becomes nontrivial, when they must fit into the existing administrative structure of a country. There are alternative ways to approach the problem, generating a constant de-
15 bate even in countries with well-established democracies such as the United States. (For a comprehensive historical overview see Balinski and Young, 1982). A new entrant to this debate, the European Commission for Democracy through Law, better known as the Venice Commission published a comprehensive guide-
book on good electoral laws in 2002. The Code of Good Practice in Electoral
20 Matters (Venice Commission, 2002a) – consequently used in reviewing Albania’s and Estonia’s electoral law in 2011 (OSCE/ODIHR, 2011; Venice Commission and OSCE/ODIHR, 2011) and forming an apparent basis to the modifications

¹Voting terminology differs from country to country: We use *electoral district* or *constituency* interchangeably to refer to a geographical area, the people and especially the voters living there who elect one or more representatives. For simplicity we focus on *single-seat* voting districts.

Hungary introduced to in its electoral law in 2012 –, contains original recommendations for a good practice of apportionment.

25 “**Equality in voting power**, where the elections are not being
held in one single constituency, requires constituency boundaries to
be drawn in such a way that seats in the lower chambers represent-
ing the people are distributed equally among the constituencies, in
accordance with a specific apportionment criterion, e.g. the number
30 of residents in the constituency, the number of resident nationals
(including minors), the number of registered electors, or possibly
the number of people actually voting ... Constituency boundaries
may also be determined on the basis of geographical criteria and
the administrative or indeed historic boundary lines, which often
35 depend on geography ... The maximum admissible departure from
the distribution criterion adopted depends on the individual situa-
tion, although it should seldom exceed 10% and never 15%, except
in really exceptional circumstances (a demographically weak admin-
istrative unit of the same importance as others with at least one
40 lower-chamber representative, or concentration of a specific national
minority).” (Venice Commission, 2002a, §§13–15 in Section 2.2)

Similar conditions are common, though not universal. In Georgia, where
the electoral law of 1999 did not set rules about the sizes of constituencies, the
number of voters per (single-seat) constituency ranged from some 3,600 voters
45 in the Lent’ekhi district or 4,200 in the Kazbegi districts to over 138,000 in
Kutaisi City giving multiple times more influence to voters from Lent’ekhi or
the Kazbegi district than to those from Kutaisi City. The report of the Venice
Commission arrived to the conclusion that huge deviations like this question
the fairness of the whole election process (Venice Commission, 2002b). On the
50 other hand, in the United States, theoretically, no deviations from the equality
of constituencies are permitted; “other common thresholds are 5 percent (e.g.,
New Zealand, Albania, and Yemen); 10 percent (e.g., Australia, Italy, and the

Ukraine); 15 percent (e.g., Armenia, Germany, and the Czech Republic) and 20 percent (e.g., Zimbabwe and Papua New Guinea). In Canada, the independent commissions charged with creating federal electoral districts are allowed to deviate by up to 25 percent from the provincial quotas, and even more under ‘extraordinary circumstances’.” (Handley, 2007). In Singapore the tolerance is 30%; a recent proposal to reform the constituency map of the United Kingdom worked with a 5% permitted deviation from the average size (Balinski et al., 2010). The draft version of the 2012 electoral law of Hungary adopted the Venice Commission’s recommendation almost word by word, but the 10-15% maximum admissible departure between the population of any two constituency turned out to be infeasible given the actual size of the parliament and the populations of counties, if the constituencies cannot extend over county borders (Biró et al., 2012; Bodnár, 2012). Even with the subsequent relaxation, allowing a 15% (at most 20%) departure from the average size of constituencies in the final version, the requirements were just met.

We take this maximum admissible departure property and compare it with the properties that had been used to evaluate and judge apportionment methods. As generally there may be more than one apportionments satisfying this property, we look for apportionments where the maximum admissible departure is the lowest, then applying the same idea to the second largest difference, and so on. The naturally emerging leximin apportionment is unique (up to symmetries) and can be calculated using an efficient algorithm.

We focus on the simple setting where (1) districts elect single representatives, (2) forming even constituencies within each of the regions is possible and (3) representation is proportional. The results generalise directly to multi-representative districts. In designing the constituencies of a region not only the obvious integer problem (such as having two equal-sized constituencies in a region with odd population), but contiguity requirements and town and township boundaries too, create additional constraints that make the actual apportionment a little less equal. How the actual borders of the constituencies are drawn is yet another question. The strategic, manipulative design of voting districts,

known as *gerrymandering* (Gul and Pesendorfer, 2010; Chambers and Miller,
85 2013) may happen with equally-sized districts, too. Last, we assume that the
role of regions, states or counties is purely administrative and that they do
not have sovereign interests. If they do, representatives are likely to vote in
blocks and the effect on voting outcomes is no longer proportional to the size
of the region’s population, hence in larger problems there is a systematic bias
90 in favour of larger regions. The literature on voting power and power indices
(Penrose, 1946, Shapley and Shubik, 1954 and Banzhaf, 1965; for a recent sur-
vey see Felsenthal and Machover, 1998) study weighted voting situations such
as the European Council of Ministers (Kóczy, 2011), Penrose’s square root law
(Penrose, 1946) or the degressive proportionality (Laslier, 2012; Koriyama et al.,
95 2013) of the base+prop method (Pukelsheim, 2007) promoted in the Cambridge
Compromise (Grimmett, 2012; Grimmett et al., 2011; Rose et al., 2012) and give
explicit recommendations on how to adjust for this. With the appropriate pop-
ulation adjustments, these problems, however, can be reduced to proportional
allocation.

100 In the following we first formalise the apportionment problem and critically
discuss the (smallest) maximum admissible departure property. In Sections 4
and 5 we introduce the leximin solution for apportionment problems and then
apply this to apportionment problems from various countries: we compare the
leximin solution with the actual apportionment as outlined by the new electoral
105 law of Hungary and make a similar comparison for the German Bundestag and
the United States House of Representatives. We conclude with a more technical
discussion of the related apportionment methods in literature.

2. The apportionment problem, mechanisms and their properties

An *apportionment problem* (\mathbf{p}, H) is a pair consisting a vector

$$\mathbf{p} = (p_1, p_2, \dots, p_n)$$

of state populations, where $P = \sum_{i=1}^n p_i$ is the population of the country and
110 H denotes the number of seats in the legislature or House. Our task is to de-

termine the non-negative integers a_1, a_2, \dots, a_n with $\sum_{i=1}^n a_i = H$ representing the number of constituencies in states $1, 2, \dots, n$.

Let $\mathbf{p} \in \mathbb{N}_+^n$ and $\mathbf{a} \in \mathbb{N}^n$ be the n -dimensional vectors that contain the population sizes and the allotted number of seats respectively (where $\mathbb{N}_+ = \{1, 2, 3, \dots\}$). An *apportionment method or rule* is a function M that assigns an allotment for each apportionment problem (\mathbf{p}, H) . The fraction $\frac{p_i}{P}H$ is called the *respective share* of state i . Throughout the paper we will employ the following notation: let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say that $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ for $i = 1, 2, \dots, n$.

In the following we introduce several properties of apportionments.

Quota. Proportional representation is seldom possible as the respective shares of the states are hardly ever integer numbers. However if such case occurs i.e. the fractions $a_i = \frac{p_i}{P}H$ are integers for all $i \in \{1, \dots, n\}$ then the allotment \mathbf{a} is said to have the *exact quota* property.

