PACIFIC JOURNAL OF MATHEMATICS
Vol. 141, No. 1, 1990

# FANO BUNDLES OVER $P^3$ AND $Q_3$

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**A vector bundle $\mathscr{E}$ is called Fano if its projectivization $P(\mathscr{E})$ is a Fano manifold. In this article we prove that Fano bundles exist only on Fano manifolds and discuss rank-2 Fano bundles over the projective space $P^3$ and a 3-dimensional smooth quadric $Q_3$.**

Fano bundles appear naturally as we strive to construct examples of Fano manifolds of dimension $\geq 3$; they form interesting yet accessible class of Fano $n$-folds. For example: among 87 types of Fano 3-folds with $b_2 \geq 2$ listed in **[13]** 22 types are ruled (i.e. obtained by projectivization of Fano bundles). Moreover some of the non-ruled manifolds listed there can be easily expressed as either finite covers of ruled 3-folds or divisors (or, more generally, complete intersections) in ruled Fano manifolds of higher dimension.

Let us mention another aspect of dealing with Fano bundles: it is how to determine whether or not a vector bundle is ample. This very fine property of a vector bundle cannot be determined by its numerical invariants, see [7]. Assuming the bundle to be stable helps to establish a sufficient condition for ampleness: **[10]**, **[17]**, which however is far from being necessary. In the present paper we take advantage of some already known facts about stable bundles with small Chern classes and determine that a bundle $\mathscr{E}$ is not ample by finding its jumping lines or sections of $\mathscr{E}(-k)$.

Let us note that some results of this paper have already been published, see remarks after the proofs of Theorems (1.6) and (2.1).

**1. Fano bundles; preliminaries.** Let $\mathscr{E}$ be a vector bundle of rank $r \geq 2$ on a smooth complex projective variety $M$. Let us recall that the tautological line bundle $\xi = \xi_{\mathscr{E}}$ on $V = P(\mathscr{E})$ is uniquely determined by the conditions $\xi_{\mathscr{E}}|F \approx \mathscr{O}_F(1)$ and $p_*\xi_{\mathscr{E}} = \mathscr{E}$. By $p$ we have denoted the projection morphism of $V = P(\mathscr{E})$ onto $M$ and by $F$—the fibre of $p$. Obviously, $F \cong P^{r-1}$ and $p\colon V \to M$ is a $P^{r-1}$-bundle. The Picard group of $V$ can be expressed as a direct sum: $\operatorname{Pic} V \cong Z\cdot\xi_{\mathscr{E}} \oplus p^*(\operatorname{Pic} M)$. Replacing $\mathscr{E}$ by its twist with a line bundle $\mathscr{L}$ on $M$ does not affect the projectivization and

$$\xi_{\mathscr{E}\otimes\mathscr{L}} = \xi_{\mathscr{E}} \otimes p^*(\mathscr{L}).$$

Moreover, $\mathscr{E}$ is generated by global sections iff $\xi_{\mathscr{E}}$ is. We have the following relative Euler sequence on $V = P(\mathscr{E})$:

$$(1.1) \qquad 0 \to \mathscr{O}_V \to p^*(\mathscr{E})^\vee \otimes \xi_{\mathscr{E}} \to T_{V|M} \to 0$$

where the latter bundle is the relative tangent bundle of $p$ and fits in the exact sequence

$$(1.2) \qquad 0 \to T_{V|M} \to TV \to p^*TM \to 0.$$

We then obtain

$$(1.3) \qquad c_1 V = p^*(c_1 M - c_1\mathscr{E}) + r\xi_{\mathscr{E}}$$

The theorem of Leray and Hirsch yields that in the cohomology ring of $V$ the following holds

$$(1.4) \qquad \xi_{\mathscr{E}}^r - p^*(c_1\mathscr{E})\xi_{\mathscr{E}}^{r-1} + p^*(c_2\mathscr{E})\xi_{\mathscr{E}}^{r-2} - \cdots \pm p^*(c_r\mathscr{E}) = 0.$$

From now on we assume in this section that $\mathscr{E}$ is a rank-$r$ Fano bundle on an $n$-fold $M$, i.e., that $\mathbb{P}(\mathscr{E})$ is a Fano manifold. We prove that such $M$ must be Fano, as well.

