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FAREY SIMPLICES IN THE SPACE OF QUATERNIONS

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1. Introduction.

In 1932 Speiser [5] proved among other results the following concerning the Diophantine approximation of quaternions (for the notation of this introduction see § 2 below):

THEOREM. *For any irrational quaternion* ξ *the inequality*

$$|\xi - pq^{-1}| < \frac{1}{(\frac{5}{2})^{\frac{1}{2}} N(q)}$$

has infinitely many solutions pq^{-1}, *where* p, q, $q \neq 0$, *are integral quaternions (in the sense of Hurwitz).*

However, it does not follow from Speiser's paper whether the constant $(\frac{5}{2})^{\frac{1}{2}}$ in his theorem is best possible or may be replaced by larger constants c. In fact, no upper bound for the set of admissible constants c has been known so far, this lack of knowledge being due to the absence of the commutative law of multiplication for quaternions, because in analogous cases where real or complex numbers are to be approximated, it is indeed very easy to obtain such upper bounds by a simple method of Perron (cf. § 7 of [4]).

In a recent paper of mine [4] Farey triangles and Farey quadrangles in the complex plane were introduced and applied to study the approximation spectra of complex numbers in the cases $\mathsf{Q}(im^{\frac{1}{2}})$, $m=1$, 2, 3 and 7. Analogously we introduce in § 2 Farey simplices in the space of quaternions and develop their basic properties in §§ 3–5.

Subsequently, in §§ 6–8 we shall apply Farey simplices in an investigation of the approximation spectrum of quaternions, i.e. the set of all approximation constants $C(\xi)$, where $C(\xi)$ for any irrational quaternion ξ is defined as

$$C(\xi) = \limsup \,(|q|\;|\xi q - p|)^{-1} ,$$

the lim sup being taken over all $p,q \in \mathsf{H}$, $q \neq 0$, where H denotes the set of integral quaternions (in the sense of Hurwitz).

It should be noted that since

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$$\{|q|\,|\xi q-p| \mid p,q \in \mathsf{H}, q \neq 0\} = \{|q|\,|q\bar{\xi}-p| \mid p,q \in \mathsf{H}, q \neq 0\},$$

the approximation of ξ by quotients pq^{-1} is equivalent to the approximation of $\bar{\xi}$ by quotients $q^{-1}p$. In fact, Speiser's way of approximating quaternions differs in this respect from ours, which for certain reasons is preferable here.

Apart from an independent proof of Speiser's theorem we find that the only approximation constant $C(\xi) < (2.51)^{\frac{1}{2}}$ is $C(\xi) = (\frac{5}{2})^{\frac{1}{2}}$. Further, the set $C^{-1}\big((\frac{5}{2})^{\frac{1}{2}}\big)$ consists of two distinct equivalence classes of quaternions represented by

$$\tfrac{1}{2} + \tfrac{1}{4}(1+5^{\frac{1}{2}})i + \tfrac{1}{4}(1-5^{\frac{1}{2}})j \quad \text{and} \quad \tfrac{1}{2} + \tfrac{1}{4}(1-5^{\frac{1}{2}})i + \tfrac{1}{4}(1+5^{\frac{1}{2}})j .$$

Here $\eta \sim \xi$ when

$$\eta = (\alpha\xi+\beta)(\gamma\xi+\delta)^{-1},$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is a unimodular matrix (see § 2).

Also, the set $C^{-1}\big((\frac{5}{2})^{\frac{1}{2}}\big)$ has a simple characterization in terms of Farey simplices.

For the convenience of the reader the sections of this paper are enumerated parallel to the sections of [4]. However, except that we make use of two of the lemmas from [4], this article may be read independently.

2. Farey simplices. Unimodular homographic maps.

We consider the non-commutative field K of quaternions

$$\mathsf{K} = \{a_1+a_2i+a_3j+a_4k \mid a_1,a_2,a_3,a_4 \in \mathsf{R}\},$$

where

$$i^2=j^2=k^2=-1, \quad ij=-ji=k, \quad jk=-kj=i, \quad ki=-ik=j .$$

For $\alpha=a_1+a_2i+a_3j+a_4k \in \mathsf{K}$ the *conjugate, norm, absolute value* and *trace* of α are defined as usual by

$$\begin{aligned} \bar{\alpha} &= a_1-a_2i-a_3j-a_4k, \\ N(\alpha) &= \alpha\bar{\alpha} = a_1^2+a_2^2+a_3^2+a_4^2, \\ |\alpha| &= (N(\alpha))^{\frac{1}{2}} = (a_1^2+a_2^2+a_3^2+a_4^2)^{\frac{1}{2}}, \\ S(\alpha) &= \alpha+\bar{\alpha} = 2a_1 . \end{aligned}$$

Representing $\alpha=a_1+a_2i+a_3j+a_4k \in \mathsf{K}$ as the point (a_1,a_2,a_3,a_4) in

Euclidean space R^4, the absolute value $|\alpha-\beta|$ thus coincides with the Euclidean distance between the points corresponding to α and β.

In the following we shall make use of a number of simple and well-known computation rules for $\alpha,\beta,\gamma \in \mathsf{K}$ and $a \in \mathsf{R}$:

$$a\alpha = \alpha a, \quad \alpha^{-1} = \bar{\alpha} N(\alpha)^{-1}, \text{ if } \alpha \neq 0, \quad \bar{\bar{\alpha}} = \alpha ,$$
$$\overline{\alpha \pm \beta} = \bar{\alpha} \pm \bar{\beta}, \quad \overline{\alpha\beta} = \bar{\beta}\bar{\alpha} ,$$
$$N(\bar{\alpha}) = N(\alpha), \quad S(\bar{\alpha}) = S(\alpha) ,$$
$$N(\alpha\beta) = N(\alpha)N(\beta), \quad S(\alpha \pm \beta) = S(\alpha) \pm S(\beta) ,$$
$$N(\alpha+\beta) = N(\alpha)+N(\beta)+S(\alpha\bar{\beta}) ,$$
$$S(a\alpha) = a\,S(\alpha), \quad S(\alpha\beta) = S(\beta\alpha) .$$

At one instance we shall need also the more complicated rule

$$S(\alpha\beta)^2+S(\alpha i\beta)^2+S(\alpha j\beta)^2+S(\alpha k\beta)^2 = 4N(\alpha)N(\beta) , \tag{2.1}$$

which is equivalent to the identity

$$(a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2 + (-a_1b_2-a_2b_1+a_3b_4-a_4b_3)^2 +$$
$$+(-a_1b_3-a_2b_4-a_3b_1+a_4b_2)^2 + (-a_1b_4+a_2b_3-a_3b_2-a_4b_1)^2$$
$$= (a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2) .$$

A quaternion $\alpha=a_1+a_2i+a_3j+a_4k$ is called *rational* if $a_1,a_2,a_3,a_4 \in \mathsf{Q}$, otherwise α is called *irrational.*

A quaternion α is called *integral* (in the sense of Hurwitz) if α is rational, $N(\alpha) \in \mathsf{Z}$ and $S(\alpha) \in \mathsf{Z}$. By this definition the set H of integers equals

$$\mathsf{H} = \{h_1+h_2i+h_3j+h_4\omega \mid h_1,h_2,h_3,h_4 \in \mathsf{Z}\} ,$$

where $\omega=\frac{1}{2}(1+i+j+k)$.

According to Hurwitz [2] H is a *Euclidean* ring, i.e. given any $\alpha,\beta \in \mathsf{H}$, $\beta \neq 0$, there exist $\gamma,\delta \in \mathsf{H}$ such that

$$N(\alpha-\gamma\beta) < N(\beta) \quad \text{and} \quad N(\alpha-\beta\delta) < N(\beta) . \tag{2.2}$$

An integral quaternion α is called a *unit* when $N(\alpha)=1$. The 24 units of H,

$$\pm 1, \quad \pm i, \quad \pm j, \quad \pm k, \quad \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k) , \tag{2.3}$$

constitute a multiplicative group U.

We shall make extensive use of the ring of 2×2 matrices over K, and we define in accordance with Study [6] for any such matrix

$$\mathfrak{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{2.4}$$

$$D(\mathfrak{M}) = N(\alpha)N(\delta) + N(\beta)N(\gamma) - S(\alpha\bar{\gamma}\delta\bar{\beta}) . \tag{2.5}$$

The function D has the following properties (cf. Study [6]):

$$D(\mathfrak{M}) \geqq 0 \quad \text{for all } \mathfrak{M} , \tag{2.6}$$

$$D(\mathfrak{M}) > 0 \quad \text{if and only if } \mathfrak{M} \text{ has a reciprocal matrix} , \tag{2.7}$$

$$D(\mathfrak{M}_1\mathfrak{M}_2) = D(\mathfrak{M}_1)\, D(\mathfrak{M}_2) \quad \text{for all } \mathfrak{M}_1, \mathfrak{M}_2 , \tag{2.8}$$

$$D\begin{pmatrix} \zeta\alpha & \zeta\beta \\ \eta\gamma & \eta\delta \end{pmatrix} = N(\zeta)\, N(\eta)\, D\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{2.9}$$

$$D\begin{pmatrix} \alpha\zeta & \beta\eta \\ \gamma\zeta & \delta\eta \end{pmatrix} = N(\zeta)\, N(\eta)\, D\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{2.10}$$

Further, D is invariant under a left row operation (i.e. a left multiple of one row is added to the other row) and under a right column operation.

A matrix $\mathfrak{M}$ of the form (2.4) is called *integral* if $\alpha,\beta,\gamma,\delta \in \mathsf{H}$. Obviously $D(\mathfrak{M}) \in \mathsf{Z}$ if $\mathfrak{M}$ is integral.

A matrix $\mathfrak{M}$ is called *unimodular* if $\mathfrak{M}$ is integral and $D(\mathfrak{M}) = 1$. According to Mahler [3] the unimodular matrices form a multiplicative group. It is worth noticing that the proof of this uses the Euclidean property of H.

