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FANO FIVEFOLDS OF INDEX TWO WITH BLOW-UP STRUCTURE

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Abstract. We classify Fano fivefolds of index two which are blow-ups of smooth manifolds along a smooth center.

1. Introduction. A smooth complex projective variety X is called *Fano* if its anticanonical bundle $-K_X$ is ample; the *index* r_X of X is the largest natural number m such that $-K_X = mH$ for some (ample) divisor H on X, while the *pseudoindex* i_X is the minimum anticanonical degree of rational curves on X.

By a theorem of Kobayashi and Ochiai [15], $r_X = \dim X + 1$ if and only if $(X, L) \simeq (\boldsymbol{P}^{\dim X}, \mathcal{O}_{\boldsymbol{P}}(1))$, and $r_X = \dim X$ if and only if $(X, L) \simeq (\boldsymbol{Q}^{\dim X}, \mathcal{O}_{\boldsymbol{Q}}(1))$, where $\boldsymbol{Q}^{\dim X}$ is a quadric hypersurface in $\boldsymbol{P}^{\dim X+1}$. Fano manifolds of index equal to $\dim X - 1$ and to $\dim X - 2$, which are called *del Pezzo* and *Mukai* manifolds respectively, have been classified, mainly by Fujita, Mukai and Mella (see [11, 18, 17]). In case of index equal to $\dim X - 3$, the classification has been completed for Fano manifolds of Picard number ρ_X greater than one and dimension greater or equal than six (see [29]).

For Fano manifolds of dimension five and index two it was proved in [1] that the Picard number is less than or equal to five, equality holding only for a product of five copies of $\boldsymbol{P}^1$. Then, in [9], the structure of the possible Mori cones of curves of those manifolds, i.e., the number and type of their extremal contractions, was described. A first step in going from the table of the cones given in [9] to the actual classification of Fano fivefolds of index two has been done in [19], where ruled Fano fivefolds of index two, i.e., fivefolds of index two with a $\boldsymbol{P}^1$-bundle structure over a smooth fourfold, were classified.

In this paper we classify Fano fivefolds of index two which are blow-ups of smooth manifolds along smooth centers. In Section 3 we recall the structure of the cones of curves of these manifolds, as described in [9], and we summarize the known results. Using previous results we are reduced to the following cases:

$\rho_X = 2$ and the two extremal rays of $\mathrm{NE}(X)$ correspond respectively to the blow-up of a smooth variety X' along a smooth surface S and to a fiber type contraction $\vartheta : X \to Y$.

$\rho_X = 3$. In this case $\mathrm{NE}(X)$ has three extremal rays: one of them is associated to the blow-up of a smooth variety along a smooth surface, one corresponds to a fiber type

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contraction, and the last one is associated either to another blow-up contraction or to another fiber type contraction.

The hardest case, which is the heart of the paper and is dealt with in Section 4, is when $\rho_X = 2$. In this case it is easy to show that the pseudoindex of X' is equal either to six or to four: if $i_{X'} = 6$ then $X' \simeq \boldsymbol{P}^5$ by results in [14], and the classification of S follows observing that S cannot have proper trisecants. In case $i_{X'} = 4$ we prove that also $r_{X'} = 4$, i.e., that X' is a del Pezzo manifold and that S is a del Pezzo surface. The classification of (X', S) then follows studying the possible conormal bundles $N^*_{S/X'}$.

In Section 5 we study the case $\rho_X = 3$; apart from one case, the target of the birational contraction is a Fano manifold, which is either a product with $\boldsymbol{P}^1$ as a factor or a $\boldsymbol{P}^3$-bundle over a surface; the classification of the center follows.

Our results are summarized in the following

THEOREM 1.1. *Let X be a Fano fivefold of index two which is the blow-up of a smooth variety X' along a smooth subvariety S. Then (X', S) is as in Table 1, where, in the last column, F denotes a fiber type extremal ray, D_i denotes a birational extremal ray whose associated contraction contracts a divisor to an i-dimensional variety and S denotes a ray whose associated contraction is small.*

In [4], Fano manifolds X obtained by blowing up a smooth variety Y along a center T of dimension $\dim T \leq i_X - 1$ were classified; the results in this paper show that the case $\dim T = i_X$ will be far more complicated.

2. Preliminaries.

2.1. Fano-Mori contractions and rational curves. Let X be a smooth Fano variety of dimension n and K_X its canonical divisor. By Mori's *Cone Theorem* the cone $\mathrm{NE}(X)$ of effective 1-cycles, which is contained in the $\boldsymbol{R}$-vector space $N_1(X)$ of 1-cyles modulo numerical equivalence, is polyhedral; a face τ of $\mathrm{NE}(X)$ is called an *extremal face* and an extremal face of dimension one is called an *extremal ray*. To every extremal face τ one can associate a morphism $\varphi : X \to Z$ with connected fibers onto a normal variety; the morphism φ contracts those curves whose numerical class lies in τ, and is usually called the *Fano-Mori contraction* (or the *extremal contraction*) associated to the face τ. A Cartier divisor D such that $D = \varphi^* A$ for an ample divisor A on Z is called a *supporting divisor* of the map φ (or of the face τ). An extremal ray R is called *numerically effective*, or of *fiber type*, if $\dim Z < \dim X$, otherwise the ray is *non nef* or *birational*. We usually denote with $E = E(\varphi) := \{x \in X \mid \dim \varphi^{-1}(\varphi(x)) > 0\}$ the *exceptional locus* of φ; if φ is of fiber type then of course $E = X$. If the exceptional locus of a birational ray R has codimension one, the ray and the associated contraction are called *divisorial*, otherwise they are called *small*.

DEFINITION 2.1. An elementary fiber type extremal contraction $\varphi : X \to Z$ is called a *scroll* (resp. a *quadric fibration*) if there exists a φ-ample line bundle $L \in \mathrm{Pic}(X)$ such that $K_X + (\dim X - \dim Z + 1)L$ (resp. $K_X + (\dim X - \dim Z)L$) is a supporting divisor of φ. An elementary fiber type extremal contraction $\varphi : X \to Z$ onto a smooth variety Y is

TABLE 1.

ρ_X	No.	X'	S	$NE(X)$
2	(a1)	$\boldsymbol{P}^5$	a point	$\langle F, D_0 \rangle$
	(b1)	$\boldsymbol{P}^5$	a linear $\boldsymbol{P}^2$	$\langle F, D_2 \rangle$
	(b2)	$\boldsymbol{P}^5$	the complete intersection of three quadrics	$\langle F, D_2 \rangle$
	(b3)	$\boldsymbol{P}^5$	$\boldsymbol{P}^1 \times \boldsymbol{P}^1$ embedded by $\mathcal{O}(1,2)$	$\langle F, D_2 \rangle$
	(b4)	$\boldsymbol{P}^5$	$\boldsymbol{F}_2$ embedded by $C_0 + 3f$	$\langle F, D_2 \rangle$
	(b5)	$\boldsymbol{P}^5$	the blow-up of $\boldsymbol{P}^2$ in four points $x_1, \ldots, x_4$ such that the line bundle $\mathcal{O}_{\boldsymbol{P}^2}(3) - \sum E_i$ is very ample	$\langle F, D_2 \rangle$
	(b6)	$\boldsymbol{P}^5$	the blow-up of $\boldsymbol{P}^2$ in seven points $x_0, \ldots, x_6$ such that the line bundle $\mathcal{O}_{\boldsymbol{P}^2}(4) - 2E_0 - \sum_{i=1}^{6} E_i$ is very ample	$\langle F, D_2 \rangle$
	(b7)	V_d (*)	the complete intersection of three general members of $\lvert\mathcal{O}_{V_d}(1)\rvert$	$\langle F, D_2 \rangle$
	(b8)	V_3	$\boldsymbol{P}^2$ with $(\mathcal{O}_{V_3}(1))_{\lvert\boldsymbol{P}^2} \simeq \mathcal{O}_{\boldsymbol{P}^2}(1)$	$\langle F, D_2 \rangle$
	(b9)	V_4	$\boldsymbol{P}^2$ with $(\mathcal{O}_{V_4}(1))_{\lvert\boldsymbol{P}^2} \simeq \mathcal{O}_{\boldsymbol{P}^2}(1)$	$\langle F, D_2 \rangle$
	(b10)	V_4	$\boldsymbol{Q}^2$ with $(\mathcal{O}_{V_4}(1))_{\lvert\boldsymbol{Q}} \simeq \mathcal{O}_{\boldsymbol{Q}}(1)$	$\langle F, D_2 \rangle$
	(b11)	V_5	a plane of bidegree $(1, 0)$ (**)	$\langle F, D_2 \rangle$
	(b12)	V_5	a quadric of bidegree $(1, 1)$	$\langle F, D_2 \rangle$
	(b13)	V_5	a surface $\boldsymbol{F}_1$ of bidegree $(2, 1)$ not contained in a $\boldsymbol{G}(1, 3)$	$\langle F, D_2 \rangle$
	(c1)	$\boldsymbol{P}^5$	a Veronese surface	$\langle D_2, D_2 \rangle$
	(c2)	$\boldsymbol{P}^5$	$\boldsymbol{F}_1$ embedded by $C_0 + 2f$	$\langle D_2, D_2 \rangle$
	(c3)	V_5	a plane of bidegree $(0, 1)$	$\langle D_2, D_2 \rangle$
	(d1)	$\boldsymbol{P}^5$	$\boldsymbol{Q}^2$ with $(\mathcal{O}_{\boldsymbol{P}}(1))_{\lvert\boldsymbol{Q}} \simeq \mathcal{O}_{\boldsymbol{Q}(1)}$	$\langle D_2, S \rangle$
3	(e1)	$\boldsymbol{P}^1 \times \boldsymbol{Q}^4$	$\boldsymbol{P}^1 \times l$ with l a line in $\boldsymbol{Q}^4$	$\langle F, F, D_2 \rangle$
	(e2)	$\boldsymbol{P}^1 \times \boldsymbol{Q}^4$	$\boldsymbol{P}^1 \times \Gamma$ with $\Gamma \subset \boldsymbol{Q}^4$ a conic not contained in a plane $\Pi \subset \boldsymbol{Q}^4$	$\langle F, F, D_2 \rangle$
	(e3)	$X' \in \lvert\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^4}(1,1)\rvert$	$\boldsymbol{P}^2$, a fiber of the projection $X' \to \boldsymbol{P}^4$	$\langle F, F, D_2 \rangle$
	(e4)	$X' \in \lvert\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^4}(1,1)\rvert$	$\boldsymbol{F}_1$, the complete intersection of X' and three general members of the linear system $\lvert\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^4}(0,1)\rvert$	$\langle F, F, D_2 \rangle$
	(f1)	$\boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)^{\oplus 3})$	$\boldsymbol{P}^2$, a section corresponding to the surjection $\mathcal{O} \oplus \mathcal{O}(1)^{\oplus 3} \to \mathcal{O}$	$\langle F, D_2, D_2 \rangle$
	(f2)	$\mathrm{Bl}_\pi(\boldsymbol{P}^5)$ (***)	$\boldsymbol{P}^2$, a non trivial fiber of $\mathrm{Bl}_\pi(\boldsymbol{P}^5) \to \boldsymbol{P}^5$	$\langle F, D_2, D_2 \rangle$
	(f3)	$\mathrm{Bl}_p(\boldsymbol{P}^5)$	$\boldsymbol{F}_1$, the strict transform of a plane in $\boldsymbol{P}^5$ through p	$\langle F, D_2, D_2 \rangle$
	(f4)	$\mathrm{Bl}_\pi(\boldsymbol{P}^5)$	$\boldsymbol{P}^2$, the strict transform of a plane in $\boldsymbol{P}^5$ not meeting π	$\langle F, D_2, D_2 \rangle$
4	(g1)	$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^3$	$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \{p\}$	$\langle F, F, F, D_2 \rangle$

(*) V_d denotes a del Pezzo fivefold of degree d.
(**) V_5 is a hyperplane section of $\boldsymbol{G}(1, 4)$. The bidegree of S is the bidegree of S in $\boldsymbol{G}(1, 4)$.
(***) $\mathrm{Bl}_\pi(\boldsymbol{P}^5)$ (resp. $\mathrm{Bl}_p(\boldsymbol{P}^5)$) denotes the blow-up of $\boldsymbol{P}^5$ along a 2-plane π (resp. along a point p).

called a $\boldsymbol{P}$*-bundle* if there exists a vector bundle $\mathcal{E}$ of rank $\dim X - \dim Z + 1$ on Z such that $X \simeq \boldsymbol{P}_Z(\mathcal{E})$; every equidimensional scroll is a $\boldsymbol{P}$-bundle by [10, Lemma 2.12].

DEFINITION 2.2. Let $\mathrm{Ratcurves}^n(X)$ be the normalized space of rational curves in X in the sense of [16]; a *family of rational curves* will be an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$. Given a rational curve $f : \boldsymbol{P}^1 \to X$ we call a *family of deformations* of f any irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ containing the equivalence class of f.

We define $\mathrm{Locus}(V)$ to be the subset of points in X which belong to a curve parametrized by V; we say that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$. Moreover, for every point $x \in \mathrm{Locus}(V)$, we will denote by V_x the subscheme of V parametrizing rational curves passing through x.

DEFINITION 2.3. Let V be a family of rational curves on X. We say that V is *unsplit* if it is proper and that V is *locally unsplit* if every component of V_x is proper for the general $x \in \mathrm{Locus}(V)$.

PROPOSITION 2.4 ([16, IV. 2.6]). *Let X be a smooth projective variety, V a family of rational curves and $x \in \mathrm{Locus}(V)$ such that every component of V_x is proper. Then*

(a) $\dim X - K_X \cdot V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1$;

(b) $-K_X \cdot V \leq \dim \mathrm{Locus}(V_x) + 1$.

In case V is the unsplit family of deformations of a minimal extremal rational curve, Proposition 2.4. gives the *fiber locus inequality*:

PROPOSITION 2.5 ([13, 30]). *Let φ be a Fano-Mori contraction of X and E its exceptional locus. Let F be an irreducible component of a (non trivial) fiber of φ. Then*

$$\dim E + \dim F \geq \dim X + l - 1$$

where $l = \min\{-K_X \cdot C \mid C$ is a rational curve in $F\}$. If φ is the contraction of a ray R, then l is called the length of the ray.

DEFINITION 2.6. We define a *Chow family of rational curves* $\mathcal{V}$ to be an irreducible component of $\mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles. If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$ is called the *Chow family associated to* V.