(1.5) LEMMA. *Let $C \subset M$ be a rational curve with a normalization $\nu: P^1 \to C$. Assume that $\nu^*(\mathscr{E}) \cong \mathscr{O}(a_1) \oplus \mathscr{O}(a_2) \oplus \cdots \oplus \mathscr{O}(a_r)$, where $a_1 \le a_2 \le \cdots \le a_r$. Then*

$$(c_1 M) \cdot C > \sum_{i=2}^{r}(a_i - a_1) \ge 0.$$

*Proof.* The right hand side inequality is obvious. To prove the left hand side inequality let us assume that $W = P(\nu^*\mathscr{E})$. The manifold $W$ is then a $P^{r-1}$-bundle over $P^1$, with a projection $\pi: W \to P^1$. We have a section $C_0$ of $\pi$ associated to the epimorphism $\nu^*\mathscr{E} \to \mathscr{O}(a_1) \to 0$, such that

$$\xi_{\nu^*\mathscr{E}}|C_0 \cong \mathscr{O}_{P^1}(a_1).$$

The normalization map $\nu: P^1 \to M$ lifts to a map $\overline{\nu}: W \to V$, making the following diagram commute

$$\begin{array}{ccc} W & \xrightarrow{\overline{\nu}} & V \\ \pi \downarrow & & \downarrow p \\ P^1 & \xrightarrow{\nu} & M \end{array}$$

By the choice of $C_0$ we have

$$\overline{\nu}^*(\xi_{\mathscr{E}}) \cdot C_0 = a_1$$

and, since $c_1 V$ is ample, we obtain by (1.3)

$$\begin{aligned} 0 < c_1 V \cdot \overline{\nu}(C_0) &= \overline{\nu}^*(c_1 V) \cdot C_0 \\ &= r \cdot \overline{\nu}^*(\xi_{\mathscr{E}}) \cdot C_0 + (\pi \circ \nu)^*(c_1 M) \cdot C_0 - (\pi \circ \nu)^*(c_1 \mathscr{E}) \cdot C_0 \\ &= r \cdot a_1 + c_1 M \cdot C - \sum_{i=1}^{r} a_i \end{aligned}$$

which yields the desired inequality.

(1.6) THEOREM. *If $\mathscr{E}$ is a Fano bundle on a manifold $M$ then $M$ is a Fano n-fold.*

*Proof.* As $c_1 V = -K_V$ is ample, the cone of curves on $V$ is spanned by the classes of extremal curves (see **[12]** for definitions and Theorem 1.2 on the cone of curves of a Fano manifold). Let us denote these curves by $l_0, l_1, \cdots, l_v$ with $l_0$ contained in $F$, a fibre of the projection $p: P(\mathscr{E}) \to M$. We see that $p^*(c_1 M) \cdot l_0 = 0$ and for $i > 0$, $p(l_i)$ is a rational curve on $M$. Therefore from (1.5) it follows that

$$0 < c_1 M \cdot p(l_i) = p^*(c_1 M) \cdot l_i \tag{1.7}$$

which means that $p^*(c_1 M)$ is numerically effective. Recall now (a conclusion from) the Kawamata-Shokurov contraction theorem, see (2.6) in **[11]**:

If $D$ is nef and $aD - K$ is ample for some $a > 0$, then $D$ is semiample, i.e., some power of $D$ is generated by global sections.

It follows that $D := p^*(c_1 M)$ is semiample. Since $p: V \to M$ is a $P^{r-1}$-bundle, we have, for any integer $k$, $p_* p^*(\mathscr{O}(kc_1 M)) = \mathscr{O}(kc_1 M)$ and the images (under $p_*$) of global sections of $p^*(\mathscr{O}(kc_1 M))$ are global sections of $\mathscr{O}(kc_1 M)$. Therefore $c_1 M$ is semiample, hence to prove that it is ample it is enough to show that $c_1 M \cdot C > 0$ for any curve $C$ in $M$.