In any other case taking one of the nearest integers to the exactly proportional share is a natural choice. An allotment \mathbf{a} satisfies *lower (upper) quotas*, if no state receives less (more) constituencies than the lower (upper) integer part of its respective share, that is $a_i \geq \lfloor \frac{p_i}{P}H \rfloor$ for all $i \in \{1, \dots, n\}$ and $a_i \leq \lceil \frac{p_i}{P}H \rceil$ for all $i \in \{1, \dots, n\}$, respectively.

An allotment satisfies the *quota* property if it satisfies both upper and lower quota. Similarly we say that an apportionment method $M(\mathbf{p}, H)$ satisfies lower (upper) quota if for any apportionment problem (p, H) , $M(\mathbf{p}, H)_i \geq \lfloor \frac{p_i}{P}H \rfloor$ or $M(\mathbf{p}, H)_i \leq \lceil \frac{p_i}{P}H \rceil$ respectively for all $i \in \{1, \dots, n\}$ and satisfies quota if it satisfies both of them.

Monotonicity. The individual states should not lose seats when more seats are available in the House. Formally:

Definition 1. An apportionment rule M is house-monotonic if $M(\mathbf{p}, H') \geq M(\mathbf{p}, H)$ for any apportionment problem (\mathbf{p}, H) and House sizes $H' > H$.

A scenario where increasing the House size would decrease the number of seats allotted to a state is often considered undesirable, perhaps even paradoxical.

140 cal. An apportionment rule where this is possible is said to exhibit the *Alabama paradox* referring to a historical occurrence of the phenomenon in the case of state Alabama. A rule is said to be *house-monotonic* if it does not suffer from such weakness.

There is a related monotonicity requirement and an associated paradox when 145 populations are considered. The *population paradox* arises when the population of two states increases at different rates. Then it is possible that the state with more rapid growth actually loses seats to the state with slower growth. Table 3 presents an example where the population paradox emerges.

Definition 2. *An apportionment rule M is population-monotonic if $M(\mathbf{p}', H)_i \geq$ 150 $M(\mathbf{p}, H)_i$ for any House size H and population sizes \mathbf{p}, \mathbf{p}' such that $p'_i > p_i$, $p'_j > p_j$ and $\frac{p'_i}{p_i} \geq \frac{p'_j}{p_j}$ while $p'_k = p_k$ for $k \in \{1, 2, \dots, n\}, k \neq i, j$.*

Note that this definition of population monotonicity is slightly stricter than the one used in the literature in general, see for instance (Lauwers and Van Puyenbroeck, 2008) or (Balinski and Young, 1982). However as we will see even this 155 strict property is violated by some rules.

Balinski and Young (1975) provided a so-called Quota-method that is house-monotonic and fulfills the quota property as well, but proved that no method that is free from both Alabama and the population paradoxes satisfies quota (Balinski and Young, 1982). In the next section we will provide examples ex- 160 hibiting these paradoxes.

Smallest maximum admissible departure property. The next property characterizes the recommendation² made by the Venice Commission (2002a). Let $\bar{a} = \frac{P}{H}$

²Although the Venice Commission is flexible on what kind of data should be the distribution criterion based on, it is clear that the difference from the average value is to be minimized. The most common interpretation is that there should be a limit on the allowable departure of the average number of registered voters per constituency (see Handley (2007)). We follow this practice as well, nevertheless our results hold in general, irrespective of the chosen reference data.

denote the average size of a constituency, let δ_i be the difference in percentage, displayed by the constituencies of state i and let d_i be its absolute value.

165 Formally

$$\delta_i = \frac{\frac{p_i}{a_i} - \bar{a}}{\bar{a}} \quad \text{and} \quad d_i = |\delta_i| \quad (1)$$

Definition 3. *An apportionment rule M satisfies the q -fixed maximum admissible departure property if $\left| \frac{\frac{p_i}{M(\mathbf{p}, H)_i} - \bar{a}}{\bar{a}} \right| \leq q$ for any apportionment problem (p, H) and for each $i \in \{1, \dots, n\}$.*

As we pointed out in the introduction European electoral laws impose a
 170 fixed limit on $\max_{i \in \{1, \dots, n\}} \{d_i\}$ rather than minimizing it. The Venice Commission follows this practice as well. It can happen, however, that, given an apportionment problem, no allotment exists that satisfy a certain limit, while an allotment with smallest maximum admissible departure always exists. Thus we focus on the latter concept. For a given apportionment problem (\mathbf{p}, H) let
 175 $\alpha_{(\mathbf{p}, H)}$ be the smallest maximum admissible departure that can be achieved with an allotment i.e.

$$\alpha_{(\mathbf{p}, H)} = \min_{\mathbf{a} \in A(n, H)} \max_{i \in \{1, \dots, n\}} \{d_i\} \quad (2)$$

where $A(n, H)$ denotes the set of n -dimensional non-negative vectors for which the sum of the coordinates is H .

Definition 4. *An apportionment rule M satisfies the smallest maximum admissible departure property if $\left| \frac{\frac{p_i}{M(\mathbf{p}, H)_i} - \bar{a}}{\bar{a}} \right| \leq \alpha_{(\mathbf{p}, H)}$ for any apportionment problem
 180 (p, H) and for each $i \in \{1, \dots, n\}$.*

The philosophy behind the quota and the smallest maximum admissible departure property is very similar, but not quite the same. The quota specifies how many seats a state should receive at least and at most. If a state gets less than
 185 its lower quota, then the allotment can be considered somewhat unfair from the

point of view of *that particular state*. The smallest maximum admissible departure property is concerned rather with *the individual voter*. If the population sizes of the constituencies differ too much so does the voters' influence. Not surprisingly, the quota property plays more central role in the U.S. where the states are large and highly independent. In Europe, where the countries consist
190 of small and in some sense uniform counties, the smallest maximum admissible departure property is more accepted.

3. The smallest maximum admissible departure property

In this section we review the basic features of the smallest maximum admissible departure property. In the following we will omit the lower index of $\alpha_{(\mathbf{p}, H)}$
195 and write simply α . First let us note that α is not monotone decreasing in the House size. To see this consider the allocation problem where $p = (100, 200)$ and let $H = 3$. Then it is possible to distribute the seats according to the exact quota thus $\alpha = 0$. However an increase of H by 1 renders both d_1 and d_2
200 positive.

3.1. Upper bounds on the smallest maximum admissible departure

Obviously d_i is the smallest if state i receives either its lower or upper quota, although it matters which one. Note that the closest integer to the respective share not always yields the smallest difference from the average. Let us elaborate on this relationship a little bit further. Let $l_i = \lfloor \frac{p_i}{P} H \rfloor$ and $u_i = \lceil \frac{p_i}{P} H \rceil$,
205 respectively denote these quotas of state i and let β_i denote the minimum difference achievable for state i . The maximum of these β_i values, denoted by β , is a natural lower bound for α . Formally:

$$\beta_i = \min \left(\left| \frac{\frac{p_i}{l_i} - \bar{a}}{\bar{a}} \right|, \left| \frac{\frac{p_i}{u_i} - \bar{a}}{\bar{a}} \right| \right), \quad \beta = \max_{i \in N} \beta_i.$$

Empirical analysis shows that, in general, increasing H results in a lower α
210 *ceteris paribus*. The problem with small House sizes is that they imply a larger average constituency size. Divisibility issues can appear for smaller states that

are only a few times as large as \bar{a} . It can happen that the average size of the constituencies of state i is equally far away from \bar{a} for both the lower and upper integer part of $\frac{P}{p_i}$, formally

$$\frac{\frac{p_i}{l_i} - \bar{a}}{\bar{a}} = \frac{\bar{a} - \frac{p_i}{u_i}}{\bar{a}}. \quad (3)$$

215 For instance, if $l_i = 2$ and $u_i = 3$ then $p_i = \frac{12}{5}\bar{a}$ and $d_i = 0.2$. A simple computation shows that, in general, if (3) holds, then $d_i = \frac{1}{2l_i+1}$. The Table 1 summarizes the problematic state population sizes.

