FAMILIES OF CODES WITH FEW DISTINCT WEIGHTS FROM SINGULAR & NON-SINGULAR,etc.

I.M. Chakravarti #1726

Families of Codes with Few Distinct Weights From Singular and Non-Singular Hermitian Varieties and Quadrics in Projective Geometries and Hadamard Difference Sets and Designs Associated with Two-Weight Codes

by

I.M. Chakravarti

0. Summary.

In this paper, we present several doubly infinite families of linear projective codes with two-, three- and four distinct non-zero Hamming weights together with the frequency distributions of their weights.

The codes have been defined as linear spaces of coordinate vectors of points on certain projective sets described in terms of Hermitian and quadratic forms - non-degenerate and singular - in projective spaces. The weight-distributions have been derived by considering the geometry of intersections of projective sets by hyperplanes in relevant projective spaces. Results from Bose and Chakravarti (1966) and Chakravarti (1971) on the Hermitian geometry and Bose (1964), Primrose (1951) and Ray-Chaudhuri (1959,1962) have been used in the enumeration of weights and their frequencies.

The paper has been organized as follows. Preliminary definitions, concepts and results on Hermitian geometry [from Bose and Chakravarti (1966) and Chakravarti (1971)] are given in Section 1.

Two families of two-weight codes $\mathscr{C}(V_{N-1})$ and $\mathscr{C}(\bar{V}_{N-1})$ over $GF(s^2)$ and associated families $\mathscr{C}'(V_{N-1})$ and $\mathscr{C}'(\bar{V}_{N-1})$ over GF(s) together with their weight-distributions are given in Section 2. Here V_{N-1} denotes a non-degenerate Hermitian variety in $PG(N,s^2)$ and $\bar{V}_{N-1}$ is its complement and a code $\mathscr{C}(S)$ is defined as the linear space of the coordinate vectors of the points in the projective set S.

The eigenvalues of the adjacency matrix $A = B_2 - B_1$ (B_i is the incidence matrix of the ith associates i=1,2) of the strongly regular graph (two-class association scheme) on $s^{2(N+1)}$ vertices defined by the two-weight code $\mathscr{C}'(V_{N-1})$ over GF(s) and the (p^i_{jk}) parameters of the two-class association scheme are given in Section 3.

In Section 4, we show that for s=2, B_2 (the association matrix of the second associates) of the two-class association scheme of Section 3, is the incidence matrix a symmetric BIB design with parameters $v = 2^{2(N+1)}$, $k=2^{2N+1} + (-2)^N$, $\lambda = 2^{2N} + (-2)^N$ and $2B_2-J$ is a Hadamard matrix of order $2^{2(N+1)}$. Similarly, $I+B_1$ is the incidence matrix of a symmetric BIB design with parameters $v = 2^{2(N+1)}$, $k=2^{2N+1} - (-2)^N$, $\lambda = 2^{2N} - (-2)^N$ and $2(I+B_1) - J$ is a Hadamard matrix of order 2^{2N+2}. Further, it is shown that the $2^{2N+1} + (-2)^N$ codewords each of weight $(2^{2N} - (-2)^N)$, which are non-adjacent to the null codeword form a Hadamard difference set (Menon 1960, Mann 1965) with parameters $v = 2^{2N+2}$, $k = 2^{2N+1} + (-2)^N$, $\lambda = 2^{2N} + (-2)^N$ and the $(2^{2N+1} - (-2)^N -1)$ codewords each of weight 2^{2N} together with the null codeword form a Hadamard difference set with parameters $v = 2^{2N+2}$, $k = 2^{2N+1} - (-2)^N$, $\lambda = 2^{2N} - (-2)^N$, for integer N.

In Section 5, a family of four-weight linear codes and the associated weight-distributions are derived. A code here is defined as the linear span of a projective set which is the intersection of a non-degenerate Hermitian variety and the complement of one of the secant hyperplanes.

In Section 6 and 7, we consider codes which are linear spans of projective sets defined in terms of degenerate Hermitian and quadratic forms in projective spaces. The motivation here is to explore how the code parameters behave when the basic projective set is not purely a subspace nor a non-degenerate Hermitian or quadric variety but an analgam of the two, which still admits a geometric description (and algebraic equations).

In section 6, the basic projective set is a degenerate Hermitian variety V^{o}_{N-2} which is the intersection of a non-degenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ with one of its tangent hyperplanes. The code $\mathscr{C}(V^{o}_{N-2})$ which is the linear space generated by the coordinate vectors of the points of V^{o}_{N-2}, is shown to be a tri-weight code. Its weight-distribution as a code over $GF(s^2)$ as well as that of its sister code over $GF(s)$ are given.

In section 7, a degenerate quadric Q^{o}_{N-1} which is the intersection of a non-degenerate quadric Q_N in $PG(N,s)$ with one of its tangent hyperplanes, is taken as the basic projective set. The code $\mathscr{C}(Q^{o}_{N-1})$ which is the linear space of the coordinate vectors of the points of Q^{o}_{N-1}, is shown to be a tri-weight code both for odd and even N. The frequency distributions of the weights are given for both odd and even N. For odd N, both the cases elliptic and hyperbolic have been considered. These families supplement those obtained by Wolfmann (1975) from non-degenerate quadrics.

1. Introduction

The geometry of Hermitian varieties in finite dimensional projective spaces have been studied by Jordan (1870), Dickson (1901), Dieudonné (1971), and recently, among others by Bose (1963, 1971), Segre (1965, 1967), Bose and Chakravarti (1966) and Chakravarti (1971). In this paper, however, we have used results given in the last two articles.

If h is any element of a Galois field $GF(s^2)$, where s is a prime or a power of a prime, then $\bar{h}=h$ is defined to be conjugate to h. Since $h^2=h$, h is conjugate to $\bar{h}$. A square matrix $H = (h_{ij})$, $i,j = 0,1,\dots,N$, with elements from $GF(s^2)$ is called Hermitian if $h_{ij} = \bar{h}_{ji}$ for all i,j. The set of all points in $PG(N,s^2)$ whose row-vectors $\underline{x}^T = (x_0,x_1,\dots,x_N)$ satisfy the equation $\underline{x}^T H \underline{x}^{(s)} = 0$ are said to form a Hermitian variety V_{N-1}, if H is Hermitian and $\underline{x}^{(s)}$ is the column vector whose transpose is $(x_0^s, x_1^s, \ldots, x_N^s)$. The variety V_{N-1} is said to be non-degenerate if H has rank N+1. The Hermitian form $\underline{x}^T H \underline{x}^{(s)}$ where H is of order N+1 and rank r can be reduced to the canonical form $y_0\bar{y}_0 + \ldots + y_r\bar{y}_r$ by a suitable non-singular linear transformation $\underline{x} = A\underline{y}$. The equation of a non-degenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ can then be taken in the canonical form $x_0^{s+1} + x_1^{s+1} + \ldots + x_N^{s+1} = 0$.

Consider a Hermitian variety V_{N-1} in $PG(N,s^2)$ with equation $\underline{x}^T H \underline{x}^{(s)} = 0$. A point C in $PG(N,s^2)$ with row-vector $\underline{c}^T = (c_0, c_1, \ldots, c_N)$ is called a <u>singular</u> point of V_{N-1} if $\underline{c}^T H = \underline{0}^T$ or equivalently, $H \underline{c}^{(s)} = \underline{0}$. A point of V_{N-1} which is not singular is called a <u>regular</u> point of V_{N-1}. Thus a non-singular point is either a regular point of V_{N-1} or a point not on V_{N-1}, in which case it is called an <u>external</u> point of $PG(N,s^2)$, with respect to V_{N-1}. It is clear that a non-degenerate V_{N-1} cannot possess a singular point. On the other hand, if V_{N-1} is degenerate and rank $H = r < N+1$, the singular points of V_{N-1} constitute a (N-r)-flat called the <u>singular space</u> of V_{N-1}.