By (2.9) and (2.10) multiplication from the left (right) of the rows (columns) of a unimodular matrix by units in H leaves it a unimodular matrix. Also, by the computation rules listed above, reversal of rows (columns) in a unimodular matrix leaves it a unimodular matrix.

In preparation for an application in § 3 we shall establish the following

LEMMA 1. *$p_0, q_0 \in \mathsf{H}$ can occur as a row (column) in a unimodular matrix if and only if p_0 and q_0 have no common left (right) divisor not a unit.*

PROOF. The necessity of the conditions follows immediately from (2.9) and (2.10).

In the proof of the sufficiency of the conditions we need by symmetry only consider the case where p_0 and q_0 have no non-trivial common right divisor. Further we may assume without restriction that $N(p_0) \leqq N(q_0)$. The proof itself is now by induction on $N(p_0) + N(q_0)$.

If $N(p_0) + N(q_0) = 1$, then by our assumptions $N(p_0) = 0$ and $N(q_0) = 1$. Hence

$$\begin{pmatrix} p_0 & 1 \\ q_0 & 0 \end{pmatrix}$$

is a unimodular matrix.

If $N(p_0) + N(q_0) > 1$, then by our assumptions $N(p_0) > 0$. Hence by the Euclidean property (2.2) of H, there exists a $\gamma \in \mathsf{H}$ such that

$$N(q_0-\gamma p_0) < N(p_0) \leqq N(q_0)\,.$$

Now p_0 and $q_0-\gamma p_0$ have no non-trivial common right divisor by the assumption of the lemma, and hence by the induction hypothesis there is a unimodular matrix of the form

$$\begin{pmatrix} p_0 & \beta \\ q_0-\gamma p_0 & \delta \end{pmatrix}.$$

Consequently

$$\begin{pmatrix} p_0 & \beta \\ q_0 & \delta+\gamma\beta \end{pmatrix}$$

is a unimodular matrix by the invariance of D under a left row operation. This proves lemma 1.

Another preparation is the important formula

$$N(\gamma)\,N(\delta)\,N(\alpha\gamma^{-1}-\beta\delta^{-1}) = D\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \text{for } \gamma,\delta \neq 0\,. \tag{2.11}$$

In fact, by the calculation rules for quaternions,

$$\begin{aligned} N(\gamma)\,N(\delta)\,N(\alpha\gamma^{-1}-\beta\delta^{-1}) &= N(\alpha\gamma^{-1}-\beta\delta^{-1})\,N(\gamma\delta) \\ &= N(\alpha\delta-\beta\delta^{-1}\gamma\delta) \\ &= N(\alpha\delta)+N(\beta\delta^{-1}\gamma\delta)-S(\alpha\delta\bar{\delta}\bar{\gamma}\overline{\delta^{-1}}\bar{\beta}) \\ &= N(\alpha)\,N(\delta)+N(\beta)\,N(\gamma)-S(\alpha\bar{\gamma}\delta\bar{\beta})\,, \end{aligned}$$

which proves (2.11) in view of the definition (2.5).

Before introducing the fundamental notion of Farey simplex it will be convenient to consider the related concept of *Farey matrix.*

Definition 1. *A* 2×5 *matrix*

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix} \tag{2.12}$$

is called a Farey matrix (FM), *if* $p_l, q_l \in \mathsf{H}$, $1 \leqq l \leqq 5$, *and*

$$D\begin{pmatrix} p_l & p_m \\ q_l & q_m \end{pmatrix} = 1, \qquad 1 \leqq l < m \leqq 5\,. \tag{2.13}$$

By a remark above it follows that any permutation of the columns and multiplication of each column from the right by a unit in H leaves a Farey matrix a Farey matrix. This motivates the following

Definition 2. *Two Farey matrices are called associated if one is ob-*

tained from the other by a permutation of the columns together with a multiplication of each column from the right by a unit in H.

In order to give a precise description of the Farey matrices existent we shall make use of the group of unimodular linear maps

$$\Phi : \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \pi \\ \varrho \end{pmatrix} \tag{2.14}$$

of $\mathsf{H} \times \mathsf{H}$ onto itself.

An immediate consequence of (2.8) is that a unimodular linear map of $\mathsf{H} \times \mathsf{H}$ onto itself in a natural way maps a Farey matrix onto a Farey matrix.

Conversely, we have the following important result:

THEOREM 1. *Any Farey matrix is associated with a Farey matrix of the form*

$$\mathfrak{M}\begin{pmatrix} 1 & 0 & 1 & \omega & \omega-k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $\mathfrak{M}$ *is a unimodular matrix.*

PROOF. Let the Farey matrix given have the form (2.12). Then

$$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^{-1} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \pi_3 & \pi_4 & \pi_5 \\ 0 & 1 & \varrho_3 & \varrho_4 & \varrho_5 \end{pmatrix},$$

where by definition 1, (2.5) and the invariance of the set of Farey matrices under a unimodular linear map Φ in particular

$$N(\pi_l) = N(\varrho_l) = 1, \qquad 3 \leqq l \leqq 5 .$$

Hence for any unit $\varepsilon \in \mathsf{H}$ the Farey matrix (2.12) is associated with

$$\begin{pmatrix} p_1\pi_3\varepsilon & p_2\varrho_3\varepsilon & p_3\varepsilon & p_4\varrho_4^{-1}\varrho_3\varepsilon & p_5\varrho_5^{-1}\varrho_3\varepsilon \\ q_1\pi_3\varepsilon & q_2\varrho_3\varepsilon & q_3\varepsilon & q_4\varrho_4^{-1}\varrho_3\varepsilon & q_5\varrho_5^{-1}\varrho_3\varepsilon \end{pmatrix} = \tag{2.15}$$

$$\begin{pmatrix} p_1\pi_3\varepsilon & p_2\varrho_3\varepsilon \\ q_1\pi_3\varepsilon & q_2\varrho_3\varepsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \varepsilon^{-1}\pi\varepsilon & \varepsilon^{-1}\pi'\varepsilon \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where

$$\pi = \pi_3^{-1}\pi_4\varrho_4^{-1}\varrho_3 \quad \text{and} \quad \pi' = \pi_3^{-1}\pi_5\varrho_5^{-1}\varrho_3 .$$

Applying again definition 1, (2.5) and the invariance of the set of Farey matrices under a unimodular linear map Φ, it follows by (2.15) that

$$N(\pi) = N(\pi') = 1, \qquad S(\pi) = S(\pi') = S(\pi\bar{\pi}') = 1 ,$$

and hence π, π' are units in H of the form

$$(\pi,\pi') = (\alpha_l,\beta_m) \quad \text{or} \quad (\pi,\pi') = (\beta_l,\alpha_m), \qquad l \neq m,\ 1 \leqq l \leqq 4,\ 1 \leqq m \leqq 4\ ,$$

where

$$\alpha_1 = \omega, \quad \alpha_2 = \omega - j - k, \quad \alpha_3 = \omega - i - k, \quad \alpha_4 = \omega - i - j\ ,$$
$$\beta_1 = \omega - i - j - k, \quad \beta_2 = \omega - i, \quad \beta_3 = \omega - j, \quad \beta_4 = \omega - k\ .$$

The sets

$$A = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}, \qquad B = \{\beta_1,\beta_2,\beta_3,\beta_4\}$$

are conjugacy classes in the multiplicative group of units (2.3) in H. Hence in case $\pi \in A$, we can choose a unit ε_1 such that

$$\varepsilon_1^{-1}\pi\varepsilon_1 = \alpha_1 = \omega\ . \tag{2.16}$$

Then

$$\varepsilon_1^{-1}\pi'\varepsilon_1 = \beta_m, \quad m = 2,\ 3 \text{ or } 4\ . \tag{2.17}$$

Finally taking $\varepsilon = \varepsilon_1\omega$, $\varepsilon_1(1-\omega)$ or ε_1 according as $m=2$, 3 or 4, we obtain by (2.16) and (2.17)

$$\varepsilon^{-1}\pi\varepsilon = \omega, \qquad \varepsilon^{-1}\pi'\varepsilon = \omega - k\ .$$

In case $\pi \in B$, we get analogously a unit ε such that

$$\varepsilon^{-1}\pi\varepsilon = \omega - k, \qquad \varepsilon^{-1}\pi'\varepsilon = \omega\ .$$

By (2.15) and definition 2 this proves theorem 1.

Definition 3. *A Farey simplex* $\mathrm{FS}(p_1q_1^{-1},p_2q_2^{-1},\ldots,p_5q_5^{-1})$ *in the space of quaternions is the convex hull of five points* $p_lq_l^{-1}$, $q_l \neq 0$, $1 \leqq l \leqq 5$, *such that the corresponding matrix* (2.12) *is a Farey matrix.*

By the definitions 1, 2 and 3 a Farey simplex corresponds to a number of classes of associated Farey matrices having no zero in the second row, and conversely every class of associated Farey matrices having no zero in the second row defines a Farey simplex.

The unimodular homographic map

$$\varphi:\ w = (\alpha z + \beta)(\gamma z + \delta)^{-1} \tag{2.18}$$

corresponding to the unimodular linear map Φ defined in (2.14) is a 1-1 map of the one point extension of the space of quaternions z onto the one point extension of the space of quaternions w.

Also, since $\varphi(\pi\varrho^{-1}) = pq^{-1}$ when $\pi,\varrho,p,q \in$ H are related by (2.14), φ is a 1-1 map of the set of quotients $\pi\varrho^{-1}$, $\pi,\varrho \in$ H, and π,ϱ without common non-trivial right divisors ($\varepsilon 0^{-1}$, where ε is a unit in H, counted among these quotients) onto itself.