$l_i - u_i$	p_i^*	β_i
1 - 2	$\frac{4}{3}\bar{a}$	0.333
2 - 3	$\frac{12}{5}\bar{a}$	0.2
3 - 4	$\frac{24}{7}\bar{a}$	0.143
4 - 5	$\frac{40}{9}\bar{a}$	0.111
5 - 6	$\frac{60}{11}\bar{a}$	0.091

Table 1: Critical state populations regarding divisibility. The first column shows the lower and upper quotas, p_i^* are the population levels where the minimal difference produced by these is maximal, and β_i is the corresponding worst case value of the minimal difference.

In other words, if there is a state with population $\frac{4}{3}$ of the average constituency size then α is at least $\frac{1}{3}$. For this value a lower d cannot be adhered to. One way to overcome this is to increase the house size H and thereby increase the number of constituencies allocated to each state, in particular, to the smallest state. Let i denote the smallest state and

$$\gamma \stackrel{def}{=} \begin{cases} \frac{1}{2l_i+1} & \text{if } l_i \neq 0, \\ \infty & \text{if } l_i = 0. \end{cases}$$

225 As the House size increases, l_i increases, and therefore γ decreases. Note that γ is an upper bound for β but there is no obvious connection between γ and α . For instance, if we are able to distribute the seats according to the exact

quota, then α is zero, but γ can be high. However let $p_1, p_2 = 200$ and $p_3 = 600$ and let the House size equal to 7. Then $\gamma = \frac{1}{3}$, but $\alpha \geq 0.4$. We will further analyze the relation of α, β and γ in Section 5 using real data.

230 *3.2. Properties*

As we mentioned earlier, the quota and smallest maximum admissible departure properties have different objectives. An apportionment method that implements the latter will distribute less seats to a state than its lower quota if the maximum admissible departure can be lowered in this way. Large states
 235 serve as puffers where superfluous seats can be allocated or seats can be acquired if there are needed elsewhere as these do not change the average size of constituencies dramatically. Table 2 demonstrates this process.

Method \Rightarrow		Smallest max. adm. dep.			quota		
State	Population	seats	$\frac{p_i}{a_i}$	δ_i	seats	$\frac{p_i}{a_i}$	δ_i
A	26	3	8.666	-0.138	2	13	0.293
B	27	3	9	-0.104	3	9	-0.104
C	28	3	9.333	-0.071	3	9.333	-0.071
D	29	3	9.666	-0.038	3	9.666	-0.038
E	91	8	11.375	0.131	9	10.111	0.006
Total	201	20	10.05		20	10.05	

Table 2: quota vs. smallest maximum admissible departure. A simple example of 5 regions and 20 seats with the number of seats a_i , the population per seat $\frac{p_i}{a_i}$ and the difference from the average δ .

In the above example the total population equals to 201 while the average constituency size is 10.05. If we insist on applying the quota then State E must
 240 receive at least 9 seats. As a result State A – the smallest one – gets only 2. The voters in State A have the least influence, nearly 30% less than on average. On the other hand, if we apply the smallest maximum admissible departure property, State A gets an extra seat and the largest bias is reduced to 13.8%.

The next table shows an occurrence of population paradox when we try to
 245 minimize the maximum admissible departure. The population of State C grows
 more rapidly than the population of State B, yet it loses a seat to State B. If
 for instance we commit ourselves to the original seat distribution, where State
 B gets 3 seats, the smallest maximum admissible departure jumps from 12.7%
 to 16.3%. It is easy to check that the only allotment that falls within the 15%
 250 limit proposed by the Venice Commission is the (3,4,7).

State	Population	seats	δ_i	Population	seats	δ_i
A	69	3	0.114	69	3	0.003
B	70	3	0.130	80	4	-0.127
C	150	8	-0.091	172	7	0.071
Total	289	14		321	14	

Table 3: Population paradox with smallest maximum admissible departure

Finally we note that the smallest maximum admissible departure prop-
 erty is not compatible with house-monotonicity either. An apportionment rule
 that minimizes the maximum admissible departure can produce the Alabama-
 paradox.

State	Population	Seats	$\frac{p_i}{a_i}$	δ_i	Seats	$\frac{p_i}{a_i}$	δ_i
A	69	3	23	0.114	4	17.250	-0.104
B	70	3	23.333	0.130	4	17.500	-0.091
C	150	8	18.750	-0.091	7	21.428	0.112
Total	289	14	20.642		15	19.266	

Table 4: House-monotonicity and smallest maximum admissible departure

255 Table 4 shows an example where increasing the House size from 14 to 15
 causes State C to lose a seat. State C is the largest state hence its average
 constituency size changes only a little when one of its seats is assigned else-
 where. A house-monotone allotment such as $\mathbf{a} = (3, 4, 8)$ would have a 19.3%
 smallest maximum admissible departure almost twice as much as the allotment

260 in the example. This also exceeds the 15% limit of the Venice Commission's recommendation, making it an unfeasible solution.

In all of the three above examples the proposed allotment was the only one that minimized the maximum admissible departure. Hence any apportionment method that satisfies the smallest maximum admissible departure property
265 would be inherently incompatible with the house- or population monotonicity properties. Therefore there is no apportionment rule that is conform with the recommendation made by the Venice Commission and is free from the Alabama paradox or complies with the quota-property. We consider this as a conflict between equality among states versus equality among voters. The Venice com-
270 mission clearly cast its vote in favour of the second.

4. The lexicographically optimal solution

The recommendation of the Venice Commission gives a strong constraint for the solution of an apportionment problem. However, the set of allowable allotments can still be large, that may leave room for arbitrary political manip-
275 ulations. This in turn can have a similar effect as gerrymandering. As Balinski and Young (1975) also argue, having a well-defined allotment rule that leads to a unique solution is the best way to avoid political issues in the apportionment process.

In this section we introduce such a unique rule based on the smallest maxi-
280 mum admissible departure property, following the recommendation of the Venice Commission and give an efficient algorithm to compute such a solution. Our definition is based on the lexicographic ordering of relative differences.

What is lexicographic ordering? The term refers to alphabetic ordering, where the words are compared letter-by-letter and the ordering is based on the first
285 difference. When it comes to real vectors the ordering is based on the first coordinates where these vectors differ. Formally vector $x \in \mathbb{R}^m$ is *lexicographically smaller* than $y \in \mathbb{R}^m$ (denoted by $x \preceq y$) if $x \neq y$ and there exists a number $1 \leq j \leq m$ such that $x_i = y_i$ if $i < j$ and $x_j < y_j$.

Returning to our model, given an apportionment problem (\mathbf{p}, H) and an
 290 allotment \mathbf{a} , let $\Delta(\mathbf{a})$ denote a nonnegative n -dimensional vector, where the
 differences $d_i(\mathbf{a})$ are contained in a non-increasing order. A solution \mathbf{a} is said to
 be *lexicographically minimal*, or simply *leximin*, if there is no other allotment \mathbf{a}'
 where $\Delta(\mathbf{a}')$ is lexicographically smaller than $\Delta(\mathbf{a})$.