Let C be a point with row vector $\underline{c}^T$. Then the <u>polar space</u> of C with respect to the Hermitian variety V_{N-1} with equation $\underline{x}^T H \underline{x}^{(s)} = 0$, is defined to be the set of points of $PG(N,s^2)$ which satisfy $\underline{x}^T H \underline{c}^{(s)} = 0$.

When C is a singular point of V_{N-1}, the polar space of C is the whole space $PG(N,s^2)$. When, however, C is either a regular point of V_{N-1} or an external point, $\underline{x}^T H \underline{c}^{(s)} = 0$ is the equation of a hyperplane which is called the <u>polar hyperplane</u> of C with respect to V_{N-1}. Let C and D be two points of $PG(N,s^2)$. If the polar hyperplane of C passes through D, then the polar hyperplane of D passes through C. Two such points C and D are said to be <u>conjugates</u> to each other with respect to V_{N-1}. Thus the points lying in the polar hyperplane of C are all the points which are conjugates to C. If C is a <u>regular</u> point of V_{N-1}, the polar hyperplane of C passes through C; C is thus self-conjugate. In this case, the polar hyperplane is called the <u>tangent hyperplane</u> to V_{N-1} at C.

When V_{N-1} is non-degenerate, there is no singular point. To every point, there corresponds a unique polar hyperplane, and at every point of V_{N-1}, there is a unique tangent hyperplane. If C is an external point, its polar hyperplane will be called a secant hyperplane.

The number of points in a non-degenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ is $\phi(N,s^2) = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)/(s^2-1)$.

A polar hyperplane $\mathcal{S}_{N-1}$ of an external point $\mathfrak{D}$ (also called a secant hyperplane) in $PG(N,s^2)$ intersects a non-degenerate Hermitian variety V_{N-1}, in a non-degenerate Hermitian variety V_{N-2} of rank N. It has $(s^N - (-1)^N)(s^{N-1} - (-1)^{N-1})/(s^2-1)$ points.

A tangent hyperplane $\mathcal{T}_{N-1}$ to a non-degenerate V_{N-1} at a point C, intersects V_{N-1} in a degenerate V_{N-2} of rank N-1. The singular space of V_{N-2} consists of the single point C.

The number of points in a degenerate Hermitian variety V_{N-1} of rank $r < N+1$ in $PG(N,s^2)$ is $(s^2-1)\ f(N-r,s^2)\ \phi(r-1,s^2) + f(N-r,s^2) + \phi(r-1,s^2)$, where $f(k,s^2) = (s^{2(k+1)}-1/(s^2-1)$. Thus the number of points in a degenerate V_{N-2} of rank N-1, is

$$(s^2-1)\ f(0,s^2)\ \phi(N-2,s^2) + f(0,s^2) + \phi(N-2,s^2)$$
$$= 1 + (s^{N-1} - (-1)^{N-1})(s^{N-2} - (-1)^{N-2})s^2/(s^2-1).$$

2. Two-weight codes from non-degenerate Hermitian varieties in projective spaces.

Consider the code $\mathscr{C}(V_{N-1})$ over $GF(s^2)$, which is the linear space generated by the coordinate vectors of the points on a non-degenerate Hermitian variety V_{N-1} in $PG(N,s^2)$. This variety has $(s^{N+1}) - (-1)^{N+1})(s^N - (-1)^N)/(s^2-1) = n$ (say) points. Thus a matrix $G = (g_{ij})$ $i=0,1,\ldots,N$, $j=1,\ldots,n$, whose columns are the coordinate vectors of the n points on V_{N-1}, is a generator matrix of the code $\mathscr{C}(V_{N-1})$ which is a projective linear code (n, k=N+1). Now, a tangent hyperplane meets V_{N-1} at

$1 + (s^{N-1} - (-1)^{N-1})(s^{N-2} - (-1)^{N-2})s^2/(s^2-1)$ points and a secant hyperplane meets V_{N-1} at $(s^N - (-1)^N)(s^{N-1} - (-1)^{N-1})/(s^2-1)$ points; hence this code $\mathscr{C}(V_{N-1})$ has only two distinct non-zero weights w_1 and w_2 (say), where

$$w_1 = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)/(s^2-1)$$
$$- (s^{N-1} - (-1)^{N-1})(s^{N-2} - (-1)^{N-2})s^2/(s^2-1)-1 = s^{2N}-1$$
$$w_2 = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)/(s^2-1)$$
$$- (s^N - (-1)^N)(s^{N-1} - (-1)^{N-1})/s^2-1 = s^{2N-1} + (-s)^{N-1}.$$

The frequency f_{w_1} of the code-words with weight w_1 is equal to the number of tangent hyperplanes (same as the number of points on V_{N-1}) multiplied by (s^2-1). The frequency f_{w_2} of codewords of weight w_2 is the number of secant hyperplanes (same as the number of external points in $PG(N,s^2)$) multiplied by (s^2-1). Thus

$$f_{w_1} = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N),$$
$$f_{w_2} = (s^{2(N+1)}-1) - (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)$$
$$= (s-1)(s^{2N+1} + (-s)^N)^* .$$

Let $\bar{V}_{N-1}$ denote the set of external points of $PG(N,s^2)$ with respect to V_{N-1}. Thus $\bar{V}_{N-1}$ is the complement of V_{N-1}. Considering the intersections of $\bar{V}_{N-1}$ by tangent and secant hyperplanes, we get another sequence $\mathscr{C}(\bar{V}_{N-1})$ of two-weight linear projective codes in s^2 symbols, with parameters

$\bar{n} = (s^{2N+1} + (-s)^N)/(s+1)$, $\bar{k} = N+1$, $\bar{w}_1 = s^{2N} - s^{2N-1}$, $\bar{w}_2 = s^{2N} - s^{2N-1} - (-s)^{N-1}$,
$f_{\bar{w}_1} = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)$, $f_{\bar{w}_2} = (s-1)(s^{2N+1} + (-s)^N)$.

---

*While presenting this result at 3ème Colloque International sur la Théorie des Graphes et Combinatoire, Marseille-Luminy, June 1986, the author learnt that Calderbank and Kantor (1986) have also obtained this family using the rank three representation of unitary groups.

Now a projective linear (n,k) code over $GF(s^r)$ with weights w_i $i=1,\dots,t$, determines a projective linear (n',k') code over GF(s) with weights w_i' $i=1,\dots,t$, $n' = n(s^r-1)/(s-1)$, $k' = kr$, $w_i' = s^{r-1}w_i$, $i=1,\dots,t$, (Delsarte, 1972). Hence the code $\mathscr{C}(V_{N-1})$ over $GF(s^2)$ determines a sister code $\mathscr{C}'(V_{N-1})$ over GF(s) with parameters

$n' = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)/(s-1)$, $k=2(N+1)$, $w_1' = s^{2N}$; $w_2' = s^{2N} - (-s)^N$,
$f_{w_1'} = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)$, $f_{w_2'} = (s-1)(s^{2N+1} + (-s)^N)$.

Similarly, the code $\mathscr{C}(\bar{V}_{N-1})$ over $GF(s^2)$ determines a code $\mathscr{C}'(\bar{V}_{N-1})$ over GF(s) with parameters $\bar{n}' = s^{2N+1} + (-s)^N$, $\bar{k}' = 2(N+1)$, $\bar{w}_1' = s^{2N+1} - s^{2N}$, $\bar{w}_2' = s^{2N+1} - s^{2N} + (-s)^N$,
$f_{w_1'} = (s^{N+1} - (-1)^{N+1}(s^N-(-1)^N)$, $f_{w_2'} = (s-1)(s^{2N+1}+(-s)^N)$.