By (2.14), $q=\gamma\pi+\delta\varrho$, and hence

$$|q| = |\gamma|\,|\varrho|\,|\pi\varrho^{-1}+\gamma^{-1}\delta|, \qquad \gamma,\varrho \neq 0 . \tag{2.19}$$

Further by (2.18)

$$\begin{aligned} w\gamma-\alpha &= (\alpha z+\beta)(\gamma z+\delta)^{-1}\gamma-\alpha \\ &= \big(\alpha z+\beta-\alpha\gamma^{-1}(\gamma z+\delta)\big)(\gamma z+\delta)^{-1}\gamma \\ &= (\beta-\alpha\gamma^{-1}\delta)(\gamma z+\delta)^{-1}\gamma , \end{aligned}$$

and hence

$$|w\gamma-\alpha| = |\beta\gamma-\alpha\gamma^{-1}\delta\gamma|\,|\gamma z+\delta|^{-1} .$$

However, since φ is unimodular,

$$N(\beta\gamma-\alpha\gamma^{-1}\delta\gamma) = 1$$

by the proof of (2.11), and consequently

$$|w\gamma-\alpha| = |\gamma z+\delta|^{-1} . \tag{2.20}$$

By the papers of Fueter [1] and Speiser [5] the Jacobi determinant of the unimodular homographic map φ defined in (2.18) equals $N(\gamma z+\delta)^{-4}$, and hence, since φ is conformal, we have for any fixed $z \in \mathsf{K}$, $z \neq -\gamma^{-1}\delta$, and arbitrary $\pi,\varrho \in \mathsf{H}$, $\varrho \neq 0$, such that $\pi\varrho^{-1} \neq -\gamma^{-1}\delta$,

$$\begin{aligned} |w-pq^{-1}| &= |\varphi(z)-\varphi(\pi\varrho^{-1})| \\ &= |z-\pi\varrho^{-1}|\,\big(N(\gamma\pi\varrho^{-1}+\delta)^{-1}+\sigma(z-\pi\varrho^{-1})\big) \\ &= |\varrho|\,|z\varrho-\pi|\,\big(N(\gamma\pi+\delta\varrho)^{-1}+N(\varrho)^{-1}\sigma(z-\pi\varrho^{-1})\big) , \end{aligned}$$

where

$$\sigma(z-\pi\varrho^{-1}) \to 0 \quad \text{as} \quad \pi\varrho^{-1} \to z . \tag{2.21}$$

Consequently, since $\gamma\pi+\delta\varrho=q$,

$$|q|\,|wq-p| = |\varrho|\,|z\varrho-\pi|\,\big(1+N(q)N(\varrho)^{-1}\sigma(z-\pi\varrho^{-1})\big) . \tag{2.22}$$

By (2.19), (2.21), (2.22) and the definitions of equivalence and approximation constant in § 1, we deduce for any irrational quaternion ξ the validity of

$$C(\xi) = C(\eta), \qquad \text{when } \xi \sim \eta . \tag{2.23}$$

The properties of a unimodular homographic map developped above together with the well-known property of being sphere-preserving (cf. [1]) allows us to investigate Farey simplices by means of their inverse images by suitably chosen unimodular homographic maps, e.g. obtained in letting the corresponding Farey matrices be of the form indicated in theorem 1. Further by the proof of theorem 1 any vertex of a given

Farey simplex may have ∞ as its inverse image by such a unimodular homographic map.

3. Fundamental properties of Farey simplices.

We shall begin this paragraph by collecting some important geometric properties connected with the ring H of integral quaternions.

Let $\mathscr{P}_{\mathsf{U}}$ denote the convex hull of the set U consisting of the 24 units (2.3) in H. It is well known that $\mathscr{P}_{\mathsf{U}}$ is a regular polytope of Schläfli symbol (3, 4, 3). Thus the boundary of $\mathscr{P}_{\mathsf{U}}$ consists of 24 vertices, 96 edges of unit length, 96 regular triangles and 24 regular octahedra.

Further let

$$\mathsf{I} = \{z \in \mathsf{H} \mid N(z) \equiv 0 \pmod 2\}\,. \tag{3.1}$$

Otherwise stated, I is the twosided ideal in H generated by $1+i$. Then it is also well known that the set of regular congruent polytopes of Schäfli symbol (3, 4, 3)

$$\{\mathscr{P}_{\mathsf{U}} + z \mid z \in \mathsf{I}\} \tag{3.2}$$

constitutes a regular tessellation of the space of quaternions.

The importance of Farey simplices as a tool to treat the approximation problem of quaternions we are dealing with is obvious already from the following

THEOREM 2. *Let ξ be a quaternion satisfying the inequality*

$$|\xi - p_0 q_0^{-1}| \leqq \frac{1}{2^{\frac{1}{2}} N(q_0)}, \tag{3.3}$$

where $p_0, q_0 \in \mathsf{H}$, $q_0 \neq 0$, and p_0, q_0 have no non-trivial common right divisor.

Then ξ belongs to a non-degenerate Farey simplex having $p_0 q_0^{-1}$ as one of its vertices.

The constant $2^{\frac{1}{2}}$ in (3.3) is least possible.

In the proof of theorem 2 we shall need the following

LEMMA 2. *Let Γ be a sphere in the space of quaternions with radius $2^{\frac{1}{2}}$ and an arbitrary centre O.*

Then O is an interior point of a polytope $\mathscr{P}$ with vertices $z_1, z_2, \ldots, z_m$, where m depends on O, $8 \leqq m \leqq 48$, such that $\mathscr{P}$ satisfies the following conditions:

$$z_l \in \mathsf{H}, \qquad 1 \leqq l \leqq m\,, \tag{i}$$

(ii) *all the n faces of $\mathscr{P}$, $16 \leqq n \leqq 192$, are regular tetrahedra of edgelength 1,*

(iii) *all the n spheres through O and the vertices of a face of $\mathscr{P}$ lie in the closed ball bounded by Γ.*

PROOF. If $O \in \mathsf{H}$, we may suppose without restriction that $O=0$. In this case $m=48$, $z_1, z_2, \ldots, z_{24}$ are the 24 units in H, and $z_{25}, z_{26}, \ldots, z_{48}$ are the 24 elements in H symmetric with 0 with respect to the 24 faces of $\mathscr{P}_{\mathsf{U}}$. (Otherwise stated, $z_{25}, z_{26}, \ldots, z_{48}$ are the $z \in \mathsf{H}$ which have $N(z)=2$.) The faces of $\mathscr{P}$ are the $n=192=24\times 8$ regular tetrahedra of edgelength 1 having one vertex z_l, $25 \leqq l \leqq 48$, and three vertices in U. Hence the conditions (i) and (ii) are satisfied, but also (iii), since the 192 spheres in question are just the 24 spheres (each sphere occurring 8 times) having diameters $0z_l$, $25 \leqq l \leqq 48$, of length $2^{\frac{1}{2}}$.

In the general case $O \notin \mathsf{H}$, we may assume without restriction (otherwise we make use of the tessellation (3.2)) that $O \in \mathscr{P}_{\mathsf{U}}$. Now the polytope $\mathscr{P}$ defined above in the special case $O=0$ may be subdivided into 24 regular crosspolytopes of edgelength 1, each crosspolytope being a double pyramid with a face of $\mathscr{P}_{\mathsf{U}}$ as a base. In the present case we define $\mathscr{P}$ as the union of those of the 24 crosspolytopes for which O is an interior point of the corresponding circumscribed spheres. Since $O \notin \mathsf{H}$, this union is non-empty. By construction $\mathscr{P}$ satisfies the conditions (i), (ii) and (iii) as well as the inequalities on m and n stated in the theorem.

This proves lemma 2.

PROOF OF THEOREM 2. The theorem states that the closed ball bounded by the sphere C with centre at $p_0 q_0^{-1}$ and radius $(N(q_0)2^{\frac{1}{2}})^{-1}$ is covered by the set of all non-degenerate Farey simplices having $p_0 q_0^{-1}$ as a vertex. In fact, it will be shown that at most 192 such Farey simplices are sufficient to cover the ball bounded by C.

Since p_0, q_0 have no non-trivial common right divisor, there exists by lemma 1 a unimodular homographic map of the form

$$\varphi:\ w = (p_0 z + P)(q_0 z + Q)^{-1} .$$

By (2.20)

$$|w - p_0 q_0^{-1}| \leqq \frac{1}{N(q_0)2^{\frac{1}{2}}} \Leftrightarrow |z + q_0^{-1}Q| \geqq 2^{\frac{1}{2}} , \tag{3.4}$$

and consequently

$$\varphi^{-1}(C) = \Gamma ,$$

where Γ is the sphere in z-space with radius $2^{\frac{1}{2}}$ and centre at $-q_0^{-1}Q$. Also by (3.4) the interior of C corresponds by φ^{-1} to the exterior of Γ. Hence by lemma 2 with $O = -q_0^{-1}Q$ the closed ball in w-space bounded by C is covered by the n, $16 \leqq n \leqq 192$, Farey simplices (and then also by the non-degenerate Farey simplices among these)

$$\mathrm{FS}\big(p_0 q_0^{-1}, \varphi(z_{\nu 1}), \varphi(z_{\nu 2}), \varphi(z_{\nu 3}), \varphi(z_{\nu 4})\big), \qquad 1 \leqq \nu \leqq n ,$$

where

$$z_{\nu 1}, z_{\nu 2}, z_{\nu 3}, z_{\nu 4}, \qquad 1 \leqq \nu \leqq n ,$$

are the vertices of the n tetrahedral faces of the polytope $\mathscr{P}$ of lemma 2.

The constant $2^{\frac{1}{2}}$ in (3.3) is best possible for $\xi = \frac{1}{2}(1+i)$ and $(p_0, q_0) = (0,1)$.

This proves theorem 2.

The following two lemmas are concerned with Farey simplices of special type.

LEMMA 3. *No Farey simplex is regular.*

PROOF. In a regular simplex in Euclidean space R^4 the acute angle u between two of its faces has $\cos u = \frac{1}{4}$, and hence we may prove the lemma by establishing that no angle between two faces of a Farey simplex can possess this property.