DEFINITION 2.7. Let X be a smooth variety, $\mathcal{V}^1, \ldots, \mathcal{V}^k$ Chow families of rational curves on X and Y a subset of X. We denote by $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ the set of points $x \in X$ that can be joined to Y by a connected chain of k cycles belonging *respectively* to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$. We denote by $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ the set of points $x \in X$ that can be joined to Y by a connected chain of at most m cycles belonging to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

DEFINITION 2.8. Let $V^1, \ldots, V^k$ be unsplit families on X. We will say that $V^1, \ldots, V^k$ are *numerically independent* if their numerical classes $[V^1], \ldots, [V^k]$ are linearly independent in the vector space $N_1(X)$. When moreover $C \subset X$ is a curve, we will say that

$V^1, \dots, V^k$ are numerically independent from C if the class of C in $N_1(X)$ is not contained in the vector subspace generated by $[V^1], \dots, [V^k]$.

LEMMA 2.9 ([1, Lemma 5.4]). *Let $Y \subset X$ be a closed subset and V an unsplit family. Assume that curves contained in Y are numerically independent from curves in V, and that $Y \cap \mathrm{Locus}(V) \neq \emptyset$. Then for a general $y \in Y \cap \mathrm{Locus}(V)$*

(a) $\dim \mathrm{Locus}(V)_Y \geq \dim(Y \cap \mathrm{Locus}(V)) + \dim \mathrm{Locus}(V_y)$;

(b) $\dim \mathrm{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1$.

Moreover, if $V^1, \dots, V^k$ are numerically independent unsplit families such that curves contained in Y are numerically independent from curves in $V^1, \dots, V^k$, then either $\mathrm{Locus}(V^1, \dots, V^k)_Y = \emptyset$ *or*

(c) $\dim \mathrm{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum(-K_X \cdot V^i) - k$.

DEFINITION 2.10. We define on X a relation of *rational connectedness with respect to* $\mathcal{V}^1, \dots, \mathcal{V}^k$ in the following way: x and y are in rc($\mathcal{V}^1, \dots, \mathcal{V}^k$)-relation if there exists a chain of rational curves in $\mathcal{V}^1, \dots, \mathcal{V}^k$ which joins x and y, i.e., if $y \in \mathrm{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_x$ for some m. If all the points of X are in rc($\mathcal{V}^1, \dots, \mathcal{V}^k$)-relation we say that X is *rc*($\mathcal{V}^1, \dots, \mathcal{V}^k$)-*connected*.

To the rc($\mathcal{V}^1, \dots, \mathcal{V}^k$)-relation we can associate a fibration, at least on an open subset of X (see [16, IV.4.16]); we will call it rc($\mathcal{V}^1, \dots, \mathcal{V}^k$)-fibration.

DEFINITION 2.11. Let $\mathcal{V}$ be the Chow family associated to a family of rational curves V. We say that V is *quasi-unsplit* if every component of any reducible cycle in $\mathcal{V}$ is numerically proportional to V.

NOTATION. Let T be a subset of X. We write $N_1^X(T) = \langle V^1, \dots, V^k \rangle$ if the numerical class in X of every curve $C \subset T$ can be written as $[C] = \sum_i a_i [C_i]$, with $a_i \in \boldsymbol{Q}$ and $C_i \in V^i$. We write $\mathrm{NE}^X(T) = \langle V^1, \dots, V^k \rangle$ (or $\mathrm{NE}^X(T) = \langle R_1, \dots, R_k \rangle$) if the numerical class in X of every curve $C \subset T$ can be written as $[C] = \sum_i a_i [C_i]$, with $a_i \in \boldsymbol{Q}_{\geq 0}$ and $C_i \in V^i$ (or $[C_i]$ in R_i).

PROPOSITION 2.12 ([1, Corollary 4.2], [9, Corollary 2.23]). *Let V be a family of rational curves and x a point in* $\mathrm{Locus}(V)$.

(a) *If V is quasi-unsplit, then* $\mathrm{NE}^X(\mathrm{ChLocus}_m(V)_x) = \langle V \rangle$ *for every $m \geq 1$;*

(b) *if V_x is unsplit, then* $\mathrm{NE}^X(\mathrm{Locus}(V_x)) = \langle V \rangle$.

Moreover, if τ is an extremal face of $\mathrm{NE}(X)$, *F is a fiber of the associated contraction and V is unsplit and independent from τ, then*

(c) $\mathrm{NE}^X(\mathrm{ChLocus}_m(V)_F) = \langle \tau, [V] \rangle$ *for every $m \geq 1$.*

2.2. Fano bundles.

DEFINITION 2.13. Let $\mathcal{E}$ be a vector bundle on a smooth complex projective variety Z. We say that $\mathcal{E}$ is a *Fano bundle* if $X = \boldsymbol{P}_Z(\mathcal{E})$ is a Fano manifold. By [27, Theorem 1.6] if $\mathcal{E}$ is a Fano bundle over Z then Z is a Fano manifold.

M. Szurek and J. Wiśniewski have classified Fano bundles over $\boldsymbol{P}^2$ ([26, 28]) and Fano bundles of rank two on surfaces [28]. What follows is a characterization of Fano bundles of rank $r \geq 2$ over del Pezzo surfaces, which generalizes some results in [28].

PROPOSITION 2.14. *Let S_k be a del Pezzo surface obtained by blowing up $k > 0$ points in $\boldsymbol{P}^2$, and let $\mathcal{E}$ be a Fano bundle of rank $r \geq 2$ over S_k; then, up to twist $\mathcal{E}$ with a suitable line bundle, the pair $(S_k, \mathcal{E})$ is one of the following*:

(i) $(S_k, \oplus\mathcal{O}^{\oplus r})$;

(ii) $(S_1, \theta^*(\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}^{\oplus(r-1)}))$;

(iii) $(S_1, \theta^*(T\boldsymbol{P}^2(-1) \oplus \mathcal{O}_{\boldsymbol{P}^2}^{\oplus(r-2)}))$,

where $\theta : S_1 \to \boldsymbol{P}^2$ is the blow-up of $\boldsymbol{P}^2$ at one point.

PROOF. Let $\mathcal{E}$ be a Fano bundle of rank $r \geq 2$ over S_k and let $X = \boldsymbol{P}_{S_k}(\mathcal{E})$; by [19, Proposition 3.4] there is a one-to-one correspondence between the extremal rays of $\mathrm{NE}(S_k)$ and the extremal rays of $\mathrm{NE}(X)$ spanning a two-dimensional face with the ray $R_{\mathcal{E}}$ corresponding to the projection $p : X \to S_k$. Let $R_{\theta_1} \subset \mathrm{NE}(S_k)$ be an extremal ray of S_k associated to a blow-up $\theta_1 : S_k \to S_{k-1}$, and call E_{θ_1} the exceptional divisor of θ_1; let R_{ϑ_1} be the corresponding ray in $\mathrm{NE}(X)$, with associated extremal contraction $\vartheta_1 : X \to X_1$. By [19, Lemma 3.5] ϑ_1 is birational and has one-dimensional fibers, hence by [3, Theorem 5.2] we have that X_1 is smooth and ϑ_1 is the blow-up of a smooth subvariety of codimension two in X_1; moreover, by [19, Lemma 3.5] and dimensional computations, $\mathrm{Exc}(R_{\vartheta_1}) = p^{-1}(E_{\theta_1})$. The divisor $E_{\vartheta_1} := \mathrm{Exc}(R_{\vartheta_1})$ has two projective bundle structures: a $\boldsymbol{P}^1$-bundle structure over the center of the blow-up and a $\boldsymbol{P}^{r-1}$-bundle structure over E_{θ_1}; by [24, Main theorem] we have that $E_{\vartheta_1} \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^{r-1}$. It follows that $\mathcal{E}|_{E_{\theta_1}} \simeq \mathcal{O}^{\oplus r}$, hence by [2, Lemma 2.9] there exists a vector bundle of rank r on S_{k-1} such that $\mathcal{E} = \theta_1^*\mathcal{E}_1$. It is now easy to prove that the induced map $\boldsymbol{P}_{S_k}(\theta_1^*\mathcal{E}_1) = X \to \boldsymbol{P}_{S_{k-1}}(\mathcal{E}_1)$ is nothing but ϑ_1, hence $X_1 = \boldsymbol{P}_{S_{k-1}}(\mathcal{E}_1)$. Since $\mathrm{NE}(E_{\vartheta_1}) = \langle R_{\mathcal{E}}, R_{\vartheta_1} \rangle$, the divisor E_{ϑ_1} cannot contain the exceptional locus of another extremal ray of X; it follows that X_1 is a Fano manifold by [30, Proposition 3.4].

We iterate the argument k times, until we find a Fano bundle $\mathcal{E}_k$ over $\boldsymbol{P}^2$ such that, denoted by θ and ϑ the composition of the contractions θ_i and ϑ_i respectively, $\mathcal{E} = \theta^*\mathcal{E}_k$. We have a commutative diagram

$$\begin{array}{ccc}
\boldsymbol{P}_{S_k}(\mathcal{E}) = X & \xrightarrow{\ \vartheta\ } & X_k = \boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{E}_k) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_k} \\
S_k & \xrightarrow[\ \theta\ ]{} & \boldsymbol{P}^2
\end{array}$$

Up to considering the tensor product of $\mathcal{E}_k$ with a suitable line bundle, we can assume that $0 \leq c_1(\mathcal{E}_k) \leq r - 1$; by [26, Proposition 2.2] we have that $\mathcal{E}_k$ is nef.

Let l be a line in $\boldsymbol{P}^2$; the restriction of $\mathcal{E}_k$ to l decomposes as a sum of nonnegative line bundles, hence we can write $(\mathcal{E}_k)|_l \simeq \bigoplus_{i=0}^{r-1} \mathcal{O}(a_i)$, with $a_0 = 0$ and $a_i \geq 0$. Let $\tilde{l}$ be the strict transform of l in S_k; since $\theta|_{\tilde{l}} : \tilde{l} \to l$ is an isomorphism we have $\mathcal{E}|_{\tilde{l}} \simeq (\mathcal{E}_k)|_l$; let $C_0 \subset X$ be a section of p over $\tilde{l}$ corresponding to a surjection $\mathcal{E}|_{\tilde{l}} \to \mathcal{O} \to 0$; we have

$$(1) \qquad 0 < -K_X \cdot C_0 = ra_0 - K_{S_k} \cdot \tilde{l} - \sum_{i=0}^{r-1} a_i = -K_{S_k} \cdot \tilde{l} - c_1(\mathcal{E}_k)\,.$$

Now if l passes through a point blown up by θ, by equation (1) we have $c_1(\mathcal{E}_k) \leq 1$. In this case, by the classification in [26], either $\mathcal{E}_k$ is trivial, or $\mathcal{E}_k \simeq \mathcal{O}(1) \oplus \mathcal{O}^{\oplus(r-1)}$, or $\mathcal{E}_k \simeq T\boldsymbol{P}^2(-1) \oplus \mathcal{O}^{\oplus(r-2)}$.

Assume that $k \geq 2$ and let l be a line in $\boldsymbol{P}^2$ joining two of the blown-up points; again by equation (1) we have $c_1(\mathcal{E}_k) = 0$, so only the first case occurs. □

PROPOSITION 2.15. *Let $\mathcal{E}$ be a Fano bundle of rank $r \geq 2$ over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$; then, up to twist $\mathcal{E}$ with a suitable line bundle, $\mathcal{E}$ is one of the following*:

(i) $\mathcal{O}^{\oplus r}$;
(ii) $\mathcal{O}(1,0) \oplus \mathcal{O}^{\oplus(r-1)}$;
(iii) $\mathcal{O}(1,1) \oplus \mathcal{O}^{\oplus(r-1)}$;
(iv) $\mathcal{O}^{\oplus(r-2)} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$;
(v) *a vector bundle fitting in the exact sequence*
$0 \to \mathcal{O}(-1,-1) \to \mathcal{O}^{\oplus(r+1)} \to \mathcal{E} \to 0.$

In all cases the cone of curves of $X = \boldsymbol{P}(\mathcal{E})$ is generated by the ray corresponding to the bundle projection and by two other extremal rays; *in case* (i) *the other rays are of fiber type*, *in case* (ii) *one of them is of fiber type and the other corresponds to a smooth blow-up*, *while in cases* (iii)–(v) *both the other rays correspond to smooth blow-ups.*

PROOF. We will show the result by induction on r, the case $r = 2$ having been established in [28, Main Theorem]. Let $X = \boldsymbol{P}(\mathcal{E})$; first of all we prove that $\mathrm{NE}(X)$ is generated by three extremal rays. Let $R_{\mathcal{E}} \subset \mathrm{NE}(X)$ be the extremal ray corresponding to the projection $p : X \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$; since $\rho_X = 3$ it is enough to prove that any other extremal ray of $\mathrm{NE}(X)$ lies in a two-dimensional face with $R_{\mathcal{E}}$.

Let R_ϑ be another extremal ray of X with associated contraction ϑ and let F be a non-trivial fiber of ϑ. We claim that $\dim F = 1$: in fact, since curves contained in F are not contracted by p, we have $\dim F \leq 2$, and, if $\dim F = 2$, we would have $X = p^{-1}(p(F))$ and $\mathrm{NE}(X) = \langle R, R_{\mathcal{E}} \rangle$ by Proposition 2.12 (c), against the fact that $\rho_X = 3$. In particular, by Proposition 2.5., ϑ cannot be a small contraction.

Let V_ϑ be a family of rational curves of minimal degree (with respect to some fixed ample line bundle) among the families which dominate the exceptional locus of R_ϑ and whose class is in R_ϑ. Such a family is quasi-unsplit by the extremality of R_ϑ and locally unsplit by the assumptions on its degree. We claim that V_ϑ is horizontal and dominating with respect to p. This is clear if the contraction ϑ associated to R_ϑ is of fiber type. Assume that ϑ is divisorial, with exceptional locus E: we cannot have $E \cdot R_{\mathcal{E}} = 0$, otherwise $E = p^*D$ for some effective divisor D in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$; but every effective divisor on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ is nef and so E would be nef, against the fact that $E \cdot R_\vartheta < 0$. It follows that $E \cdot R_{\mathcal{E}} > 0$, so E dominates $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and thus V_ϑ is horizontal and dominating with respect to p, and the claim is proved. We can now apply [9, Lemma 2.4] and conclude that $[V_\vartheta]$ and $R_\mathcal{E}$ lie in a two-dimensional extremal face of $\mathrm{NE}(X)$.

We have thus proved that every extremal ray different from $R_\mathcal{E}$ lies in a two-dimensional face with $R_\mathcal{E}$; therefore $\mathrm{NE}(X)$ is generated by three extremal rays. We will call R_{ϑ_1} and R_{ϑ_2} the two rays different from $R_\mathcal{E}$, i.e., $\mathrm{NE}(X) = \langle R_\mathcal{E}, R_{\vartheta_1}, R_{\vartheta_2} \rangle$.