Let $C$ be an irreducible curve in $M$. Taking an appropriate component from an intersection of the inverse image $p^{-1}(C)$ with general $r-1$ divisors from a very ample linear system, we can produce an irreducible curve $C_1 \subset V$, such that $p(C_1) = C$. Then $C_1$ is numerically equivalent to a linear combination $\sum a_i l_i$ with at least one $a_i$ different from zero for $i > 0$. Let $d$ be the degree of the map $p|C_1 : C_1 \to C$.

Now the inequality (1.7) gives

$$c_1M \cdot C = \frac{1}{d} \cdot p^*(c_1M) \cdot C_1 = \frac{1}{d} \cdot \left(\sum a_i \cdot p^*(c_1M) \cdot l_i\right) > 0,$$

which concludes the proof of the theorem.

REMARK. Theorem (1.6) has already been known for bundles of rank 2 on surfaces [4] and 3-folds [1].

**2. Rank-2 Fano bundles on $P^3$.** The results stated below (Theorem (2.1)) can be understood as one more example of an exceptional character of the null-correlation bundle (see e.g. [3] or [15] for the definition of the null-correlation bundle).

(2.1) THEOREM. *The only rank-2 Fano bundles with $c_1 = 0, -1$, on $\mathbb{P}^3$ are*

(1) $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$,
(2) $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)$,
(3) $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(+1)$,
(4) $\mathcal{E} = \mathcal{O}(-2) \oplus \mathcal{O}(+1)$,
(5) *the null-correlation bundle* $\mathcal{N}$.

*Proof.* Let $V = P(\mathcal{E})$. We then have

$$(2.2) \quad -K_V = 2\xi + (4 - c_1\mathcal{E})H = \begin{cases} 2\xi + 4H = 2\xi_{\mathcal{E}(1)} + 2H = 2\xi_{\mathcal{E}(2)} & \text{if } c_1 = 0, \\ 2\xi + 5H = 2\xi_{\mathcal{E}(2)} + H = 2\xi_{\mathcal{E}(3)} - H & \text{if } c_1 = -1, \end{cases}$$

and we see that any of the bundles listed above is Fano. Indeed, if $\mathcal{E}$ is one of those listed as (1), (3) or (5) (respectively: (2) or (4)) then $c_1\mathcal{E} = 0$ (resp. $c_1\mathcal{E} = -1$) and $\mathcal{E}(1)$ (resp. $\mathcal{E}(2)$) is generated by its global sections. Now, since $\rho(V) = 2$, it follows from (2.2) that $c_1V$ is ample as the sum of two non-proportional nef divisors.

An easy corollary follows.

(2.3) For a normalized Fano bundle $\mathcal{E}$ of rank 2 on $P^3$:
if $c_1\mathcal{E} = 0$, then $\mathcal{E}(2)$ is ample,
if $c_1\mathcal{E} = -1$, then $\mathcal{E}(3)$ is ample.
We shall discuss the two cases separately.

*Case* $c_1 = 0$. A straightforward consequence of the theorem of Leray and Hirsch (1.4) yields that in the cohomology ring of $V$ the following holds

$$\xi^2 + c_2H^2 = 0.$$

Since $H^4 = 0$ and $H^3\xi = 1$, the above formula then gives

$$(2.4) \qquad H^2\xi^2 = 0, \qquad H\xi^3 = -c_2, \qquad \xi^4 = 0,$$

so that $(-K_V)^4 = (2\xi + 4H)^4 = 128(4 - c_2)$ and we see that $c_2 < 4$.

Assume first $\mathscr{E}$ is not semistable, i.e., $H^0(\mathscr{E}(-1)) \neq 0$. Let $s$ be a non-zero section of $\mathscr{E}(-1)$. We claim that $s$ does not vanish anywhere. Indeed, if $Z = \{s = 0\}$ were not empty, then for a line $L$ meeting $Z$ in a finite number of points we would have

$$\mathscr{E}(-1)|L = \mathscr{O}(d) \oplus \mathscr{O}(e) \quad \text{with } d \geq 1, d + e = -2,$$

contradicting (2.3). Therefore $s$ does not vanish and thus $\mathscr{E}(-1) = \mathscr{O} \oplus \mathscr{O}(-2)$, hence $\mathscr{E}$ is as in (3) of the theorem.