Greedy leximin algorithm

295 Let us refer to $\mathbf{a} \in \mathbb{N}^n$ as a *pre-allotment* if the $\sum_i a_i = H$ condition is
 relaxed. Let \mathbf{a}^{i+} denote a pre-allotment adjusted from \mathbf{a} , where $a_i^{i+} = a_i + 1$
 and $a_j^{i+} = a_j$ for each $j \neq i$. Similarly, let \mathbf{a}^{i-} denote a pre-allotment, where
 $a_i^{i-} = a_i - 1$ and $a_j^{i-} = a_j$ for each $j \neq i$. For simplicity, and to ensure the
 uniqueness of the solution, we assume that $d_i(\mathbf{a})$ is not equal to $d_j(\mathbf{a})$ for any
 300 strictly positive pre-allotment \mathbf{a} and pair of states i and j . (Note that this
 condition can be always satisfied if we perturb \mathbf{p} , and it does not affect the
 optimality of the solution.)

Phase 1: Let $\mathbf{a}[0]$ be a pre-allotment such that $d_i(\mathbf{a}[0])$ is minimal for each state
 i (i.e. equal to β_i). Let the total number of seats allocated in $\mathbf{a}[0]$ be
 305 $l = \sum_{i=1}^n a_i[0]$. If $l = H$ then STOP, $\mathbf{a}[0]$ is the leximin allotment.

Phase 2: If $l < H$ then for each $t = 0, 1 \dots H - l - 1$ do the following adjustment.
 Let $\mathbf{a}[t + 1] = \mathbf{a}^{i+}[t]$ for $i \in \{1, \dots, n\}$ such that $d_i(\mathbf{a}^{i+}[t])$ is minimal.
 If $l > H$ then for each $t = 0, 1 \dots l - H - 1$ do the following adjustment.
 Let $\mathbf{a}[t + 1] = \mathbf{a}^{i-}[t]$ for $i \in \{1, \dots, n\}$ such that $d_i(\mathbf{a}^{i-}[t])$ is minimal.

310

That is, we first find a pre-allotment $\mathbf{a}[0]$ that is lexicographically minimal
 and then we simply increase (or decrease) the number of seats in a greedy way,
 we add (or remove) a seat to (or from) state i if the increased difference is the
 smallest for this state. In what follows we show that these greedy adjustments
 315 lead to leximin pre-allotments in each step, and therefore a leximin allotment
 at the end of the process.

Theorem 5. *The greedy leximin algorithm results in the leximin solution for the apportionment problem.*

Proof. If $\sum_{i=1}^n a_i[0] = l = H$ then $\mathbf{a}[0]$ is the leximin allotment, obviously. Since
320 $a_i[0] = \lfloor \frac{p_i}{a} \rfloor$ or $a_i[0] = \lceil \frac{p_i}{a} \rceil$ for each state i , the difference $|l - H|$ must be less
than or equal to n .

Suppose that $l < H$, the case of $l > H$ can be proved in a similar way. Let us
show by induction for $t = 0, 1, \dots, H - l$, that $\mathbf{a}[t]$ is the leximin pre-allotment
when $l + t$ seats are available, so in particular, $\mathbf{a}[H - l]$ is the leximin allotment
325 for the original problem. The statement is true for $t = 0$, suppose that it is true
for an arbitrary $t : 0 \leq t < H - l$ and let us verify the statement for $t + 1$.

Suppose for a contradiction that there exists a pre-allotment \mathbf{b} where the to-
tal number of seats allocated is $l + t + 1$ and \mathbf{b} is leximin among the pre-allotments
with the same number of seats, so in particular, $\Delta(\mathbf{b})$ is lexicographically smaller
330 than $\Delta(\mathbf{a}[t + 1])$.

Let $\mathbf{a}[0]_{\leq} = \{\mathbf{a} : \mathbf{a}[0] \leq \mathbf{a}\}$, that is set of pre-allotments where each state has
at least as many seats as in $\mathbf{a}[0]$. First we shall observe that for any $\mathbf{a} \in \mathbf{a}[0]_{\leq}$,
 $d_i(\mathbf{a}) < d_i(\mathbf{a}^{i+})$, i.e., when we increase the number of seats in any state i the
absolute difference from the average size in state i can only increase. This implies
335 that if $\mathbf{a}[0] \leq \mathbf{a} \leq \mathbf{a}'$ and $\mathbf{a} \neq \mathbf{a}'$ then $\Delta(\mathbf{a}) \prec \Delta(\mathbf{a}')$. The greedy algorithm
allocates the remaining seats gradually, therefore $\mathbf{a}[0] \leq \mathbf{a}[1] \leq \dots \leq \mathbf{a}[H - l]$.
Furthermore, $\mathbf{b} \in \mathbf{a}[0]_{\leq}$ must hold, because if $b_i < a_i[0]$ for a state i then there
must be another state j such that $b_j > a_j[0]$, and since $d_i(\mathbf{b}) > d_i(\mathbf{b}^{i+}) \geq$
 $d_i(\mathbf{a}[0])$ and $d_j(\mathbf{b}) > d_j(\mathbf{b}^{j-}) \geq d_j(\mathbf{a}[0])$, we could decrease the departure of \mathbf{b}
340 from the average size in both states i and j by moving one seat from j to i ,
contradicting to the optimality of \mathbf{b} .

First we prove that $\Delta(\mathbf{a}[t]) \prec \Delta(\mathbf{b})$. Let i be a state where $b_i > a_i[t]$
(there must be such a state since \mathbf{b} allocates one more seat than $\mathbf{a}[t]$). Then
 $\mathbf{a}[0] \leq \mathbf{b}^{i-} \leq \mathbf{b}$ implies $\Delta(\mathbf{b}^{i-}) \prec \Delta(\mathbf{b})$. Therefore $\Delta(\mathbf{b}) \prec \Delta(\mathbf{a}[t])$ would
345 imply $\Delta(\mathbf{b}^{i-}) \prec \Delta(\mathbf{a}[t])$, which is in contradiction with our assumption since
 \mathbf{b}^{i-} is an allotment with $l + t$ seats.

Let us now assume that when adjusting the pre-allotment $\mathbf{a}[t]$ to $\mathbf{a}[t+1]$ in the greedy algorithm we increase the number of seats in country i . Suppose that the difference $d_i(\mathbf{a}[t+1])$ is the r th largest, i.e., $d_i(\mathbf{a}[t+1])$ is the r th entry of vector $\Delta(\mathbf{a}[t+1])$. The first $r-1$ entries of $\Delta(\mathbf{a}[t])$ and $\Delta(\mathbf{a}[t+1])$ are the same, so $\Delta(\mathbf{a}[t]) \prec \Delta(\mathbf{b}) \prec \Delta(\mathbf{a}[t+1])$ implies that the first $r-1$ entries of \mathbf{b} are also the same, so in each of the corresponding $r-1$ states all these three pre-allotments assign the same number of seats (thus they are identical for these $r-1$ states). Regarding state i , it must be the case that $d_i(\mathbf{b}) \leq d_i(\mathbf{a}[t+1])$ (otherwise \mathbf{b} would not be lexicographically smaller than $\mathbf{a}[t+1]$), therefore $b_i \leq a_i[t+1]$. From the latter it follows that among the rest of the $n-r$ states there must be one, say j , where $b_j > a_j[t+1] = a_j[t]$ since both \mathbf{b} and $\mathbf{a}[t+1]$ allocate $l+t+1$ seats, and they are not identical. Therefore $b_j \geq a_j[t] + 1$, which implies $d_j(\mathbf{a}[t]^{j+}) \leq d_j(\mathbf{b})$. But $\Delta(\mathbf{b}) \prec \Delta(\mathbf{a}[t+1])$ also implies $d_j(\mathbf{a}[t]^{j+}) \leq d_j(\mathbf{b}) < d_i(\mathbf{a}[t+1]) = d_i(\mathbf{a}[t]^{i+})$, leading to a contradiction with the selection of i in the greedy algorithm. \square

Note that both Phase 1 and Phase 2 can be conducted in n^2 steps, if one step means a comparison of two differences.