The family $\mathscr{C}'(V_{N-1})$ of two-weight codes over GF(s) is a subfamily of the one derived by Wolfmann (1978) from non-degenerate quadrics. This is because a non-degenerate Hermitian form over $PG(N,s^2)$ becomes a non-degenerate quadric over PG(2N+1,s) (Dickson, 1958, p. 144). It is elliptic if N is even and hyperbolic if N is odd (see, for instance, Heft, 1971).

3. Strongly regular graphs of Latin square and negative Latin square types, from two-weight codes.

Delsarte (1972) has shown that a two-weight projective (n,k) code over GF(s) determines a strongly regular graph on $v = s^k$ vertices (a two-class association scheme) and that the eigenvalues ρ_i of the adjacency matrix $A = B_2-B_1$ of the graph (B_i is the association matrix of the ith associates, i=1,2,: see, for instance, Bose and Mesner 1959) are given by

$$(w_2-w_1)\rho_0 = 2m\ v/s - (w_1+w_2)(v-1),$$
$$(w_2-w_1)\rho_1 = w_1+w_2 - (1+(-1)^i)v/s,\ i=1,2,$$

with $v = s^k$, $m = n(s-1)$, w_1 and w_2 are the two distinct non-zero weights of codewords. One can calculate the parameters p^i_{jk}, i,j,k = 1,2, of the two-class association scheme from ρ_1, ρ_2, n_1 and n_2, where n_i is the number of codewords of weight w_i, i=1,2.

Writing $(\rho_1+\rho_2)/2 = -\gamma$, $(\rho_1-\rho)/2 = \sqrt{\Delta}$ and $\Delta^2 = \gamma^2+ 2\beta+1$, one can show that $(\beta+\gamma)/2 = p^2_{12}$ and $(\beta-\gamma)/2 = p^1_{12}$. Then $p^1_{11} = n_1-1-p^1_{12}$, $p^1_{22} = n_2-p^1_{12}$, $p^2_{11} = n_1 - p^2_{12}$, $p^2_{22} = n_2-1-p^2_{12}$ (see, for instance, Bose and Mesner 1959).

Thus the graph on $s^{2(N+1)}$ vertices corresponding to the code $\mathscr{C}'(V_{N-1})$ over GF(s) is strongly regular and the eigenvalues of its adjacency matrix $A = B_2-B_1$, are $\rho_0 = (s^{2N+1} + (-s)^N)(s-1) - (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N) = n_2-n_1$, $\rho_1 = 1-2(-s)^N$ and $\rho_2 = 1+2(s-1)(-s)^N$. As a two-class association scheme its parameters are $n_1 = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)$, $n_2 = (s-1)(s^{2N+1} + (-s)^N)$, $p^1_{12} = (s-1)s^{2N}$, $p^2_{12} = (s-1)s^{2N} - (s-2)(-s)^N - 1$, $p^1_{11} = s^{2N}-(s-1)(-s)^N-2$, $p^2_{11} = s^{2N}-(-s)^N$, $p^1_{22} = s^{2N}(s-1)^2 + (-s)^N(s-1)$, $p^2_{22} = s^{2N}(s-1)^2 + (-s)^N(2s-3)$.

For N=2, this family gives the two-class negative Latin square association scheme $NL_g(s^3)$ with $g = s^2-1$, given by Mesner (1967, pp. 579-580). Mesner considered two points on an $EG(3,s^2)$ to be first associates if the line joining the two points, shared a point with a non-degenerate Hermitian curve in the ideal plane of the $PG(3,s^2)$ in which $EG(3,s^2)$ is embedded. Thus at first sight our construction generalizes Mesner's family based on the Hermitian curve. However, it is to be noted that for every odd N, our family is the same as the hyperbolic family which is called pseudo-Latin square $L_g(s^{N+1})$, $g = s^N+1$ and for every even N, our family is the same as the elliptic family which Mesner called the negative Latin square $NL_g(s^{N+1})$, $g = s^N-1$, (Mesner 1967, p. 578). See also Hubaut (1975, pp. 374-379, C. $12^{\pm}$).

Similarly, one can verify that the eigenvalues of the adjacency matrix $\bar{A} = \bar{B}_2-\bar{B}_1$ of the strongly regular graph on $v = s^{2(N+1)}$ vertices corresponding to the two-weight projective code $\mathscr{C}'(\bar{V}_{N-1})$ over GF(s) are $\bar{\rho}_0 = A_{w_2'} - A_{w_1'}$, $\bar{\rho}_1 = 1+2(s-1)(-s)^N$ and $\rho_2 = 1-2(-s)^N$. Thus it is clear that the sequence of two-weights codes $\bar{\mathscr{C}}'(\bar{V}_N)$ over GF(s) gives rise to essentially the same two sequences of (Latin square type $L_g(n)$ and negative Latin square type $NL_g(n)$) two-class association schemes, as derived from the sequence of codes $\mathscr{C}'(V_{N-1})$.

4. Hadamard difference sets, Hadamard matrices and symmetric BIB designs from the association schemes.

For s=2, the parameters of the two-class association scheme corresponding to the two-weight binary projective code $\mathscr{C}'(V_{N-1})$ are $n_1 = (2^{N+1} - (-1)^{N+1})(2^N-(-1)^N)$, $n_2 = 2^{2N+1} + (-2)^N$, $p_{12}^1 = 2^{2N}$, $p_{12}^2 = 2^{2N} - 1$, $p_{11}^1 = 2^{2N}-(-2)^N-2$, $p_{11}^2 = 2^{2N}-(-2)^N$, $p_{22}^1 = 2^{2N}+(-2)^N$, $p_{22}^2 = 2^{2N}+(-2)^N$.

The equality $p_{22}^1 = p_{22}^2 = 2^{2N}+(-2)^N$ implies that the matrix B_2- the association matrix of the second associates (non-adjacent vertices) is the incidence matrix of a symmetric BIB design with parameters $v = 2^{2(N+1)}$, $k = 2^{2N+1}+(-2)^N$, $\lambda = 2^{2N}+(-2)^N$ and $2B_2-J$ is a Hadamard matrix of order $2^{2(N+1)}$ which corresponds to the Hadamard difference set $v = 2^{2N+2}$, $k=2^{2N+1}+(-2)^N$, $\lambda = 2^{2N}+(-2)^N$.

Consider the $(2^{2N+1}+(-2)^N)$ codewords each of weight $(2^{2N}-(-2)^N)$ which are non-adjacent to (second associates of) the null codeword. The null codeword and a codeword of weight 2^{2N} are adjacent to (first associates of) each other and among the non-adjacent vertices of the null codeword, there are $p_{22}^1 = 2^{2N}+(-2)^N$ codewords which are also non-adjacent to the given codeword of weight 2^{2N}. This implies that the given code vector can be expressed as the differences of $2^{2N}+(-2)^N$ pairs of code vectors each of weight $(2^{2N}-(-2)^N)$. Similarly, since $p_{22}^2 = 2^{2N}+(-2)^N$ if follows that every codeword of weight $2^{2N}-(-2)^N$ can be expressed as the differences of $2^{2N}+(-2)^N$ pairs of codevectors each of weight $(2^{2N}-(-2)^N)$.

Thus the codewords of weight $2^{2N}-(-2)^N$ form a difference set with $v = 2^{2N+2}$, $k = 2^{2N+1} + (-2)^N$, $\lambda = 2^{2N} + (-2)^N$.

Again, $p_{11}^1+2 = p_{11}^2 = 2^{2N}-(-2)^N$. Thus the $(2^{2N+1}-(-2)^N-1)$ codewords each of weight 2^{2N} together with the null codeword form a difference set with $v = 2^{2N+2}$, $k = 2^{2N+1}-(-2)^N$, $\lambda = 2^{2N}-(-2)^N$.

