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Falling fuzzy quasi-associative ideals of BCI-algebras

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Abstract. As a generalization of a fuzzy quasi-associative ideal of a BCI-algebra, the notion of a falling fuzzy quasi-associative ideal of a BCI-algebra is introduced by using the theory of a falling shadow. Relations between falling fuzzy ideals and falling fuzzy quasi-associative ideals are given. Conditions for a falling fuzzy ideal to be a falling fuzzy quasi-associative ideal are provided.

1. Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [1] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [9] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [8]. Tan et al. [6, 7] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [5] discussed the notion of a falling fuzzy subalgebra/ideal of a BCK/BCI-algebra. Jun and Kang [3] studied the falling fuzzy positive implicative ideals of a BCK-algebra. They provided relations between falling fuzzy positive implicative ideals and falling fuzzy ideals. They also considered relations between fuzzy positive implicative ideals and falling fuzzy positive implicative ideals. Jun and Kang [4] considered the fuzzification of generalized Tarski filters of generalized Tarski algebras, and investigated related properties. They established characterizations of a fuzzy generalized Tarski filter, and introduced the notion of falling fuzzy generalized Tarski filters in generalized Tarski algebras based on the theory of falling shadows. They provided relations between fuzzy generalized Tarski filters and falling fuzzy generalized Tarski filters, and established a characterization of a falling fuzzy generalized Tarski filter. They showed that every falling fuzzy generalized Tarski filter is a T_m-fuzzy generalized Tarski filter.

In this paper, we use the theory of falling shadows to establish a falling fuzzy quasi-associative ideal in a BCI-algebra as a generalization of a fuzzy quasi-associative ideal in BCI-algebras. We provide relations between falling fuzzy quasi-associative ideals and falling fuzzy ideals. We also consider relations between

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fuzzy quasi-associative ideals and falling fuzzy quasi-associative ideals. We give conditions for a falling fuzzy ideal to be a falling fuzzy quasi-associative ideal. We show that every falling fuzzy quasi-associative ideal is a T_m-fuzzy quasi-associative ideal.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

Any BCI-algebra X satisfies the following conditions:

(a1) $(\forall x \in X)\ (x * 0 = x)$,

(a2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,

(a3) $(\forall x, y, z \in X)\ (0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))$,

(a4) $(\forall x, y \in X)\ (0 * (0 * (x * y)) = (0 * y) * (0 * x))$.

A subset I of a BCI-algebra X is called an *ideal* of X if it satisfies:

(b1) $0 \in I$.
(b2) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

Every ideal I of a BCI-algebra X has the following assertion:

$$(\forall x \in X)\ (\forall y \in I)\ (x \leq y \Rightarrow x \in I). \tag{1}$$

A subset I of a BCI-algebra X is called a quasi-associative ideal (briefly, QA-ideal) of X if it satisfies (b1) and

(b3) $(\forall x, y, z \in X)\ (x * (y * z) \in I,\ y \in I \Rightarrow x * z \in I)$.

Note that every quasi-associative ideal is both an ideal and a subalgebra (see [10]).
A fuzzy set μ in a BCI-algebra X is called a *fuzzy ideal* of X if it satisfies:

(c1) $(\forall x \in X)\ (\mu(0) \geq \mu(x))$.
(c2) $(\forall x, y \in X)\ (\mu(x) \geq \min\{\mu(x * y), \mu(y)\})$.

A fuzzy set μ in a BCI-algebra X is called a *fuzzy quasi-associative ideal* (briefly, fuzzy QA-ideal) of X (see [2]) if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)\ (\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\})$.

Proposition 2.1. ([2]) *Let μ be a fuzzy set in a BCI-algebra X. Then μ is a fuzzy QA-ideal of X if and only if*

$$(\forall t \in [0, 1])\ (\mu_t \neq \emptyset \Rightarrow \mu_t \text{ is a } QA\text{-ideal of } X)$$

where $\mu_t := \{x \in X \mid \mu(x) \geq t\}$.

Note that every fuzzy QA-ideal is a fuzzy ideal, but the converse is not true (see [2, Theorem 2.5 and Example 2.7]).

