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Proceedings of the American Mathematical Society
DOI: 10.1090/proc/14209

Accepted Manuscript

FAITH'S PROBLEM ON R-PROJECTIVITY IS UNDECIDABLE

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In memory of Gena Puninski.

ABSTRACT. In [7], Faith asked for what rings R does the Dual Baer Criterion hold in Mod–R, that is, when does R-projectivity imply projectivity for all right R-modules? Such rings R were called right testing. Sandomierski proved that all right perfect rings are right testing. Puninski et al. [1] have recently shown for a number of non-right perfect rings that they are not right testing, and noticed that [17] proved consistency with ZFC of the statement 'each right testing ring is right perfect' (the proof used Shelah's uniformization).

Here, we prove the complementing consistency result: the existence of a right testing, but not right perfect ring is also consistent with ZFC (our proof uses Jensen-functions). Thus the answer to the Faith's question above is undecidable in ZFC. We also provide examples of non-right perfect rings such that the Dual Baer Criterion holds for small modules (where small means countably generated, or $\leq 2^{\aleph_0}$-presented of projective dimension ≤ 1).

1. Introduction

The classic Baer Criterion for Injectivity [3] says that a (right R-) module M is injective, if and only if it is *R-injective*, that is, each homomorphism from any right ideal I of R into M extends to R. This criterion is the key tool for classification of injective modules over particular rings.

A module M is called *R-projective* provided that each homomorphism from M into R/I where I is any right ideal, factors through the canonical projection $\pi : R \to R/I$ [2, p.184]. One can formulate the *Dual Baer Criterion* as follows: a module M is projective, if and only if it is R-projective. The rings R such that this criterion holds true are called right *testing*, [1, Definition 2.2].

Dualizations are often possible over perfect rings. Indeed, Sandomierski proved that each right perfect ring is right testing [15]. The question of existence of non-right perfect right testing rings is much harder. Faith [7, p.175] says that "the characterization of all such rings is still an open problem" – we call it the Faith's problem here.

Note that if R is not right perfect, then it is consistent with ZFC + GCH that R is not right testing. Indeed, as observed in [1], [17, Lemma 2.4] (or [16]) implies that there is a κ^+-presented module N of projective dimension 1 such that $\operatorname{Ext}^1_R(N, I) = 0$ for each right ideal I of R (and hence N is R-projective, but not projective) in the extension of ZFC satisfying GCH and Shelah's Uniformization Principle UP_κ for an uncountable cardinal κ such that $\operatorname{card}(R) < \kappa$ and $\operatorname{cf}(\kappa) = \aleph_0$. In particular, attempts [4] to prove the existence of non-right perfect testing rings in ZFC could not be successful.

Date: February 9, 2018.

1991 *Mathematics Subject Classification.* Primary: 16D40, 03E35. Secondary: 16E30, 16E50, 03E45, 18G05.

Key words and phrases. Baer Criterion, Jensen-functions, non-perfect rings, projective module, R-projective module.

Research supported by GAČR 17-23112S.

Moreover, in the extension of ZFC + GCH satisfying UP_κ for all uncountable cardinals κ such that $cf(\kappa) = \aleph_0$ [6], all right testing rings are right perfect. So it is consistent with ZFC + GCH that all right testing rings are right perfect.

For many non-right perfect rings R, one can actually prove that R is not right testing in ZFC: this is the case for all commutative noetherian rings [10, Theorem 1], all semilocal right noetherian rings [1, Proposition 2.11], and all commutative domains (see Lemma 2.1 below).

It is easy to see that all finitely generated R-projective modules are projective, that is, the Dual Baer Criterion holds for all finitely generated modules over any ring. So in order to find examples of R-projective modules which are not projective, one has to deal with infinitely generated modules. The task is quite complex in general: in Section 2, we will show that there exist non-right perfect rings such that the Dual Baer Criterion holds for all countably generated modules, or for all $\leq 2^{\aleph_0}$-presented modules of projective dimension ≤ 1.