5. A family of four weight codes. Code defined as a linear span of a projective set which is the intersection of a non-degenerate Hermitian variety and the complement of one of the secant hyperplanes.

Let $\mathcal{S}^{o}_{N-1}$ be a hyperplane in $PG(N,s^2)$ which is not one of the tangent hyperplanes of the non-degenerate Hermitian variety V_{N-1}. Then $\mathcal{S}^{o}_{N-1}$ intersects V_{N-1} in a non-degenerate Hermitian variety V^{o}_{N-2}. Let X denote the set of points on V_{N-1} which are not on V^{o}_{N-2}, that is, $X = V_{N-1} \backslash V^{o}_{N-2}$. Then the number points in X is

$$|X| = [(s^{N+1}-(-1)^{N+1})(s^{N}-(-1)^{N}) - (s^{N}-(-1)^{N})(s^{N-1}-(-1)^{N-1})](s^2-1)$$
$$= s^{2N-1} + (-s)^{N-1}.$$

Let $\mathscr{C}(X)$ be the code over $GF(s^2)$, which is the linear space generated by the coordinate vectors of the points in X. Then $\mathscr{C}(X)$ has $n = s^{2N-1}+(-s)^{N-1}$, $k = N+1$.

Let $\mathcal{S}_{N-1}$ be another hyperplane (distinct from $\mathcal{S}^{o}_{N-1}$) which is not a tangent to V_{N-1}. Then S_{N-1} intersects V_{N-1} in a non-degenerate Hermitian variety V_{N-2} and meets $\mathcal{S}^{o}_{N-1}$ in a (N-2)-flat $\mathcal{S}^{o}_{N-2}$. $\mathcal{S}^{o}_{N-2}$ intersects V^{o}_{N-2} in a non-degenerate V^{o}_{N-3}. Thus every non-tangent $\mathcal{S}_{N-1}$ distinct from $\mathcal{S}^{o}_{N-1}$ meets X in $s^{2N-3} + (-s)^{N-2}$ points. Let

$$w_1 = |X| - |X \cap S_{N-1}| = (s+1)[s^{2N-2}-s^{2N-3}-(-s)^{N-2}].$$

Now the number of hyperplanes in $PG(N,s^2)$, which are not tangent to V_{N-1} is

$$(s^{2N+2}-1)/(s^2-1) - (s^{N+1}-(-1)^{N+1})/(s^2-1) = (s^{2N+2}-s^{2N+1}-(-s)^{N+1}-(-s)^{N})/(s^2-1) =$$
$$= (s^{2N+1}+(-s)^{N})/(s+1).$$

Thus it follows that in $\mathscr{C}(X)$, there are $f_{w_1} = (s^{2N+1}+(-s)^{N})(s-1) -(s^2-1)$ codewords each of weight w_1 and $f_{w_4} = s^2-1$ codewords each of weight $w_4 = s^{2N-1}+(-s)^{N-1}$.

Let C be a point of V^{o}_{N-2} and let $\mathcal{T}_{N-1}(C)$ be the hyperplane tangent to V_{N-1} at C. Then the intersection of $\mathcal{T}_{N-1}(C)$ and V_{N-1} is a degenerate Hermitian variety V_{N-2} with $1 + (s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})(s^2-1)$ points. But $\mathcal{T}_{N-1}(C)$ meets $\mathcal{S}^{o}_{N-1}$ in an (N-2)-flat $\mathcal{T}^{o}_{N-2}(C)$ which is tangent to V^{o}_{N-2} at C. Thus $\mathcal{T}^{o}_{N-2}$ meets V^{o}_{N-2} in a

degenerate V^o_{N-3} consisting of
$1 + (s^{N-2}-(-1)^{N-2})(s^{N-3}-(-1)^{N-3})s^2/(s^2-1)$ points. Thus

$$|X \cap \mathcal{T}_{N-1}(C)| = |V_{N-1} \cap \mathcal{T}_{N-1}(C)| - |V^o_{N-2} \cap \mathcal{T}^o_{N-2}(C) = s^{2N-3} + (-s)^{N-1}.$$

Let

$$w_2 = |X| - |X \cap \mathcal{T}_{N-1}(C)| = s^{2N-1}-s^{2N-3} = s^{2N-3}(s^2-1)$$

and let $f_{w_2} = (s^2-1)$. (no. of points on V^o_{N-2})

$$= (s^N-(-1)^N)(s^{N-1}-(-1)^{N-1}).$$

Then in $\mathscr{C}(X)$ there are f_{w_2} codewords each of weight w_2.

Let $\mathcal{T}_{N-1}(P)$ be the tangent hyperplane to V_{N-1} at a point P which is not on V^o_{N-2}. Then $\mathcal{T}_{N-1}(P)$ meets V_{N-1} in a degenerate V_{N-2} consisting of

$$1 + (s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})s^2/(s^2-1)$$

points. But $\mathcal{T}_{N-1}(P)$ intersects $\mathcal{S}^o_{N-1}$ in an (N-2)-flat $\mathcal{S}^o_{N-2}$ which is not a tangent to V^o_{N-2}. Thus $\mathcal{S}^o_{N-2}$ meets V^o_{N-2} in a non-degenerate V^o_{N-3} consisting of $(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})/(s^2-1)$ points. Hence

$$|X \cap \mathcal{T}_{N-1}(P)| = |V_{N-1} \cap \mathcal{T}_{N-1}(P)| - |V^o_{N-2} \cap S^o_{N-2}|$$
$$= s^{2N-3}+(-s)^{N-1}+(-s)^{N-2}$$

Let $w_3 = |X| - |X \cap \mathcal{T}_{N-1}(P) = s^{2N-1} - s^{2N-3}-(-s)^{N-2}$,
and $f_{w_3} = (s^2-1)$. (No. of points in X)

$$= (s^2-1)(s^{2N-1})+(-s)^{N-1}).$$

Thus in $\mathscr{C}(X)$ there are f_{w_3} codewords each of weight w_3. Thus the linear projective code $\mathscr{C}(X)$ over $GF(s^2)$ has four distinct non-zero weights w_i with corresponding frequencies f_{w_i} i=1,2,3,4. It follows that $\mathscr{C}(X)$ determines a linear projective code $\mathscr{C}'(X)$ over GF(s) with parameters $n' = n(s+1) = (s+1)(s^{2N-1}+(-s)^{N-1})$, $k' = 2k = 2(N+1)$, $w'_1 = sw_1 = (s+1)(s^{2N-1}-s^{2N-2}+(-s)^{N-1})$, $w'_2 = sw_2 = s^{2N+2}(s^2-1)$, $w'_3 = sw_3 = s^{2N}-s^{2N-2}+(-s)^{N-1}$ and $w'_4 = sw_4 = s^{2N}-(-s)^N$ and the associated frequencies f_{w_i} i=1,...,4 remain unchanged.

6. A family of three-weight codes from degenerate Hermitian varieties in projective spaces.

Consider the section of a non-degenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ with the tangent space $\mathcal{T}_{N-1}(C)$ at a point C on V_{N-1}. This section is a degenerate $V^o_{N-2}(C)$ of rank N-1 and its singular space consists of the single point C.

A (N-2)-space $\mathcal{S}_{N-2}$ contained in $\mathcal{T}_{N-2}(C)$ but not containing C, intersects $V^o_{N-2}(C)$ in a non-degenerate Hermitian variety V_{N-3} of rank N-1. Every point of $V^o_{N-2}(C)$ lies on some line joining C to a point of V_{N-3}. Thus the number of points on $V^o_{N-2}(C)$ $= 1 + (s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})s^2/(s^2-1)$.