Suppose on the contrary that a certain Farey simplex has two faces with the above property. Then, since the two faces in question have normals with all coordinates in Q, there are

$$a_l, b_l \in \mathsf{Z}, \qquad 1 \leqq l \leqq 4 ,$$

such that

$$\text{g.c.d.}(a_1, a_2, a_3, a_4) = \text{g.c.d.}(b_1, b_2, b_3, b_4) = 1 , \tag{3.5}$$

and

$$16 \left(\sum_1^4 a_l b_l \right)^2 = \sum_1^4 a_l^2 \sum_1^4 b_l^2 . \tag{3.6}$$

By (3.6) we may assume without restriction that

$$\sum_1^4 a_l^2 \equiv 0 \pmod 4 ,$$

and since at least one a_l is odd by (3.5),

$$a_l \equiv 1 \pmod 2, \qquad 1 \leqq l \leqq 4 ,$$

and hence

$$\sum_1^4 a_l^2 \equiv 4 \pmod 8 .$$

Consequently, by (3.5) and (3.6) analogous congruences hold with a_l replaced by b_l, $1 \leqq l \leqq 4$.

However, this leads to a contradiction, since the left hand side of (3.6) is then divisible by 64, while the right hand side of (3.6) is not divisible by 32.

This proves lemma 3.

The *norm* N of a Farey simplex $\mathrm{FS}(p_1q_1^{-1}, p_2q_2^{-1}, \ldots, p_5q_5^{-1})$ is defined as

$$(3.7)\qquad N = N(\mathrm{FS}) = \sum_1^5 N_l, \quad \text{where } N_l = N(q_l), \qquad 1 \leqq l \leqq 5 .$$

From this definition $N(\mathrm{FS}) \geqq 5$ for any Farey simplex FS. By lemma 3 there is no Farey simplex of norm 5. However, in the next lemma we shall see that Farey simplices of norm 6 are particularly interesting.

LEMMA 4. *The Farey simplices of norm* 6 *constitute a tessellation of the space of quaternions.*

PROOF. In the first place we shall construct a tessellation of the space of quaternions into Farey simplices of norm 6. In order to do this we subdivide $\mathscr{P}_{\mathsf{U}}$ into 24 pyramids with apex 0 and the octahedral faces of $\mathscr{P}_{\mathsf{U}}$ as bases, one of these bases having vertices

$$A_1 = 1, \quad A_2 = i, \quad B_1 = \omega, \quad B_2 = \omega - j - k,$$
$$C_1 = \omega - k, \quad C_2 = \omega - j ,$$

where A_1 and A_2 etc. are opposite vertices. The centre of this octahedron is $(1-i)^{-1}$. Each of these 24 pyramids is subdivisible into 8 Farey simplices of norm 6, in particular the pyramid with the base indicated above is subdivisible into the 8 Farey simplices of norm 6,

$$\mathrm{FS}(0, A_{l_1}, B_{l_2}, C_{l_3}, (1-i)^{-1}) \quad \text{with} \quad l_\nu = 1 \text{ or } 2, \quad 1 \leqq \nu \leqq 3 .$$

Finally, the set of Farey simplices obtained by considering all translates $z \in \mathsf{I}$ of the $192 = 8 \times 24$ Farey simplices of norm 6 subdividing $\mathscr{P}_{\mathsf{U}}$ yields a tessellation of the space of quaternions into Farey simplices of norm 6.

Secondly, we must show that each Farey simplex of norm 6 occurs in the tessellation of the space of quaternions thus constructed. We note first that the norms of the "denominators" in a Farey simplex of norm 6 are 1, 1, 1, 1, 2. Hence if a Farey simplex of norm 6 has 0 as a vertex, three of the other vertices are units in H spanning a regular triangle of side 1, and the fifth vertex is then determined except for a symmetry in the hyperplane through the first four vertices. However, since $\mathscr{P}_{\mathsf{U}}$ has 96 regular triangles of side 1 on the boundary, there are at most $192 = 2 \times 96$ Farey simplices of norm 6 having 0 as a vertex. Hence the tessellation constructed above, which in fact contains exactly 192 Farey simplices of norm 6 with 0 as a vertex, must contain all such Farey simplices.

In an arbitrary Farey simplex of norm 6 four of the vertices are in H, and span a regular tetrahedron of edge 1. Hence the four vertices in H

are in different residue classes modulo I, and since H/I consists of exactly 4 elements (represented by $0, 1, \omega, \omega - k$), one of the vertices of the Farey simplex considered is in I. Consequently an arbitrary Farey simplex of norm 6 is the translate by a $z \in$ I of a Farey simplex of norm 6 with 0 as a vertex. This, however, implies that the tessellation constructed above contains all Farey simplices of norm 6.

This proves lemma 4.

By theorem 2 it is important to get a description of the set of non-degenerate Farey simplices containing a fixed quaternion ξ. Incidentally we know from lemma 4 that this set contains at least one Farey simplex of norm 6, and so is non-empty. The central idea in our description is that of *subdivision* of a given Farey simplex into a finite number of Farey simplices. Here subdivision is to be taken in a combinatorial sense rather than a geometric one. In fact, the subdivisions of a Farey simplex we are going to consider give in general only a covering of the given Farey simplex by the Farey simplices in the subdivisions.

THEOREM 3. *Every Farey simplex* FS *with* $N(\mathrm{FS}) > 6$ *is in two different ways subdivisible into* 31 *Farey simplices. The vertices of each subdivision all lie on one side of the circumscribed sphere* (*hyperplane*) *of* FS, *and the vertices of the two subdivisions are inverse* (*symmetric*) *with respect to this sphere* (*hyperplane*).

Every Farey simplex FS *with* $N(\mathrm{FS}) = 6$ *is in one way subdivisible into* 31 *Farey simplices. The vertices of this subdivision all lie inside the circumscribed sphere of* FS.

The graph of a subdivision of FS *together with* FS *itself is isomorphic to the graph of the boundary of the* 5-*dimensional crosspolytope.*

PROOF. By theorem 1 we may suppose that the Farey simplex FS is of the form $\mathrm{FS}(p_1 q_1^{-1}, p_2 q_2^{-1}, \ldots, p_5 q_5^{-1})$, where

$$(3.8) \qquad \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \omega & \omega - k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Now we consider the following two 2×5 matrices

$$(3.9) \qquad \begin{pmatrix} p_1' & p_2' & p_3' & p_4' & p_5' \\ q_1' & q_2' & q_3' & q_4' & q_5' \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 & 1+i & i & \omega - j - k & \omega - j \\ 1 - i & 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$(3.10)\quad \begin{pmatrix} p_1^* & p_2^* & p_3^* & p_4^* & p_5^* \\ q_1^* & q_2^* & q_3^* & q_4^* & q_5^* \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 & 1+j & j & \omega-i-k & \omega-i \\ 1-j & 1 & 1 & 1 & 1 \end{pmatrix}.$$

By the Farey matrix-preserving property of a unimodular linear map the $31+31$ 2×5 matrices obtained by replacing 1, 2, 3, 4 or all columns in (3.8) by the corresponding columns in (3.9) or in (3.10) are all as well Farey matrices. Further it is easily seen that any 2×5 matrix over H, which is related to the Farey matrix (3.8) in this way, is obtained from the Farey matrices (3.9) or (3.10) by multiplication of each column from the right by a unit in H. Of course, we require in addition that such a matrix should have all columns different from right multiples by units in H of the corresponding columns in (3.8).

To indicate how this is seen, we write the first column in such a matrix in the form

$$(3.11)\quad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} \pi \\ \varrho \end{pmatrix},$$

and $\pi, \varrho \in \mathsf{H}$ are then subject to the requirement that

$$\begin{pmatrix} \pi & 0 & 1 & \omega & \omega-k \\ \varrho & 1 & 1 & 1 & 1 \end{pmatrix}$$

is a Farey matrix. However, by the additional requirement above this shows that

$$(3.12)\quad \begin{pmatrix} \pi \\ \varrho \end{pmatrix} = \begin{pmatrix} \varepsilon \\ (1-i)\varepsilon \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \pi \\ \varrho \end{pmatrix} = \begin{pmatrix} \varepsilon \\ (1-j)\varepsilon \end{pmatrix},$$

ε being a unit in H, and hence by (3.11) and (3.12)

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_1'\varepsilon \\ q_1'\varepsilon \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_1^*\varepsilon \\ q_1^*\varepsilon \end{pmatrix}.$$

Similarly, the other columns in such a 2×5 matrix are right multiples by a unit in H of the corresponding columns in (3.9) or (3.10). Finally the result follows from the observation that

$$\begin{pmatrix} p_l' & p_m^* \\ q_l' & q_m^* \end{pmatrix}$$

by (3.9) and (3.10) is not a unimodular matrix for $l \neq m$, $1 \leqq l \leqq 5$, $1 \leqq m \leqq 5$.

Now assume that none of the Farey matrices (3.9) and (3.10) have zeros in the second row. Then, by the properties of the unimodular homographic map

(3.13) $$\varphi : w = (p_1 z + p_2)(q_1 z + q_2)^{-1}$$

deduced in § 2, there are precisely two subdivions of FS of the kind described in the formulation of theorem 3. The 31 Farey simplices in each of the subdivisions are obtained by exchanging 1, 2, 3, 4 or all vertices in FS by the corresponding vertices in

(3.14) $$\mathrm{FS}(p_1' q_1'^{-1}, p_2' q_2'^{-1}, \ldots, p_5' q_5'^{-1})$$

or in

(3.15) $$\mathrm{FS}(p_1^* q_1^{*-1}, p_2^* q_2^{*-1}, \ldots, p_5^* q_5^{*-1}) .$$

The statement in theorem 3 about the position of the vertices in the two subdivisions of FS follows from the properties of the unimodular homographic map φ together with the observation that the two point sets

$$\{(1-i)^{-1}, 1+i, i, \omega-j-k, \omega-j\} , \qquad \{(1-j)^{-1}, 1+j, j, \omega-i-k, \omega-i\}$$

lie on opposite sides of and are symmetric with respect to the hyperplane through 0, 1, ω, $\omega - k$.