Some questions related to the vanishing of Ext, such as the Whitehead problem, are known to be undecidable in ZFC, cf. [5]. In Section 3, we will prove that this is also true for the existence of non-right perfect right testing rings. To this purpose, we will employ Gödel's Axiom of Constructibility V = L, or rather its combinatorial consequence, the existence of Jensen-functions (see [5, §VI.1] and [8, §18.2]). Our main result, Theorem 3.3 below, says that the existence of Jensen-functions implies that a particular subring of K^ω (where K is a field of cardinality $\leq 2^\omega$) is testing, but not perfect.

For unexplained terminology, we refer the reader to [2], [5], [8] and [9].

2. R-projectivity versus projectivity

It is easy to see that for each R-projective module M, each submodule $N \subseteq R^n$ and each $f \in \mathrm{Hom}_R(M, R^n/N)$, there exists $g \in \mathrm{Hom}_R(M, R^n)$ such that $f = \pi_N g$ where $\pi_N : R^n \to R^n/N$ is the projection (see e.g. [2, Proposition 16.12(2)]). In particular, all finitely generated R-projective modules are projective.

This not true of countable generated R-projective modules in general - for example, by the following lemma, the abelian group $\mathbb{Q}$ is $\mathbb{Z}$-projective, but not projective:

Lemma 2.1. *Let R be a commutative domain. Then each divisible module is R-projective. So R is testing, iff R is a field.*

Proof. Assume R is testing and possesses a non-trivial ideal I. Let M be any divisible module. If $0 \neq \mathrm{Hom}_R(M, R/I)$, then R/I contains a non-zero divisible submodule of the form J/I for an ideal $I \subsetneq J \subseteq R$. Let $0 \neq r \in I$. The r-divisibility of J/I yields $Jr + I = J$, but $Jr \subseteq I$, a contradiction. So $\mathrm{Hom}_R(M, R/I) = 0$, and M is R-projective. In particular, if R is testing, then each injective module is projective, so R is a commutative QF-domain, hence a field. □

However, there do exist rings such that all countably generated R-projective modules are projective. We will now examine one such class of rings that will be relevant for proving the independence result in Section 3:

Definition 2.2. Let K be a field, and R the unital K-subalgebra of K^ω generated by $K^{(\omega)}$. In other words, R is the subalgebra of K^ω consisting of all eventually constant sequences in K^ω.

For each $i < \omega$, we let e_i be the idempotent in K^ω whose ith component is 1 and all the other components are 0. Notice that $\{e_i \mid i < \omega\}$ is a set of pairwise orthogonal idempotents in R, so R is not perfect.

First, we note basic ring and module theoretic properties of this particular setting:

Lemma 2.3. *Let R be as in Definition 2.2.*

(1) *R is a commutative von Neumann regular semiartinian ring of Loewy length 2, with $\mathrm{Soc}(R) = \sum_{i<\omega} e_i R = K^{(\omega)}$ and $R/\mathrm{Soc}(R) \cong K$.*
(2) *If I is an ideal of R, then either $I = I_A = \sum_{i\in A} e_i R$ for a subset $A \subseteq \omega$ and I is semisimple and projective, or else $I = fR$ for an idempotent $f \in R$ such that f is eventually 1. In particular, R is hereditary.*
(3) *$\{e_i R \mid i < \omega\} \cup \{S\}$ is a representative set of all simple modules, where $S = R/\mathrm{Soc}(R)$. All these modules are $\sum$-injective, and all but S are projective.*
(4) *Let $M \in \mathrm{Mod}\text{–}R$. Then there are unique cardinals κ, κ_i ($i < \omega$) and λ such that $M \cong S^{(\kappa)} \oplus N$, $\mathrm{Soc}(N) \cong \bigoplus_{i<\omega}(e_i R)^{(\kappa_i)}$, and $N/\mathrm{Soc}(N) \cong S^{(\lambda)}$.*
   If $N = R^{(\mu)}/I$, then
   $$\mathrm{Soc}(N) = (\mathrm{Soc}(R^{(\mu)}) + I)/I \cong \mathrm{Soc}(R^{(\mu)})/(\mathrm{Soc}(R^{(\mu)}) \cap I)$$
   and $N/\mathrm{Soc}(N) \cong R^{(\mu)}/(\mathrm{Soc}(R^{(\mu)}) + I)$. Hence for each $i < \omega$, κ_i is the codimension of the $e_i R$-homogenous component of $\mathrm{Soc}(R^{(\mu)}) \cap I$ in $\mathrm{Soc}(R^{(\mu)})$, while λ is the codimension of $(\mathrm{Soc}(R^{(\mu)}) + I)/\mathrm{Soc}(R^{(\mu)})$ in $R^{(\mu)}/\mathrm{Soc}(R^{(\mu)}) \cong S^{(\mu)}$.