Let now $\mathscr{E}$ be semistable but not stable: $H^0(\mathscr{E}(-1)) = 0, H^0(\mathscr{E}) \neq 0$. If a non-zero section of $\mathscr{E}$ does not vanish anywhere, $\mathscr{E}$ must then be $\mathscr{O} \oplus \mathscr{O}$. Otherwise a section vanishes on a curve. If the curve is not a single line then cutting it by a line leads to a contradiction, as above. But if a single line $L$ was a zero set of a section of $\mathscr{E}$ then, by the adjunction formula, the degree of the canonical divisor of $L$ would be

$$\deg(K_L) = (K_{P^3} + c_1\mathscr{E}) \cdot L = -4,$$

which is impossible. Because of Bogomolov's inequality $c_1^2 < 4c_2$ for stable bundles, **[15]**, it remains then to study stable bundles with $c_1 = 0$ and $c_2 = 1, 2, 3$. In the first case $\mathscr{E}$ is the null-correlation bundle $\mathscr{N}$, for which $\mathscr{N}(2)$ is ample; $\mathscr{N}$ is then Fano.

In the remaining cases we know that $\mathscr{E}$ has multiple jumping lines, i.e. such lines $L$ for which $\mathscr{E}|L = \mathscr{O}_L(-2) \oplus \mathscr{O}_L(2)$, see **[8]**, Proposition 9.11, and **[18]**, respectively. In virtue of (2.3), such bundles cannot be Fano.

*Case* $c_1 = -1$. The multiplication table is now:

$$(2.5) \qquad H^4 = 0, \qquad H^3\xi = 1, \qquad H^2\xi^2 = -1,$$
$$H\xi^3 = -c_2 + 1, \qquad \xi^4 = 2c_2 - 1$$

and from

$$(-K_V)^4 = (2\xi + 5H)^4 = 32(-4c_2 + 17) > 0$$

we obtain that the only possible non-negative values for $c_2$ are 0, 2 or 4 (recall that Schwarzenberger's condition says $c_1c_2 \equiv 0 \pmod 2$). Assume $H^0(\mathscr{E}(-1)) \neq 0$. As above, we show that no section $s \neq 0$

vanishes: if $Z = \{\text{zero}(s)\}$ were not empty, for a line $L$ meeting $Z$ at finitely many points we would have

$$\mathscr{E}(-1)|L = \mathscr{O}_L(d) \oplus \mathscr{O}_L(e) \quad \text{with } d \geq 1, d+e=-3,$$

contradicting (2.3). Therefore the sections $\mathscr{E}(-1)$ do not vanish anywhere, so that $\mathscr{E}$ is as in (4) of Theorem (2.1).

Let then $H^0(\mathscr{E}(-1)) = 0$, $H^0(\mathscr{E}) \neq 0$. The zero set $Z$ of a non-zero section is then a curve (if not empty). Again, if $Z$ were anything different from a single line, for a line $L$ that cuts $Z$ at a finite number $\geq 2$ of points we would have

$$\mathscr{E}|L = \mathscr{O}_L(d) \oplus \mathscr{O}_L(e), \qquad d \geq 2, d+e=-1,$$

contradicting the ampleness of $\mathscr{E}(3)$. But $c_1c_2$ is even so that the case $c_1 = -1$, $c_2 = 1$ does not hold, hence $Z$ is not a line. The non-zero sections of $\mathscr{E}$ do not vanish, hence $\mathscr{E} = \mathscr{O} \oplus \mathscr{O}(1)$.

It remains to exclude the cases of stable vector bundles with $c_1 = -1$ and $c_2 = 2$ or 4. In the former case $\mathscr{E}$ has multiple jumping lines, [**9**], Proposition 4.1, i.e., those for which $\mathscr{E}|L = \mathscr{O}_L(-3) \oplus \mathscr{O}_L(2)$, hence $\mathscr{E}$ cannot be Fano in view of (2.3). In the latter one $\mathscr{E}(2)$ has a section, see [**2**], Lemma 1, and $2H + \xi$ is effective with

$$(2H+\xi)(c_1P(\xi))^3 = (2H+\xi)(2\xi+5H)^3 = -17.$$

These bundles are then not Fano.