It remains to be proved that one of the Farey matrices (3.9) or (3.10) can have a zero in the second row only in case $N(\mathrm{FS}) = 6$. By a remark at the end of § 2 we may suppose that $q_1' = 0$ or $q_1^* = 0$. However, by (2.19) this assumption is equivalent to

(3.16) $$-q_1^{-1} q_2 = (1-i)^{-1} \text{ or } (1-j)^{-1} .$$

Since q_1, q_2 by lemma 1 have no non-trivial common left divisor, it follows by (3.16) that $N(q_2) = 1$ and $N(q_1) = 2$. Using (2.19) again we see that $N(q_3) = N(q_4) = N(q_5) = 1$, and hence $N(\mathrm{FS}) = 6$.

Conversely, for a Farey matrix FS with $N(\mathrm{FS}) = 6$ we may assume, by the remark at the end of § 2, that $N(q_1) = 2$, $N(q_l) = 1$ for $2 \leqq l \leqq 5$, and hence

$$\begin{pmatrix} 1 & p_2 & p_3 & p_4 & p_5 \\ 0 & q_2 & q_3 & q_4 & q_5 \end{pmatrix}$$

is a Farey matrix, and consequently $q_1' = 0$ or $q_1^* = 0$. Hence for a Farey matrix FS of norm 6 there is exactly one subdivision arising from (3.9) or (3.10) according as $-q_1^{-1} q_2 = (1-j)^{-1}$ or $(1-i)^{-1}$.

For a non-degenerate Farey simplex FS we shall call a subdivision *inner* or *outer* according as the vertices in the subdivision are all inside or outside the circumscribed sphere of FS. Obviously, with the notation above (3.14) or (3.15) is the *central* Farey simplex in the inner subdivision of FS according as $-q_1^{-1} q_2$ lies in the open halfspace bounded by the hyperplane through 0, 1, ω, $\omega - k$ and containing $(1-j)^{-1}$ or

$(1-i)^{-1}$, respectively. In particular we note that a Farey simplex FS of norm 6 always has an inner subdivision (but no outer subdivision). This completes the proof of theorem 3.

In the following it is convenient to let (3.14) and (3.15) always denote the central Farey simplices in the inner and outer subdivisions of a non-degenerate Farey simplex FS, respectively. This amounts to interchanging the primes and asterisks in (3.9) and (3.10) in case $-q_1^{-1}q_2$ lies in the open halfspace bounded by the hyperplane through 0, 1, ω, $\omega-k$ and containing $(1-i)^{-1}$. With this convention it follows immediately from (2.19) that

$$N(q_l') > N(q_l^*), \qquad 1 \leqq l \leqq 5 . \tag{3.17}$$

In the case of a degenerate Farey simplex FS we cannot distinguish between inner and outer subdivision, and we note that in this case (again by (2.19))

$$N(q_l') = N(q_l^*), \qquad 1 \leqq l \leqq 5 . \tag{3.18}$$

The following lemma gives important supplementary information to theorem 3:

LEMMA 5. *Let $\xi \in \mathsf{K}$ belong to the non-degenerate Farey simplex* FS. *Then there is a non-degenerate Farey simplex* FS′ *among the Farey simplices in the inner subdivision of* FS, *such that*

$$\xi \in \mathrm{FS}' \quad \textit{and} \quad N(\mathrm{FS}') > N(\mathrm{FS}) .$$

In case $N(\mathrm{FS}) > 6$, there is a non-degenerate Farey simplex FS* *among the Farey simplices in the outer subdivision of* FS, *such that*

$$\xi \in \mathrm{FS}^* \quad \textit{and} \quad N(\mathrm{FS}^*) < N(\mathrm{FS}) .$$

PROOF. Using the notation in the proof of theorem 3, we may suppose by symmetry that $-q_1^{-1}q_2$ lies in the open halfspace bounded by the hyperplane through 0, 1, ω, $\omega-k$ and containing $(1-j)^{-1}$.

The central Farey simplex $\mathrm{FS}(p_1'q_1'^{-1}, p_2'q_2'^{-1}, \ldots, p_5'q_5'^{-1})$ in the inner subdivision of FS is then given by (3.9), and hence, by (2.19) applied to the unimodular homographic map φ defined in (3.13),

$$|q_1'| = |q_1|\,|1-i|\,|(1-i)^{-1} + q_1^{-1}q_2| .$$

Hence $N(q_1') \leqq N(q_1)$ if and only if

$$N((1-i)^{-1} + q_1^{-1}q_2) \leqq \tfrac{1}{2} ,$$

that is, $-q_1^{-1}q_2$ lies inside or on the boundary of the sphere with centre at $(1-i)^{-1}$ and passing through 0, 1, ω, $\omega-k$ and $1+i$, i, $\omega-j-k$, $\omega-j$. Equivalently the points $p_l' q_l'^{-1}$, $1 \leqq l \leqq 5$, all lie in the closed halfspace bounded by the hyperplane through $p_l q_l^{-1}$, $2 \leqq l \leqq 5$, and not containing $p_1 q_1^{-1}$.

Consequently, if $N(q_1') \leqq N(q_1)$, none of the 16 Farey simplices in the inner subdivision of FS, which have $p_1' q_1'^{-1}$ as a vertex, contain interior points of FS. By the remark at the end of § 2 the same result holds then for any l, $1 \leqq l \leqq 5$, for which $N(q_l') \leqq N(q_l)$, and hence FS is covered by non-degenerate Farey simplices in the inner subdivision of FS with norms strictly greater than $N(\mathrm{FS})$.

This proves the first part of lemma 5.

Similarly the central Farey simplex $\mathrm{FS}(p_1^* q_1^{*-1}, p_2^* q_2^{*-1}, \ldots, p_5^* q_5^{*-1})$ in the outer subdivision of FS is given by (3.10), and

$$|q_1^*| = |q_1|\,|1-j|\,|(1-j)^{-1} + q_1^{-1} q_2| \,.$$

Hence $N(q_1^*) \geqq N(q_1)$ if and only if

$$N\big((1-j)^{-1} + q_1^{-1} q_2\big) \geqq \tfrac{1}{2} \,,$$

that is, $-q_1^{-1} q_2$ lies outside or on the boundary of the sphere with centre at $(1-j)^{-1}$ and passing through 0, 1, ω, $\omega-k$ and $1+j$, j, $\omega-i-k$, $\omega-j$. Equivalently the points $p_l^* q_l^{*-1}$, $1 \leqq l \leqq 5$, all lie in the closed halfspace bounded by the hyperplane through $p_l q_l^{-1}$, $2 \leqq l \leqq 5$, and not containing $p_1 q_1^{-1}$.

Consequently, if $N(q_1^*) \geqq N(q_1)$, none of the 16 Farey simplices in the outer subdivision of FS, which have $p_1^* q_1^{*-1}$ as a vertex, contain interior points of FS. By the remark at the end of § 2 the same result holds then for any l, $1 \leqq l \leqq 5$, for which $N(q_l^*) \geqq N(q_l)$, and hence FS is covered by non-degenerate Farey simplices in the outer subdivision of FS with norms strictly less than $N(\mathrm{FS})$.

This finishes the proof of lemma 5.

DEFINITION 4. *A chain of Farey simplices containing $\xi \in \mathsf{K}$ is an infinite sequence of non-degenerate Farey simplices*

(3.19) $$\mathrm{FS}^{(0)}, \mathrm{FS}^{(1)}, \ldots, \mathrm{FS}^{(n)}, \ldots$$

such that

(i) $\xi \in \mathrm{FS}^{(n)}$ *for all* $n \geqq 0$,

(ii) $\mathrm{FS}^{(n+1)}$ *is one of the Farey simplices in the inner subdivision of* $\mathrm{FS}^{(n)}$, $n \geqq 0$,

(iii) $N(\mathrm{FS}^{(0)}) = 6$,

(iv) $N(\mathrm{FS}^{(n+1)}) > N(\mathrm{FS}^{(n)})$ *for all* $n \geqq 0$.

THEOREM 4. *For every $\xi \in \mathsf{K}$ and any non-degenerate Farey simplex* FS *containing ξ there exists a chain* (3.19) *of Farey simplices containing ξ such that* $\mathrm{FS}=\mathrm{FS}^{(n)}$ *for some $n \geqq 0$.*

For every chain (3.19) *of Farey simplices containing $\xi \in \mathsf{K}$*

$$\lim_{n\to\infty} p_l^{(n)} q_l^{(n)^{-1}} = \xi, \qquad 1 \leqq l \leqq 5 . \tag{3.20}$$

COROLLARY 1. *If there exists a chain of Farey simplices* (3.19) *containing ξ, and such that ξ is an interior point of* $\mathrm{FS}^{(n)}$ *for all $n \geqq 0$, then* (3.19) *is the only chain of Farey simplices containing ξ.*

COROLLARY 2. *If $\xi = a_1 + a_2 i + a_3 j + a_4 k$, where 1, a_1, a_2, a_3 and a_4 are linearly independent over* Q, *there is a unique chain of Farey simplices containing ξ.*

PROOF. By lemma 3 the norm of a Farey simplex is at least 6, hence the first (and difficult) part of theorem 4 follows by repeated application of lemma 5. It is important to notice that this result together with lemma 4 shows that every $\xi \in \mathsf{K}$ is contained in at least one chain of Farey simplices.

By the definition of a Farey simplex FS and formula (2.11), the diameter of FS satisfies the following inequality (we assume for simplicity that $N_1 \geqq N_2 \geqq N_3 \geqq N_4 \geqq N_5$):

$$\begin{aligned} \mathrm{diam}(\mathrm{FS}) &= |p_4 q_4^{-1} - p_5 q_5^{-1}| \\ &\leqq |p_1 q_1^{-1} - p_4 q_4^{-1}| + |p_1 q_1^{-1} - p_5 q_5^{-1}| \\ &= (N_1 N_4)^{-\frac{1}{2}} + (N_1 N_5)^{-\frac{1}{2}} \\ &\leqq 2N_1^{-\frac{1}{2}} \\ &\leqq 2(\tfrac{1}{5} N(\mathrm{FS}))^{-\frac{1}{2}} . \end{aligned}$$

From this inequality and condition (iv) of definition 4

$$\lim_{n\to\infty} \mathrm{diam}(\mathrm{FS}^{(n)}) = 0 ,$$

and consequently (3.20) follows from condition (i) of definition 4.