Proof. (1) Clearly, R is commutative, and if $r \in R$, then all non-zero components of r are invertible in K, so there exists $s \in R$ with $rsr = r$, i.e., R is von Neumann regular.

For each $i < \omega$, $e_i R = e_i K^\omega$ is a simple projective module, whence $J = \sum_{i<\omega} e_i R \subseteq \mathrm{Soc}(R)$. Moreover, $R/J \cong K$ is a simple non-projective module. So R is semiartinian of Loewy length 2, and $J = \mathrm{Soc}(R)$ is a maximal ideal of R.

(2) If $I \subseteq \mathrm{Soc}(R)$, then I is a direct summand in the semisimple projective module $\mathrm{Soc}(R)$. Since the simple projective modules $\{e_i R \mid i < \omega\}$ are pairwise non-isomorphic, $I \cong I_A = \sum_{i\in A} e_i R$, and hence $I = I_A$, for a subset $A \subseteq \omega$.

If $I \nsubseteq \mathrm{Soc}(R)$, then there is an idempotent $e \in I \setminus \mathrm{Soc}(R)$ and $eR + \mathrm{Soc}(R) = R$. Note that e is eventually 1, so in particular, $eR \supseteq \sum_{i\in B} e_i R$ where $B \subseteq \omega$ is the (cofinite) set of all indices i such that the ith component of e is 1. Then $I = eR \oplus (\sum_{i\notin B} e_i R \cap I)$. The latter direct summand equals I_A for a (finite) subset $A \subseteq \omega \setminus B$, and $I = fR$ for the idempotent $f = e + \sum_{i\in A} e_i$.

In either case, I is projective, hence R is hereditary.

(3) By part (2), the maximal spectrum $\mathrm{mSpec}(R) = \{I_\omega\} \cup \{(1 - e_i)R \mid i < \omega\}$. The $\sum$-injectivity of all simple modules follows from part (1) and [9, Proposition 6.18]. The simple module S is not projective because I_ω is not finitely generated.

(4) These (unique) cardinals are determined as follows: κ is the dimension of the S-homogenous component of M, and κ_i the dimension of its $e_i R$-homogenous component ($i < \omega$). The semisimple module $\bar{M} = M/\mathrm{Soc}(M) \cong N/\mathrm{Soc}(N)$ is isomorphic to a direct sum of copies of the unique non-projective simple module S; λ is the (S-) dimension of $\bar{M}$.

The final claim follows from the fact that $P = (\mathrm{Soc}(R^{(\mu)}) + I)/I$ is a direct sum of projective simple modules, while $R^{(\mu)}/(\mathrm{Soc}(R^{(\mu)}) + I)$ a direct sum of copies of S, so $\{0, P, N\}$ is the socle sequence of N. □

Next we turn to R-projectivity:

Lemma 2.4. *Let R be as in Definition 2.2.*

(1) *A module M is R-projective, iff it is projective w.r.t. the projection $\pi : R \to R/\mathrm{Soc}(R)$.*
(2) *The class of all R-projective modules is closed under submodules. If $M \in \mathrm{Mod}\text{–}R$ is R-projective, then all countably generated submodules of M are projective. In particular, the Dual Baer Criterion holds for all countably generated modules.*

Proof. (1) First, note that by part (2) of Lemma 2.3, the only ideals I such that R/I is not projective, are of the form $I = I_A$ where A is an infinite subset of ω (and hence $I \subseteq \mathrm{Soc}(R) = I_\omega$). So it suffices to prove that if M is projective w.r.t. the projection $\pi : R \to R/\mathrm{Soc}(R)$, then it is projective w.r.t. all the projections $\pi_{I_A} : R \to R/I_A$ such that $A \subseteq \omega$ is infinite.