Let $\mathcal{C}(V^o_{N-2})$ be the code over $GF(s^2)$ which is the linear space of the coordinates of the points on $V^o_{N-2}(C)$. This is a linear projective code with $n = 1 + (s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})s^2/(s^2-1)$ and $k=N$.

There are $[(s^2)^N-1] - [(s^2)^{N-1}-1]/(s^2-1) = (s^2)^{N-1}$ (N-2)-spaces S_{N-2} in $S_{N-1}(C)$ which do not pass through C. Each such S_{N-2} meets $V^o_{N-2}(C)$ in a non-degenerate V_{N-3} with $((s)^{N-1}-(-1)^{N-1})((s)^{N-2}-(-1)^{N-2})/(s^2-1)$ points. Thus in $\mathcal{C}(V^o_{N-2})$ there are

$$f_{w_1} = (s^2-1)(s^2)^{N-1} = s^{2N} - s^{2N-2}$$

codewords each of weight

$$\begin{aligned} w_1 &= 1 + s^2(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})/(s^2-1) \\ &\quad - (s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})/(s^2-1) \\ &= s^{2N-3} + (-s)^{N-1} + (-s)^{N-2}. \end{aligned}$$

Let $\mathcal{S}^*_{N-2}$ be a fixed (N-2)-flat contained in $\mathcal{T}_{N-1}(C)$, not passing through C and let V^*_{N-3} a non-degenerate variety of rank N-1, denote the intersection of $\mathcal{S}^*_{N-2}$ and $V^o_{N-2}(C)$. Let $\mathcal{T}_{N-2}(D)$ be the tangent space to V^*_{N-3} at D which is a regular point of $V^o_{N-2}(C)$. Then the (N-2)-space $\mathcal{T}_{N-2}(C,D)$ which is the join of the point C and $\mathcal{T}_{N-3}(D)$, is the tangent space to $V^o_{N-2}(C)$ at D and to every point on the line joining D to C.

The tangent space $\mathcal{T}_{N-2}(C,D)$ intersects $V^o_{N-2}(C)$ in a degenerate variety $V^o_{N-3}(C,D)$ of rank N-3 and its singular space is the line joining C and D.

Number of points on such $V^o_{N-3}(C,D)$ (degenerate of order 2) is

$$(s^2-1)(s^2+1)\ (s^{N-3}-(-1)^{N-3})(s^{N-4}-(-1)^{N-4})/(s^2-1)$$
$$+ s^2 + 1 + (s^{N-3}-(-1)^{N-3})(s^{N-4}-(-1)^{N-4})/(s^2-1)$$
$$= 1 + s^2 + s^4(s^{2N-7} + (-s)^{N-3} + (-s)^{N-4}-1)/(s^2-1)$$

Thus $\quad w_2 = 1 + s^2(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})/(s^2-1)$

$$-1 - s^2 - s^2(s^{2N-5} + (-s)^{N-1} + (-s)^{N-2}-1)/(s^2-1) = s^{2N-3}$$

is the weight of a codeword which corresponds to the tangent space $\mathcal{T}_{N-2}(C,D)$ in $\mathcal{T}_{N-1}(C)$. The number of such $\mathcal{T}_{N-2}(C,D)$ is equal to the number of regular points D on $V^o_{N-2}(C)$, that is the number of points on V^*_{N-2} a non-degenerate Hermitian variety. Thus the frequency f_{w_2} of the codewords in $\mathcal{C}(V^o_{N-2})$ of weight w_2 is

$f_2 = (s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})$.

Let $\mathcal{S}^*_{N-2}$ be a (N-3)-flat in S^*_{N-2}, which is not a tangent to V^*_{N-3}. Let $\mathcal{S}_{N-2}$ be the join of C and $\mathcal{S}_{N-3}$. Since $\mathcal{S}_{N-3}$ meets V^*_{N-3} in a non-degenerate Hermitian variety V_{N-4}, the section of $V^o_{N-2}(C)$ with $\mathcal{S}_{N-2}$ is a degenerate V^*_{N-3} which consists of all the points on the lines joining C to the points of V_{N-4}.

Let $\quad w_3 = |V^o_{N-2}(C)| - |V^*_{N-3}| = 1 + s^2(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})/(s^2-1)$

$$-1 - s^2 - s^2(s^{N-2} -(-1)^{N-2})(s^{N-3}-(-1)^{N-3})/(s^2-1)$$
$$= s^{2N-3} + (-s)^{N-1}.$$

Thus w_3 is the weight of a codeword which corresponds to an (N-2)-flat $\mathcal{S}_{N-2}$ which passes through C, but is not a tangent to $V^o_{N-2}(C)$. Number of such (N-2)-flats

$\mathcal{S}_{N-2}$ = Number of $\mathcal{S}_{N-3}$ in $\mathcal{S}^*_{N-2}$ - Number $\mathcal{S}_{N-3}$ in $\mathcal{S}^*_{N-2}$, which are tangents to V^*_{N-3}

$$= ((s^2)^{N-1}-1)/(s^2-1) - (s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})/(s^2-1)$$
$$= (s^{2N-3}+(-s)^{N-2})/(s+1)$$

Thus the frequency f_3 of codewords of weight w_3

$= (s^2-1)(s^{2N-3}+(-s)^{N-2})/(s+1)$

$= (s-1)(s^{2N-3}+(-s)^{N-2})$.

It follows that $\mathscr{C}(V^o_{N-2})$ determines a linear projective code $\mathscr{C}'(V^o_{N-2})$ over GF(s) with parameters $n' = n(s+1) = (s+1) + \{s^2(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})\}/(s-1)$, $k' = 2k$, $w_1' = sw_1 = s^{2N-2}-(-s)^N - (-s)^{N-1}$, $w_2' = sw_2 = s^{2N-2}$, $w_3' = sw_3 = s^{2N-2}-(-s)^N$ and the frequencies f_{w_i} i=1,2,3 remain unchanged.

7. Families of three-weight codes from degenerate quadrics (cones) in projective spaces.

Let Q_N be a non-degenerate quadric in PG(N,s) and let $\mathscr{T}_{N-1}(P)$ be the hyperplane tangent to Q_N at P. Then the intersection of $\mathscr{T}_{N-1}(P)$ and Q_N is a cone $Q^o_{N-1}(P)$ of rank N and order 1 and its vertex consists of the single point P.

A (N-2)-flat $\mathscr{S}_{N-2}$ contained in $\mathscr{T}_{N-1}(P)$ but not passing through P, intersects Q_N in a non-degenerate quadric Q_{N-2}. The points of $Q^o_{N-1}(P) = \mathscr{T}_{N-1}(P) \cap Q_N$, are then all the points in the lines joining P to the points of Q^o_{N-2}. $Q_{N-1}(P)$ has then $1 + s\,\psi(N-2,0)$ points where $\psi(N-2,0)$ is the number of points on Q_{N-2}.

Let $\mathscr{S}^*_{N-3}$ be a (N-3)-flat contained in a $\mathscr{S}^*_{N-2}$ which is one of the (N-2)-flats of $\mathscr{T}_{N-1}(P)$, not passing through P. Suppose that $\mathscr{S}^*_{N-3}$ is not a tangent to $Q^*_{N-2} = \mathscr{S}^*_{N-2} \cap Q^o_{N-1}(P)$. Then $\mathscr{S}^*_{N-3}$ intersects Q^*_{N-2} (a non-degenerate quadric) in a non-degenerate Q^*_{N-3}. Then the join of P and $\mathscr{S}^*_{N-3}$ which is a (N-2)-flat $\mathscr{S}_{N-2}(P) = P \mathbin{\dot\cup} \mathscr{S}^*_{N-3}$ passing through P, intersects $Q^o_{N-1}(P)$, in a cone Q^o_{N-2} of order 1 with its vertex a single point P.