5. Data Analysis

In this section we first evaluate the 2011 Electoral Law of Hungary that triggered our interest in the recommendations of the Venice Commission at the first place. Then we look at the United States House of Representatives and the German Bundestag and discuss the allocation of seats according to the leximin method.

5.1. Hungary

The 2011 Electoral Law of Hungary drastically decreased the number of seats in the parliament and fixed the number of constituencies at 106. The law also proposed a seat distribution among the counties. Although the apportionment method was not provided, the law prescribed some principles for subsequent

375 redistribution of seats. These conditions closely followed the directives of the
 Venice Commission. The law requires that the difference between the population
 of any constituency and the average constituency size should be within 15%.
 The only exception is if a constituency would extend over the county border
 or its connectivity could not be ensured. In these cases higher difference is
 380 allowed, but if it ever exceeds 20% then a new allotment should be provided.
 Table 7 compares the seat distribution proposed by the law with the one that
 is produced by the leximin algorithm³.

County	Voters	Seats		Difference (δ_i)	
		law	leximin	law	leximin
Budapest	1 407 470	18	17	1	6.95
Baranya	325 943	4	4	5.26	5.26
Bács-Kiskun	438 352	6	6	-5.63	-5.63
Békés	308 471	4	4	-0.38	-0.38
Borsod-Abaúj-Zemplén	567 910	7	7	4.8	4.8
Csongrád	345 945	4	5	11.72	-10.63
Fejér	351 237	5	5	-9.26	-9.26
Győr-Moson-Sopron	364 894	5	5	-5.73	-5.73
Hajdú-Bihar	439 618	6	6	-5.35	-5.35
Heves	257 490	3	3	10.87	10.87
Jász-Nagykun-Szolnok	324 869	4	4	4.91	4.91
Komárom-Esztergom	255 396	3	3	9.97	9.97
Nógrád	170 463	2	2	10.1	10.10
Pest	973 668	12	12	4.81	4.81
Somogy	268 844	4	4	-13.18	-13.18
Szabolcs-Szatmár-Bereg	450 556	6	6	-3	-3
Tolna	196 751	3	3	-15.28	-15.28
Vas	215 773	3	3	-7.09	-7.09

³To calculate δ_i we used the demographic data of the 2010 election.

Veszprém	300 081	4	4	-3.09	-3.09
Zala	242 236	3	3	4.3	4.3
Total	8 205 967	106	106		

Table 5: The seat distribution and the differences from the average district size by the Electoral Law and by the leximin algorithm

Note that only two out of 20 counties have a different number of seats allotted. The average constituency size in Heves County is 85830 which is 30.87% higher than the average constituency size of Tolna. Therefore voters in Tolna have 30.87% more influence than those living in Heves. If we allow 20% discrepancy from the average constituency size then the difference between voters' influence can be as high as 50%. Interestingly, it is not these counties where the apportionment by law differs from the results of the 7 common methods calculated by Bodnár (2012), but Pest and Somogy.

Upper bounds on the smallest maximum admissible departure

The following figure shows how the smallest maximum admissible departure from the average constituency size (α) changes as we increase the House size from 50 to 180. To calculate the smallest maximum admissible departure the leximin method was used.

Increasing House size indeed implies smaller α , although it is far from being monotone. The upper bounds imposed by γ are clearly visible. The graph never crosses 33.33%, and for higher H values the upper limits are 20% and 14.28%. This implies that α coincides with β in most of the cases. A deeper analysis shows that $\alpha = \beta$ is true for a broader range of H . From the [50, 400] interval there are only two exceptions, namely, when the House size equals to 87 and 88. But even for these values it is true that $\alpha < \gamma$. Our conjecture is that for real life data α rarely differs from β , therefore γ can be an effective upper bound for both. That means that if one would like to meet the Venice Commission's recommendation, then the House size should be set so high that the lower quota

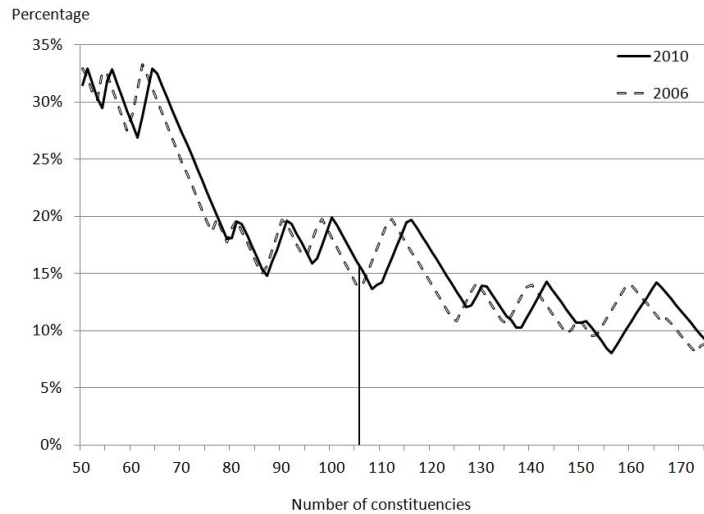


Figure 1: The decline of smallest maximum admissible departure compared to increasing House size using voter data from 2006 and 2010.

of the smallest county is at least 3 for the strict 15% limit.

Due to the demographic changes the local minimum of α shifted from 106 to 108 in four years. It can easily happen that in the near future 106 seats would mean the local maximum for α . A solution for this issue would be to choose
 410 the House size from an interval rather than fixing it. Although this seems to lead to an unpredictable system, in reality it would imply only a minor change from one election to the next as there would be one or two counties that would receive extra seats or have to give up one.

Monotonicity

415 Figure 2 shows how frequently the Alabama-paradox occurs as the House size changes.

The anomaly occurs only in the two largest counties⁴. As we mentioned earlier, the explanation is simple: large counties behave as buffers. They can store

⁴For higher House sizes the paradox occurs in the next largest county, Borsod-Abaúj-Zemplén as well.

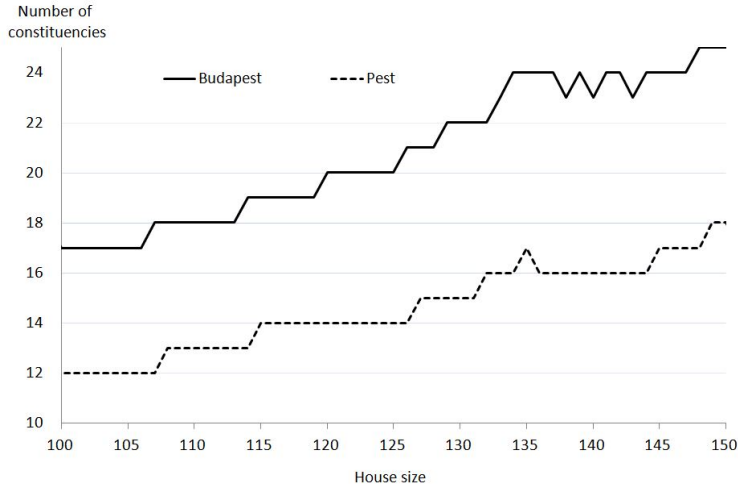


Figure 2: The number of constituencies in Budapest and in Pest county in view of House size

constituencies without affecting the leximin ordering too much and ‘borrow’
 420 seats for smaller counties that are crucial for the leximin ordering.

