Families of Riemann Surfaces and Weil-Petersson Geometry

Conference Board of the Mathematical Sciences

CBMS

Regional Conference Series in Mathematics

Number 113

Families of Riemann Surfaces and Weil-Petersson Geometry

Scott A. Wolpert

Published for the

Conference Board of the Mathematical Sciences by the



American Mathematical Society
Providence, Rhode Island
with support from the
National Science Foundation



NSF-CBMS Regional Research Conferences in the Mathematical Sciences on Families of Riemann Surfaces and Weil-Petersson Geometry held at Central Connecticut State University, New Britain, CT July 20-24, 2009

Partially supported by the National Science Foundation. The author acknowledges support from the Conference Board of the Mathematical Sciences and NSF Grant DMS 0834134.

2000 Mathematics Subject Classification. Primary 20F67, 30F60, 32G15, 32G15, 37F30; Secondary 11F41, 14H15, 32Q05, 32Q45.

For additional information and updates on this book, visit ${\bf www.ams.org/bookpages/cbms-113}$

Library of Congress Cataloging-in-Publication Data

Wolpert, Scott A.

Families of Riemann surfaces and Weil-Petersson geometry / Scott A. Wolpert.

p. cm. – (CBMS regional conference series in mathematics; no. 113)

Includes bibliographical references and index.

ISBN 978-0-8218-4986-6 (alk. paper)

1. Riemann surfaces. 2. Teichmüller spaces. 3. Hyperbolic groups. 4. Ergodic theory. 5. Geometry, Riemannian. I. Title.

 $\begin{array}{c} {\rm QA337.W65\ 2010} \\ {\rm 515'.93-dc22} \end{array}$

2010011413

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294 USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2010 by the author. All rights reserved. Printed in the United States of America.

The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

 Visit the AMS home page at http://www.ams.org/

10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

Contents

Prefa	ce	vii
Chap	eter 1. Preliminaries	1
1.	Riemann surfaces and line bundles	1
2.	Introduction of first-order deformations	2
3.	Hyperbolic geometry	4
4.	Standard cusps and collars	6
5.	Uniformization, $PSL(2;\mathbb{R})$ representation spaces and Mumford	
	compactness	7
6.	Collars converging to cusp pairs, version 1.0	8
7.	Holomorphic plumbing fixture - collars converging to cusps,	
	version 2.0	9
8.	Further readings	11
Chap	eter 2. Teichmüller Space and Horizontal Strip Deformations	13
1.	Definition of Teichmüller space	13
2.	Deformations of concentric annuli and horizontal strips	15
3.	Variational formulas for a horizontal strip	17
4.	Plumbing family tangents and cotangents	20
5.	Further readings	21
Chap	eter 3. Geodesic-Lengths, Twists and Symplectic Geometry	23
1.	Basics of geodesic-lengths and twists	23
2.	Twist derivatives and Riera's formula	25
3.	Hessian of geodesic-length	27
4.	Fenchel-Nielsen coordinates are canonical	29
5.	Further readings	30
Chap	eter 4. Geometry of the Augmented Teichmüller Space, Part 1	33
1.	Augmented Teichmüller space	33
2.	Second order Masur type expansions	34
3.	Model metric comparison	38
4.	Teichmüller metric	39
5.	Further readings	40
Chap	eter 5. Geometry of the Augmented Teichmüller Space, Part 2	43
1.	$CAT(0)$ geometry and geodesics on $\overline{\mathcal{T}}$	43
2.	Properties of Bers regions	46

vi CONTENTS

3.	Further readings	49		
Chap	ter 6. Geometry of the Augmented Teichmüller Space, Part 3	51		
1.				
2.	Visual spheres	53		
3.	Ending laminations for geodesics in \mathcal{T}	55		
4.	Alexandrov tangent cone	57		
5.	Teichmüller-Coxeter complex	60		
6.	Further readings	62		
Chap	ter 7. Deformations of hyperbolic metrics and the curvature			
	tensor	65		
1.	Prescribed curvature equation	65		
2.	Variational formulas	67		
3.	Plumbing expansion - collars converging to cusps, version 3.0	69		
4.	Further readings	71		
Chap	ter 8. Collar expansions and exponential-distance sums	75		
1.	Example sums and expansions	75		
2.	Collar principle and the distant sum estimate	77		
3.	Bounds for single and double coset sums	78		
4.	Further readings	81		
Chap	ter 9. Train tracks and the Mirzakhani volume recursion	83		
1.	Measured geodesic laminations and train tracks	83		
2.	McShane-Mirzakhani length identity	86		
3.	Mirzakhani volume recursion	87		
4.	Moduli volumes, symplectic reduction and tautological classes	92		
5.	Virasoro constraint equations and Witten-Kontsevich theory	96		
Chap	ter 10. Mirzakhani prime simple geodesic theorem	99		
1.	Prime geodesic theorems	99		
2.	Counting integral multi curves	102		
3.	Finding the scaled orbit limit measure	104		
4.	Multi curve constants and Thurston volume integrals	106		
Biblio	ography	109		
Index		117		

Preface

These written lectures are the companion to the NSF-CBMS Regional Research Conference in the Mathematical Sciences, organized July 20-24, 2009 at Central Connecticut State University, by Eran Makover and Jeffrey McGowan. My goal for the lectures is a generally self-contained course for graduate students and postgraduates. The topics run across current research areas. By plan the approach is didactic. Concepts are developed across multiple lectures. Opportunities are taken to introduce general concepts, to present recurring methods and to generally provide proofs. Guides to the research literature are included.

The study of Riemann surfaces continues to be an interface for algebra, analysis, geometry and topology. I hope that in part I am able to suggest the interaction to the audience and reader. The lectures are not intended as a proper research summary or history of the field. A collection of current and important topics are not included. Material is not always presented following the historical development of concepts. The references are not all inclusive but are intended only as a lead-in to the literature. Further readings are provided at the ends of chapters.

I thank the Conference Board of the Mathematical Sciences and the National Science Foundation for supporting the undertaking. NSF Grant DMS 0834134 supported the Regional Conference and lectures. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

First and foremost, I would like to thank Eran Makover and Jeffrey McGowan. The idea for a conference, the planning and all the arrangements were smoothly and efficiently handled by Eran and Jeff. On behalf of the participants I would like to express appreciation to Central Connecticut State University for providing arrangements and facilities. I am most appreciative for the conference participants' engagement and feedback. Also, I especially would like to thank Bill Goldman, Zheng (Zeno) Huang, Zachary McGuirk, Babak Modami, Kunio Obitsu, Athanase Papadopoulos and Mike Wolf for detailed comments, feedback and contributions.

Scott A. Wolpert January, 2010 College Park, Maryland