Now suppose that $\mathscr{T}^*_{N-3}(R)$ an (N-3)-flat contained in S^o_{N-2}, is a tangent to Q^*_{N-2} at R. Then $\mathscr{T}^*_{N-3}(R)$ intersects Q^*_{N-2} in a cone Q^o_{N-3} of order 1 with R as its vertex. Hence the join of P and $\mathscr{T}^*_{N-3}(R)$ which is an (N-2)-flat $\mathscr{T}^*_{N-2}(P,R)$ (say) intersects $Q^o_{N-1}(P)$ in a cone $Q^o_{N-2}(P,R)$ of order 2 and the line joining P and R is its vertex.

Let $\mathscr{C}(Q^o_{N-1}(P))$ be the code over GF(s), which is the linear space generated by the coordinate vectors of the points on the cone $Q^o_{N-1}(P)$. The weight-distributions of the linear projective codes are next derived treating the two cases (i) N=2t and (ii) N = 2t-1 separately.

(i) <u>N = 2t.</u> In this case, Q_{2t} is a non-degenerate quadric in PG(2t,s) and $Q^o_{2t-1}(P)$ is the intersection of Q_{2t} with its tangent hyperplane $\mathscr{T}_{2t-1}(P)$ at a point P of Q_{2t}. Thus $Q^o_{2t-1}(P)$ is a cone of order 1 with P as its vertex. Thus

$$\begin{aligned} |Q^o_{2t-1}(P)| &= \text{number of points on } Q^o_{2t-1}(P) \\ &= 1 + s\,\psi(2t-2,0) = 1 + s(s^{2t-2}-1)/(s-1) \\ &= (s^{2t-1}-1)/(s-1). \end{aligned}$$

The code $\mathscr{C}(Q^o_{2t-1}(P))$ which is the linear space generated by the coordinate vectors of $Q^o_{2t-1}(P)$ has then the parameters $n = (s^{2t-1})/(s-1)$, k=2t.

A (2t-2) flat $\mathscr{S}_{2t-2}$ contained in $\mathscr{T}_{2t-1}(P)$ but not passing through P intersects $Q^o_{2t-1}(P)$ in a non-degenerate quadric Q_{2t-2}. Thus

$$w_1 = (s^{2t-1}-1)/(s-1) - (s^{2t-2}-1)/(s-1) = s^{2t-2}$$

is the number of points of $Q^o_{2t-1}(P)$ which are not on such a (2t-2)-flat. The number f_{w_1} of such (2t-2)-flats in $\mathscr{T}_{2t-1}(P)$ is given by

$$f_{w_1} = (s^{2t}-1)/(s-1) - (s^{2t-1}-1)/(s-1) = s^{2t-1}.$$

Let $\mathscr{S}^*_{2t-3}$ be a (2t-3)-flat contained in $\mathscr{S}^*_{2t-2}$ which is one of the (2t-2)-flats of $\mathscr{T}_{2t-1}(P)$, not passing through P and assume further $\mathscr{S}^*_{2t-3}$ is not a tangent to $Q^*_{2t-2} = \mathscr{S}^*_{2t-2} \cap Q^o_{2t-1}(P)$. Then the join of P and $\mathscr{S}^*_{2t-3}$ is a (2t-2)-flat $S_{2t-2}(P)$ of $\mathscr{T}_{2t-1}(P)$, passing through P. $\mathscr{S}_{2t-2}(P)$ meets $Q^o_{2t-1}(P)$ in a cone $Q^o_{2t-2}(P)$ of order 1 with P as its vertex and $Q^o_{2t-2}(P)$ consists of all the points on the lines joining P to the points of a non-degenerate Q^*_{2t-3} which is the intersection of $\mathscr{S}^*_{2t-3}$ and Q^*_{2t-2}.

Number of points on $Q^o_{2t-2}(P) = |Q^o_{2t-2}(P)| = 1 + s\,\psi(2t-3,0)$,

where $\psi(2t-3,0)$ is the number of points on Q^*_{2t-3}. Thus

$$|Q^o_{2t-2}(P)| = 1 + s(s^{t-1}+1)(s^{t-2}-1)/(s-1), \text{ if } Q^*_{2t-3} \text{ is elliptic,}$$
$$= 1 + s(s^{t-1}-1)(s^{t-2}+1)/(s-1) \text{ if } Q^*_{2t-3} \text{ is hyperbolic.}$$

Then $w_2 = (s^{2t-1}-1)/(s-1) - 1 - s(s^{t-1}+1)(s^{t-2})/(s-1) = s^{2t-2} + s^{t-1}$ is the number of points of $Q^o_{2t-1}(P)$ which are not on $S^*_{2t-2}(P)$, if Q^*_{2t-3} is elliptic. Let f_{w_2} denote the number of such $S^*_{2t-2}(P)$ flats.

If Q^*_{2t-3} is hyperbolic, then

$$w_3 = (s^{2t-1}-1)/(s-1) - 1 - s(s^{t-1}-1)(s^{t-2}+1)/(s-1) = s^{2t-2} - s^{t-1},$$

is the number of points on $Q^o_{2t-1}(P)$ which are not an $\mathcal{S}^*_{2t-2}(P)$. Let f_{w_3} denote the number of such (2t-2)-flats $\mathcal{S}^*_{2t-2}(P)$.

Let $\mathcal{T}_{2t-3}(R)$ be a (2t-3)-flat in $\mathcal{S}^*_{2t-2}$ which is a tangent to Q^*_{2t-2} at a point R in Q^*_{2t-2}. Hence $\mathcal{T}_{2t-3}(R)$ meets Q^*_{2t-2} in a cone $Q^o_{2t-3}(R)$ of order 1 and its vertex is the point R. The join of P and $\mathcal{T}_{2t-3}(R)$ which is a (2t-2)-flat $\mathcal{T}_{2t-2}(P,R)$ intersects $Q^o_{2t-1}(P)$ in a cone $Q^o_{2t-2}(P,R)$ of order 2 and its vertex is the line joining P and R. Then $|Q^o_{2t-2}(P,R)| = (s^2-1)/(s-1) + s^2\ \psi(2t-4,0) = (s+1) + s^2(s^{2t-4}-1)/(s-1) = (s^{2t-2}-1)/(s-1)$. Thus $w_4 = (s^{2t-1}-1)/(s-1) - (s^{2t-2}-1)/(s-1) = s^{2t-2}$ is the number of points of $Q^o_{2t-1}(P)$ which are not on $\mathcal{T}_{2t-2}(P,R)$. Then the number f_{w_4} of such flats, which is equal to the number of points on Q^*_{2t-2}, is given by $f_{w_4} = (s^{2t-2}-1)/(s-1)$. Now, $f_{w_2} + f_{w_3}$ = (number of (2t-2)-flats in $\mathcal{T}_{2t-1}(P)$, passing through P) - (number of (2t-3)-flats in a (2t-2)-flat S^*_{2t-2} of $\mathcal{T}_{2t-1}(P)$, not passing through P and tangents to Q^*_{2t-2}) $= (s^{2t-1}-1)/(s-1) - (s^{2t-2}-1)/(s-1) = s^{2t-2}$).