REMARK. Theorem (2.1) (in a somewhat weaker form) was first announced by Artiushkin, [**1**]. His proof was, however, incorrect: in line 36 on page 14 if $E$ is a normalized bundle on $P^3$, then the tautological divisor $\xi_E = L$ in op. cit. need not to be effective, therefore $(-K)^3 \cdot L$ need not to be positive. Our actual proof is more complicated.

Let us conclude this section by proving that $P(\mathscr{N})$ has a $P^1$-bundle structure over a 3-dimensional quadric $Q_3$. To see this, first let us recall that $\mathscr{N}(1)$ can be defined as the bundle fitting in the following exact sequence on $P^3$

$$0 \to \mathscr{O} \to \Omega P^3(2) \to \mathscr{N}(1) \to 0.$$

Note that $P(\Omega P^3(2))$ is the incidence variety

$$I = \{(x, l) \in P^3 \times \text{Grass}(1,3) : x \in l\}$$

and Grass(1, 3) is isomorphic to a 4-dimensional quadratic. Now, from the above exact sequence it follows that $P(\mathscr{N}(1))$ is a divisor in $I$ which is an inverse image of a hyperplane section of Grass(1, 3).

Therefore:

(2.6) PROPOSITION. *The Fano 4-fold $P(\mathscr{N}(1))$ is a projectivization of a rank-2 vector bundle on smooth quadratic $Q_3 \subset$ Grass(1, 3), obtained by restricting to $Q_3$ the universal quotient bundle from* Grass(1, 3).

**3. Bundles over $Q_3$.** Let us recall that the cohomology ring of $Q_3$ is generated by the classes of $[H] \in H^2(Q_3, Z)$, $[L] \in H^4(Q_3, Z)$, and $[P] \in H^6(Q_3, Z)$ where $H$, $L$ and $P$ are a quadratic surface, a line and a point, respectively. There are the following relationships: $[H]^2 = 2L$, $[H][L] = [P]$ and hence $[H]^3 = 2[P]$. If $\mathscr{F}$ is a coherent sheaf on $Q_3$ with the Chern polynomial

$$1 + c_1(\mathscr{F})[H]t + c_2(\mathscr{F})[L]t^2 + c_3(\mathscr{F})[P]t^3,$$

then the numbers $c_i$ are called the Chern classes of $\mathscr{F}$.

Recall the Riemann-Roch formula for $\mathscr{F}$, [5]

$$\chi(\mathscr{F}) = \tfrac{1}{6}(2c_1^3 - 3c_1c_2 + 3c_3) + \tfrac{3}{2}(c_1^2 - c_2) + \tfrac{13}{6}c_1 + \operatorname{rank}\mathscr{F}.$$

Let now $\mathscr{E}$ be a rank-2 vector bundle on $Q_3$. The theorem of Leray and Hirsch (1.4) gives the following relations between the generators of $\operatorname{Pic}(P(\mathscr{E})) \cong Z \oplus Z$

$$\begin{cases} \text{if } c_1 = 0, \text{ then } \xi^2 + \frac{1}{2}c_2(\mathscr{E})H^2 = 0; \\ \text{if } c_1 = -1, \text{ then } \xi^2 + \xi H + \frac{1}{2}c_2(\mathscr{E})H^2 = 0. \end{cases}$$

Because $H^4 = 0$ and $H^3\xi = 2$, we obtain:

if $c_1 = 0$, then $H^2\xi^2 = 0$, $H\xi^3 = -c_2$, $\xi^4 = 0$;

if $c_1 = -1$, then $H^2\xi^2 = -2$, $H\xi^3 = 2 - c_2$, $\xi^4 = 2c_2 - 2$.