Changing the size of the regions

Finally, another way to lower the smallest maximum admissible departure
 is to increase the size of the administrative units that bundles the constituen-
 cies. Instead of counties we can use regions requiring only that no constituency
 425 extends over the region border. Table 6 summarizes the results for regions.

For instance, Northern Hungary consists of Borsod-Abaúj-Zemplén, Heves
 and Nógrád counties. By the law 7, 3 and 2 seats are assigned to them respec-
 tively, altogether 12. Heves produces the highest difference from the average:
 10.87%. However if we treat these three counties as one administrative unit
 430 then it receives 13 seats and the sizes of its constituencies will be 76605, only
 1.05% lower than the average. In this way Western Transdanubia generates the
 highest average 3.37% which is only a fraction of the 15.28% that Tolna county
 produces.

Region	Voters	Number of seats		Difference. (%)	
		law	leximin	law	leximin
Northern Hungary	995 863	12	13	10.87	1.05
Northern Great Plain	1 215 043	16	16	5.35	1.90
Southern Great Plain	1 092 768	14	14	11.72	0.83
Central Hungary	2 381 138	30	30	4.81	2.53
Central Transdanubia	906 714	12	12	9.97	2.40
Western Transdanubia	822 903	11	11	7.09	3.37
Southern Transdanubia	791 538	11	10	15.28	2.25
Total	8 205 967	106	106		

Table 6: The optimal seat distribution where no constituency extends over the region border

5.2. The German Bundestag

435 Similarly to the Hungarian election system, Germans voters may cast two votes⁵. The first vote decides – with simple relative majority – the fate of 299 seats. With this vote the residents can choose their candidate of choice in their own district. The second vote is the more important one, with it the voters can decide which party to support from the regional electoral list. Based on the
440 proportion of second votes, the rest of the mandates – another 299 seats – are distributed to the parties who have achieved at least 5 percent of valid second votes. It may happen that according to the second votes a party should receive less seats than it actually gained with the first votes. In such cases the party is
445 so called *overhang* seats. Indeed, after the 2013 elections the Bundestag was formed with 631 representatives, and this number can theoretically be as high as 897.

Here we analyze the seats that are elected directly with the first votes using data from the 2013 elections (?). The average constituency size is 207 180.

⁵The resemblance is not by chance. The German election system was a case model when the Hungarian Electoral Law was first scripted in 1989.

450 Although the leximin algorithm refines the seat distribution a little, it cannot
change the enormous bias caused by Bremen, where the average constituency
size is 241 912 (+16.76%). Furthermore, the constituencies of Bremen are not of
the same size – the larger of the two has 256 547 voters (+23.83%). Interestingly,
this does not even produce the largest difference. The Hamburg-Mitte district
455 – the central district of Hamburg – has an even higher number of voters, 256
862 voters (+23.98%), while Deggendorf in Bayern has 155 082 or -25.15%
that is even larger difference. These examples show that creating equally sized
constituencies *within* an administrative region may cause further complications.
For more on the apportionment issues related to the German election system
460 see (?).

County	Voters	Seats		Difference (δ_i)	
		law	leximin	law	leximin
Baden-Württemberg	7 689 895	38	37	-2.32	0.32
Bayern	9 472 738	45	46	1.60	-0.60
Berlin	2 505 718	12	12	0.79	0.79
Brandenburg	2 065 944	10	10	-0.28	-0.28
Bremen	483 823	2	2	16.76	16.76
Hamburg	1 281 918	6	6	3.12	3.12
Hessen	4 413 271	22	21	-3.17	1.44
Mecklenburg-Vorpommern	1 350 705	6	7	8.66	-6.86
Niedersachsen	6 117 473	30	30	-1.58	-1.58
Nordrhein-Westfalen	13 253 554	64	64	-0.05	-0.05
Rheinland-Pfalz	3 092 424	15	15	-0.49	-0.49
Saarland	796 072	4	4	-3.94	-3.94
Sachsen	3 406 430	16	16	2.76	2.76
Sachsen-Anhalt	1 930 880	9	9	3.55	3.55
Schleswig-Holstein	2 251 796	11	11	-1.19	-1.19
Thüringen	1 834 259	9	9	-1.63	-1.63
Total	61 946 900	299	299		

Table 7: The seat distribution and the differences from the average district size in the Bundestag in the 2013 elections compared with the same data induced by the leximin algorithm

5.3. The United States House of Representatives

Much of the literature of apportionment is based on the problems encountered at the regular updates of seat allocation in the United States House of Representatives. In the following we explain how and why our recommended allocation for the US House of Representatives differs and how the current method
465 fares in general when compared with our leximin approach.

5.3.1. The leximin vs. the equal proportions method

To further illustrate the properties of the leximin rule let us compare it with the equal proportion (EP) method (Huntington, 1921), that is used to distribute
470 the seats of the US House of Representatives. The method of Equal Proportions first distributes one seat to each state, then the remaining seats are allocated one at a time, to the state with the highest ‘priority number’. Priority of state i is determined by the formula $\frac{p_i}{\sqrt{k(k+1)}}$, where k runs from 1 until all the seats are distributed. It is a house-monotone apportionment rule, but it does not
475 satisfy quota (although it rarely produces a non-quota solution). The table of the apportionment of the 2010 US census compared with the result of the leximin algorithm can be found in the appendix Appendix A; Figure 3 provides a visual summary.

The two resulting allotments are very similar. In fact there are only two
480 states where the solutions differ: California and Montana. The scenario is the same we have seen before. The largest state lends a seat to one of the smaller ones and the smallest maximum admissible departure drops by almost 10%. It is quite surprising that the voters of Rhode Island - where the average constituency size is the smallest - have 88% more influence than the voters of
485 Montana. Although the leximin allotment reduces this gap somewhat, the only

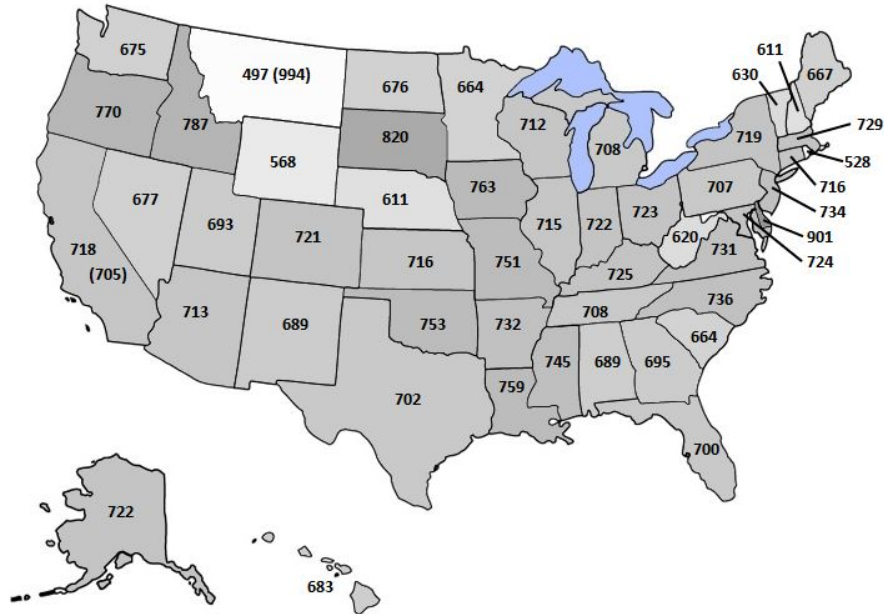


Figure 3: The number of citizens per representative according to the leximin method (in thousands). In parentheses the same figure for the EP method (where different). Note the dramatic swing for Montana.

efficient solution would be to drastically increase the House size. As there are 50 states and seven among them end up with only one representative each, the size of the House of Representatives can be considered rather small. Figure 4 shows how the smallest maximum admissible departure changes for higher House sizes.