Now $Q^o_{2t-1}(P)$ has $(s^{2t-1}-1)/(s-1)$ points and each point is on $(s^{2t-1}-1)/(s-1)$ (2t-2)-flats contained in $\mathcal{T}_{2t-1}(P)$. Hence counting (point, (2t-2)-flat) pairs in two ways, one gets

$$f_{w_1} = (s^{2t-2}-1)/(s-1) + f_{w_2}(s^{2t-2}-1)/(s-1) + f_{w_2}(s^{2t-2}-s^t+s^{t-1}-1)/(s-1)$$
$$+ f_{w_3}(s^{2t-2} + s^t - s^{t-1}-1)/(s-1)$$

$$= (s^{2t-1}-1)/(s-1) \times (s^{2t-1}-1)/(s-1),$$

$$f_{w_2}+f_{w_2} = s^{2t-2},\ f_{w_1} = s^{2t-1},\ f_{w_4} = (s^{2t-2}-1)/(s-1).$$

Solving these equations, one gets

$$f_{w_2} = (s^{2t-2} - s^{t-1})/2,\ f_{w_3} = (s^{2t-2}+s^{t-1})/2.$$

Thus the weight-distribution of the $\mathscr{C}(Q^o_{2t-1}(P))$ is

weight	frequency
0	1
s^{2t-2}	$(s-1)\{s^{2t-1} + (s^{2t-2}-1)/s-1\} = s^{2t}-s^{2t-1}+s^{2t-2}-1$
$s^{2t-2} + s^{t-1}$	$(s-1)(s^{2t-2}-s^{t-1})/2 = (s^{2t-1}-s^{2t-2}-s^{t}+s^{t-1})/2$
$s^{2t-2} - s^{t-1}$	$(s-1)(s^{2t-2}-s^{t-1})/2 = (s^{2t-1}-s^{2t-2}-s^{t}+s^{t-1})/2$
	s^{2t}

Thus this is a tri-weight linear projective code over GF(s) with $n = (s^{2t-1}-1)/(s-1)$ and $k = 2t$.

(ii) <u>$N = 2t-1$.</u> In this case Q_{2t-1} is a non-degenerate quadric in PG(2t-1,s) and $Q^o_{2t-2}(P)$ is the intersection of Q_{2t-1} and its tangent hyperplane $\mathcal{T}_{2t-2}(P)$ at a point P. Thus $Q^o_{2t-2}(P)$ is a cone of order 1 and P is its vertex. Then the number of points on $Q^o_{2t-2}(P)$ is

$$|Q^o_{2t-2}(P)| = 1 + s\ \psi(2t-3,0)$$

$$= 1 + s(s^{t-1}+1)(s^{t-2}-1)/(s-1) = (s^{2t-2} - s^{t} + s^{t-1}-1)/(s-1),$$

if Q_{2t-1} is elliptic;

$$= 1 + s(s^{t-1}-1)(s^{t-2}+1)/(s-1) = (s^{2t-2} + s^{t} - s^{t-1}-1)/(s-1),$$

if Q_{2t-1} is hyperbolic.

Thus the code $\mathscr{C}(Q^o_{2t-2}(P))$ which is the linear space generated by the coordinate vectors of the points of $Q^o_{2t-2}(P)$ is a linear projective code over GF(s) with $n = |Q^o_{2t-2}(P)|$ and $k = 2t-1$.

A (2t-3)-flat $\mathcal{S}_{2t-3}$ contained in $\mathcal{T}_{2t-2}(P)$ but not passing through P, intersects Q_{2t-1} in a non-degenerate quadric Q_{2t-3} which is elliptic (hyperbolic) if Q_{2t-1} is elliptic (hyperbolic). Thus the intersection of such a $\mathcal{S}_{2t-3}$ and $Q^o_{2t-2}(P)$ is a non-degenerate Q_{2t-3}.

Hence if Q_{2t-3} is elliptic, the number of points on $Q^o_{2t-2}(P)$ which are not $\mathcal{S}_{2t-3}$ is

$$w_1 \text{ (ellip.)} = (s^{2t-2} - s^t + s^{t-1}-1)/(s-1) - (s^{t-1}+1)(s^{t-2}-1)/(s-1)$$
$$= s^{2t-3} - s^{t-1} + s^{t-2}$$

On the other hand if Q_{2t-3} is hyperbolic, the number of points on $Q^o_{2t-2}(P)$ which are not on S_{2t-3} is

$$w_1 \text{ (hyperbol.)} = (s^{2t-2} + s^t - s^{t-1}-1)/(s-1) - (s^{t-1}-1)(s^{t-2}-1)/(s-1)$$
$$= s^{2t-3} + s^{t-1} - s^{t-2}.$$

The number f_{w_1} of such flats is s^{2t-2}.

Let $\mathcal{S}^*_{2t-3}$ be one of the (2t-3)-flats of $\mathcal{T}_{2t-2}(P)$, not passing through P and let the non-degenerate Q^*_{2t-3} denote the intersection of $\mathcal{S}^*_{2t-3}$ and $Q^o_{2t-2}(P)$. Let $\mathcal{S}^*_{2t-4}$ be a (2t-4)-flat of $\mathcal{S}^*_{2t-3}$, which is not a tangent to Q^*_{2t-3}. Let $\mathcal{S}_{2t-3}(P)$ be the join of P and $\mathcal{S}^*_{2t-3}$. Then $\mathcal{S}_{2t-3}(P)$ intersects $Q^o_{2t-2}(P)$ in a cone $Q^o_{2t-3}(P)$ of order 1 with P as its vertex and it is the join of P and a non-degenerate quadric Q^*_{2t-4} where $Q^*_{2t-4} = \mathcal{S}_{2t-4} \cap Q^*_{2t-3}$.

Now, the number of points on $Q^o_{2t-3}(P)$ is

$$|Q^o_{2t-3}(P)| = 1 + s\,\psi(2t-4,0)$$
$$= 1 + s(s^{2t-4}-1)/(s-1) = (s^{2t-3} - 1)/(s-1).$$

Thus if Q_{2t-1} is elliptic, the number of points on $Q^o_{2t-2}(P)$ which are not on such a $\mathcal{S}_{2t-3}(P)$ flat is

$$w_2 \text{ (ellip.)} = (s^{2t-2} - s^t + s^{t-1}-1)/(s-1) - (s^{2t-3}-1)/(s-1)$$
$$= s^{2t-3} - s^{t-1}$$

If, however, Q_{2t-1} is hyperbolic, the number of points on $Q^o_{2t-2}(P)$ which are not on such a $\mathscr{S}_{2t-3}(P)$ flat is

$$w_2 \text{ (hyperbol.)} = (s^{2t-2} + s^t - s^{t-1}-1)/(s-1) - (s^{2t-3}-1)/(s-1)$$
$$= s^{2t-3} + s^{t-1}.$$

The number f_{w_2} of such $\mathscr{S}_{2t-3}(P)$ flats if Q_{2t-1} is elliptic is given by

$$f_{w_2} \text{ (ellip.)} = (s^{2t-2}-1)/(s-1) - (s^{t-1}+1)(s^{t-2}-1)/(s-1)$$
$$= s^{2t-3} + s^{t-2}.$$

If Q_{2t-1} is hyperbolic, the number f_{w_2} of such $\mathscr{S}_{2t-3}(P)$ flats is

$$f_{w_2} \text{ (hyperbol.)} = (s^{2t-2}-1)/(s-1) - (s^{t-1}-1)(s^{t-2}+1)/(s-1)$$
$$= s^{2t-3} - s^{t-2}.$$

Let $\mathscr{T}_{2t-4}(R)$ be a (2t-4)-flat in $\mathscr{S}^*_{2t-3}$ and a tangent to Q^*_{2t-3} at the point R. Then $\mathscr{T}_{2t-4}(R)$ meets Q^*_{2t-3} in a cone $Q^o_{2t-4}(R)$ of order 1 with the point R as its vertex. The join of P and $\mathscr{T}_{2t-4}(R)$ is a (2t-3)-flat $\mathscr{T}_{2t-3}(P,R)$ which intersects $Q_{2t-2}(P)$ in a cone $Q^o_{2t-3}(P,R)$ of order 2 and its vertex is the line joining the points P and R. It is clear that $Q^o_{2t-3}(P,R)$ is the join of the line PR and a non-degenerate quadric Q_{2t-5}.