By [19, Proposition 3.4], for every $i = 1, 2$ we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \vartheta_i\ } & Z_i \\ {\scriptstyle p}\downarrow & \searrow^{\psi_i} & \downarrow \\ \boldsymbol{P}^1 \times \boldsymbol{P}^1 & \xrightarrow[\ \theta_i\ ]{} & \boldsymbol{P}^1 \end{array}$$

where ψ_i is the contraction of the face of $\mathrm{NE}(X)$ spanned by $R_\mathcal{E}$ and R_{ϑ_i}.

Let $x \in \boldsymbol{P}^1$ and let f_x^i be the fiber of θ_i over x; since we can factor ψ_i as $\psi_i = \theta_i \circ p$, the fiber of ψ_i over x is $\boldsymbol{P}(\mathcal{E}|_{f_x^i})$. By the smoothness of ψ_i and adjunction, $\boldsymbol{P}(\mathcal{E}|_{f_x^i})$ is a Fano manifold, hence either $\mathcal{E}|_{f_x^i} \simeq \mathcal{O}(a)^{\oplus r}$ or $\mathcal{E}|_{f_x^i} \simeq \mathcal{O}(a+1) \oplus \mathcal{O}(a)^{\oplus(r-1)}$. Since the degree of $\mathcal{E}$ does not change as x varies in $\boldsymbol{P}^1$ we have that, for a fixed $i = 1, 2$, the splitting type of $\mathcal{E}$ along the fibers of θ_i is constantly $(a, \dots, a)$ or $(a+1, a, \dots, a)$. Up to twist $\mathcal{E}$ with a line bundle we can assume that its splitting type along the fibers of θ_i is constantly $(0, \dots, 0)$ or $(1, 0, \dots, 0)$

If for some $i = 1, 2$ the splitting type of $\mathcal{E}$ on the fibers of θ_i is $(0, \dots, 0)$ then $\mathcal{E} \simeq \theta_i^* \mathcal{E}'$, with $\mathcal{E}'$ a vector bundle on $\boldsymbol{P}^1$; hence $\mathcal{E}$ is decomposable and we are in case (i) or (ii).

Assume now that the splitting type of $\mathcal{E}$ on the fibers of θ_i is $(1, 0, \dots, 0)$ for $i = 1$ and $i = 2$, and thus $c_1(\mathcal{E}) = (1, 1)$. We claim that in this case the contractions $\vartheta_i : X \to Z_i$ are birational. Assume by contradiction that for some i, say $i = 1$, the contraction ϑ_1 is of fiber type. Let $x \in \boldsymbol{P}^1$ be a general point; the fiber of $Z_1 \to \boldsymbol{P}^1$ has dimension strictly smaller than the dimension of $\psi_1^{-1}(x)$. It follows that both the restrictions of ϑ_1 and p to $\psi_1^{-1}(x)$ are of fiber type, yet $\psi_1^{-1}(x) \simeq \mathrm{Bl}_{\boldsymbol{P}^{r-2}}(\boldsymbol{P}^r)$, so it has only one fiber type contraction.

We have already proved that the nontrivial fibers of the contractions ϑ_i are one dimensional, hence for every $i = 1, 2$ the variety Z_i is smooth and ϑ_i is the blow-up of a smooth subvariety of codimension two in Z_i by [3, Theorem 5.2]. Consider one of the birational contractions of X, say $\vartheta_1 : X \to Z_1$, and let E_1 be its exceptional locus. For every fiber f_x of θ_1 the restriction of E_1 to $\boldsymbol{P}_{f_x}(\mathcal{E}|_{f_x})$ is a non nef divisor, hence it is the exceptional divisor of the contraction $\boldsymbol{P}_{f_x}(\mathcal{E}|_{f_x}) \to \boldsymbol{P}^r$. In particular $E_1 \cdot R_\mathcal{E} = 1$ and E_1 does not contain any fiber of p. By [10, Lemma 2.12] the restriction of p makes E_1 a projective bundle over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, that is $E_1 = \boldsymbol{P}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(\mathcal{E}')$ with $\mathcal{E}'$ a rank $r-1$ vector bundle over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. We will now split the proof in two cases, depending on the sign of the intersection number of E_1 with R_{ϑ_2}.

Case 1. $E_1 \cdot R_{\vartheta_2} \le 0$.

In this case the line bundle $-K_X - E_1$ is ample on X; therefore its restriction to E_1 is ample, E_1 is a Fano manifold and $\mathcal{E}'$ is a Fano bundle of rank $r-1$ over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Note also that E_1 has a fiber type contraction different from the bundle projection onto $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, coming from the blow-up contraction ϑ_1, so, by induction, either $\mathcal{E}'$ is trivial or $\mathcal{E}' \simeq \mathcal{O}(1,0) \oplus \mathcal{O}^{\oplus(r-2)}$. The injection $\boldsymbol{P}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(\mathcal{E}') \hookrightarrow \boldsymbol{P}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(\mathcal{E})$ gives an exact sequence of bundles on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$

$$0 \to \mathcal{O}(a,b) \to \mathcal{E} \to \mathcal{E}' \to 0\,,$$

with $E_1 = \xi_{\mathcal{E}} + p^*\mathcal{O}(-a,-b)$. Computing the intersection numbers of E_1 with R_{ϑ_1} and R_{ϑ_2} and recalling the splitting type of $\mathcal{E}$ we have the following possibilities:

$$0 \to \mathcal{O}(0,1) \to \mathcal{E} \to \mathcal{O}^{\oplus(r-2)} \oplus \mathcal{O}(1,0) \to 0\,;$$

$$0 \to \mathcal{O}(1,1) \to \mathcal{E} \to \mathcal{O}^{\oplus(r-1)} \to 0\,.$$

Both these sequences split, so we are in cases (iii) or (iv).

Case 2. $E_1 \cdot R_{\vartheta_2} > 0$.

By [30, Proposition 3.4] Z_1 is a Fano manifold. Z_1 has a fiber type elementary contraction onto $\boldsymbol{P}^1$. For a general $x \in \boldsymbol{P}^1$ the fiber $\psi_1^{-1}(x) = \boldsymbol{P}(\mathcal{E}|_{f_x^i})$ is isomorphic to $\mathrm{Bl}_{\boldsymbol{P}^{r-2}}(\boldsymbol{P}^r)$, hence the fiber of $Z_1 \to \boldsymbol{P}^1$ over x is isomorphic to $\boldsymbol{P}^r$. It follows that Z_1 has a projective bundle structure over $\boldsymbol{P}^1$ (cfr. [19, Lemma 2.17]), so either $Z_1 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^r$ or $Z_1 \simeq \mathrm{Bl}_{\boldsymbol{P}^{r-1}}(\boldsymbol{P}^{r+1})$.

The second case cannot happen: in fact, let $\psi : X \to \boldsymbol{P}^{r+1}$ be the contraction of the face spanned by R_{ϑ_1} and R_{ϑ_2}. Denoting by E the exceptional divisor of the contraction $Z_1 \to \boldsymbol{P}^r$, by $\tilde{E}$ its strict transform in X, and applying twice the canonical bundle formula for blow-ups we have

$$K_X = \vartheta_1^* K_{Z_1} + E_1 = \psi^* K_{\boldsymbol{P}^{r+1}} + \vartheta_1^* E + E_1 = \psi^* K_{\boldsymbol{P}^{r+1}} + \tilde{E} + kE_1\,.$$

Since $K_X \cdot R_{\vartheta_2} = -1$ and $\psi^* K_{\boldsymbol{P}^{r+1}} \cdot R_{\vartheta_2} = 0$ we have $\tilde{E} \cdot R_{\vartheta_2} < 0$. This implies that $\tilde{E} = E_2$, and thus $\tilde{E} \cdot R_{\vartheta_2} = -1$, yielding $E_1 \cdot R_{\vartheta_2} = 0$, a contradiction.

Note that the minimal extremal curves contracted by ϑ_i are the minimal sections (those corresponding to the trivial summands) of $p : \boldsymbol{P}(\mathcal{E}|_{f_x^i}) \to \boldsymbol{P}^1$ along the fibers of θ_i; therefore $\xi_{\mathcal{E}} \cdot R_{\vartheta_i} = 0$ for $i = 1, 2$. Being trivial on the face spanned by R_{ϑ_1} and R_{ϑ_2} and positive on $R_{\mathcal{E}}$ the line bundle $\xi_{\mathcal{E}}$ is nef. Let ψ be the contraction of the face spanned by R_{ϑ_1} and R_{ϑ_2}; this contraction factors through $Z_1 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^r$ and therefore is onto $\boldsymbol{P}^r$, since it does not contract curves in $R_{\mathcal{E}}$. The line bundle $\xi_{\mathcal{E}}$ restricts to $\mathcal{O}(1)$ on the fibers of p, hence $\xi_{\mathcal{E}} = \psi^*\mathcal{O}_{\boldsymbol{P}^r}(1)$. Therefore $\xi_{\mathcal{E}}$ (and so $\mathcal{E}$) is spanned and we have an exact sequence on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$:

$$0 \to \mathcal{O}(a,b) \to \mathcal{O}^{\oplus(r+1)} \to \mathcal{E} \to 0\,,$$

Computing the first Chern class we have $a = -1, b = -1$ and we are in case (v). In this case $X = \boldsymbol{P}(\mathcal{E})$ is a divisor in the linear system $\mathcal{O}(1,1,1)$ in $\boldsymbol{P}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(\mathcal{O}^{\oplus(r+1)}) \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^r$.

□

2.3. Surfaces in $\boldsymbol{G}(1,4)$. Let $\boldsymbol{G}(r,n)$ be the Grassmann variety of projective r-spaces in $\boldsymbol{P}^n$, embedded in $\boldsymbol{P}^N$ via the Plücker embedding. We will denote a point in $\boldsymbol{G}(r,n)$ by a capital letter, and the corresponding linear space in $\boldsymbol{P}^n$ by the same small letter.

Consider the Schubert cycles $\Omega_1 := \Omega(0,1,\ldots,r-1,r+2)$ and $\Omega_2 := \Omega(0,1,\ldots,r-2,r,r+1)$; the cohomology class of a surface $S \subset \boldsymbol{G}(r,n)$ can be written as $\alpha\Omega_1 + \beta\Omega_2$. Recalling that the class of an hyperplane section of $\boldsymbol{G}(r,n)$ is the class of the Schubert cycle $\Omega_H := \Omega(n-r-1, n-r, \ldots, n-2, n)$, we obtain that the degree of S as a subvariety of $\boldsymbol{P}^N$ is given by

$$\deg(S) = \alpha\Omega_1\Omega_H^2 + \beta\Omega_2\Omega_H^2 = \alpha + \beta\,.$$

The integer α is the number of linear spaces parametrized by S which meet a general $(n-r-2)$-space in $\boldsymbol{P}^n$, as one can see intersecting with the Schubert cycle $\Omega(n-r-2, n-r+1, n-r+2, \ldots, n)$; it is called the *order* of S and denoted by $\mathrm{ord}(S)$. The integer β is the number of linear spaces parametrized by S which meet a general $n-r$ space in a line, as one can see intersecting with the Schubert cycle $\Omega(n-r-1, n-r, n-r+2, \ldots, n)$; it is called the *class* of S and denoted by $\mathrm{cl}(S)$.

DEFINITION 2.16. The *bidegree* of S is the pair $(\mathrm{ord}(S), \mathrm{cl}(S))$. By the discussion above we have that $\deg S = \mathrm{ord}(S) + \mathrm{cl}(S)$.

REMARK 2.17. A 2-plane Λ^2_π in $\boldsymbol{G}(1,4)$ which parametrizes the family of lines which are contained in a given 2-plane $\pi \subset \boldsymbol{P}^4$, classically called a *$\rho$-plane*, has bidegree $(0,1)$. Moreover, given a point $L \in \boldsymbol{G}(1,4)$ there exists a line in $\boldsymbol{G}(1,4)$ joining Λ^2_π and L if and only if the corresponding line $l \subset \boldsymbol{P}^4$ has nonempty intersection with π.

REMARK 2.18. The family of lines through a given point p in $\boldsymbol{P}^4$ is parametrized by a three-dimensional linear space $\Lambda^3_p \subset \boldsymbol{G}(1,4)$, classically called a Σ-solid. A two-dimensional linear subspace of a Σ-solid, classically called a *σ-plane*, parametrizes the family of lines through a given point in $\boldsymbol{P}^4$ which lie in a given hyperplane H, and has bidegree $(1,0)$; we will denote it by $\Lambda^2_{p,H}$. Given a σ-plane $\Lambda^2_{p,H}$ and a point $L \in \boldsymbol{G}(1,4)$ there exists always a line in $\boldsymbol{G}(1,4)$ joining $\Lambda^2_{p,H}$ and L. This is clear if L is contained in the Σ-solid Λ^3_p; otherwise, let π be the plane $\subset \boldsymbol{P}^4$ spanned by l and p and let q be $l \cap H$ if $l \not\subset H$ or any point of l if $l \subset H$: the pencil of lines in π with center q is represented by a line in $\boldsymbol{G}(1,4)$ passing through L and meeting $\Lambda^2_{p,H}$.

EXAMPLE 2.19. If Λ^2_π is a 2-plane of bidegree $(0,1)$ (a ρ-plane) then the blow-up of $\boldsymbol{G}(1,4)$ along Λ^2_π is a Fano manifold whose other contraction is the blow-up of $\boldsymbol{P}^6$ along a cubic threefold contained in a hyperplane (see [25, Theorem XLI]). If else $\Lambda^2_{p,H}$ is a 2-plane of bidegree $(1,0)$ (a σ-plane) the linear system $|\mathcal{O}_{\boldsymbol{G}}(1) \otimes \mathcal{I}_{\Lambda^2_{p,H}}|$ defines a rational map $\boldsymbol{G} \dashrightarrow \boldsymbol{P}^6$ whose image is a quadric cone in $\boldsymbol{P}^6$ with zero-dimensional vertex; the blow-up of $\boldsymbol{G}(1,4)$ along $\Lambda^2_{p,H}$ is a Fano manifold whose other contraction is of fiber type onto this quadric cone. This can be checked by direct computation.