Let $f \in \mathrm{Hom}_R(M, R/I_A)$. If $\mathrm{Im}(f) \subseteq \mathrm{Soc}(R)/I_A$, then there exists a homomorphism $h \in \mathrm{Hom}_R(\mathrm{Soc}(R)/I_A, \mathrm{Soc}(R))$ such that $\pi_{I_A} h = id$, whence $g = hf$ yields a factorization of f through π_{I_A}. Otherwise, let $\rho : R/I_A \to R/\mathrm{Soc}(R)$ be the projection. By assumption, there is $g \in \mathrm{Hom}_R(M, R)$ such that $\rho f = \pi g$. So $\rho(f - \pi_{I_A} g) = 0$, and $\mathrm{Im}(f - \pi_{I_A} g) \subseteq \mathrm{Soc}(R)/I_A$. Then $f - \pi_{I_A} g$ factorizes through π_{I_A} by the above, and so does f.

(2) The closure of the class of all R-projective modules under submodules follows from part (1) and from the injectivity of $S = R/\mathrm{Soc}(R)$ (see part (3) of Lemma 2.3). So it only remains to prove that each countably generated R-projective module is projective. However, as remarked above, for any ring R, each finitely generated R-projective module is projective. Since R is hereditary and von Neumann regular, [17, Lemma 3.4] applies and gives that also all countably generated R-projective modules are projective. □

We finish this section by presenting two more classes of non-right perfect rings over which small modules satisfy the Dual Baer Criterion.

In both cases, the rings will be von Neumann regular and right self-injective. Apart from classic facts about these rings from [9, §10], we will also need the following easy observation (valid for any right self-injective ring R, see [1, Proposition 2.6]): a module M is R-projective, iff $\mathrm{Ext}^1_R(M, I) = 0$ for each right ideal I of R.

Example 2.5. Let R be a right self-injective von Neumann regular ring such that R has primitive factors artinian, but R is not artinian (e.g., let R be an infinite direct product of skew-fields). Then all R-projective modules are non-singular, and the Dual Baer Criterion holds for all countably generated modules.

For the first claim, let M be R-projective and assume there is an essential right ideal $I \subsetneq R$ such that R/I embeds into M. Let J be a maximal right ideal containing I. By [9, Proposition 6.18], the simple module R/J is injective, so the projection $\rho : R/I \to R/J$ extends to some $f \in \mathrm{Hom}_R(M, R/J)$. The R-projectivity of M yields $g \in \mathrm{Hom}_R(M, R)$ such that $f = \pi g$ where $\pi : R \to R/J$ is the projection. Then g restricts to a non-zero homomorphism from R/I into the non-singular module R, a contradiction. Thus, M is non-singular.

For the second claim, we recall from [11, Example 6.8], that for von Neumann regular right self-injective rings, non-singular modules coincide with the $\aleph_1$-projective modules. However, each countably generated $\aleph_1$-projective module (over any ring) is projective. Thus each countably generated R-projective module is projective.

Example 2.6. Let R be a von Neumann regular right self-injective ring which is *purely infinite* in the sense of [9, Definition on p.116]. That is, there exists no central idempotent $0 \neq e \in R$ such that the ring eRe is directly finite (where a ring R is *directly finite* in case $xy = 1$ implies $yx = 1$ for all $x, y \in R$.)

For example, the endomorphism ring of any infinite dimensional right vector space over a skew-field has this property, see [9, p. 116].