Let $\mathscr{E}$ be a normalized rank-2 vector bundle on $Q_3$ and $V = P(\mathscr{E})$ its projectivization. We then have

$$(3.1) \qquad c_1V = -K_V = \begin{cases} 2\xi + 3H & \text{when } c_1 = 0, \\ 2\xi + 4H & \text{for } c_1 = -1. \end{cases}$$

*Case of non-stable bundles.* Assume $\mathscr{E}$ is non-stable with $c_1(\mathscr{E}) = -1$. If a non-zero section from $H^0(\mathscr{E}(-1))$ vanishes at some point, let us consider a line $L$ passing through this point and not contained in the zero set entirely. Then $\mathscr{E}(-1)|L = \mathscr{O}(d) \oplus \mathscr{O}(e)$ with $d \geq 1$, $d + e = -3$ that contradicts the ampleness of $\mathscr{E}(2)$, (3.1).

Assume $H^0(\mathscr{E}(-1)) = 0$, $H^0(\mathscr{E}) \neq 0$. Then a non-zero section of $\mathscr{E}$ either does not vanish anywhere or it vanishes on a set of pure dimension 1. The divisor $\xi_{\mathscr{E}}$ is effective on $P(\mathscr{E})$ and

$$\xi \cdot (-K_V)^3 = 8\xi(\xi + 2H)^3 = 16(-2c_2 + 1),$$

and we see that $c_2 \leq 0$. But then sections of $\mathscr{E}$ do not have zeros, hence $\mathscr{E} = \mathscr{O}(-1) \oplus \mathscr{O}$. Finally, we easily check that $\mathscr{O}(-1) \oplus \mathscr{O}$ is a Fano bundle (because $\mathscr{O}(1) \oplus \mathscr{O}(2)$ is ample).

In case $c_1 = 0$ we exclude non-semistable bundles in a very similar way. Finally, if $\mathscr{E}$ is semistable but not stable, that is $H^0(\mathscr{E}) \neq 0 = H^0(\mathscr{E}(-1))$, the divisor $\xi_{\mathscr{E}}$ is effective and

$$0 < \xi(2\xi + 3H)^3 = 18(-2c_2 + 3)$$

so that $c_2 \leq 0$ (recall that $c_2 \equiv 0 \bmod 2$, see **[5]**, §1). If so, a non-zero section of $\mathscr{E}$ does not vanish anywhere and $\mathscr{E}$ must then be $\mathscr{O} \oplus \mathscr{O}$.

*Case of stable bundles with* $c_1 = 0$. From the condition $K^4 > 0$ we easily obtain that if $V = P(\mathscr{E})$ is Fano, then $c_2 \leq 4$, and since $c_2 \equiv 0$ (mod 2) it follows that either $c_2 = 2$ or 4. We believe that there is no Fano bundle on $Q_3$ with $c_1 = 0$, $c_2 = 4$, however we do not have enough information on these bundles to prove it.

In case of $c_2 = 2$, one can easily check that the pull-back $\pi^*(\mathscr{N})$ of the null-correlation bundle, under a double covering $\pi\colon Q_3 \to P^3$, is Fano. Indeed, $\pi^*(\mathscr{N})(1)$ is then spanned on $Q_3$, therefore $-K_{P(\pi^*(\mathscr{N}))} = 2\xi_{\pi^*(\mathscr{N})(1)} + H$ is ample. On the other hand we have

(3.2) PROPOSITION. *If $\mathscr{E}$ is a stable bundle on $Q_3$ with $c_1 = 0$, $c_2 = 2$ such that $\mathscr{E}(1)$ is spanned by global sections then $\mathscr{E}$ is a pull-back $\pi^*(\mathscr{N})$ of a null-correlation bundle $\mathscr{N}$, under a double covering $\pi\colon Q_3 \to P^3$.*

*Proof.* The argument is based on the following fact: for any two disjoint lines on $P^3$ there exists a section of a twisted null-correlation bundle $\mathscr{N}(1)$ vanishing exactly on these lines. Therefore, if we prove that a section of $\mathscr{E}(1)$ vanishes on a set being a pullback, via a double covering $\pi\colon Q_3 \to P^3$, of two disjoint lines on $P^3$, then in view of Theorem 1.1 and Remark 1.1.1 from **[8]**, $\mathscr{E}(1)$ is a pullback of $\mathscr{N}(1)$; if $Z$ is the union of two disjoint lines and $Y$ its pullback then it is easy to check that every isomorphism between $\omega_Q(-2)|Y$ and $\omega_Y$ comes from $\omega_{\mathbf{P}}(-2)|Z \simeq \omega_Z$.