LEMMA 2.20. *Let S be a surface in* $\boldsymbol{G}(1,4)$. *If* $\mathrm{ord}(S) = 0$, *then S is a plane of bidegree* $(0,1)$, *while if* $\mathrm{cl}(S) = 0$, *then S is contained in a Σ-solid.*

PROOF. Let $I \subset \boldsymbol{G}(1,4) \times \boldsymbol{P}^4$ be the incidence variety. Denote by $p_1 : I \to \boldsymbol{G}(1,4)$ and $p_2 : I \to \boldsymbol{P}^4$ the projections and let $\mathrm{Locus}(S) = p_2(p_1^{-1}(S))$. If $\mathrm{ord}(S) = 0$, then the general line of $\boldsymbol{P}^4$ does not meet $\mathrm{Locus}(S)$; therefore $\mathrm{Locus}(S)$ is two-dimensional. Moreover, since $p_1^{-1}(S)$ is irreducible, also $\mathrm{Locus}(S)$ is irreducible. Therefore $\mathrm{Locus}(S)$ is an irreducible surface in $\boldsymbol{P}^4$ which contains a two-parameter family of lines. It is easy to prove that $\mathrm{Locus}(S)$ is a plane, hence S is the ρ-plane which parametrizes the lines of $\mathrm{Locus}(S)$.

Assume now that $\mathrm{cl}(S) = 0$. Since we can identify $\boldsymbol{G}(1,4)$ with the Grassmannian $\boldsymbol{G}(2,4)$ of planes in the dual space $\boldsymbol{P}^{4*}$, S can be viewed as a surface which parametrizes a two-dimensional family of planes in $\boldsymbol{P}^{4*}$. The duality exchanges order and class, so S, as a subvariety of $\boldsymbol{G}(2,4)$, has order zero, i.e., through a general point of $\boldsymbol{P}^{4*}$ there are no planes parametrized by S. Denote by $I^* \subset \boldsymbol{G}(2,4) \times \boldsymbol{P}^{4*}$ the incidence variety, by $p_1^* : I^* \to \boldsymbol{G}(2,4)$ and $p_2^* : I^* \to \boldsymbol{P}^{4*}$ the projections and define $\mathrm{Locus}^*(S) = p_2^*(p_1^{*-1}(S))$. Then $\dim \mathrm{Locus}^*(S) \leq 3$. Therefore $\mathrm{Locus}^*(S) \subset \boldsymbol{P}^{4*}$ is an irreducible threefold which contains a two-parameter family of planes. It is easy to prove that in this case $\mathrm{Locus}^*(S)$ is a hyperplane of $\boldsymbol{P}^{4*}$. It follows that S parametrizes a family of planes in $\boldsymbol{P}^{4*}$ contained in a hyperplane, and hence, by duality, S parametrizes a two-dimensional family of lines passing through a point of $\boldsymbol{P}^4$, and it is therefore contained in a Σ-solid. □

LEMMA 2.21. *Let S be a surface in* $\boldsymbol{G}(1,3) \subset \boldsymbol{P}^5$. *If* $\mathrm{ord}(S) \geq 2$ *or* $\mathrm{cl}(S) \geq 2$, *then there exist proper secant lines of S which are contained in* $\boldsymbol{G}(1,3)$.

PROOF. Let $p \in \boldsymbol{P}^3$ be a general point. The order of S is the number of lines parametrized by S which pass through p. Hence, if $\mathrm{ord}(S) \geq 2$, there exist at least two lines l_1, l_2 parametrized by S containing p. The pencil of lines generated by l_1 and l_2 corresponds to a line in $\boldsymbol{G}(1,3)$ joining the points $L_1, L_2 \in S$. Since p is general, the general member of the pencil is not a line parametrized by S, and hence the corresponding secant is not contained in S.

Let $\pi \subset \boldsymbol{P}^3$ be a general plane; the class of S is the number of lines parametrized by S contained in π. So if $\mathrm{cl}(S) \geq 2$ there exist $l_1, l_2 \subset \pi$, and the pencil of lines generated by l_1 and l_2 corresponds to a line in $\boldsymbol{G}(1,3)$ joining the points L_1 and L_2. Since π is general, the general member of the pencil is not a line parametrized by S, and hence the corresponding secant is not contained in S. □

COROLLARY 2.22. *If* $S \subset \boldsymbol{G}(1,3)$ *and* $\deg S \geq 3$ *then there exist proper secant lines of S which are contained in* $\boldsymbol{G}(1,3)$.

PROPOSITION 2.23. *Let* $\mathcal{Q} \subset \boldsymbol{G}(1,4) \subset \boldsymbol{P}^9$ *be a two-dimensional smooth quadric such that no proper secant of $\mathcal{Q}$ is contained in* $\boldsymbol{G}(1,4)$; *then $\mathcal{Q}$ is contained in a* $\boldsymbol{G}(1,3)$ *and has bidegree* $(1,1)$. *In particular, $\mathcal{Q}$ parametrizes the family of lines which lie in a hyperplane* $H \subset \boldsymbol{P}^4$ *and meet two skew lines* $r, s \subset H$.

Proof. We have $2 = \deg(\mathcal{Q}) = \mathrm{ord}(\mathcal{Q}) + \mathrm{cl}(\mathcal{Q})$; by Lemma 2.20 we cannot have $\mathrm{ord}(S) = 0$. If $\mathrm{ord}(S) = 2$ then $\mathrm{cl}(S) = 0$ and the same Lemma yields that $\mathcal{Q}$ is contained in a Σ-solid, and in this case all the lines in the Σ-solid meet $\mathcal{Q}$ and are contained in $\boldsymbol{G}(1,4)$. Therefore $\mathrm{ord}(\mathcal{Q}) = 1$ and the statement follows by [22, Main Theorem]. □

Proposition 2.24. *Let $S \subset \boldsymbol{G}(1,4)$ be a surface of degree three such that no proper secant of S is contained in $\boldsymbol{G}(1,4)$; then the bidegree of S is $(2,1)$ and S is not contained in any $\boldsymbol{G}(1,3)$.*

Proof. We have $3 = \deg(S) = \mathrm{ord}(S) + \mathrm{cl}(S)$; we cannot have $\mathrm{ord}(S) = 0$ by Lemma 2.20. By the same lemma, if $\mathrm{ord}(S) = 3$ then S is contained in a Σ-solid, and in this case all the lines in the Σ-solid are secant to S and lie in $\boldsymbol{G}(1,4)$. If $S \subset \boldsymbol{G}(1,3)$ then S has proper secants contained in $\boldsymbol{G}(1,3)$ by Lemma 2.21. Moreover if $\mathrm{ord}(S) = 1$ then $S \subset \boldsymbol{G}(1,3)$ by [22, Main Theorem]. □

Proposition 2.25. *Let $\mathcal{S} \subset \boldsymbol{G}(1,4)$ be a surface of bidegree $(2,1)$ not contained in a subgrassmannian $\boldsymbol{G}(1,3)$. Then $\mathcal{S}$ parametrizes lines which are contained in a family F_1 of planes of a quadric cone $\mathcal{C} \subset \boldsymbol{P}^4$ with zero-dimensional vertex and meet a given line m which lies in a plane $\pi_m \in F_2$, where F_2 is the other family of planes of $\mathcal{C}$.*

Proof. Identifying $\boldsymbol{G}(1,4)$ with the Grassmannian $\boldsymbol{G}(2,4)$ of planes in the dual space $\boldsymbol{P}^{4^*}$, $\mathcal{S}$ can be viewed as a surface which parametrizes a two-dimensional family of planes in $\boldsymbol{P}^{4^*}$. The duality exchanges order and class, so $\mathcal{S}$, as a subvariety of $\boldsymbol{G}(2,4)$, has bidegree $(1,2)$. We apply [22, Main Theorem] and we have the following description of $\mathcal{S}$:

Let $\beta : \mathrm{Bl}_{M^*}(\boldsymbol{P}^{4^*}) \to \boldsymbol{P}^{4^*}$ be the blow-up of $\boldsymbol{P}^{4^*}$ along a plane $M^* \subset \boldsymbol{P}^{4^*}$. We can write $\mathrm{Bl}_{M^*}(\boldsymbol{P}^{4^*}) = \boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{E})$, where $\mathcal{E} := \mathcal{O}^3_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$; denote by p the projection $\mathrm{Bl}_{M^*}(\boldsymbol{P}^{4^*}) \to \boldsymbol{P}^1$. Let $\mathcal{F}$ be a quotient of $\mathcal{E}$ with $\mathrm{rk}(\mathcal{F}) = \deg \mathcal{F} = 2$ and denote by $p_0 := p|_{\boldsymbol{P}(\mathcal{F})}$.

$$\begin{array}{ccc} & \mathrm{Bl}_{M^*}(\boldsymbol{P}^{4^*}) & \\ {}^{\beta}\swarrow & & \searrow^{p} \\ \boldsymbol{P}^{4^*} & & \boldsymbol{P}^1 \end{array}$$

Then

$$\mathcal{S} = \mathcal{S}(M^*, \mathcal{F}) := \{\pi \in \boldsymbol{G}(2,4) \mid \beta(p_0^{-1}(x)) \subset \pi \subset \beta(p^{-1}(x)) \text{ for some } x \in \boldsymbol{P}^1\}.$$

Since $\mathcal{E}$ is nef also $\mathcal{F}$ is, so $\mathcal{F} = \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(b)$ with $a, b \geq 0$ and $a + b = 2$. Therefore two cases can occur:

(i) $a = 1$, $b = 1$, i.e., $\boldsymbol{P}(\mathcal{F}) \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$. In this case the tautological bundle $\xi_{\mathcal{E}}$ restricts to $\mathcal{F}$ as $\mathcal{O}(1,1)$, so the image $\beta(\boldsymbol{P}(\mathcal{F})) \subset \boldsymbol{P}^{4^*}$ is a smooth quadric $\mathcal{Q}$. The plane M^* contains a line in one ruling of the quadric, and $\mathcal{S}(M^*, \mathcal{F})$ parametrizes planes in $\boldsymbol{P}^{4^*}$ which intersect M^* along this line and contain a line belonging to the other ruling of $\mathcal{Q}$. Passing to the dual we have the claimed description of $\mathcal{S}$, where m is the dual line to the plane M^*.

(ii) $a = 0$, $b = 2$, i.e., $\boldsymbol{P}(\mathcal{F}) \simeq \boldsymbol{F}_2$. In this case the tautological bundle $\xi_{\mathcal{E}}$ restricts to $\mathcal{F}$ as $C_0 + 2f$, so the image $\beta(\boldsymbol{P}(\mathcal{F})) \subset \boldsymbol{P}^{4^*}$ is a quadric cone whose vertex is a point $h^* \in M^*$, therefore all the planes parametrized by $\mathcal{S}$ pass through h^*. It follows that all the lines parametrized by $\mathcal{S} \subset \boldsymbol{G}(1,4)$ are contained in the hyperplane H, dual to h^*; in particular, $\mathcal{S}$ is contained in $\boldsymbol{G}_H(1,3)$. This contradicts our hypothesis and thus exclude this case. □

3. Getting started.

REMARK 3.1. Let X be a Fano fivefold with Picard number $\rho_X \geq 2$ and index $r_X = 2$; then X has pseudoindex two. In fact, by [1], the generalized Mukai conjecture

$$\rho_X(i_X - 1) \leq \dim X$$

holds for a Fano fivefold, hence we have that i_X cannot be a multiple of $r_X = 2$.

LEMMA 3.2. *Let X be a Fano fivefold of index two and $\sigma : X \to X'$ a birational extremal contraction of X which contracts a divisor to a surface. Then σ is a smooth blow-up.*

PROOF. Let R_σ be the extremal ray in $\mathrm{NE}(X)$ corresponding to σ. From the fiber locus inequality we have $l(R_\sigma) = 2$, since the general fiber of σ is two-dimensional. Let A' be a very ample line bundle on X'; the line bundle $A = H \otimes \sigma^* A'$ is relatively ample and $K_X + 2A = 2\sigma^* A'$ is a supporting divisor for σ. We can thus apply [5, Corollary 5.8.1] to get that σ is equidimensional and the statement then follows from [3, Theorem 5.2]. □

PROPOSITION 3.3. *Let X be a Fano fivefold of index two which is the blow-up of a smooth variety X' along a smooth center T; then the cone of curves of X is one among those listed in the following table, where F denotes a fiber type extremal ray, D_i denotes a birational extremal ray whose associated contraction contracts a divisor to an i-dimensional variety and S denotes a ray whose associated contraction is small*:

ρ_X	R_1	R_2	R_3	R_4	
2	F	D_0			(a)
	F	D_2			(b)
	D_2	D_2			(c)
	D_2	S			(d)
3	F	F	D_2		(e)
	F	D_2	D_2		(f)
4	F	F	F	D_2	(g)

PROOF. The result will follow from the list in [9, Theorem 1.1], once we have proved that X has no contractions of type D_1. Let $\sigma : X \to X'$ be the blow-up of X' along T, let E be the exceptional divisor and let l be a line in a fiber of σ. Let H be the fundamental divisor of X; from the canonical bundle formula

$$-2H = K_X = \sigma^* K_{X'} + (\operatorname{codim} T - 1)E$$

we know that $-2H \cdot l = (\operatorname{codim} T - 1)E \cdot l$, so the codimension of T is odd. It follows that either T is a surface or T is a point. □

In this paper we will deal with cases (b), (e) and (f), since the other cases have already been classified; in particular:

- in case (a) $X' \simeq \boldsymbol{P}^5$ by [8, Théorème 1].
- As noted in the introduction of [9], for a Fano fivefold of pseudoindex 2 possessing a quasi-unsplit locally unsplit dominating family of rational curves is equivalent to have a fiber type elementary contraction, so, in cases (c) and (d), we can apply [9, Theorem 1.2] and see that either $X' \simeq \boldsymbol{P}^5$ and T is

(c1) a Veronese surface,

(c2) $\boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{O}(1) \oplus \mathcal{O}(2))$ embedded in a hyperplane of $\boldsymbol{P}^5$ by the tautological bundle (a cubic scroll),

(d1) a two-dimensional smooth quadric (a section of $\mathcal{O}(2)$ in a linear $\boldsymbol{P}^3 \subset \boldsymbol{P}^5$),

or X' is a del Pezzo manifold of degree five and T is a plane of bidegree (0, 1). This corresponds to case (c3) which arises as the other extremal contraction of case (c2); for a detailed description see [9, Section 3, Example e1].

- In case (g) $X' \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^3$ and $T \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \{p\}$ by [19, Corollary 5.3].

4. Case (b). 4.1. Classification of X'. We will now prove that if X is as in case (b) then X' is either the projective space of dimension five or a del Pezzo manifold of degree ≤ 5.