490 *Smallest maximum admissible departures*

As it can be anticipated the smallest maximum admissible departure of the leximin solution never exceeds 33.3% however for the EP there is no such limit. To make certain that the smallest maximum admissible departure is below 20% we have to ensure that the smallest state, Wyoming a) receives at least two
 495 representatives and b) the constituency size obtained this way is within 20% of the average. A simple calculation shows that the smallest House size that guarantees these two criteria is 871 - a little more than twice its current size. As it is unlikely that the House of Representatives will be expanded in such

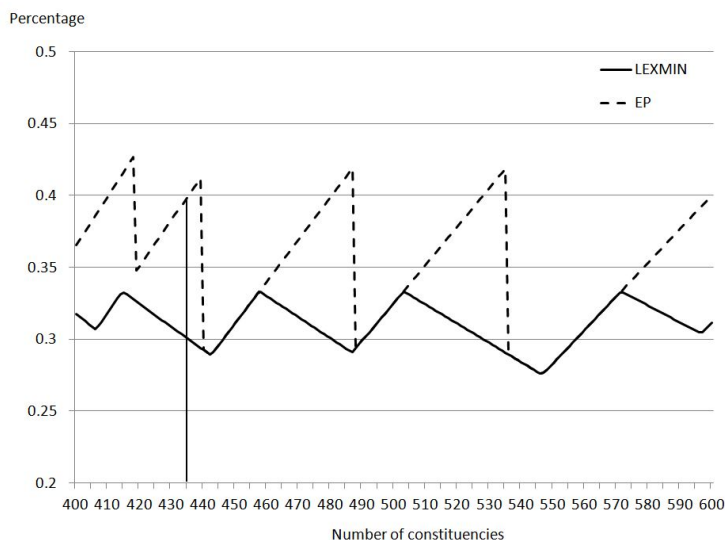


Figure 4: The smallest maximum admissible departure in view of the House size

fashion the influence of the voters will continue to vary from state to state. A
 500 temporary solution would be to increase the number of representatives by seven.
 The smallest maximum admissible departure for both the leximin and the EP
 solution meets its minimum at House size 442. In that case the highest gap
 between voters influence is ‘only’ 55.19%.

6. Conclusion

505 More and more countries adopt fairness measures in their electoral law that is
 based on, or similar to the recommendation of the Venice Commission (2002a).
 Based on the smallest maximum admissible departure property we introduce
 the well-defined Leximin Rule.

510 Our apportionment method is not the first. The problem of apportionment
 goes centuries back, the problem has been around ever since the new member
 states and population changes required a new seat allocation in the US House of
 Representatives. Balinski and Young (1982) give an illuminating theoretical and
 historical overview of the problem of apportionment and the political debates

that arose due to it. Methods like Hamilton’s (also called the Method of Largest
515 Remainders), Jefferson’s (Method of Greatest Divisors, but in Europe often
referenced as the d’Hondt method), or the Huntingdon-Hill or Equal Proportions
method, the currently used method in the US House of Representatives have
all been developed as responses to practical problems with apportionment such
as the emergence of one or another paradox. Lauwers and Van Puyenbroeck
520 (2006) compare some of these methods.

Apportionment problems are most often used for allocating seats among
administrative or political regions based on the population size of these regions:
states in the US House of Representatives, countries in the EU parliament and
so on. Our paper focuses on these applications. Apportionment is also used
525 for the allotment of seats to parties based on the outcome of an election, in
fact, sometimes both segmentations appear at the same time; the so-called bi-
apportionment is used in some European countries and the problem has been
studied by Demange (2012) and Serafini and Simeone (2012).

The Lexicographic Rule is, to the best of our knowledge, an original appor-
530 tionment method, although lexicographic solution concepts have already been
proposed by Gambarelli (1999) and Gambarelli and Palestini (2007). The clos-
est model is by Serafini and Simeone (2012), where the relative differences from
the target quotas are lexicographically minimized in the bi-apportionment prob-
lem. However, their target quotas are not the same as ours (when restricted to
535 a one-dimensional case), and their methods proposed are more complex, since
they are designed for the more general bi-apportionment problem.

There are also papers on minimizing the *relative* difference over pairs of
constituencies. Burt and Harris (1963) proposed this concept in for the US
House of Representatives, but then it got criticized by Gilbert and Schatz (1964).
540 A recent overview on this concept is given by Edelman (2006). Our problem
is different from this one, and it is easy to construct an example where the
solutions minimizing the relative difference of any two constituencies and the
maximum departure from the average size differ. So far, it seems, none of these
models are compatible with the recommendation of the Venice Commission.

545 The smallest maximum admissible departure property is very natural and
provides greater equality among citizens concerning their voting power than
other apportionment principles. Unfortunately, the property and therefore the
Leximin Rule turn out to be incompatible with the quota, the population- and
house-monotoncities over the class of apportionment problems, so that the Al-
550 abama and population paradoxes may arise when using it. It seems we have
just introduced a new method that fails all existing industry standards! Indeed
these properties emerged over century-long debates over the election rules for
the US House of Representatives and cannot be just ignored. On the other hand
the proposal of the Venice Commission is a very natural and plausible one, it is
555 comparable to the quota, while the violations of monotonicity are hardly sur-
prising in an integer problem. The fact that more and more countries consider
the smallest maximum admissible departure property as its *de facto* standard
to evaluate, improve and design apportionment among regions may be seen as
an indication that this is a natural and valid property whose incompatibility
560 with some old thoughts from the New World is interesting to be aware of, but
is perhaps time to update them in a representative democracy. And finally, a
third reading is just to note the emergence of two incompatible schools of ap-
portionment that may coexist thanks to minor differences in our understanding
of representative democracy.

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Appendix A. The seat distribution of the US House of Representatives by the equal proportion method and by the leximin algorithm

State	Voters	Number of seats		Difference (%)	
		EP	leximin	EP	leximin
Alabama	4 802 982	7	7	3.46	3.46
Alaska	721 523	1	1	1.51	1.51
Arizona	6 412 700	9	9	0.24	0.24
Arkansas	2 926 229	4	4	2.92	2.92
California	37341989	53	52	0.87	1.03
Colorado	5 044 930	7	7	1.39	1.39
Connecticut	3 581 628	5	5	0.78	0.78
Delaware	900 877	1	1	26.74	26.74
Florida	18 900 773	27	27	1.51	1.51
Georgia	9 727 566	14	14	2.24	2.24
Hawaii	1 366 862	2	2	3.84	3.84
Idaho	1 573 499	2	2	10.69	10.69
Illinois	12 864 380	18	18	0.55	0.55
Indiana	6 501 582	9	9	1.63	1.63
Iowa	3 053 787	4	4	7.41	7.41
Kansas	2 863 813	4	4	0.72	0.72
Kentucky	4 350 606	6	6	2.01	2.01
Louisiana	4 553 962	6	6	6.78	6.78
Maine	1 333 074	2	2	6.22	6.22
Maryland	5 789 929	8	8	1.82	1.82
Massachusetts	6 559 644	9	9	2.54	2.54
Michigan	9 911 626	14	14	0.39	0.39
Minnesota	5 314 879	8	8	6.52	6.52
Mississippi	2 978 240	4	4	4.75	4.75