Assume $\mathscr{E}$ is stable with $c_1(\mathscr{E}) = 0$, $c_2(\mathscr{E}) = 2$ on $Q_3$. We easily compute the following cohomology table of $h^i(\mathscr{E}(-m))$

| | $m=3$ | $m=2$ | $m=1$ | $m=0$ |
|---|---|---|---|---|
| $h^3$ | 0 | 0 | 0 | 0 |
| $h^2$ | 1 | 1 | 0 | 0 |
| $h^1$ | 0 | 0 | 1 | 1 |
| $h^0$ | 0 | 0 | 0 | 0 |

Indeed, vanishing of the lower and upper row is a consequence of the stability (plus Serre's duality) and the "spectrum" technique, namely Corollary 2.4 in [5], gives

$$h^1(\mathscr{E}(-2)) = h^1(\mathscr{E}(-3)) = h^2(\mathscr{E}) = h^2(\mathscr{E}(-1)) = 0$$

and the remaining part of the table follows from computing the Euler-Poincaré characteristic.

Since $\chi(\mathscr{E}(1)) = 5$ and $h^2(\mathscr{E}(1)) = h^1(\mathscr{E}(-4)) = 0$ by Corollary 2.4 in [5], we see $h^0(\mathscr{E}(1)) \geq 5$. Let $Y$ be the zero of a generic section.

Since $H^0(\mathscr{E}) = 0$ and $\mathscr{E}(1)$ is assumed to be globally generated, $Y$ is a smooth (not necessarily connected) curve. From the diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \to & \mathscr{O}(-2) & \to & \mathscr{E}(-1) & \to & J_Y & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \mathscr{O}_{Q_3} & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathscr{O}_Y & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

we calculate, with the aid of the cohomology table above, that $h^0(\mathscr{O}_Y) = 2$, i.e., $Y$ consists of two connected components, say $Y_1$ and $Y_2$.

*Claim.* $Y_1$ and $Y_2$ are conics.

*Proof of claim.* Since $c_2\mathscr{E}(1) = 4$ and both $Y_i$ are smooth (therefore reduced) we have only to exclude the possibility that one of them is a line $L$. But then by the adjunction formula we would obtain

$$\deg(K_L) = (K_{Q_3} + c_1\mathscr{E}(1)) \cdot L = -1$$

which is impossible.

Let now $H_i$ be the plane containing $Y_i$, $i = 1, 2$; clearly $Q_3 \cap H_i = Y_i$ and $H_1$, $H_2$ meet at one point in $P^4$ off $Q_3$. Projecting $Q_3 \subset P^4$ from this point onto a hyperplane $H$ in $P^4$ is a double covering of $H$ and the images of $Y_1$ and $Y_2$ are two skew lines, say $L_1$ and $L_2$. It then follows that $\mathscr{E}(1)$ is the pull-back of the null-correlation bundle $\mathscr{N}(1)$ corresponding to $L_1$ and $L_2$.

REMARK. It is not entirely clear whether or not any stable bundle on $Q_3$ with $c_1 = 0$ and $c_2 = 2$ enjoys the property stated in (3.2).

*Case of stable bundles with* $c_1 = -1$, $c_2 = 1$. Here a more detailed description of Fano bundles can be given. Let $\mathscr{E}$ be a stable bundle on $Q_3$ with $c_1 = -1$, $c_2 = 1$.

(3.3) The cohomology of such a bundle are the following:
(1) $h^0(\mathscr{E}(m)) = 0$ for $m \leq 0$,
(2) $h^0(\mathscr{E}(1)) = 4$,
(3) $h^1(\mathscr{E}(m)) = h^2(\mathscr{E}(m)) = 0$ for all $m$,
(4) $h^3(\mathscr{E}(m)) = 0$ for $m \geq -2$.