To prove corollary 1 suppose there were a chain $\widetilde{\mathrm{FS}}^{(n)}$, $n \geqq 0$, of Farey simplices containing ξ and an $n_0 \geqq 0$ such that

$$\widetilde{\mathrm{FS}}^{(n)} = \mathrm{FS}^{(n)}, \qquad 0 \leqq n < n_0 ,$$

but

$$\widetilde{\mathrm{FS}}^{(n_0)} \neq \mathrm{FS}^{(n_0)} .$$

Then

$$\xi \in \widetilde{\mathrm{FS}}^{(n_0)} \cap \mathrm{FS}^{(n_0)} ,$$

and hence ξ were a boundary point of $\mathrm{FS}^{(n_0)}$ by lemma 4 in case $n_0=0$, and by theorem 3 together with the conditions (i) and (ii) of definition 4 for $n=n_0-1$ in case $n_0>0$. This contradiction proves corollary 1.

As to corollary 2, we know already by a remark above that, ξ is contained in at least one chain of Farey simplices. However, by assumption ξ does not lie on the boundary of any non-degenerate Farey simplex, thus the uniqueness follows in fact from corollary 1.

4. Linear norm relations.

In this section we shall continue the investigation of the subdivisions of a Farey simplex described in the preceding section. The linear norm relations connected with these subdivisions will be deduced by means of the following

LEMMA 6. *Let*

$$\Phi(\pi_l,\varrho_l) = (p_l,q_l), \qquad 1\leqq l\leqq n\,,$$

where Φ is a unimodular linear map.

Suppose the following linear relations hold:

$$\sum_{l=1}^{n} b_l N(\pi_l) = \sum_{l=1}^{n} b_l N(\varrho_l) = \sum_{l=1}^{n} b_l \pi_l \bar{\varrho}_l = 0,\ b_l\in \mathsf{R},\ 1\leqq l\leqq n\,.$$

Then the corresponding linear relations

$$\sum_{l=1}^{n} b_l N(p_l) = \sum_{l=1}^{n} b_l N(q_l) = \sum_{l=1}^{n} b_l p_l \bar{q}_l = 0$$

are also valid.

PROOF. See the proof of lemma 2 in [4].

For any Farey simplex at least one of the corresponding Farey matrices has the form announced in theorem 1. Hence the points $p_l q_l^{-1}$ in the subdivisions of a Farey simplex have the form

$$(p_l,q_l) = \Phi(\pi_l,\varrho_l)\,,$$

where Φ is a unimodular linear map, and the (π_l,ϱ_l) involved are listed in table 1 below, the indices being in agreement with the notation used in § 3.

Alternatively, the primes and asterisks in table 1 should be interchanged.

l	π_l	ϱ_l	$N(\pi_l)$	$N(\varrho_l)$	$\pi_l\bar{\varrho}_l$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	1	1	1	1
4	ω	1	1	1	ω
5	$\omega-k$	1	1	1	$\omega-k$
1′	1	$1-i$	1	2	$1+i$
2′	$1+i$	1	2	1	$1+i$
3′	i	1	1	1	i
4′	$\omega-j-k$	1	1	1	$\omega-j-k$
5′	$\omega-j$	1	1	1	$\omega-j$
1*	1	$1-j$	1	2	$1+j$
2*	$1+j$	1	2	1	$1+j$
3*	j	1	1	1	j
4*	$\omega-i-k$	1	1	1	$\omega-i-k$
5*	$\omega-i$	1	1	1	$\omega-i$

Table 1.

From table 1 or the alternative one the following linear norm relations connected with the subdivisions of a Farey simplex are deduced by means of lemma 6:

$$N'+N^* = 3N, \tag{4.1}$$

$$N_1+N_1' = N_2+N_2' = \ldots = N_5+N_5', \tag{4.2}$$

$$N_1+N_1^* = N_2+N_2^* = \ldots = N_5+N_5^*. \tag{4.3}$$

In these norm relations

$$N = \sum_{l=1}^{5} N_l, \quad \text{where} \quad N_l=N(q_l), \qquad 1 \leqq l \leqq 5,$$

and N_l', N_l^*, N', N^* are similarly defined.

Of course, by lemma 6 all the norm relations above are valid with $N_l=N(p_l)$ as well.

It should be noticed that the number of independent norm relations listed above equals the number of points involved minus six $(15-6=9)$, which is the maximal number of linearly independent relations obtainable by means of lemma 6, since $\mathsf{R}\times\mathsf{R}\times\mathsf{K}$ is a 6-dimensional vector space over R.

Now, given a Farey simplex, only five independent norms are known, and hence there is one norm relation missing in order that the N_l in the subdivisions should be determined. It follows from the definition of a

Farey simplex that once all the N_l in the subdivisions are known, the points $p_l q_l^{-1}$ themselves are determined geometrically.

5. A non-linear norm relation.

It was pointed out in the preceding paragraph that, given a Farey simplex, the inner and outer subdivisions are not completely determined by the linear norm relations found in that paragraph, but that there is one norm relation missing. It was also motivated that this norm relation must be non-linear.

Now we shall deduce such a norm relation, namely

$$N' = \tfrac{3}{2}N + \tfrac{5}{2}(N^2-4N^{(2)})^{\frac{1}{2}}, \tag{5.1}$$

where N and N' as usual are the norms of the given Farey simplex and the central Farey simplex in its inner subdivision, respectively, while

$$N^{(2)} = \sum_{l=1}^{5} N_l^2 . \tag{5.2}$$

By theorem 1 at least one of the Farey matrices corresponding to the Farey simplex given has the form

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \omega & \omega-k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} . \tag{5.3}$$

Further by the proof of theorem 3 (cf. table 1 in § 4) the inner and outer central Farey simplices in the subdivisions of the given Farey simplex are represented by the Farey matrices

$$\begin{pmatrix} p_1' & p_2' & p_3' & p_4' & p_5' \\ q_1' & q_2' & q_3' & q_4' & q_5' \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 & 1+i & i & \omega-j-k & \omega-j \\ 1-i & 1 & 1 & 1 & 1 \end{pmatrix} \tag{5.4}$$

and

$$\begin{pmatrix} p_1^* & p_2^* & p_3^* & p_4^* & p_5^* \\ q_1^* & q_2^* & q_3^* & q_4^* & q_5^* \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 & 1+j & j & \omega-i-k & \omega-i \\ 1-j & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{5.5}$$

or alternatively the same formulae with primes and asterisks interchanged. From (5.3) we get using (5.2) and the simple calculation rules for quaternions listed in § 2

$$N = 4(N_1+N_2) + S(q_1(2+i+j)\bar{q}_2)$$

and

$$N^{(2)} = 4(N_1^2+N_2^2) + 6N_1N_2 + 2(N_1+N_2)\,S(q_1(2+i+j)\bar{q}_2) + \\ + S^2(q_1\bar{q}_2) + S^2(q_1\omega\bar{q}_2) + S^2(q_1(\omega-k)\bar{q}_2) ,$$

whence

$$\begin{aligned} N^2 - 4N^{(2)} &= 8N_1N_2 + S^2(q_1(2+i+j)\bar{q}_2) - 4S^2(q_1\bar{q}_2) - \\ &\quad - S^2(q_1(1+i+j+k)\bar{q}_2) - S^2(q_1(1+i+j-k)\bar{q}_2) \\ &= 8N_1N_2 + S^2(q_1(i-j)\bar{q}_2) - \\ &\quad - 2\big(S^2(q_1\bar{q}_2) + S^2(q_1 i\bar{q}_2) + S^2(q_1 j\bar{q}_2) + S^2(q_1 k\bar{q}_2)\big) , \end{aligned}$$

and hence by (2.1)

$$N^2 - 4N^{(2)} = S^2(q_1(i-j)\bar{q}_2) . \tag{5.6}$$

Similarly, we get from (5.4) and (5.5)

$$\begin{aligned} N' &= 6(N_1+N_2) + S(q_1(3+4i-j)\bar{q}_2) , \\ N^* &= 6(N_1+N_2) + S(q_1(3-i+4j)\bar{q}_2) , \end{aligned}$$

or alternatively the same formulae with primes and asterisks interchanged. Hence in either case we get

$$(N'-N^*)^2 = 25\, S^2(q_1(i-j)\bar{q}_2) . \tag{5.7}$$

Since, by (3.17) and (3.18), $N' \geqq N^*$, a combination of (5.6) and (5.7) yields

$$N' - N^* = 5(N^2 - 4N^{(2)})^{\frac{1}{2}} .$$

Finally, this formula together with (4.1) proves (5.1).

6. Approximation lemmas.

In this section we shall deduce a number of important approximation lemmas of a purely geometric nature, however formulated by means of quaternions. The degree of approximation of a quotient pq^{-1} of quaternions to a quaternion $\xi \neq pq^{-1}$ will in these lemmas be measured by means of the real number c defined as follows

$$c = (|q|\,|\xi q - p|)^{-1} . \tag{6.1}$$

We shall begin with two preparatory lemmas of independent interest.