We claim that the Dual Baer Criterion holds for all $\leq 2^{\aleph_0}$-presented modules M of projective dimension ≤ 1. Indeed, assume that such module M is R-projective. By [9, Theorem 10.19], R contains a right ideal J which is a free module of rank $2^{\aleph_0}$. If the projective dimension of M equals 1, then there is a non-split presentation $0 \to K \to L \to M \to 0$ where K and L are free of rank $\leq 2^{\aleph_0}$. Thus $\mathrm{Ext}^1_R(M, J) \neq 0$, in contradiction with the R-projectivity of M. This shows that M is projective.

In particular, if the global dimension of R is 2, and all right ideals of R are $\leq 2^{\aleph_0}$-presented (which is the case when R is the endomorphism ring of a right vector space of dimension $\aleph_0$ over any field of cardinality $\leq 2^{\aleph_0}$ under CH - see [13]), then the Dual Baer Criterion holds for all right ideals of R.

Remark 2.7. As mentioned in the Introduction, for any non-right perfect ring R, Shelah's Uniformization Principle UP_κ (for an uncountable cardinal κ such that $\mathrm{card}(R) < \kappa$ and $\mathrm{cf}(\kappa) = \aleph_0$) and GCH imply the existence of a κ^+-presented R-projective module N of projective dimension equal to 1.

If we choose R to be the endomorphism ring of a right vector space of infinite dimension $< \aleph_\omega$ over a skew-field of cardinality $< \aleph_\omega$, then, on one hand, we can take $\kappa = \aleph_\omega$, so the module N above can be chosen $\aleph_\omega^+$-presented. On the other hand, Example 2.6 gives a lower bound for the possible size of N: it has to be $> 2^{\aleph_0}$-presented.

3. The consistency of existence of non-perfect testing rings

In this section, we return to the setting of Definition 2.2, so K will denote a field, and R the subalgebra of K^ω consisting of all eventually constant sequences in K^ω. In order to prove that it is consistent with ZFC that R is testing, we will employ the notion of Jensen-functions, cf. [12] and [8, §18.2]:

Definition 3.1. Let κ be a regular uncountable cardinal.

(1) A subset $C \subseteq \kappa$ is called a *club* provided that C is *closed* in κ (i.e., $\sup(D) \in C$ for each subset $D \subseteq C$ such that $\sup(D) < \kappa$) and C is *unbounded* (i.e., $\sup(C) = \kappa$). Equivalently, there exists a strictly increasing continuous function $f : \kappa \to \kappa$ whose image is C.
(2) A subset $E \subseteq \kappa$ is *stationary* provided that $E \cap C \neq \emptyset$ for each club $C \subseteq \kappa$.
(3) Let A be a set of cardinality $\leq \kappa$. An increasing continuous chain, $\{A_\alpha \mid \alpha < \kappa\}$, consisting of subsets of A of cardinality $< \kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, is called a κ-*filtration* of the set A.
Similarly [5, IV.1.3.], if M is a $\leq \kappa$-generated module, then an increasing continuous chain, $(M_\alpha \mid \alpha < \kappa)$, consisting of $< \kappa$-generated submodules of M such that $M = \bigcup_{\alpha<\kappa} M_\alpha$, is called a κ-*filtration* of the module M.
(4) Let E be a stationary subset of κ. Let A and B be sets of cardinality $\leq \kappa$. Let $\{A_\alpha \mid \alpha < \kappa\}$ and $\{B_\alpha \mid \alpha < \kappa\}$) be κ-filtrations of A and B, respectively. For each $\alpha < \kappa$, let $c_\alpha : A_\alpha \to B_\alpha$ be a map. Then $(c_\alpha \mid \alpha < \kappa)$ are called *Jensen-functions* provided that for each map $c : A \to B$, the set $E(c) = \{\alpha \in E \mid c \restriction A_\alpha = c_\alpha\}$ is stationary in κ.