We now display the basic theory on falling shadows. We refer the reader to the papers [1, 6–9] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathscr{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2}$$

and for each $E \in \mathscr{P}(U)$, let

$$\dot{E} := \{\dot{u} \mid u \in E\}. \tag{3}$$

An ordered pair $(\mathscr{P}(U), \mathscr{B})$ is said to be a hyper-measurable structure on U if $\mathscr{B}$ is a σ-field in $\mathscr{P}(U)$ and $\dot{U} \subseteq \mathscr{B}$. Given a probability space $(\Omega, \mathscr{A}, P)$ and a hyper-measurable structure $(\mathscr{P}(U), \mathscr{B})$ on U, a random set on U is defined to be a mapping $\xi : \Omega \to \mathscr{P}(U)$ which is $\mathscr{A}$-$\mathscr{B}$ measurable, that is,

$$(\forall C \in \mathscr{B})\,(\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathscr{A}).$$

Suppose that ξ is a random set on U. Let $\tilde{H}(u) := P(\omega \mid u \in \xi(\omega))$ for each $u \in U$. Then $\tilde{H}$ is a kind of fuzzy set in U. We call $\tilde{H}$ a falling shadow of the random set ξ, and ξ is called a cloud of $\tilde{H}$.

For example, $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t-cut of $\tilde{H}$. Then

$$\xi : [0,1] \to \mathscr{P}(U),\ t \mapsto \tilde{H}_t$$

is a random set and ξ is a cloud of $\tilde{H}$. We shall call ξ defined above as the cut-cloud of $\tilde{H}$ (see [1]).

3. Falling fuzzy quasi-associative ideals

In what follows let X denote a BCI-algebra unless otherwise.

Definition 3.1. ([5])Let $(\Omega, \mathscr{A}, P)$ be a probability space, and let $\xi : \Omega \to \mathscr{P}(X)$ be a random set. If $\xi(\omega)$ is an ideal (resp. a subalgebra) of X for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ, i.e.,

$$\tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$$

is called a *falling fuzzy ideal* (resp. falling fuzzy subalgebra) of X.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $F(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}$, where X is a BCI-algebra. Define an operation $\circledast$ on $F(X)$ by

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega))$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$. It can be easily to check that $(F(X); \circledast, \theta)$ is a BCI-algebra (see [5]).

Definition 3.2. Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $\xi : \Omega \to \mathscr{P}(X)$ be a random set. If $\xi(\omega)$ is a QA-ideal of X for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ, i.e., $\tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$ is called a *falling fuzzy QA-ideal* of X.

Example 3.3. Consider a BCI-algebra $X = \{0, a, b\}$ with a Cayley table which is given by Table 1. Take $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ as a probability space and define a random set $\xi : \Omega \to \mathscr{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \{0, a\} & \text{if } t \in [0, 0.6), \\ X & \text{if } t \in [0.6, 1]. \end{cases}$$

Table 1: Cayley table

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Then the falling shadow $\tilde{H}$ of ξ is represented as follows:

$$\tilde{H}(x) = \begin{cases} 1 & \text{if } x \in \{0, a\} \\ 0.4 & \text{if } x = b, \end{cases}$$

and $\tilde{H}$ is a falling fuzzy QA-ideal of X.

Example 3.4. Let $X = \mathbb{Q} - \{0\}$ where $\mathbb{Q}$ is the set of all rational numbers. Then $(X, \div, 1)$ is a BCI-algebra where $\div$ is the division as general. Take $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$ as a probability space and define a random set $\xi : \Omega \to \mathscr{P}(X)$ as follows:

$$\xi(t) := \begin{cases} A & \text{if } t \in [0, 0.3), \\ X & \text{if } t \in [0.3, 1], \end{cases}$$

where A is a QA-ideal of X. Then the falling shadow $\tilde{H}$ of ξ is represented as follows:

$$\tilde{H}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0.7 & \text{if } x \in X - A, \end{cases}$$

and $\tilde{H}$ is a falling fuzzy QA-ideal of X.