Number of points on $Q^o_{2t-3}(P,R)$

$$= (s+1) + s^2\, \psi(2t-5,0)$$
$$= (s^{2t-3} - s^t + s^{t-1}-1)/(s-1) \text{ if } Q_{2t-1} \text{ is elliptic}$$
$$= (s^{2t-3} + s^t - s^{t-1} - 1)/(s-1) \text{ if } Q_{2t-1} \text{ is hyperbolic.}$$

Thus the number of points on $Q^o_{2t-2}(P)$ which are not on such flats $\mathscr{T}_{2t-3}(P,R)$ is

$$w_3 = (s^{2t-2}-s^t+s^{t-1}-1)/(s-1) - (s^{2t-3}-s^t+s^{t-1}-1)/(s-1)$$
$$= s^{2t-3} \text{ if } Q_{2t-1} \text{ is elliptic.}$$

It is easy to check that $w_3 = s^{2t-3}$ also if Q_{2t-1} is hyperbolic. Number f_{w_3} of such

$\mathcal{T}_{2t-3}(P,R)$ (2t-3)-flats is equal to the number of points on a non degenerate quadric Q_{2t-3}.

Thus f_{w_3} (ellip.) $= (s^{t-1}+1)(s^{t-2}-1)/(s-1)$ if Q_{2t-1} is elliptic,

and f_{w_3} (hyperbol.) $= (s^{t-1}-1)(s^{t-2}+1)/(s-1)$ if Q_{2t-1} is hyperbolic.

Thus the weight-distributions of the code $\mathcal{C}(Q^{o}_{2t-2}(P))$ for the two cases - Q_{2t-1} elliptic and Q_{2t-1} hyperbolic are given below.

Q_{2t-1} elliptic		:	Q_{2t-1} hyperbolic	
weight	frequency	:	weight	frequency
0	1		0	1
$s^{2t-3}-s^{t-1}+s^{t-2}$	$(s-1)s^{2t-2}$		$s^{2t-3}+s^{t-1}-s^{t-2}$	$(s-1)s^{2t-2}$
$s^{2t-3}-s^{t-1}$	$(s-1)(s^{2t-3}+s^{t-2})$		$s^{2t-3}+s^{t-1}$	$(s-1)(s^{2t-3}-s^{t-2})$
s^{2t-3}	$s^{2t-3}-s^{t-1}+s^{t-2}-1$		s^{2t-3}	$s^{2t-3}+s^{t-1}-s^{t-2}-1$
	s^{2t-1}			s^{2t-1}

References

Bose, R.C. (1962). Lecture Notes on "Combinatorial Problems of Experimental Design", Department of Statistics, University of North Carolina at Chapel Hill.

Bose, R.C. (1963). On the application of finite projective geometry for deriving a certain series of balanced Kirkman arrangements, Calcutta Math. Soc. Golden Jubilee Comm. vol. Part II (1958-59), 341-356.

Bose, R.C. (1971). Self-conjugate tetrahedra with respect to the Hermitian variety $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ in $PG(3,2^2)$ and a representation of PG(3,3). Proc. Symp. on Pure Math. 19 (Amer. Math. Soc. Providence R.I.), 27-37.

Bose, R.C. and Chakravarti, I.M. (1966). Hermitian varieties in a finite projective space $PG(N,q^2)$, Canad. J. Math 18, 1161-1182.

Bose, R.C. and Mesner, D.M. (1959). On linear associative algebras corresponding to association schemes of partially balanced designs. Ann. Math. Statist., 30, 21-38.

Brouwer, A.E. (1985). Some new two-weight codes and strongly regular graphs. Discrete Appl. Math., 10, 111-114.

Calderbank, R. and Kantor, W.M. (1986). The geometry of two-weight codes. Bull. London Math. Soc., 18, 97-122.

Chakravarti, I.M. (1971). Some properties and applications of Hermitian varieties in $PG(N,q^2)$ in the construction of strongly regular graphs (two-class association schemes) and block designs. Journal of Comb. Theory, Series B, 11(3), 268-283.

Chakravarti, I.M. (1987). The generalized Goppa codes and related discrete designs from hermitian varieties. Institute of Statistics Mimeo Series 1713. Department of Statistics. University of North Carolina at Chapel Hill.

Delsarte, P. (1972). Weights of linear codes and strongly regular normed spaces. Discrete Math. 3, 47-64.

Delsarte, P. (1973). An algebraic approach to the association schemes of coding theory. Philips. Res. Rep. Suppl, 19.

Dembowski, P. (1968). Finite Geometries, Springer-Verlay 1968.

Dickson, L.E. (1901,1958). Linear Groups with an Exposition of the Galois Field Theory. Teubner 1901, Dover Publications Inc., New York 1958.

Dieudonné, J. (1971). La Géométrie des Groupes Classiques, Springer-Verlag, Berin, Troisième Edition.

Dowling, T.A. (1969). A class of tri-weight codes. Institute of Statistics Mimeo series No. 600.3. University of North Carolina at Chapel Hill, Department of Statistics.

Games, R.A. (1986). The geometry of quadrics and correlations of sequences. IEEE Trans. Inf. Th. IT-32, 423-426.

Heft, S.M. (1971). Spreads in Projective Geometry and Associated Designs, Ph.D. dissertation submitted to the University of North Carolina, Dept. of Statistics, Chapel Hill.

Higman, D.J. and McLaughlin, J.E. (1965). Rank 3 subgroups of finite symplectic and unitary groups. J. Reine Angew. Math. 218, 174-189.

Hubaut, Xavier L., (1975). Strongly regular graphs. Discrete Mathematics 13, 357-381.

Jordan, C. (1870). Traité des Substitutions et des Équations Algébriques, Gauthier-Villans, Paris.

MacWilliams, F.J. and Sloane, N.J.A. (1977). The Theory of Error - Correcting Codes, North Holland.

MacWilliams, F.J., Odlyzko, A.M., Sloane, N.J.A. and Ward, H.N. (1978). Self-dual codes over GF(4). J. Comb. Th. A25, 288-318.

Mann, H.B. (1965). Addition Theorems. John Wiley & Sons, Inc.

Menon, P.K. (1960). Difference sets in Abelian groups. Proc. Amer. Math. Soc. 11, 368-376.

Mesner, D.M. (1967). A new family of partially balanced incomplete block designs with some latin square design properties. Ann. Math. Statist., 38, 571-581.

Primrose, E.J.F. (1951). Quadrics in finite geometries. Proc. Comb. Phil. Soc. 47, 299-304.

Ray-Chaudhuri, D.K. (1959). On the application of the geometry of quadrics to the construction of partially balanced incomplete block designs and error correcting codes. Ph.D. dissertation submitted to the University of North Carolina at Chapel Hill.

.ay-Chaudhuri, D.K. (1962). Some results on quadrics in finite projective geometry based on Galois fields. Canad. J. Math., 14, 129-138.

Segre, B. (1965). Forme e geometries hermitiane, con particolare riguardo al caso finito. Ann. Math. Pure Appl., 70 1,202.

Segre, B. (1967). Introduction to Galois Geometries, Atti della Acc. Nazionale dei Lincei, Roma, 8(5) 137-236.

Wolfman, J. (1975). Codes projectifs à deux ou trois poids associés aux hyperquadriques d'une géométrie finie. Discrete Mathematics, 13, 185-211.

Wolfman, J. (1977). Codes projectifs à deux poids, "caps" complets et ensembles de différences. J. Combin. Theory, 23A, 208-222.