Missouri	6 011 478	8	8	5.72	5.72
Montana	994416	1	2	39.90	30.04
Nebraska	1 831 825	3	3	14.09	14.09
Nevada	2 709 432	4	4	4.70	4.70
New Hampshire	1 321 445	2	2	7.04	7.04
New Jersey	8 807 501	12	12	3.26	3.26
New Mexico	2 067 273	3	3	3.04	3.04
New York	19 421 055	27	27	1.20	1.20
North Carolina	9 565 781	13	13	3.52	3.52
North Dakota	675 905	1	1	4.90	4.90
Ohio	11 568 495	16	16	1.72	1.72
Oklahoma	3 764 882	5	5	5.93	5.93
Oregon	3 848 606	5	5	8.29	8.29
Pennsylvania	12 734 905	18	18	0.46	0.46
Rhode Island	1 055 247	2	2	25.76	25.76
South Carolina	4 645 975	7	7	6.62	6.62
South Dakota	819 761	1	1	15.33	15.33
Tennessee	6 375 431	9	9	0.33	0.33
Texas	25 268 418	36	36	1.24	1.24
Utah	2 770 765	4	4	2.54	2.54
Vermont	630 337	1	1	11.31	11.31
Virginia	8 037 736	11	11	2.80	2.80
Washington	6 753 369	10	10	4.98	4.98
West Virginia	1 859 815	3	3	12.77	12.77
Wisconsin	5 698 230	8	8	0.21	0.21
Wyoming	568 300	1	1	20.04	20.04
Total	309 183 463	435	435	max : 39.9	max : 30.04

575 Balinski, M., Johnston, R., McLean, I., Young, P., Cummine, A., 2010. Drawing a New Constituency Map for the United Kingdom. Technical Report. British

Academy, Policy Centre.

Balinski, M., Young, H.P., 1975. The quota method of apportionment. *American Mathematical Monthly* 82, 701–730.

580 Balinski, M., Young, H.P., 1982. *Fair Representation: Meeting the Ideal of One Man, One Vote*. Yale University Press, New Haven.

Banzhaf, J.F., 1965. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review* 19, 317–343.

Biró, P., Kóczy, L.A., Sziklai, B., 2012. Választókerzetek igazságosan?
585 *Közgazdasági Szemle* 59, 1165–1186.

Bodnár, E., 2012. Alkotmányjogi dilemmák az új országgyűlési választási törvénnyel kapcsolatban. *Közjogi Szemle* 5, 40–48.

Burt, O.R., Harris, C.C., 1963. Apportionment of the US House of Representatives: A minimum range, integer solution, allocation problem. *Operations*
590 *Research* 11, 648–652.

Chambers, C.P., Miller, A.D., 2013. Measuring legislative boundaries. *Mathematical Social Sciences* , 1–8.

Demange, G., 2012. On party-proportional representation under district distortions. *Mathematical Social Sciences* 63, 181–191.

595 Edelman, P.H., 2006. Minimum Total Deviation Apportionments, in: Simeone, B., Pukelsheim, F. (Eds.), *Mathematics and Democracy*. Springer, Berlin Heidelberg, pp. 55–64.

Felsenthal, D.S., Machover, M., 1998. *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*. Edward Elgar, Cheltenham.

600 Gambarelli, G., 1999. Minimax Apportionments. *Group Decision and Negotiation* 8, 441–461.

- Gambarelli, G., Palestini, A., 2007. Minimax Multi-District Apportionments. *Homo Oeconomicus* 24, 335–356.
- Gilbert, E.J., Schatz, J.A., 1964. An Ill-Conceived Proposal for apportionment
605 of the US House of Representatives. *Operations Research* 12, 768–773.
- Grimmett, G.R., 2012. European apportionment via the Cambridge Compromise. *Mathematical Social Sciences* 63, 68–73.
- Grimmett, G.R., Oelbermann, K.F., Pukelsheim, F., 2011. A power-weighted variant of the EU27 Cambridge Compromise. *ArXiv* 1108. [arXiv:1108.1315](https://arxiv.org/abs/1108.1315).
- 610 Gul, F., Pesendorfer, W., 2010. Strategic redistricting. *American Economic Review* 100, 1616–1641.
- Handley, L., 2007. Boundary Delimitation, in: *Challenging the Norms and Standards of Election Administration*. IFES, pp. 59–74.
- Huntington, E., 1921. The mathematical theory of the apportionment of representatives. *Proceedings of the National Academy of Sciences of the United States of America* 7, 123–127.
615
- Kóczy, L.A., 2011. Beyond Lisbon: Demographic trends and voting power in the European Union Council of Ministers. *Mathematical Social Sciences* 63, 158–152.
- 620 Koriyama, Y., Laslier, J.F., Macé, A., Treibich, R., 2013. Optimal apportionment. *Journal of Political Economy* 121, 584–608.
- Kubiak, W., 2009. *Proportional Optimization and Fairness*. Springer.
- Laslier, J.F., 2012. Why not proportional? *Mathematical Social Sciences* 63, 90–93.
- 625 Lauwers, L., Van Puyenbroeck, T., 2006. The Hamilton Apportionment Method Is Between the Adams Method and the Jefferson Method. *Mathematics of Operations Research* 31, 390–397.

- Lauwers, L., Van Puyenbroeck, T., 2008. Minimally Disproportional Representation: Generalized Entropy and Stolarsky Mean-Divisor Methods of Apportionment. Working Papers 2008/24. Hogeschool-Universiteit Brussel, Faculteit Economie en Management.
- OSCE/ODIHR, 2011. Estonia Parliamentary Elections 6 March 2011. Election Assessment Mission Report. Organization for Security and Co-operation in Europe, Office for Democratic Institutions and Human Rights. Warsaw.
- Penrose, L.S., 1946. The elementary statistics of majority voting. *Journal of the Royal Statistical Society* 109, 53–57.
- Pukelsheim, F., 2007. Putting Citizens First : Representation and Power in the European Union, in: Cichocki, M., Życzkowski, K. (Eds.), *Institutional Design and Voting Power in the European Union*. Ashgate, London. chapter 14, pp. 235–253.
- Pukelsheim, F., 2014. *Proportional Representation*.
- Rose, R., Bernhagen, P., Borz, G., 2012. Evaluating competing criteria for allocating parliamentary seats. *Mathematical Social Sciences* 63, 85–89.
- Serafini, P., Simeone, B., 2012. Parametric maximum flow methods for minimax approximation of target quotas in biproportional apportionment. *Networks* 59, 191–208.
- Shapley, L.S., Shubik, M., 1954. A method for evaluating the distribution of power in a committee system. *American Political Science Review* 48, 787–792.
- Venice Commission, 2002a. Code of Good Practice in Electoral Matters. CDL-AD 23, 1–33.
- Venice Commission, 2002b. Opinion on the Unified election Code of Georgia. CDL-AD 9, 1–19.

Venice Commission, OSCE/ODIHR, 2011. On the Electoral Law and the Electoral Practice of Albania. Joint Opinion. Venice Commission and Organization for Security and Co-operation in Europe, Office for Democratic Institutions and Human Rights. Strasbourg.