Assume throughout the section that X is a Fano fivefold of index two with $-K_X = 2H$ and Mori cone $\mathrm{NE}(X) = \langle R_\vartheta, R_\sigma \rangle$, where $\vartheta : X \to Y$ is a fiber type contraction and $\sigma : X \to X'$ is a blow-down with center a smooth surface $S \subset X'$ and exceptional divisor E. By [7, Theorem 1] we know that X' is a smooth Fano variety with $\rho_{X'} = 1$ and $i_{X'} \geq 2$; moreover by the canonical bundle formula

$$K_X = \sigma^* K_{X'} + 2E$$

we have that $r_{X'}$ is even.

LEMMA 4.1. *Let V' be a minimal dominating family for X', V a family of deformations of the strict transform of a general curve in V' and $\mathcal{V}$ the Chow family associated to V. Then $E \cdot V = 0$, the family $\mathcal{V}$ is not quasi-unsplit and $-K_{X'} \cdot V' = 4$ or 6.*

PROOF. By [16, II.3.7], the general curve in V' does not intersect S, so $E \cdot V = 0$. It follows that

$$-K_X \cdot V = -K_{X'} \cdot V' \leq \dim X' + 1 = 6\,. \tag{2}$$

The family V is dominating and it is not extremal, otherwise E would be non positive on the whole cone of X. This implies by [9, Lemma 2.4] that X is rc$\mathcal{V}$-connected; in particular, since

$\rho_X = 2$, the family $\mathcal{V}$ is not quasi-unsplit. Therefore $-K_{X'} \cdot V' = -K_X \cdot V \geq 4$ so, recalling that $r_{X'}$ is even, the lemma is proved. □

If the anticanonical degree of the minimal dominating family V' is equal to $6 = \dim X' + 1$ then $X' \simeq \boldsymbol{P}^5$ by [14, Theorem 1.1] (Note that the assumptions of the quoted result are different, but the proof actually works in our case since for a very general $x' \in X'$ the pointed family $(V')_{x'}$ has the properties 1–3 in [14, Theorem 2.1]).

We are thus left with the case $-K_{X'} \cdot V' = 4$, which requires some more work. First of all we will analyze the families of rational curves on X; as a consequence we will prove that the exceptional divisor E of the blow-up is a Fano manifold and that the fiber type extremal contraction of X restricts to an extremal contraction of E with the same target Y. Using the classification of Fano bundles over a surface, given in [26] and [28] and completed in Section 2.2 of the present paper, we will find a line bundle on Y whose pullback to X has degree one on the fibers of the blow-up, and this implies the existence of a line bundle on X' which has degree one on the rational curves of minimal degree in X'. In this way we will be able to show that X' is a del Pezzo manifold.

LEMMA 4.2. *Let D be an effective divisor of X; then D contains curves whose numerical class is in R_σ.*

PROOF. We can assume that $D \neq E$, otherwise the statement is trivial. The image of D via σ is an effective divisor in X', hence it is ample since $\rho_{X'} = 1$; therefore $\sigma(D) \cap S \neq \emptyset$ and so $D \cap E \neq \emptyset$. Let x be a point in $D \cap E$ and let F_x be the fiber of σ through x; since $\dim F_x = 2$ then $D \cap F_x$ contains a curve in F_x. □

LEMMA 4.3. *Let W be an unsplit family of rational curves on X such that* $\mathrm{Locus}(W) \subseteq E$; *then* $[W] \in R_\sigma$.

PROOF. Let F be a fiber of σ such that $F \cap \mathrm{Locus}(W) \neq \emptyset$; we have $\mathrm{Locus}(W)_F \subseteq \mathrm{Locus}(W) \subseteq E$. Assume that $[W] \notin R_\sigma$; we can apply Lemma 2.9 to get $\dim \mathrm{Locus}(W)_F = 4$, so in this case $E = \mathrm{Locus}(W)_F = \mathrm{Locus}(W)$ and $\mathrm{NE}^X(E) = \langle [W], R_\sigma \rangle$ by Proposition 2.12 (c). It follows that E contains two independent unsplit dominating families, and it is easy to prove that their degree with respect to $-K_E$ is equal to three; we can therefore apply [20, Theorem 1] and obtain that $E \simeq \boldsymbol{P}^2 \times \boldsymbol{P}^2$. The effective divisor E, being negative on R_σ, must be positive on R_ϑ, so E dominates Y; since $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ is a toric variety, by [21, Theorem 1] we have that $Y \simeq \boldsymbol{P}^4$. Moreover $\vartheta : X \to \boldsymbol{P}^4$ is a $\boldsymbol{P}^1$-bundle by [19, Corollary 2.15]; by [19, Theorem 1.2] it must be $X \simeq \boldsymbol{P}_{\boldsymbol{P}^4}(\mathcal{O} \oplus \mathcal{O}(a))$ with $a = 1$ or $a = 3$, and in these cases X is not a blow-up along a surface, a contradiction. □

LEMMA 4.4. *There does not exist on X any unsplit family of rational curves W which satisfies all the following conditions*:

(i) $-K_X \cdot W = 2$;

(ii) $[W]$ *is not extremal in* $\mathrm{NE}(X)$;

(iii) $D_W := \mathrm{Locus}(W)$ *has dimension* 4;

(iv) $\mathrm{NE}^X(D_W) \subset \langle R_\sigma, [W]\rangle$.

PROOF. Assume by contradiction that such a family exists. In this case we have $D_W \cdot R_\sigma \geq 0$ (otherwise we would have $D_W = E$ and $[W] \in R_\sigma$ by Lemma 4.3, against assumption (ii)) and $D_W \cdot R_\vartheta > 0$ (otherwise D_W would contain curves in R_ϑ, against assumption (iv)); this implies that D_W is nef, and that it possibly vanishes only on R_σ. By [19, Corollary 2.15] the contraction $\vartheta : X \to Y$ is a $\boldsymbol{P}^1$-bundle, i.e., $X = \boldsymbol{P}_Y(\mathcal{E} = \vartheta_* H)$; by the classification in [19, Theorem 1.3] (note that we are in case $\rho_X = 2$) this is possible only if Y is a Fano manifold of index one and pseudoindex two or three; in fact in none of the other cases of [19, Theorem 1.3] X is the blow-up of a smooth variety along a (smooth) surface.

Let V_Y be a family of rational curves on Y with $-K_Y \cdot V_Y = i_Y$ and let $\nu : \boldsymbol{P}^1 \to Y$ be the normalization of a curve in V_Y; the pull-back $\nu^*\mathcal{E}$ splits as $\mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$ in case $i_Y = 2$, and as $\mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2)$ in case $i_Y = 3$. We have a commutative diagram

$$\begin{array}{ccc} S := \boldsymbol{P}(\nu^*\mathcal{E}) & \xrightarrow{\bar{\nu}} & X \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \vartheta} \\ \boldsymbol{P}^1 & \xrightarrow[\nu]{} & Y \end{array}$$

Let $C \subset S$ be a section corresponding to a surjection $\nu^*\mathcal{E} \to \mathcal{O}_{\boldsymbol{P}^1}(1) \to 0$, and let V_C be the family of deformations of $\bar{\nu}(C)$; since $H \cdot \bar{\nu}(C) = \mathcal{O}_{\boldsymbol{P}(\nu^*\mathcal{E})}(1) \cdot C = 1$ the family V_C has anticanonical degree two and is unsplit.

We claim that the numerical class of W lies in the interior of the cone spanned by $[V_C]$ and R_ϑ; this is trivial if $[V_C] \in R_\sigma$, so we can assume that this is not the case. The cone of curves of S is generated by the numerical class of a fiber and the numerical class of C, i.e., $\mathrm{NE}(S) = \langle [C], [f]\rangle$. The morphism $\bar{\nu}$ induces a map $N_1(S) \to N_1(X)$ which allows us to identify $\mathrm{NE}(S)$ with the subcone of $\mathrm{NE}(X)$ generated by $[V_C]$ and R_ϑ. The divisor D_W is positive on this subcone, hence the effective divisor $\Gamma = \bar{\nu}^* D_W$ is ample on S. It follows that Γ lies in the interior of $\mathrm{NE}(S)$, hence $\bar{\nu}(\Gamma)$, which is a curve in D_W, lies in the interior of the cone generated by $[V_C]$ and R_ϑ. Therefore also $[W]$ lies in the interior of the cone generated by $[V_C]$ and R_ϑ by assumption (iv), and we can write

$$[W] = a[C_\vartheta] + b[V_C] \qquad \text{with} \quad a, b > 0\,,$$

where C_ϑ is a minimal curve in R_ϑ. Intersecting with H we get $a + b = 1$, and intersecting with $-\vartheta^* K_Y$ we have

$$-\vartheta^* K_Y \cdot W = b i_Y < i_Y\,;$$

therefore if C_W is a curve in W we have $-K_Y \cdot \vartheta_*(C_W) < i_Y$, a contradiction. □

PROPOSITION 4.5. *Let V' be a minimal dominating family for X', V a family of deformations of the strict transform of a curve in V' and $\mathcal{V}$ the Chow family associated to V. Assume that $-K_{X'} \cdot V' = 4$. Then any irreducible component of a reducible cycle in $\mathcal{V}$ which is not numerically proportional to V is a minimal extremal curve.*

PROOF. Let $\Gamma = \sum \Gamma_i$ be a reducible cycle in $\mathcal{V}$ with $[\Gamma_1] \neq \lambda[V]$; since $r_X = 2$, Γ has exactly two irreducible components. Denote by W and $\bar{W}$ their families of deformations, which have anticanonical degree two and so are unsplit. Since by Lemma 4.1 $E \cdot V = 0$, we can assume that $E \cdot W < 0$, hence by Lemma 4.3 we have that $[W] \in R_\sigma$.

As a consequence, note that if $\Gamma' = \Gamma'_1 + \Gamma'_2$ is another reducible cycle in $\mathcal{V}$, then either Γ'_1 and Γ'_2 are numerically proportional to V or, denoted by W' and $\bar{W}'$ their families of deformations, we can assume that $[W'] = [W]$ and $[\bar{W}'] = [\bar{W}]$.

We claim that $[\bar{W}]$ is extremal.

Case 1. V is not locally unsplit.

Let $\{\bar{W}^i\}_{i=1,\dots,n}$ be the families of deformations of the irreducible components of cycles in $\mathcal{V}$ such that $[\bar{W}^i] = [\bar{W}]$; since V is not locally unsplit, for some index i the family $\bar{W}^i$ is dominating. We can then apply [9, Lemma 2.4].

Case 2. V is locally unsplit.

Assume by contradiction that $[\bar{W}]$ is not extremal. By the argument in the proof of Case 1 we have that $\bar{W}^i$ is not dominating for every i. By inequality 2.4 (a) we have that $\dim \mathrm{Locus}(\bar{W}^i) = 3$ or 4; we distinguish two cases:

(i) There exists an index i such that $\dim \mathrm{Locus}(\bar{W}^i) = 4$.

Let $D = \mathrm{Locus}(\bar{W}^i)$; if $D \cdot V = 0$ then D is negative on an extremal ray of $\mathrm{NE}(X)$, hence on R_σ, but this implies $D = E$, against Lemma 4.3. Therefore $D \cdot V > 0$, hence $D \cap \mathrm{Locus}(V_x) \neq \emptyset$ for a general $x \in X$. Since we are assuming that V is locally unsplit, we have that $\dim \mathrm{Locus}(V_x) \geq 3$ and $\mathrm{NE}^X(\mathrm{Locus}(V_x)) = \langle V \rangle$ by Proposition 2.12 (b), so $\dim \mathrm{Locus}(\bar{W}^i)_{\mathrm{Locus}(V_x)} \geq 4$ by Lemma 2.9 (b) and $D = \mathrm{Locus}(\bar{W}^i)_{\mathrm{Locus}(V_x)}$. It follows by [20, Lemma 1] that every curve in D can be written as $aC_V + bC_{\bar{W}^i}$ with $a \geq 0$, C_V a curve contained in $\mathrm{Locus}(V_x)$ and $C_{\bar{W}^i}$ a curve in $\bar{W}^i$. Therefore $\mathrm{NE}^X(D) \subset \langle R_\sigma, [\bar{W}^i] \rangle$, but this is excluded by Lemma 4.4.

(ii) For every i we have $\dim \mathrm{Locus}(\bar{W}^i) = 3$.

By inequality 2.4 (a) we have $\dim \mathrm{Locus}(\bar{W}_x) = 3$ for every $x \in \mathrm{Locus}(\bar{W})$. Let

$$\Omega = \bigcup_i (\mathrm{Locus}(W^i) \cup \mathrm{Locus}(\bar{W}^i)) = E \cup \bigcup_i \mathrm{Locus}(\bar{W}^i),$$

and take a point y outside Ω; since X is rc$\mathcal{V}$-connected we can join y and Ω with a chain of cycles in $\mathcal{V}$. Let C be the first irreducible component of these cycles which meets Ω. Clearly C cannot belong to any family W^i or $\bar{W}^i$ because it is not contained in Ω, so it belongs either to V or to a family λV which is numerically proportional to V; by [1, Lemma 9.1] we have that either $C \subset \mathrm{Locus}(V_z)$ for some z such that V_z is unsplit or $C \subset \mathrm{Locus}(\lambda V)$. Moreover, since $E \cdot V = 0$ the intersection $C \cap \Omega$ is contained in $\Omega \setminus E$. Let t be a point in $C \cap \Omega$ and let $\Omega_j = \mathrm{Locus}(\bar{W}^j)$ be the irreducible component of Ω which contains t. If $C \subset \mathrm{Locus}(V_z)$ we have $\dim(\mathrm{Locus}(V_z) \cap \Omega_j) \geq 1$, against the fact that $N_1^X(V_z) = \langle [V] \rangle$ and $N_1^X(\Omega_j) = \langle [\bar{W}^j] \rangle$. If else $C \subset \mathrm{Locus}(\lambda V)$ we have that $\dim \mathrm{Locus}(\lambda V)_{\Omega_j} \geq 4$ by Lemma 2.9 (b) and that $\mathrm{NE}^X(\mathrm{Locus}(\lambda V)_{\Omega_j}) \subset \langle [\lambda V], R_\vartheta \rangle$ by [20, Lemma 1]; this is clearly impossible if $\mathrm{Locus}(\lambda V)_{\Omega_j} = X$, and it contradicts Lemma 4.2 if $\dim \mathrm{Locus}(\lambda V)_{\Omega_j} = 4$.