*Proof.* (1) is a criterion of stability, (4) is dual to (1), (2) will follow from (3), (4) and the Riemann-Roch formula. Corollary 2.4 in **[5]** gives $h^1(\mathscr{E}(m)) = 0$ for $m \leq -1$. By duality, $h^2(\mathscr{E}(m)) = 0$ for $m \geq -1$ so that $h^1(\mathscr{E}) = \chi(\mathscr{E}) = 0$. The Castelnuovo criterion (see e.g. Lecture 14 in **[14]**) now yields that $\mathscr{E}(m)$ are generated by global sections if $m \geq 1$ and that all cohomology $H^i(\mathscr{E}(m))$ vanish for $i \geq 1$, $i + m \geq 1$. Now by duality (3) follows for any integer $m$.

Note that from the Castelnuovo criterion it follows that $\mathscr{E}(1)$ is spanned; therefore $\mathscr{E}(2)$ is ample and $\mathscr{E}$ is Fano.

Now we prove that such $\mathscr{E}$ is the one from (2.6). Since the bundle $\mathscr{E}(1)$ is spanned and $h^0(\mathscr{E}(1)) = 4$ it follows that the linear system $|H + \xi|$ is base point free and of dimension 3. Let $\varphi\colon P(\mathscr{E}) \to P^3$ be the map associated with this system.

(3.4). PROPOSITION. $\varphi\colon P(\mathscr{E}) \to P^3$ *is a* $P^1$*-bundle which is the projectivization of a null-correlation bundle.*

*Proof.* First note that a general divisor $D$ in the linear system $|2H+\xi|$ is a Fano 3-fold listed as n° 17 in Table 2 **[13]**. The map $\varphi|_D$ is a blow-down morphism from $D$ onto $P^3$.

We claim that $\varphi$ has no fibre of dimension $\geq 2$. Assume that $S$ is such a fibre. Then $f := D \cap S$ is isomorphic to $P^1$ and $\mathscr{O}_f(H) \cong \mathscr{O}_{P^1}(1)$. In view of Theorem 2.1b′, **[6]** we see that $S \cong P^2$ and $\mathscr{O}_S(H) \cong \mathscr{O}_{P^2}(1)$. But in this case $p\colon S \to Q_3$ is a plane embedding of $P^2$ in $Q_3$, which is impossible.

Now any fibre of $\varphi$ is numerically equivalent to $(H + \xi)^3$ and, since $H \cdot (H + \xi)^3 = 1$, it follows that it must be isomorphic to $P^1$. The push-forward $\varphi_*(\mathscr{O}(H))$ is a rank-2 Fano bundle on $P^3$. From the results of §2 we see that it is a null-correlation bundle.

COROLLARY. *Any stable rank-*2 *bundle on* $Q_3$ *with Chern classes* $c_1 = -1$, $c_2 = 1$ *is a pull-back of the universal quotient bundle on*

Grass(1, 3) *via some hyperplane embedding*

$$Q_3 \to \mathrm{Grass}(1,3) = Q_4 \subset P^5.$$

REMARK 1. The above example shows that the Horrocks splitting principle, as it stands on $P^n$ (see e.g. [15]), cannot be applied literally to bundles on $Q_3$ (see [16] for an analogue of the Horrocks splitting principle on $Q_n$). Let us also notice that the bundle discussed above is uniform: its decomposition type is the same on all lines and smooth conics in $Q_3$.

REMARK 2. It is proved in [19] that $V = P(\mathcal{N}) = P(\mathcal{E})$ (where $\mathcal{E}$ is the bundle discussed above and $\mathcal{N}$ is the null-correlation bundle on $P^3$) is the only ruled Fano 4-fold of index 2 obtained from a non-decomposable bundle.

*Added in the proof.* Together with Ignacio Sols we have concluded the case of rank-2 Fano bundles on $Q_3$. Firstly, we have proved that the first twist of a stable bundle with $c_1 = 0$, $c_2 = 2$ is spanned by global sections (see Proposition (3.2) and the subsequent remark). Secondly, we have decided that bundles with $c_1 = 0$, $c_2 = 4$ are not Fano (see the discussion preceding (3.2)).

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Received January 26, 1988.


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