For any subset A of X and $f \in F(X)$, let $A_f := \{\omega \in \Omega \mid f(\omega) \in A\}$ and

$$\xi : \Omega \to \mathscr{P}(F(X)),\ \omega \mapsto \{f \in F(X) \mid f(\omega) \in A\}.$$

Then $A_f \in \mathscr{A}$.

Theorem 3.5. *If A is a QA-ideal of X, then $\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\}$ is a QA-ideal of $F(X)$.*

Proof. Assume that A is a QA-ideal of X and let $\omega \in \Omega$. Since $\theta(\omega) = 0 \in A$, we have $\theta \in \xi(\omega)$. Let $f, g, h \in F(X)$ be such that $g \in \xi(\omega)$ and $f \circledast (g \circledast h) \in \xi(\omega)$. Then $g(\omega) \in A$ and

$$f(\omega) * (g(\omega) * h(\omega)) = (f \circledast (g \circledast h))(\omega) \in A.$$

Since A is a QA-ideal of X, we get $(f \circledast h)(\omega) \in A$, that is, $f \circledast h \in \xi(\omega)$. Hence $\xi(\omega)$ is a QA-ideal of $F(X)$. □

Since $\xi^{-1}(\dot{f}) = \{\omega \in \Omega \mid f \in \xi(\omega)\} = \{\omega \in \Omega \mid f(\omega) \in A\} = A_f \in \mathscr{A}$, we see that ξ is a random set on $F(X)$. Let $\tilde{H}(f) = P(\omega \mid f(\omega) \in A)$. Then $\tilde{H}$ is a falling fuzzy QA-ideal of $F(X)$.

Theorem 3.6. *Every fuzzy QA-ideal of X is a falling fuzzy QA-ideal of X.*

Proof. Consider the probability space $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesque measure. Let μ be a fuzzy QA-ideal of X. Then μ_t is a QA-ideal of X for all $t \in [0, 1]$ by Proposition 2.1. Let $\xi : [0, 1] \to \mathscr{P}(X)$ be a random set and $\xi(t) = \mu_t$ for every $t \in [0, 1]$. Then μ is a falling fuzzy QA-ideal of X. □

Theorem 3.7. *Every falling fuzzy QA-ideal is both a falling fuzzy ideal and a falling fuzzy subalgebra.*

Proof. Let $\tilde{H}$ be a falling fuzzy QA-ideal of X. Then $\xi(\omega)$ is a QA-ideal of X, and hence it is both an ideal and a subalgebra of X. Thus $\tilde{H}$ is both a falling fuzzy ideal and a falling fuzzy subalgebra of X. □

The converse of Theorem 3.7 is not true in general as shown by the following examples.

Example 3.8. Consider a BCI-algebra $X = \{0, a, b, c\}$ with a Cayley table which is given by Table 2.

Table 2: Cayley table

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Take $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$ as a probability space and define a random set $\xi : \Omega \to \mathscr{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \{0\} & \text{if } t \in [0, 0.4), \\ X & \text{if } t \in [0.4, 1]. \end{cases}$$

Then $\xi(t)$ is both a subalgebra and an ideal of X for all $t \in [0, 1]$. But if $t \in [0, 0.4)$ then $\xi(t)$ is not a QA-ideal of X since $c * (0 * a) = c * c = 0 \in \xi(t)$ and $c * a = b \notin \xi(t)$. Therefore the falling shadow $\tilde{H}$ of ξ is both a falling fuzzy ideal and a falling fuzzy subalgebra of X, but it is not a falling fuzzy QA-ideal of X.

Example 3.9. Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with a Cayley table which is given by Table 3. Take $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$ as a probability space and define a random set $\xi : \Omega \to \mathscr{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \{0, 1\} & \text{if } t \in [0, 0.4), \\ \{0, 2\} & \text{if } t \in [0.4, 0.65), \\ \{0, 1, 2\} & \text{if } t \in [0.65, 1]. \end{cases}$$

Then $\xi(t)$ is an ideal of X for all $t \in [0, 1]$. Hence the falling shadow $\tilde{H}$ of ξ is a falling fuzzy ideal of X. If we take $t \in [0.4, 0.65)$, then $\xi(t) = \{0, 2\}$ is not a QA-ideal of X since $b*(0*a) = b*b = 0 \in \{0, 2\}$ and $b*a = a \notin \{0, 2\}$. Therefore $\tilde{H}$ is not a falling fuzzy QA-ideal of X.