Lemma 7. *Let p', p'', q', q'' be quaternions, q', $q'' \neq 0$, such that*

$$\Delta^2 = |p'q'^{-1} - p''q''^{-1}|^2\, N(q')\, N(q'') > 0 .$$

Further let ξ be any quaternion different from $p'q'^{-1}$ and $p''q''^{-1}$. The real numbers c' and c'' are given by (6.1), *and the angle u, $0 \leqq u \leqq \pi$, is the angle $p'q'^{-1}, \xi, p''q''^{-1}$. Also let $f = \max(|q''|/|q'|, |q'|/|q''|)$.*

Suppose $f \geqq f_0 \geqq 1$, $u \geqq u_0 \geqq \frac{1}{2}\pi$, then

$$\max(c', c'') \geqq (f_0^2 + 1/f_0^2 - 2\cos u_0)^{\frac{1}{2}}/\Delta ,$$

where the equality sign occurs if and only if simultaneously

$$f = f_0, \quad u = u_0, \quad |\xi q' - p'|/|\xi q'' - p''| = |q''/q'| ,$$

in which case

$$c' = c'' = (f_0^2 + 1/f_0^2 - 2\cos u_0)^{\frac{1}{2}}/\Delta .$$

PROOF. See the proof of lemma 3 in [4].

LEMMA 8. *Let $n+1$ unit vectors $\boldsymbol{a}_l$, $1 \leqq l \leqq n+1$, $n \geqq 2$, initiating from $\boldsymbol{o}$ be situated in Euclidean space R^n such that the convex hull of the points $\boldsymbol{a}_l$, $1 \leqq l \leqq n+1$, contains $\boldsymbol{o}$. The angle $u_{l,m}$, $0 \leqq u_{l,m} \leqq \pi$, is the angle between $\boldsymbol{a}_l$ and $\boldsymbol{a}_m$.*

Then for any δ, $0 \leqq \delta < (n^2-n)^{-1}$, either

$$\text{(i)} \qquad \min_{1 \leqq l < m \leqq n+1} \cos u_{l,m} < -(1/n + \delta) ,$$

or

$$\text{(ii)} \qquad \cos u_{l,m} = -(1/n + \delta_{l,m})$$

with

$$\delta_{l,m} \geqq -\frac{(n-1)(n+1)^2}{2n} \frac{\delta}{\left(1-((n^2-n)\delta)^{\frac{1}{2}}\right)^2}, \qquad 1 \leqq l < m \leqq n+1 .$$

PROOF. Suppose that we are not in case (i), i.e.

$$\text{(6.2)} \qquad \delta_{l,m} \leqq \delta, \qquad 1 \leqq l < m \leqq n+1 .$$

If $\delta_{l,m} \geqq 0$ for $1 \leqq l < m \leqq n+1$, there is nothing to prove. Consequently, in addition to (6.2) we may suppose without restriction that

$$\text{(6.3)} \qquad \delta_{1,2} < 0 ,$$

and by symmetry it suffices to prove the inequality in (ii) for $(l,m) = (1,2)$. By the assumptions of the lemma, $\boldsymbol{o}$ is representable (possibly in several ways) in the form

$$\boldsymbol{o} = \sum_{l=1}^{n+1} \alpha_l \boldsymbol{a}_l ,$$

where

$$\text{(6.4)} \qquad \sum_{l=1}^{n+1} \alpha_l = 1 \quad \text{and} \quad \alpha_l \geqq 0, \qquad 1 \leqq l \leqq n+1 .$$

Now

$$\begin{aligned} 0 &= \left(\sum_{l=1}^{n+1} \alpha_l \boldsymbol{a}_l\right) \cdot \left(\sum_{l=1}^{n+1} \alpha_l \boldsymbol{a}_l\right) \\ &= \sum_{l=1}^{n+1} \alpha_l^2 + \sum_{1 \leqq l < m \leqq n+1} 2\cos u_{l,m} \alpha_l \alpha_m \\ &= \sum_{l=1}^{n+1} \alpha_l^2 - \sum_{1 \leqq l < m \leqq n+1} 2(1/n + \delta_{l,m}) \alpha_l \alpha_m \\ &= \sum_{l=1}^{n+1} \alpha_l^2 - (1/n) \sum_{1 \leqq l < m \leqq n+1} (\alpha_l^2 + \alpha_m^2) + (1/n) \sum_{1 \leqq l < m \leqq n+1} (\alpha_l - \alpha_m)^2 - \sum_{1 \leqq l < m \leqq n+1} 2\delta_{l,m} \alpha_l \alpha_m , \end{aligned}$$

whence

$$(6.5)\qquad (1/n)\sum_{1\le l<m\le n+1}(\alpha_l-\alpha_m)^2 = \sum_{1\le l<m\le n+1} 2\delta_{l,m}\alpha_l\alpha_m .$$

By (6.2), (6.3), (6.4) and (6.5)

$$\begin{aligned}(1/n)\sum_{1\le l<m\le n+1}(\alpha_l-\alpha_m)^2 &\le \sum_{\substack{1\le l<m\le n+1\\ m\ne 2}} 2\delta\alpha_l\alpha_m\\ &= \delta\left\{\left(\sum_{l=1}^{n+1}\alpha_l\right)^2-\sum_{l=1}^{n+1}\alpha_l^2-2\alpha_1\alpha_2\right\}\\ &= \delta\left\{1-(\alpha_1+\alpha_2)^2-\sum_{l=3}^{n+1}\alpha_l^2\right\} \le \delta(1-1/n) ,\end{aligned}$$

and hence

$$\sum_{1\le l<m\le n+1}(\alpha_l-\alpha_m)^2 \le (n-1)\delta ,$$

whence

$$\sum_{\substack{m=1\\ m\ne l}}^{n+1}(\alpha_l-\alpha_m)^2 \le (n-1)\delta, \quad 1\le l\le n+1 .$$

Consequently, by Cauchy–Schwarz' inequality

$$\left|(n+1)\alpha_l-\sum_{m=1}^{n+1}\alpha_m\right| \le \sum_{\substack{m=1\\ m\ne l}}^{n+1}|\alpha_l-\alpha_m| \le \left((n-1)\delta\right)^{\frac12}n^{\frac12} ,$$

and hence by (6.4)

$$(6.6)\qquad |\alpha_l-(n+1)^{-1}| \le \left((n^2-n)\delta\right)^{\frac12}(n+1)^{-1}, \quad 1\le l\le n+1 .$$

Finally, by (6.5) and an estimation above

$$2\delta_{1,2}\alpha_1\alpha_2 \ge -\sum_{\substack{1\le l<m\le n+1\\ m\ne 2}} 2\delta_{l,m}\alpha_l\alpha_m \ge -(1-1/n)\delta ,$$

and hence by (6.6) and the assumptions on δ

$$\delta_{1,2} \ge -\frac{(n-1)(n+1)^2}{2n}\,\frac{\delta}{\left(1-\left((n^2-n)\delta\right)^{\frac12}\right)^2}.$$

This proves lemma 8.

The following two approximation lemmas are now obtained by combining lemma 7 and lemma 8.

LEMMA 9. *Let $p_1,p_2,\ldots,p_5,q_1,q_2,\ldots,q_5$ be quaternions, $q_1,q_2,\ldots,q_5\ne 0$, such that*

$$\Delta_{l,m}^2 = |p_lq_l^{-1}-p_mq_m^{-1}|^2\, N(q_l)N(q_m) = 1, \qquad 1\le l<m\le 5 .$$

Further let ξ be any quaternion different from $p_l q_l^{-1}$, $1 \leqq l \leqq 5$, and lying in the closed simplex S (possibly degenerate) with vertices $p_1 q_1^{-1}, p_2 q_2^{-1}, \ldots, p_5 q_5^{-1}$. Then

$$\max(c_1, c_2, \ldots, c_5) \geqq (\tfrac{5}{2})^{\frac{1}{2}}$$

with strict inequality unless S is a regular 4-simplex, and ξ is its centre, in which case $c_1 = c_2 = \ldots = c_5 = (\frac{5}{2})^{\frac{1}{2}}$.

PROOF. Let $u_{l,m}$, $0 \leqq u_{l,m} \leqq \pi$, be the angle $p_l q_l^{-1} \xi p_m q_m^{-1}$, $1 \leqq l < m \leqq 5$. If $\cos u_{l,m} < -\frac{1}{4}$ for some pair (l,m), $1 \leqq l < m \leqq 5$, then $\max(c_l, c_m) > (\frac{5}{2})^{\frac{1}{2}}$ by lemma 7 with $\Delta = 1$,

$$f = \max\{|q_l|/|q_m|, |q_m|/|q_l|\} \geqq f_0 = 1 ,$$

$$u_{l,m} > u_0 = \operatorname{Arccos}(-\tfrac{1}{4}) .$$

If $\cos u_{l,m} \geqq -\frac{1}{4}$ for $1 \leqq l < m \leqq 5$, then $\cos u_{l,m} = -\frac{1}{4}$ for $1 \leqq l < m \leqq 5$ by lemma 8 with $n = 4$, $\delta = 0$, and hence if for some pair (l,m), $1 \leqq l < m \leqq 5$,

$$f = \max\{|q_l|/|q_m|, |q_m|/|q_l|\} > 1 ,$$

then $\max(c_l, c_m) > (\frac{5}{2})^{\frac{1}{2}}$ by lemma 7 with $\Delta = 1$, $f > f_0 = 1$, $u_{l,m} = u_0 = \operatorname{Arccos}(-\frac{1}{4})$.

In the remaining case $|q_1| = |q_2| = \ldots = |q_5|$, that is, S is a regular 4-simplex by the assumptions of the lemma, and $\cos u_{l,m} = -\frac{1}{4}$ for $1 \leqq l < m \leqq 5$, that is, ξ is the centre of S. Evidently $c_1 = c_2 = \ldots = c_5 = (\frac{5}{2})^{\frac{1}{2}}$ in this particular case.

LEMMA 10. *Let $p_1, p_2, \ldots, p_5, q_1, q_2, \ldots, q_5$ be quaternions, $q_1, q_2, \ldots, q_5 \neq 0$, such that*

$$\Delta_{l,m}^2 = |p_l q_l^{-1} - p_m q_m^{-1}|^2 N(q_l) N(q_m) = 1, \qquad 1 \leqq l < m \leqq 5 ,$$

and

$$N(q_1) \leqq N(q_2) \leqq \ldots \leqq N(q_5) .$$

Further let ξ be any quaternion different from $p_l q_l^{-1}$, $1 \leqq l \leqq 5$, and lying in the closed simplex (possibly degenerate) with vertices $p_1 q_1^{-1}, p_2 q_2^{-1}, \ldots, p_5 q_5^{-1}$.