Jensen [12] proved the following (cf. [8, Theorem 18.9])

Theorem 3.2. *Assume Gödel's Axiom of Constructibility ($V = L$). Let κ be a regular infinite cardinal, $E \subseteq \kappa$ a stationary subset of κ, and A and B sets of cardinality $\leq \kappa$. Let $\{A_\alpha \mid \alpha < \kappa\}$ and $\{B_\alpha \mid \alpha < \kappa\}$) be κ-filtrations of A and B, respectively. Then there exist Jensen-functions $(c_\alpha \mid \alpha < \kappa)$.*

Now, we can prove our main result:

Theorem 3.3. *Assume $V = L$. Let K be a field of cardinality $\leq 2^{\aleph_0}$. Then all R-projective modules are projective.*

Proof. Let M be an R-projective module. By induction on the minimal number of generators, κ, of M, we will prove that M is projective. For $\kappa \leq \aleph_0$, we appeal to part (2) of Lemma 2.4, and for κ a singular cardinal, we apply [17, Corollary 3.11].

Assume κ is a regular uncountable cardinal. Let $G = \{m_\alpha \mid \alpha < \kappa\}$ be a minimal set of R-generators of M. For each $\alpha < \kappa$, let $G_\alpha = \{m_\beta \mid \beta < \alpha\}$. Let M_α be the submodule of M generated by G_α. Then $\mathcal{M} = (M_\alpha \mid \alpha < \kappa)$ is a κ-filtration of the module M. Possibly skipping some terms of $\mathcal{M}$, we can w.l.o.g. assume that $\mathcal{M}$ has the following property for each $\alpha < \kappa$: if M_β/M_α is not R-projective for some $\alpha < \beta < \kappa$, then also $M_{\alpha+1}/M_\alpha$ is not R-projective. Let E be the set of all $\alpha < \kappa$ such that $M_{\alpha+1}/M_\alpha$ is not R-projective.

We claim that E is not stationary in κ. If our claim is true, then there is a club C in κ such that $C \cap E = \emptyset$. Let $f : \kappa \to \kappa$ be a strictly increasing continuous function whose image is C. For each $\alpha < \kappa$, let $N_\alpha = M_{f(\alpha)}$. Then $(N_\alpha \mid \alpha < \kappa)$ is a κ-filtration of the module M such that $N_{\alpha+1}/N_\alpha$ is R-projective for all $\alpha < \kappa$. By the inductive premise, $N_{\alpha+1}/N_\alpha$ is projective, hence $N_{\alpha+1} = N_\alpha \oplus P_\alpha$ for a projective module P_α, for each $\alpha < \kappa$. Then $M = N_0 \oplus \bigoplus_{\alpha<\kappa} P_\alpha$ is projective, too.

Assume our claim is not true. We will make use of Theorem 3.2 in the following setting. We let $A = G$ and $B = R$. The relevant κ-filtration of A will be $(G_\alpha \mid \alpha < \kappa)$. For B, we consider any κ-filtration $(R_\alpha \mid \alpha < \kappa)$ of the additive group $(R, +)$ consisting of subgroups of $(R, +)$ (which exists since $\mathrm{card}(K) \leq \aleph_1$ implies $\mathrm{card}(R) \leq \aleph_1 \leq \kappa$; if $\mathrm{card}(K)$ is countable, the filtration can even be taken constant $= R$). By Theorem 3.2, there exist Jensen-functions $c_\alpha : G_\alpha \to R_\alpha$ ($\alpha < \kappa$) such that for each function $c : G \to R$, the set $E(c) = \{\alpha \in E \mid c_\alpha = c \restriction G_\alpha\}$ is stationary in κ.