Table 3: Cayley table

$*$	0	1	2	a	b
0	0	0	0	b	a
1	1	0	1	b	a
2	2	2	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

Example 3.10. Consider the BCI-algebra $(X, \div, 1)$ which is given in Example 3.4. Take $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ as a probability space and define a random set $\xi : \Omega \to \mathscr{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \mathbb{Z} - \{0\} & \text{if } t \in [0, 0.2), \\ X & \text{if } t \in [0.2, 1]. \end{cases}$$

Then $\xi(t)$ is an ideal of X for all $t \in [0,1]$. Thus the falling shadow $\tilde{H}$ of ξ is a falling fuzzy ideal of X. If we take $t \in [0, 0.2)$, then $\xi(t) = \mathbb{Z} - \{0\}$ is not a QA-ideal of X since $2 \div (6 \div 3) = 1 \in \mathbb{Z} - \{0\}$ and $6 \in \mathbb{Z} - \{0\}$, but $2 \div 3 \notin \mathbb{Z} - \{0\}$. Hence $\tilde{H}$ is not a falling fuzzy QA-ideal of X.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\tilde{H}$ a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. For any $x \in X$, let $\Omega(x;\xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}$. Then $\Omega(x;\xi) \in \mathscr{A}$.

Proposition 3.11. *Let $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. If $\tilde{H}$ is a falling fuzzy QA-ideal of X, then*

$$(\forall x, y, z \in X)\,(\Omega(x * (y * z);\xi) \cap \Omega(y;\xi) \subseteq \Omega(x * z;\xi))\,. \tag{4}$$

Proof. Let $\omega \in \Omega(x * (y * z);\xi) \cap \Omega(y;\xi)$. Then $x * (y * z) \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a QA-ideal of X, it follows that $x * z \in \xi(\omega)$ so that $\omega \in \Omega(x * z;\xi)$. Hence (4) is valid. □

Corollary 3.12. *Let $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. If $\tilde{H}$ is a falling fuzzy QA-ideal of X, then*

(1) $(\forall x, y \in X)\,(\Omega(x * y;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x;\xi))\,.$
(2) $(\forall x, y \in X)\,(\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x * y;\xi))\,.$

Proof. Putting $z = 0$ in (4) and using (a1), we obtained the result (1). Taking $z = y$ in (4) and using (III), (a1) induces (2). □

We provide conditions for a falling fuzzy ideal to be a falling fuzzy QA-ideal.

Theorem 3.13. *Let a falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathscr{P}(X)$ be a falling fuzzy ideal of X that satisfies the following condition:*

$$(\forall x, y \in X)\,(\Omega(x;\xi) \subseteq \Omega(x * y;\xi))\,. \tag{5}$$

Then $\tilde{H}$ is a falling fuzzy QA-ideal of X.

Proof. Let $x, y, z \in X$ be such that $y \in \xi(\omega)$ and $x * (y * z) \in \xi(\omega)$. Then $\omega \in \Omega(y;\xi)$ and $\omega \in \Omega(x * (y * z);\xi)$. Using (5) and (a2), we have $\omega \in \Omega(y * z;\xi)$ and

$$\omega \in \Omega((x * (y * z)) * z;\xi) = \Omega((x * z) * (y * z);\xi).$$

Thus $y * z \in \xi(\omega)$ and $(x * z) * (y * z) \in \xi(\omega)$, which imply that $x * z \in \xi(\omega)$. Hence $\tilde{H}$ is a falling fuzzy QA-ideal of X. □

Theorem 3.14. *Let a falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathscr{P}(X)$ be a falling fuzzy ideal of X. Then the following are equivalent:*