Finally, since $-K_X \cdot W^i = -K_X \cdot \bar{W}^i = 2$ we also have that the curves of W^i and $\bar{W}^i$ are minimal in R_σ and R_ϑ respectively. □

COROLLARY 4.6. *In the assumptions of Proposition* 4.5, *denoting as usual by C_σ and C_ϑ minimal rational curves in the rays R_σ and R_ϑ, we have, in* $\mathrm{NE}(X)$, $[V] = [C_\sigma] + [C_\vartheta]$; *in particular we have $H \cdot C_\vartheta = 1$.*

PROPOSITION 4.7. *Let V' be a minimal dominating family for X', let V be a family of deformations of the strict transform of a curve in V' and assume that $-K_{X'} \cdot V' = 4$. Then E is a Fano manifold and X' is a del Pezzo manifold.*

PROOF. By Lemma 4.1 we have $E \cdot V = 0$, hence $E \cdot C_\vartheta = -E \cdot C_\sigma = 1$ by Corollary 4.6; It follows that

$$(-K_X - E) \cdot C_\sigma = 2 + 1 = 3$$
$$(-K_X - E) \cdot C_\vartheta = 2 - 1 = 1\,,$$

hence $-K_X - E$ is ample on X by Kleiman criterion. By adjunction $-K_E = (-K_X - E)|_E$ is ample on E and E is a Fano manifold.

We note that E contains curves of R_ϑ: otherwise the fiber type contraction ϑ would be a $\boldsymbol{P}^1$-bundle by [19, Lemma 2.13], and since $E \cdot C_\vartheta = 1$ it follows that E would be a section of ϑ, against the fact that $\rho_Y = 1$ and $\rho_E = \rho_S + 1 \geq 2$. Consider the divisor $D = H - E$: it is nef and vanishes on R_ϑ, so it is a supporting divisor for ϑ. The restriction $D|_E$ is nef but not ample, since E contains curves of R_ϑ, so $D|_E$ is associated to an extremal face of $\mathrm{NE}(E)$ and to an extremal contraction $\vartheta_E : E \to Z$ and we have a commutative diagram:

$$\begin{array}{ccc} E & \longrightarrow & X \\ {\scriptstyle \vartheta_E}\downarrow & \searrow^{\vartheta|_E} & \downarrow{\scriptstyle \vartheta} \\ Z & \longrightarrow & Y \end{array}$$

We will prove that, for every $m \in \boldsymbol{N}$, the restriction map $H^0(X, mD) \to H^0(E, mD|_E)$ is an isomorphism, hence $\vartheta|_E = \vartheta_E$ and $Z = Y$. Consider the exact sequence

$$0 \to \mathcal{O}_X(mD - E) \to \mathcal{O}_X(mD) \to \mathcal{O}_E(mD|_E) \to 0\,.$$

Since E is not contracted by ϑ we have that $h^0(mD - E) = 0$; moreover, we can write

$$mD - E = K_X + (m-1)D + 3H - 2E\,.$$

By Kleiman criterion $3H - 2E$ is ample on X and, being $(m-1)D$ nef, the divisor $(m - 1)D + 3H - 2E$ is ample, too. By the Kodaira Vanishing Theorem $h^1(mD - E) = 0$. We have proved that E is a Fano manifold, and we know that it has a $\boldsymbol{P}^2$-bundle structure over S, i.e., $E \simeq \boldsymbol{P}_S(\mathcal{E})$ with $\mathcal{E}$ a Fano bundle of rank three over S. This implies that S is a del Pezzo surface.

Let L_Y be the ample generator of $\mathrm{Pic}(Y)$; by Proposition 2.14, Proposition 2.15 and the classification in [26], the pull-back of L_Y has degree one on the fibers of the $\boldsymbol{P}^2$-bundle.

The line bundle $H - E$ has degree two on the fibers of the $\boldsymbol{P}^2$-bundle and is trivial on the fibers of ϑ, hence $H - E = 2\vartheta^* L_Y$ and so $H - \vartheta^* L_Y$ is trivial on the fibers of σ, i.e., $H - \vartheta^* L_Y = \sigma^* H_{X'}$ for some $H_{X'} \in \mathrm{Pic}(X')$. By the canonical bundle formula we have

$$-\sigma^* K_{X'} = -K_X + 2E = 2(H + E) = 4H - 4\vartheta^* L_Y = 4\sigma^* H_{X'}\,, \tag{3}$$

i.e., $r_{X'} = 4$ and so X' is a del Pezzo fivefold. □

COROLLARY 4.8. *By the classification of del Pezzo manifolds given by Fujita* [11], *denoting by* $d := H^5_{X'}$ *the degree of* X' *and recalling that* $\rho_{X'} = 1$, *we have the following possibilities*:
(i) If $d = 1$ then $X' \simeq V_1$ is a degree six hypersurface in the weighted projective space $\boldsymbol{P}(3, 2, 1, \ldots, 1)$;
(ii) if $d = 2$ then $X' \simeq V_2$ is a double cover of $\boldsymbol{P}^5$ branched along a smooth quartic hypersurface;
(iii) if $d = 3$ then $X' \simeq V_3$ is a cubic hypersurface in $\boldsymbol{P}^6$;
(iv) if $d = 4$ then $X' \simeq V_4$ is the complete intersection of two quadrics in $\boldsymbol{P}^7$;
(v) if $d = 5$ then $X' \simeq V_5$ is a linear section of the grassmannian $\boldsymbol{G}(1, 4) \subset \boldsymbol{P}^9$.

4.2. Classification of S.

THEOREM 4.9. *If* $X' \simeq \boldsymbol{P}^5$ *then* S *is as in Theorem* 1.1, *cases* (b1)–(b6).

PROOF. Let H be a hyperplane of $\boldsymbol{P}^5$, let $\tilde{H} \subset X$ be its strict transform via σ and let $\mathcal{H} = \sigma^* H$. We know that $\tilde{H}$ is an effective divisor different from E, hence it is nef; moreover if $S \subset H$ we can write $\tilde{H} = \mathcal{H} - kE$ with $k > 0$. Let Γ be a proper bisecant of S, and let $\tilde{\Gamma}$ be its strict transform; if $S \subset H$ we have

$$0 \le \tilde{H} \cdot \tilde{\Gamma} \le 1 - 2k\,;$$

it follows that S has no proper bisecants, i.e., S is a linear subspace of $\boldsymbol{P}^5$ and we are in case (b1). If else S is not contained in any hyperplane, note that S cannot be the Veronese surface, since the blow-up of $\boldsymbol{P}^5$ along a Veronese surface has two birational contractions; therefore the secant variety of S fills $\boldsymbol{P}^5$.

Let l be a line in $\boldsymbol{P}^5$ not contained in S and $\tilde{l}$ its strict transform; we have

$$-K_X \cdot \tilde{l} = \sigma^* \mathcal{O}_{\boldsymbol{P}^5}(6) \cdot \tilde{l} - 2E \cdot \tilde{l} = 6 - 2(\sharp(S \cap l)) > 0\,;$$

therefore if l is a proper bisecant of S we have $-K_X \cdot \tilde{l} = 2$; moreover S cannot have (proper) trisecant lines. In the notation of [6], the condition on the trisecants is equivalent to the fact that the trisecant variety of S (which consists of all lines contained in S and of the proper trisecants) is contained in S, so by the description in [6] (see in particular Theorem 7, Section 4 and Appendix A2) we have the possibilities (b2)–(b6).

We now show that in all these cases the blow-up of X' along S is a Fano manifold with the prescribed cone of curves. The linear system $\mathcal{L} = |\mathcal{O}_{\boldsymbol{P}^5}(2) \otimes \mathcal{I}_S|$ of the quadrics in $\boldsymbol{P}^5$ containing S has S as its base locus scheme (see [12]), so $\sigma^*\mathcal{L}$ defines a morphism $\vartheta : X \to \boldsymbol{P}(\mathcal{L})$. Since $2\mathcal{H} - E$ is nef and vanishes on the strict transforms of the bisecants of S, it follows that the numerical class of these curves is extremal in $\mathrm{NE}(X)$, and since $-K_X$ is positive on these curves, we can conclude that X is a Fano manifold. Moreover since S is neither degenerate nor the Veronese surface, the bisecants to S cover $\boldsymbol{P}^5$ and so ϑ is of fiber type. □

LEMMA 4.10. *Assume that X' is a del Pezzo fivefold. Let $H_{X'} = \mathcal{O}_{X'}(1)$ and $H_S = (H_{X'})|_S$. Then*

(i) *If* $\dim Y = 2$ *then* $H_S^2 = \deg X' = -K_S \cdot H_S$.
(ii) *If* $\dim Y = 3$ *then* $\deg X' = -K_S \cdot H_S$ *and* $\deg X' - H_S^2 \geq 2$.
(iii) *If* $\dim Y = 4$ *then* $\deg X' > -K_S \cdot H_S$.

PROOF. Denote by $\mathcal{N}$ the normal bundle of S in X' and by $\mathcal{N}^*$ the conormal bundle; let $C = \det \mathcal{N}^* \in \mathrm{Pic}(S)$. Recall that $E = \boldsymbol{P}_S(\mathcal{N}^*)$ and that $-E|_E = \xi_{\mathcal{N}^*}$. Let $\mathcal{H} = \sigma^* H_{X'}$; we have

$$\mathcal{H}^5 = (H_{X'})^5 = \deg X' =: d\,,$$

and since the intersection of three or more sections of a very ample multiple of $H_{X'}$ does not meet S, we have also

$$\mathcal{H}^4 E = \mathcal{H}^3 E^2 = 0\,.$$

Then we have

$$\begin{aligned} K_S &= (K_{X'} + \det \mathcal{N})|_S = -4H_S - C\,,\\ \mathcal{H}^2 E^3 &= (\mathcal{H}^2 E^2)|_E = H_S^2\,,\\ \mathcal{H} E^4 &= (\mathcal{H} E^3)|_E = (-\mathcal{H}\xi^3_{\mathcal{N}^*})|_E = -C \cdot H_S\,. \end{aligned}$$

Let $L := \mathcal{H} - E$; from the above equalities it follows that

$$L^4\mathcal{H} = \mathcal{H}^5 - 4\mathcal{H}^2E^3 + \mathcal{H}E^4 = d + K_S \cdot H_S\,; \tag{4}$$

$$L^3\mathcal{H}^2 = \mathcal{H}^5 - \mathcal{H}^2E^3 = d - H_S^2\,. \tag{5}$$

By Corollary 4.6 we have that $H \cdot C_\vartheta = 1$; then equation (3) yields that $\mathcal{H} \cdot R_\vartheta = E \cdot R_\vartheta = 1$, hence L is trivial on the fibers of ϑ and therefore $L = \vartheta^* L_Y$.

(i) If $\dim Y = 2$ we have $L^4\mathcal{H} = L^3\mathcal{H}^2 = 0$, so it follows from (4) and (5) that

$$0 = d + K_S \cdot H_S = d - H_S^2\,.$$

(ii) If $\dim Y = 3$ then $L^4\mathcal{H} = 0$, and so by (4) we have

$$d + K_S \cdot H_S = 0\,.$$

The contraction ϑ is a quadric fibration (see Definition 2.1) and $\mathcal{H}|_F = \mathcal{O}_F(1)$ for a general fiber F of ϑ; hence $L^3\mathcal{H}^2 = (L_Y^3)(\mathcal{H}_F^2) \geq 2$, and (5) yields that

$$d - H_S^2 \geq 2\,.$$

(iii) Finally, if $\dim Y = 4$ the general fiber F of ϑ is one-dimensional and $\mathcal{H} \cdot F = 1$, hence $L^4\mathcal{H} = L_Y^4 > 0$; again by (4) we have that

$$d + K_S \cdot H_S > 0\,.$$ □

LEMMA 4.11. *If* $\dim Y > 2$ *then* S *is* $\boldsymbol{P}^2$, *a smooth quadric* $\mathcal{Q}$ *or the ruled surface* $\boldsymbol{F}_1$, *i.e. the blow-up of* $\boldsymbol{P}^2$ *at a point.*

PROOF. By Proposition 4.7 E is a Fano manifold and, by the proof of the same Proposition, we know that the restriction $\vartheta|_E : E \to Y$ is an extremal contraction of E. Moreover, by the classification in Proposition 2.14 we know that for every del Pezzo surface S_k with $k \geq 2$ the exceptional divisor E is isomorphic to $S_k \times \boldsymbol{P}^2$, and in this case E has no maps on a variety with Picard number one and dimension greater than two. $\square$

THEOREM 4.12. *If* X' *is a del Pezzo fivefold then the pairs* (X', S) *are as in Theorem* 1.1, *cases* (b7)–(b13).

PROOF. The contraction $\vartheta : X \to Y$ is supported by $\mathcal{H} - E$, and is the resolution of the rational map $\theta : X' \dashrightarrow Y$ defined by the linear system $\mathcal{L} := \sigma_*|\vartheta^* L_Y|$, where L_Y is the ample generator of $\mathrm{Pic}(Y)$; since $|\vartheta^* L_Y|$ is base point free we have $\mathrm{Bs}\,\mathcal{L} \subseteq S$; on the other hand $\mathcal{L} \subseteq |H_{X'} \otimes \mathcal{I}_S|$, therefore $\mathrm{Bs}\,\mathcal{L} \supseteq S$ and so $\mathrm{Bs}\,\mathcal{L} = S$. It follows that the strict transforms of curves of degree one with respect to $H_{X'}$ which meet S are contracted by ϑ. Moreover, since $\mathcal{H} - E$ is nef, no curves of degree one with respect to $H_{X'}$ and not contained in S can meet S in more than one point.

• If $\dim Y = 2$ then ϑ is equidimensional and by [5, Corollary 1.4] we have that Y is smooth; moreover $\rho_Y = 1$ and Y is dominated by a Fano manifold, so $Y \simeq \boldsymbol{P}^2$. Therefore $\dim \mathcal{L} = 3$, so S is the complete intersection of three general sections in $|H_{X'}|$ and we are in case (b7).

• In case $\dim Y = 3$, if $S \simeq \boldsymbol{P}^2$ then $H_S \simeq \mathcal{O}_{\boldsymbol{P}^2}(a)$, with $a > 0$. By Lemma 4.10 (ii) we have $d = -K_{\boldsymbol{P}^2} \cdot H_{\boldsymbol{P}^2} = 3a$; recalling that $d \leq 5$ we find $H_S = \mathcal{O}_{\boldsymbol{P}^2}(1)$ and $d = 3$ (case (b8)). If $S \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ then $H_S \simeq \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(a, b)$, with $a, b > 0$. By Lemma 4.10 (ii) we have $d = -K_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \cdot H_{\boldsymbol{P}^2} = 2a + 2b$; recalling that $d \leq 5$ we find $H_S = \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1, 1)$ and $d = 4$ (case (b10)). For $S \simeq \boldsymbol{F}_1$ we have $-K_{\boldsymbol{F}_1} \cdot C \geq 5$ for every ample $C \in \mathrm{Pic}(\boldsymbol{F}_1)$, equality holding if and only if $C = C_0 + 2f$; hence, by Lemma 4.10 (ii) we have $d = -K_{\boldsymbol{F}_1} \cdot H_{\boldsymbol{F}_1} = 5$ and $H_S = C_0 + 2f$. Since all the bisecants of S which are contained in $\boldsymbol{G}(1, 4)$ are also contained in a linear section V_5, it follows by Proposition 2.24 that S is as in case (b13).