By induction on $\alpha < \kappa$, we will define a sequence $(g_\alpha \mid \alpha < \kappa)$ such that $g_\alpha \in \mathrm{Hom}_R(M_\alpha, S)$ as follows: $g_0 = 0$; if $\alpha < \kappa$ and g_α is defined, we distinguish two cases:

(I) $\alpha \in E$, and there exist $h_{\alpha+1} \in \mathrm{Hom}_R(M_{\alpha+1}, S)$ and $y_{\alpha+1} \in \mathrm{Hom}_R(M_{\alpha+1}, R)$, such that $h_{\alpha+1} \restriction M_\alpha = g_\alpha$, $h_{\alpha+1} = \pi y_{\alpha+1}$ and $y_{\alpha+1} \restriction G_\alpha = c_\alpha$. In this case we define $g_{\alpha+1} = h_{\alpha+1} + f_{\alpha+1}\rho_{\alpha+1}$, where $\rho_{\alpha+1} : M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$ is the projection and $f_{\alpha+1} \in \mathrm{Hom}_R(M_{\alpha+1}/M_\alpha, S)$ is chosen so that it does not factorize through π (such $f_{\alpha+1}$ exists because $\alpha \in E$ by part (1) of Lemma 2.4. Note that $g_{\alpha+1} \restriction M_\alpha = h_{\alpha+1} \restriction M_\alpha = g_\alpha$.

(II) otherwise. In this case, we let $g_{\alpha+1} \in \mathrm{Hom}_R(M_{\alpha+1}, S)$ be any extension of g_α to $M_{\alpha+1}$ (which exists by the injectivity of S).

If $\alpha < \kappa$ is a limit ordinal, we let $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$. Finally, we define $g = \bigcup_{\alpha<\kappa} g_\alpha$. We will prove that g does not factorize through π. This will contradict the R-projectivity of M, and prove our claim.

Assume there is $x \in \mathrm{Hom}_R(M, R)$ such that $g = \pi x$. Then the set of all $\alpha < \kappa$ such that $x \restriction G_\alpha$ maps into R_α is a closed and unbounded subset of the regular uncountable cardinal κ, so it contains some element $\alpha \in E(x \restriction G)$. For such α, we have $g_{\alpha+1} = \pi x \restriction M_{\alpha+1}$ and $x \restriction G_\alpha = c_\alpha$. So α is in case (I), because $g_{\alpha+1} \restriction M_\alpha = g_\alpha$, $g_{\alpha+1} = \pi(x \restriction M_{\alpha+1})$, and $(x \restriction M_{\alpha+1}) \restriction G_\alpha = c_\alpha$.

Let $z_{\alpha+1} = x \restriction M_{\alpha+1} - y_{\alpha+1}$. Then $z_{\alpha+1} \restriction G_\alpha = x \restriction G_\alpha - y_{\alpha+1} \restriction G_\alpha = c_\alpha - c_\alpha = 0$. So there exists $u_{\alpha+1} \in \mathrm{Hom}_R(M_{\alpha+1}/M_\alpha, R)$ such that $z_{\alpha+1} = u_{\alpha+1}\rho_{\alpha+1}$. Moreover,

$$\pi u_{\alpha+1}\rho_{\alpha+1} = \pi z_{\alpha+1} = \pi x \restriction M_{\alpha+1} - \pi y_{\alpha+1} = g_{\alpha+1} - h_{\alpha+1} = f_{\alpha+1}\rho_{\alpha+1}.$$

Since $\rho_{\alpha+1}$ is surjective, we conclude that $\pi u_{\alpha+1} = f_{\alpha+1}$, in contradiction with our choice of the homomorphism $f_{\alpha+1}$. □

Corollary 3.4. *Let K be a field of cardinality $\leq 2^{\aleph_0}$. Then the statement 'R is a testing ring' is independent of ZFC + GCH. Hence Faith's problem is undecidable in ZFC + GCH.*

Proof. Assume UP_κ for some κ such that $\mathrm{card}(R) < \kappa$ and $\mathrm{cf}(\kappa) = \aleph_0$. Then R is not testing by [17, Lemmas 2.2 and 2.4] (see also [16, Theorem 1.5] and [1, Theorem 2.7]).

Assume V = L. Then R is testing by Theorems 3.2 and 3.3. □

Acknowledgement: I owe my thanks to Gena Puninski for drawing my attention to Faith's problem, and for sharing his manuscript [14] (later incorporated in [1]). I deeply regret Gena's sudden departure in April 2017.

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