(1) *$\tilde{H}$ is a falling fuzzy QA-ideal of X.*
(2) $(\forall x, y \in \Omega)\,(\Omega(x * (0 * y);\xi) \subseteq \Omega(x * y;\xi))\,.$
(3) $(\forall x, y, z \in \Omega)\,(\Omega(x * (y * z);\xi) \subseteq \Omega((x * y) * z;\xi))\,.$

Proof. Assume that $\tilde{H}$ is a falling fuzzy QA-ideal of X and let $\omega \in \Omega(x * (0 * y); \xi)$. Then $x * (0 * y) \in \xi(\omega)$, and so $x * y \in \xi(\omega)$ by (b3). Hence $\omega \in \Omega(x * y; \xi)$ for all $x, y \in X$.

Now suppose that (2) is valid and let $\omega \in \Omega(x*(y*z); \xi)$. Then $x*(y*z) \in \xi(\omega)$. Since $(x*y)*(0*z) \leq x*(y*z)$ for all $x, y, z \in X$, it follows from (1) that $(x * y) * (0 * z) \in \xi(\omega)$, that is, $\omega \in \Omega((x * y) * (0 * z); \xi)$ so from (2) that $\omega \in \Omega((x * y) * z; \xi)$. Hence $(\Omega(x * (0 * y); \xi) \subseteq \Omega(x * y; \xi))$ for all $x, y, z \in X$.

Finally, we assume that (3) is true and let $x, y, z \in X$ be such that $y \in \xi(\omega)$ and $x * (y * z) \in \xi(\omega)$. Then $\omega \in \Omega(x * (y * z); \xi) \subseteq \Omega((x * y) * z; \xi)$, and so $(x * z) * y = (x * y) * z \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X, it follows that $x * z \in \xi(\omega)$. Hence $\xi(\omega)$ is a QA-ideal of X, and therefore $\tilde{H}$ is a falling fuzzy QA-ideal of X. □

Theorem 3.15. *Let $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. If $\tilde{H}$ is a falling fuzzy QA-ideal of X, then*

$$(\forall x, y, z \in X)\,(\tilde{H}(x) \geq T_m(\tilde{H}((x * z) * (y * z)), \tilde{H}(y)))$$

where $T_m(s, t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0, 1]$.

Proof. By Definition 3.2, $\xi(\omega)$ is a QA-ideal of X for any $\omega \in \Omega$. Hence

$$\{\omega \in \Omega \mid (x * z) * (y * z) \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \in \xi(\omega)\},$$

and thus

$$\begin{aligned} \tilde{H}(x) &= P(\omega \mid x \in \xi(\omega)) \\ &\geq P(\{\omega \mid (x * z) * (y * z) \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\}) \\ &\geq P(\omega \mid (x * z) * (y * z) \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) - P(\omega \mid (x * z) * (y * z) \in \xi(\omega) \text{ or } y \in \xi(\omega)) \\ &\geq \tilde{H}((x * z) * (y * z)) + \tilde{H}(y) - 1. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{H}(x) &\geq \max\{\tilde{H}((x * z) * (y * z)) + \tilde{H}(y) - 1, 0\} \\ &= T_m(\tilde{H}((x * z) * (y * z)), \tilde{H}(y)). \end{aligned}$$

This completes the proof. □

Theorem 3.15 means that every falling fuzzy QA-ideal of X is a T_m-fuzzy QA-ideal of X.

4. Conclusion

We have established some connections between fuzzy mathematics and probability theory via the fuzzy QA-ideal of BCI-algebras. As an algebraic approach of the theory of falling shadows, we have introduced the notion of falling fuzzy QA-ideals in BCI-algebras. We have discussed relations between falling fuzzy ideals and falling fuzzy QA-ideals. We have considered conditions for a falling fuzzy ideal to be a falling fuzzy QA-ideal. We have shown that every falling fuzzy QA-ideal is a T_m-fuzzy QA-ideal. Based on these results, we will apply the theory of falling shadows to the related algebraic structures, for example, MV-algebras, MTL-algebras, R_0-algebras and BL-algebras, etc., in the future study.

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