• Finally, in case $\dim Y = 4$ we can apply Lemma 4.10 (iii) and get: if $S \simeq \boldsymbol{P}^2$ then $H_S = \mathcal{O}(1)$ and $H_S^2 = 1$, so $d = 4$ (case (b9)) or $d = 5$; in the latter case, being ϑ of fiber type, we exclude the case of a plane of bidegree (0, 1) in view of Remark 2.19 and we are in case (b11). If $S \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ the bound $-K_S \cdot H_S \geq 4$ gives $H_S = \mathcal{O}(1, 1)$ and $H_S^2 = 2$, hence $d = 5$; in this case S has bidegree (1, 1) by Proposition 2.23 and we are in case (b12). The center of the blow-up cannot be $\boldsymbol{F}_1$ since $-K_{\boldsymbol{F}_1} \cdot H_{\boldsymbol{F}_1} \geq 5$, which contradicts Lemma 4.10 (iii).

We show now that in all these cases the blow-up of X' along S is a Fano manifold with the prescribed cone of curves. Let (X', S) be a pair as in the theorem and denote by $H_{X'}$ the fundamental divisor of X'. We claim that the linear system $|H_{X'} \otimes \mathcal{I}_S|$ has S as its base locus scheme; this is clear apart from cases (b10), which is described in Proposition 4.13, and (b12) and (b13), which are treated in Proposition 4.14. Therefore the linear system $|\sigma^* H_{X'} - E|$

defines a morphism $\vartheta : X \to \boldsymbol{P}(|\sigma^* H_{X'} - E|)$. Since $\sigma^* H_{X'} - E$ is nef and vanishes on the strict transforms of the rational curves of degree one in X' which meet S, it follows that the numerical class of these curves is extremal in $\mathrm{NE}(X)$. Being $-K_X$ positive on these curves, we can conclude that X is a Fano manifold. Finally, since the curves of degree one with respect to $H_{X'}$ which meet S cover X', we have that ϑ is a fiber type contraction. □

PROPOSITION 4.13. *Let $\mathcal{Q}$ be a smooth two-dimensional quadric in $V_4 \subset \boldsymbol{P}^7$. Then $\mathcal{Q}$ is the intersection of V_4 and the hyperplanes of $\boldsymbol{P}^7$ which contain $\mathcal{Q}$.*

PROOF. Let $\mathcal{Q}$ be a smooth two-dimensional quadric in $V_4 = \boldsymbol{Q} \cap \boldsymbol{Q}' \subset \boldsymbol{P}^7$, and let $\Lambda^3_{\mathcal{Q}}$ be the three-dimensional linear subspace of $\boldsymbol{P}^7$ which contains $\mathcal{Q}$. We claim that $\Lambda^3_{\mathcal{Q}}$ is contained in one of the two quadrics $\boldsymbol{Q}$, $\boldsymbol{Q}'$. From [23, Proposition 2.1] we know that the intersection of two quadrics is smooth if and only if there exist coordinates in $\boldsymbol{P}^n$ such that

$$\boldsymbol{Q} = \left\{\sum x_i^2 = 0\right\}, \qquad \boldsymbol{Q}' = \left\{\sum \lambda_i x_i^2 = 0\right\}$$

with $\lambda_i \neq \lambda_j$ for every $i \neq j$. So assume by contradiction that $\Lambda^3_{\mathcal{Q}} \not\subset \boldsymbol{Q} \cup \boldsymbol{Q}'$; in this case $\Lambda^3_{\mathcal{Q}} \cap \boldsymbol{Q} = \Lambda^3_{\mathcal{Q}} \cap \boldsymbol{Q}' = \mathcal{Q}$, so it must be

$$\left(\sum (1-\lambda_i) x_i^2\right)|_{\Lambda^3_{\mathcal{Q}}} \equiv 0\,.$$

But there is at most one index i such that $\lambda_i = 1$, so the kernel of the quadratic form $\sum (1-\lambda_i)x_i^2$ is at most one-dimensional and we reach a contradiction. □

PROPOSITION 4.14. *Let S be a smooth two-dimensional quadric of bidegree* $(1,1)$ *or a surface of bidegree* $(2,1)$ *not contained in a* $\boldsymbol{G}(1,3)$, *in $V_5 \subset \boldsymbol{P}^8$. Then S is the intersection of V_5 and the hyperplanes of $\boldsymbol{P}^8$ which contain S.*

PROOF. Since V_5 is an hyperplane section of $\boldsymbol{G}(1,4)$ we will show that $S \subset \boldsymbol{G}(1,4) \subset \boldsymbol{P}^9$ is the intersection of $\boldsymbol{G}(1,4)$ and the hyperplanes of $\boldsymbol{P}^9$ which contain S, by finding explicitly its equations. By Proposition 2.23, if S is a quadric of bidegree $(1,1)$, then it parametrizes lines in $\boldsymbol{P}^4$ which meet two given skew lines r, s. Up to a change of coordinates in $\boldsymbol{P}^4$, we can assume that r and s have equations

$$r:\ x_0 = x_1 = x_2 = 0\,, \qquad s:\ x_0 = x_3 = x_4 = 0\,,$$

so H is the hyperplane of equation $x_0 = 0$; in this case the equations of S in $\boldsymbol{G}$ are

$$\begin{cases} y_0 = \cdots = y_4 = y_9 = 0 \\ y_5 y_8 = y_6 y_7 \end{cases}$$

and S is the intersection of $\boldsymbol{G}$ with the three-dimensional linear subspace $\Lambda^3 \subset \boldsymbol{P}^9$ of equations

$$y_0 = \cdots = y_4 = y_9 = 0\,.$$

Let now $S \subset \boldsymbol{G}$ be a surface of bidegree $(2,1)$ not contained in a $\boldsymbol{G}(1,3)$, as described in Proposition 2.25. Up to a coordinate change in $\boldsymbol{P}^4$, assume that $\mathcal{C}$ is the cone of vertex

$(0:0:0:0:1)$ on the quadric of equations

$$x_0x_2 = x_1x_3\,, \qquad x_4 = 0\,,$$

and that m is the line of equations $x_0 = x_1 = x_4 = 0$. The two families of planes contained in $\mathcal{C}$ have equations

$$F_1 = \begin{cases} \lambda x_0 = \mu x_1 \\ \lambda x_3 = \mu x_2\,, \end{cases} \qquad F_2 = \begin{cases} \lambda x_0 = \mu x_3 \\ \lambda x_1 = \mu x_2\,, \end{cases}$$

and m lies in the plane $\pi_m \in F_2$ of equations $x_0 = x_1 = 0$. The equations of the scroll $S \subset \boldsymbol{G}$ are

$$\begin{cases} y_0 = y_3 = y_6 = y_7 = 0 \\ y_1 = y_5 \\ y_1^2 = y_2y_4 \\ y_1y_8 = y_4y_9 \\ y_1y_9 = y_2y_8\,. \end{cases}$$

In particular, S is the intersection of $\boldsymbol{G}$ with the four-dimensional linear space Λ^4_S of equations $y_0 = y_3 = y_6 = y_7 = 0$, $y_1 = y_5$. □

5. Cases (e)–(f). Setup. Throughout the section, let X be a Fano fivefold whose cone of curves is as in cases (e)–(f), and let $\sigma : X \to X'$ be an extremal contraction of X which is the blow-up of X' along a smooth surface.

PROPOSITION 5.1. *Let X be as above. Then either $X = \boldsymbol{P}_{\boldsymbol{P}^2\times\boldsymbol{P}^2}(\mathcal{O}\oplus\mathcal{O}(1,1))$ or X' is a Fano manifold of even index.*

PROOF. Let E be the exceptional locus of σ; by [30, Proposition 3.4] X' is a Fano manifold unless E contains the exceptional locus of another extremal ray; this is clearly possible only if X has another birational contraction, i.e., in case (f). Note that in this case both the birational contractions of X are smooth blow-ups by Lemma 3.2. Let $\bar{\sigma}$ be the other blow-up contraction of X, denote by R_σ and $R_{\bar{\sigma}}$ the extremal rays corresponding to σ and $\bar{\sigma}$ and by R_ϑ the extremal ray corresponding to the fiber type contraction $\vartheta : X \to Y$. Let F be a fiber of σ; by Lemma 2.9 (a) we have $\dim \mathrm{Locus}(R_{\bar{\sigma}})_F \geq 4$, hence $E = \mathrm{Locus}(R_{\bar{\sigma}})_F$ and $\mathrm{NE}^X(E) = \langle R_\sigma, R_{\bar{\sigma}}\rangle$ by Proposition 2.12. Moreover $E \cdot R_\sigma < 0$ and $E \cdot R_{\bar{\sigma}} < 0$, hence $E \cdot R_\vartheta > 0$ and ϑ is a $\boldsymbol{P}^1$-bundle by [19, Corollary 2.15]. We can thus apply [19, Theorem 1.1], noting that the only Fano manifold in the list given in that result with two birational contractions with the same exceptional locus is $X = \boldsymbol{P}_{\boldsymbol{P}^2\times\boldsymbol{P}^2}(\mathcal{O}\oplus\mathcal{O}(1,1))$. The claim about the index of X' follows from the canonical bundle formula for σ. □

LEMMA 5.2. *Let X be a Fano fivefold whose cone of curves is as in case (f); denote by R_σ and $R_{\bar{\sigma}}$ the divisorial extremal rays of* $\mathrm{NE}(X)$, *by R_ϑ the fiber type extremal ray and by E (resp. $\bar{E}$) the exceptional locus of R_σ (resp. $R_{\bar{\sigma}}$). Then either $E \cdot R_\vartheta > 0$, or $\bar{E} \cdot R_\vartheta > 0$.*

PROOF. Consider a minimal horizontal dominating family V for ϑ.

CLAIM. *The numerical class of V belongs to a two-dimensional extremal face of* $\mathrm{NE}(X)$ *which contains R_ϑ.*

If V is unsplit, since $\rho_X = 3$ the claim follows from [9, Lemma 2.4].

Denote by V_ϑ the family of deformations of a minimal curve in R_ϑ. If V is not unsplit, for a general $x \in \mathrm{Locus}(V)$ we have that $\dim \mathrm{Locus}(V_x) \geq 3$ by Proposition 2.4, $\mathrm{NE}^X(\mathrm{Locus}(V_x)) = \langle V \rangle$ by Proposition 2.12 and $\dim \mathrm{Locus}(V_\vartheta, V)_x \geq 4$ by Lemma 2.9 (c). Call $D = \mathrm{Locus}(V_\vartheta, V)_x$; then ${\mathrm{N}_1}^X(D) = \langle R_\vartheta, V \rangle$ by [20, Lemma 1], so D is a divisor since $\rho_X = 3$. It cannot be $D \cdot R_\vartheta > 0$, otherwise we could write $X = \mathrm{ChLocus}(V_\vartheta, V)_x$ and we would have $\rho_X = 2$; so it must be $D \cdot R_\vartheta = 0$. This implies that D is positive on a birational ray, say R_σ, hence $\dim(D \cap F) \geq 1$ for every fiber F of σ; since ${\mathrm{N}_1}^X(D) = \langle R_\vartheta, V \rangle$ and $\mathrm{NE}^X(F) = \langle R_\sigma \rangle$, the claim is proved.

It follows that $E \cdot R_\vartheta > 0$. In fact, if $E \cdot R_\vartheta = 0$ then $E \cdot V < 0$, since curves of V are not contracted by ϑ and so they do not belong to R_ϑ. But then we would have $\mathrm{Locus}(V) \subset E$ and V would not be dominating for ϑ, a contradiction. □

PROPOSITION 5.3. *Let X be a Fano fivefold whose cone of curves is as in cases* (e)–(f), *and let $\sigma : X \to X'$ be the blow-up of X' along a smooth surface*; *assume that E is positive on a fiber type extremal ray of X. If X' is a Fano manifold, then either $X' \simeq \boldsymbol{P}^1 \times \boldsymbol{Q}^4$, and in this case either $S \simeq \boldsymbol{P}^1 \times l$ with l a line in $\boldsymbol{Q}^4$ or $S \simeq \boldsymbol{P}^1 \times \Gamma$ with Γ a conic not contained in a plane $\pi \subset \boldsymbol{Q}^4$, or X' is a $\boldsymbol{P}^3$-bundle over $\boldsymbol{P}^2$ and S dominates $\boldsymbol{P}^2$ via the bundle projection.*

PROOF. Let R_ϑ be the extremal ray on which E is positive, and let $\vartheta : X \to Y$ be its associated contraction; let $\psi : X \to W$ be the contraction of the face spanned by R_σ and R_ϑ. Then ψ factors through σ and a morphism $\theta : X' \to W$, and we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \vartheta\ } & Y \\ {\scriptstyle \sigma}\big\downarrow & \overset{\psi}{\searrow} & \big\downarrow \\ X' & \xrightarrow[\ \theta\ ]{} & W \end{array}$$

The contractions σ and ψ have connected fibers, so the same is true for θ; moreover W is a normal variety with $\rho_W = \rho_{X'} - 1$ and $\dim W < \dim X'$. It follows that θ is an extremal elementary fiber type contraction of the Fano manifold X'; denote by R_θ the corresponding extremal ray in $\mathrm{NE}(X')$.

Let V'_θ be a dominating family of rational curves whose numerical class belongs to R_θ and whose degree with respect to some ample line bundle is minimal among the degrees of the families with this property. In particular, by the minimality assumption, such a family is locally unsplit. Let V be the family of deformations of the strict transform in X of a general curve in V'_θ. Since curves of V are contracted by ψ, the numerical class of V in $\mathrm{NE}(X)$ lies in the face spanned by R_σ and R_ϑ. By [16, II.3.7], the general curve in V'_θ does not intersect the center S of the blow-up, so $E \cdot V = 0$; it follows that $[V] \notin R_\vartheta$. Clearly we cannot have $[V] \in R_\sigma$, being $E \cdot R_\sigma < 0$, so the class $[V]$ does not generate an extremal ray of X. In particular, since V is dominating and X has no small contractions, V cannot be unsplit in view of [9, Lemma 2.29], hence

$$4 \leq -K_X \cdot V = -K_{X'} \cdot V'_\theta .$$

For a general $x \in X'$ we have, by Proposition 2.4 (b), that $\dim \mathrm{Locus}(V'_\theta)_x \geq 3$, so a general fiber of θ is at least three-dimensional and $\dim W \leq 2$.

If $\dim W = 1$ then the contraction of the extremal ray of X different from R_σ and R_ϑ is a $\boldsymbol{P}^1$-bundle by [19, Corollary 2.15] (take a fiber of ψ for D). Now we apply [19, Lemma 4.1], to get that X is a product with $\boldsymbol{P}^1$ as a factor; looking at the classification table in [19, Appendix] we find that the only products with $\rho_X = 3$ and a blow-down contraction of type D_2 are $X \simeq \boldsymbol{P}^1 \times \mathrm{Bl}_l(\boldsymbol{Q}^4)$ or $X \simeq \boldsymbol{P}^1 \times \mathrm{Bl}_\Gamma(\boldsymbol{Q}^4)$; the description of X' and S follows.

If $\dim W = 2$ we claim that X' is a $\boldsymbol{P}^3$-bundle over $\boldsymbol{P}^2$. We would like to use [19, Lemma 2.18], but we do not know that the length of the ray R_θ is $\dim X' - 1$. However we notice that, in the proof of the quoted result, the assumption on the length is used only to prove that the general fiber of the contraction is a projective space, so we will prove in a different way that this is the case in our situation.

Let x be a general point in X' and denote by F_x the fiber of θ containing x; by Proposition 2.4 (b) we have $\dim \mathrm{Locus}(V'_\theta)_x \geq 3$, hence $F_x = \mathrm{Locus}(V'_\theta)_x$. Moreover, since V'_θ is locally unsplit, by Proposition 2.12 (b), we have $\rho_{F_x} = 1$. Now we can conclude $F_x \simeq \boldsymbol{P}^3$ either by the classification of Fano threefolds or by applying [14, Theorem 1.1] as in the proof of Lemma 4.1.

Therefore, by the proof of [19, Lemma 2.18], X' is a $\boldsymbol{P}^3$-bundle over $\boldsymbol{P}^2$; E is positive on the fiber type ray R_ϑ, so the image via σ of every curve in R_ϑ is a curve contracted by θ which meets S. Since ϑ is a fiber type contraction, we know that curves in R_ϑ dominate X, hence curves contracted by θ which meet S dominate X'. Therefore S dominates $\boldsymbol{P}^2$. □

THEOREM 5.4. *Let X be a Fano fivefold whose cone of curves is as in cases* (e)–(f), *and let $\sigma : X \to X'$ be the blow-up of X' along a smooth surface S. Then the pairs (X', S) are as in Theorem* 1.1, *cases* (e1)–(e4) *or* (f1)–(f4).

PROOF. By Proposition 5.1, either $X \simeq \boldsymbol{P}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(\mathcal{O} \oplus \mathcal{O}(1,1))$ and therefore (X', S) is as in case (f1) or we can apply Proposition 5.3: in fact, in case (e) the positivity of E on a fiber type ray of $\mathrm{NE}(X)$ is trivial, otherwise it follows from Lemma 5.2. Therefore either (X', S) is as in cases (e1)–(e2) or, up to exchange σ with $\bar{\sigma}$, we have that X' is a $\boldsymbol{P}^3$-bundle over $\boldsymbol{P}^2$. In this case, the classification in [26] yields that X' is either the blow-up of $\boldsymbol{P}^5$ along a plane π_1 or $X' \simeq \boldsymbol{P}_{\boldsymbol{P}^2}(T\boldsymbol{P}^2(-1) \oplus \mathcal{O}^{\oplus 2})$. Considering the exact sequence

$$0 \to \mathcal{O}_{\boldsymbol{P}^2}(-1) \to \mathcal{O}_{\boldsymbol{P}^2}^{\oplus 5} \to T\boldsymbol{P}^2(-1) \oplus \mathcal{O}_{\boldsymbol{P}^2}^{\oplus 2} \to 0\,,$$

we see that $X' = \boldsymbol{P}_{\boldsymbol{P}^2}(T\boldsymbol{P}^2(-1) \oplus \mathcal{O}^{\oplus 2})$ embeds in $\boldsymbol{P}^2 \times \boldsymbol{P}^4$ as a section of $\mathcal{O}(1,1)$.

Let $l \subset X'$ be a line in a fiber of the $\boldsymbol{P}^3$-bundle not contained in S, and let $\tilde{l} \subset X$ be its strict transform; by the canonical bundle formula

$$(6) \qquad -K_X \cdot \tilde{l} = -\sigma^* K_{X'} \cdot \tilde{l} - 2E \cdot \tilde{l} \leq 4 - 2\#(S \cap l)\,;$$

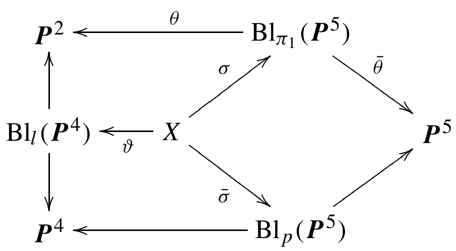
since X is Fano it must be $\#(S \cap l) \leq 1$.

Let $R_{\bar{\theta}} \subset \mathrm{NE}(X')$ be the extremal ray of X' not associated to the $\boldsymbol{P}^3$-bundle contraction. Let C be a minimal extremal curve in $R_{\bar{\theta}}$ not contained in S and let $\tilde{C} \subset X$ be its strict transform. Again by the canonical bundle formula

$$-K_X \cdot \tilde{C} = -\sigma^* K_{X'} \cdot \tilde{C} - 2E \cdot \tilde{C} \leq 2 - 2\#(S \cap C)\,,$$

hence $S \cap C = \emptyset$. Therefore, if S meets a two-dimensional fiber $F_{\bar{\theta}}$ of $\bar{\theta}$ then $S = F_{\bar{\theta}}$.

• In case $X' \simeq \mathrm{Bl}_{\pi_1}(\boldsymbol{P}^5)$, the map $\bar{\theta}$ is the blow-up map, so denoted by E' the exceptional divisor of $\bar{\theta}$ we have that either S is a fiber of $\bar{\theta}$ and we are in case (f2), or $S \cap E' = \emptyset$; in particular S cannot meet a fiber of the $\boldsymbol{P}^3$-bundle in a curve. In the first case, X has another blow-down contraction $\bar{\sigma} : X \to \mathrm{Bl}_p(\boldsymbol{P}^5)$, whose center is the strict transform of a plane passing through p; this corresponds to case (f3). In fact, X can be described as follows: let Y be the blow-up of $\boldsymbol{P}^4$ along a line, let E_Y be the exceptional divisor, let H_Y be the pullback of $\mathcal{O}_{\boldsymbol{P}^4}(1)$ and let $\mathcal{E} = (2H_Y + E_Y) \oplus (3H_Y + E_Y)$. Then $X = \boldsymbol{P}_Y(\mathcal{E})$, and the following diagram shows the extremal contractions of X:



In case $S \cap E' = \emptyset$, equation (6) yields that S is a section of the $\boldsymbol{P}^3$-bundle contraction of X'; therefore it corresponds to a surjection $\mathcal{O}^3 \oplus \mathcal{O}(1) \to \mathcal{O}(1)$, the image of S in $\boldsymbol{P}^5$ is a plane π_2 not meeting π_1 and we are in case (f4). In this case $X \simeq \boldsymbol{P}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(\mathcal{O}(0,1) \oplus \mathcal{O}(1,0))$.

• If $X' \simeq \boldsymbol{P}_{\boldsymbol{P}^2}(T\boldsymbol{P}^2(-1) \oplus \mathcal{O}^{\oplus 2})$ the contraction $\bar{\theta}$ is of fiber type; it follows that S is the union of all the fibers of $\bar{\theta}$ which have nonempty intersection with S itself. In particular, either S is a two-dimensional fiber of $\bar{\theta}$, i.e., a section corresponding to a surjection $T\boldsymbol{P}^2(-1) \oplus \mathcal{O}^{\oplus 2} \to \mathcal{O}$, and we are in case (e3), or $\bar{\theta}$ is a $\boldsymbol{P}^1$-bundle and S contains a one-parameter family of fibers isomorphic to $\boldsymbol{P}^1$. In this last case, the restriction of $\bar{\theta}$ to S is a morphism from S to a curve, and therefore $S \not\simeq \boldsymbol{P}^2$; so S cannot be a section of the natural projection $p : X' \to \boldsymbol{P}^2$. By equation (6) the restriction of p to S is a birational morphism $p|_S : S \to \boldsymbol{P}^2$, and the only surface which is birational to $\boldsymbol{P}^2$ and has a morphism on a curve all whose fibers are isomorphic to $\boldsymbol{P}^1$ is the Hirzebruch surface $\boldsymbol{F}_1$. In particular, the exceptional curve of S is a line in a fiber of p, therefore $\bar{\theta}(\boldsymbol{F}_1) = \bar{\theta}(C_0)$ is a line $l \subset \boldsymbol{P}^4$ and S is the intersection of the pullback of three hyperplanes in $\boldsymbol{P}^4$ meeting along l (case (e4)).

To conclude, we prove the effectiveness of X in these last two cases: in case (e3) let Y be a general member of $|\mathcal{O}_{\boldsymbol{P}^2\times\boldsymbol{P}^3}(1,1)|$ and let $\mathcal{E} = \mathcal{O}_Y(1,1) \oplus \mathcal{O}_Y(1,2)$; then $X \simeq \boldsymbol{P}_Y(\mathcal{E})$, as proved in [19, Proposition 7.3], and X is a $\boldsymbol{P}^1$-ruled Fano manifold. In case (e4) X can be realized as follows: let $Z = \mathrm{Bl}_l(\boldsymbol{P}^4)$, and let H_Z be the pullback of $\mathcal{O}_{\boldsymbol{P}^4}(1)$; then X is a general section in the linear system $|p_1^*\mathcal{O}_{\boldsymbol{P}^2}(1) + p_2^*H|$ in $\boldsymbol{P}^2 \times Z$, where p_1 and p_2 denote the projections onto the factors. □

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REFERENCES

[1] M. ANDREATTA, E. CHIERICI AND G. OCCHETTA, Generalized Mukai conjecture for special Fano varieties, Cent. Eur. J. Math. 2 (2004), 272–293.

[2] M. ANDREATTA AND G. OCCHETTA, Extending extremal contractions from an ample section, Adv. Geom. 2 (2002), 133–146.

[3] M. ANDREATTA AND G. OCCHETTA, Special rays in the Mori cone of a projective variety, Nagoya Math. J. 168 (2002), 127–137.

[4] M. ANDREATTA AND G. OCCHETTA, Fano manifolds with long extremal rays, Asian J. Math. 9 (2005), 523–544.

[5] M. ANDREATTA AND J. A. WIŚNIEWSKI, On contractions of smooth varieties, J. Algebraic Geom. 7 (1998), 253–312.

[6] I. BAUER, The classification of surfaces in $\boldsymbol{P}^5$ having few trisecants, Rend. Sem. Mat. Univ. Politec. Torino 56 (1998), 1–20.

[7] L. BONAVERO, Pseudo-index of Fano manifolds and smooth blow-ups, Geom. Dedicata 114 (2005), 79–86.

[8] L. BONAVERO, F. CAMPANA AND J. A. WIŚNIEWSKI, Variétés complexes dont l'éclatée en un point est de Fano, C. R. Math. Acad. Sci. Paris 334 (2002), 463–468.

[9] E. CHIERICI AND G. OCCHETTA, The cone of curves of Fano varieties of coindex four, Internat. J. Math. 17 (2006), 1195–1221.

[10] T. FUJITA, On polarized manifold whose adjoint bundles are not semipositive, Algebraic geometry, Sendai, Adv. Stud. in Pure Math. 10 (1987), 167–178.

[11] T. FUJITA, Classification theories of polarized varieties, volume 155 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1990.

[12] P. IONESCU, Embedded projective varieties of small invariants, In Algebraic geometry, Bucharest 1982, volume 1056 of Lecture Notes in Math. Springer-Verlag, 1984, 142–186.

[13] P. IONESCU, Generalized adjunction and applications, Math. Proc. Camb. Phil. Soc., 99 (1986), 457–472.

[14] S. KEBEKUS, Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron, In Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, 147–155.

[15] S. KOBAYASHI AND T. OCHIAI, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ. 13 (1973), 31–47.

[16] J. KOLLÁR, Rational curves on algebraic varieties, volume 32 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1996.

[17] M. MELLA, Existence of good divisors on Mukai varieties, J. Algebraic Geom. 8 (1999), 197–206.

[18] S. MUKAI, Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, Proc. Nat. Acad. Sci. U.S.A. 86 (1989), 3000–3002.

[19] C. Novelli and G. Occhetta, Ruled Fano fivefolds of index two, Indiana Univ. Math. J. 56 (2007), 207–241.

[20] G. Occhetta, A characterization of products of projective spaces, Canad. Math. Bull. 49 (2006), 270–280.

[21] G. Occhetta and J. A. Wiśniewski, On Euler-Jaczewski sequence and Remmert-Van de Ven problem for toric varieties, Math. Z. 241 (2002), 35–44.

[22] Z. Ran, Surfaces of order 1 in Grassmannians, J. Reine Angew. Math. 368 (1986), 119–126.

[23] M. Reid, The intersection of two quadrics, Ph. D. Thesis, Cambridge University, 1972. www.maths.warwick.ac.uk/ miles/3folds/qu.ps.

[24] E. Sato, Varieties which have two projective space bundle structures, J. Math. Kyoto Univ. 25 (1985), 445–457.

[25] J. G. Semple and L. Roth, Introduction to algebraic geometry, Reprint of the 1949 original, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1985.

[26] M. Szurek and J. A. Wiśniewski, On Fano manifolds, which are $\boldsymbol{P}^k$-bundles over $\boldsymbol{P}^2$, Nagoya Math. J. 120 (1990), 89–101.

[27] M. Szurek and J. A. Wiśniewski, Fano bundles over P^3 and Q_3, Pacific J. Math. 141 (1990), 197–208.

[28] M. Szurek and J. A. Wiśniewski, Fano bundles of rank 2 on surfaces, Compositio Math. 76 (1990), 295–305.

[29] J. A. Wiśniewski, On Fano manifolds of large index, Manuscripta Math. 70 (1991), 145–152.

[30] J. A. Wiśniewski, On contractions of extremal rays of Fano manifolds, J. Reine Angew. Math. 417 (1991), 141–157.

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