Suppose that for some δ, $0 \leqq \delta < \frac{1}{15}$,

$$\max(c_1, c_2, \ldots, c_5) \leqq (\tfrac{5}{2} + 2\delta)^{\frac{1}{2}} , \tag{6.7}$$

then

$$N(q_5)/N(q_1) \leqq 1 + \tfrac{1}{2}\{h + (h^2 + 4h)^{\frac{1}{2}}\} , \tag{6.8}$$

where

$$h = 2\delta\left\{1 + \frac{75}{8(1 - (12\delta)^{\frac{1}{2}})^2}\right\}.$$

PROOF. Let $u_{l,m}$, $0 \leqq u_{l,m} \leqq \pi$, be the angle $p_l q_l^{-1}, \xi, p_m q_m^{-1}$, $1 \leqq l < m \leqq 5$. If for some pair (l,m), $1 \leqq l < m \leqq 5$, $\cos u_{l,m} < -(\frac{1}{4}+\delta)$, then $\max(c_l, c_m) > (\frac{5}{2}+2\delta)^{\frac{1}{2}}$ by lemma 7 with $\Delta = 1$,

$$f = \max\{|q_l|/|q_m|, |q_m|/|q_l|\} \geqq f_0 = 1 ,$$
$$u_{l,m} > u_0 = \operatorname{Arccos}\left(-(\tfrac{1}{4}+\delta)\right) .$$

Consequently the assumption (6.7) of the lemma implies that $\cos u_{l,m} \geqq -(\frac{1}{4}+\delta)$ for $1 \leqq l < m \leqq 5$. Hence (with the notation of lemma 8 and the present lemma) it follows from lemma 8 with $n = 4$ and $(l,m) = (1,5)$ that

$$\delta_{1,5} \geqq \delta - \tfrac{1}{2}h . \tag{6.9}$$

Finally, by (6.7), (6.9) and lemma 7 with $\Delta = 1$,

$$f = f_0 = |q_5|/|q_1| ,$$
$$u = u_{1,5} \geqq u_0 = \operatorname{Arccos}\left(-(\tfrac{1}{4}-\tfrac{1}{2}h+\delta)\right) ,$$

we obtain

$$N(q_5)/N(q_1) + N(q_1)/N(q_5) + 2(\tfrac{1}{4}-\tfrac{1}{2}h+\delta) \leqq \tfrac{5}{2} + 2\delta ,$$

whence (6.8) by an easy calculation.

This proves lemma 10.

7. Evaluation of approximation constants.

We begin this paragraph by the observation that Speiser's theorem is an immediate consequence of previous lemmas and theorems.

In fact, by lemma 4 and theorem 4 every irrational quaternion ξ is contained in a chain (3.19) of Farey simplices. Further by lemma 3 none of the $\mathrm{FS}^{(n)}$, $n \geqq 0$, in the chain are regular, and hence by lemma 9, which is applicable to a Farey simplex by formula (2.11),

$$\max_{1 \leqq l \leqq 5} c_l^{(n)} > (\tfrac{5}{2})^{\frac{1}{2}}, \quad n \geqq 0 , \tag{7.1}$$

where

$$c_l^{(n)} = \left(|q_l^{(n)}|\ |\xi q_l^{(n)} - p_l^{(n)}|\right)^{-1} \tag{7.2}$$

in agreement with (6.1).

Now Speiser's theorem follows from (7.1) together with (3.20), since ξ is an irrational quaternion.

Of course, by Speiser's theorem the approximation constant of any irrational quaternion ξ satisfies the inequality

$$C(\xi) \geqq (\tfrac{5}{2})^{\frac{1}{2}} . \tag{7.3}$$

The following important theorem gives a formula for the approximation constant $C(\xi)$ of an irrational quaternion ξ in terms of the Farey simplices containing ξ:

THEOREM 5. *For any irrational quaternion ξ the approximation constant*

$$C(\xi) = \limsup(|q|\,|\xi q-p|)^{-1},$$

where the lim sup *is taken over all $p,q \in \mathsf{H}$, $q \neq 0$, such that pq^{-1} is a vertex of a non-degenerate Farey simplex containing ξ.*

COROLLARY. *If $\xi \in \mathsf{K}$ is contained in a chain* (3.19) *of Farey simplices such that ξ is an interior point of* $\mathrm{FS}^{(n)}$ *for all $n \geqq 0$, then ξ is an irrational quaternion, and*

(7.4) $$C(\xi) = \limsup c_l^{(n)},$$

where $c_l^{(n)}$ is given by (7.2), *and the* lim sup *is taken over all $n \geqq 0$ and all l,* $1 \leqq l \leqq 5$.

PROOF. The theorem follows immediately from theorem 2 and (7.3), since $2^{\frac{1}{2}} < (\frac{5}{2})^{\frac{1}{2}}$.

To prove the corollary we notice first that by (7.1), (3.20) and the assumption on ξ, the inequality

$$|\xi - pq^{-1}| < \left((\tfrac{5}{2})^{\frac{1}{2}} N(q)\right)^{-1}$$

has infinitely many solutions $p,q \in \mathsf{H}$, $q \neq 0$, p,q without non-trivial common right divisors, and consequently ξ is an irrational quaternion.

Finally, the formula (7.4) is in fact identical with the general formula for $C(\xi)$ in theorem 5 by the first part of theorem 4 and corollary 1 of that theorem.

In the following let the irrational quaternion ξ be contained in a chain of Farey simplices

$$\mathrm{FS}^{(n)} = \mathrm{FS}(p_1^{(n)}q_1^{(n)^{-1}},\ldots,p_5^{(n)}q_5^{(n)^{-1}}), \quad n \geqq 0.$$

By theorem 1 we may suppose, that the corresponding sequence $\mathfrak{F}^{(n)}$ of Farey matrices is of the form

(7.5)
$$\mathfrak{F}^{(n)} = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} & p_3^{(n)} & p_4^{(n)} & p_5^{(n)} \\ q_1^{(n)} & q_2^{(n)} & q_3^{(n)} & q_4^{(n)} & q_5^{(n)} \end{pmatrix} = \mathfrak{M}^{(n)} \begin{pmatrix} 1 & 0 & 1 & \omega & \omega-k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad n \geqq 0,$$

where $\mathfrak{M}^{(n)}$ is the unimodular matrix

(7.6) $$\mathfrak{M}^{(n)} = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} \\ q_1^{(n)} & q_2^{(n)} \end{pmatrix}, \quad n \geqq 0.$$

Further by theorem 1 and theorem 3 we may assume, that the transition from $\mathfrak{F}^{(n)}$ to $\mathfrak{F}^{(n+1)}$ is expressed in

$$(7.7) \qquad \mathfrak{F}^{(n+1)} = \mathfrak{M}^{(n)}\,\mathfrak{S}^{(n)}\begin{pmatrix}1 & 0 & 1 & \omega & \omega-k\\ 0 & 1 & 1 & 1 & 1\end{pmatrix}, \quad n \geqq 0\,,$$

where the transition matrix $\mathfrak{S}^{(n)}$, $n \geqq 0$, is one of $62 = 2\times 31$ unimodular matrices obtainable from table 1 of § 4 or the alternative table. Now by (7.5) and (7.7)

$$\mathfrak{M}^{(n+1)} = \mathfrak{M}^{(n)}\,\mathfrak{S}^{(n)}, \quad n \geqq 0\,,$$

and hence

$$(7.8) \qquad \mathfrak{M}^{(n+1)} = \mathfrak{M}^{(0)}\,\mathfrak{S}^{(0)}\,\mathfrak{S}^{(1)}\ldots\mathfrak{S}^{(n)}, \quad n \geqq 0\,.$$

We consider in particular the important special case, where the sequence $\mathfrak{S}^{(0)}, \mathfrak{S}^{(1)}, \ldots$ of transition matrices is periodic with period λ. Then by (7.8) for some $n_0 \geqq 0$

$$(7.9) \qquad \mathfrak{M}^{(n_0+\lambda\nu)} = \mathfrak{M}^{(n_0)}\mathfrak{S}^{\nu}, \quad \nu \geqq 0\,,$$

where

$$(7.10) \qquad \mathfrak{S} = \mathfrak{S}^{(n_0)}\,\mathfrak{S}^{(n_0+1)}\ldots\mathfrak{S}^{(n_0+\lambda-1)}\,.$$

For the unimodular matrices $\mathfrak{M}^{(n_0)}$ and $\mathfrak{S}$ we shall use the notation

$$(7.11) \qquad \mathfrak{M}^{(n_0)} = \begin{pmatrix}\alpha_0 & \beta_0\\ \gamma_0 & \delta_0\end{pmatrix} \quad \text{and} \quad \mathfrak{S} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

By (3.20), (7.6), (7.9), (7.10) and (7.11)

$$(7.12) \qquad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^{\nu} = \begin{pmatrix}\eta\,\sigma_\nu & \eta\,\tau_\nu\\ (1+\delta_\nu')\sigma_\nu & (1+\delta_\nu'')\tau_\nu\end{pmatrix}, \quad \nu \geqq 0\,,$$

where

$$(7.13) \qquad \lim_{\nu\to\infty}\delta_\nu' = \lim_{\nu\to\infty}\delta_\nu'' = 0\,,$$

and

$$(7.14) \qquad \xi = (\alpha_0\eta+\beta_0)(\gamma_0\eta+\delta_0)^{-1}\,.$$

From (7.12) we get for $\nu \geqq 0$

$$\begin{pmatrix}\eta\,\sigma_{\nu+1} & \eta\,\tau_{\nu+1}\\ (1+\delta'_{\nu+1})\sigma_{\nu+1} & (1+\delta''_{\nu+1})\tau_{\nu+1}\end{pmatrix} = \begin{pmatrix}(\alpha\eta+\beta(1+\delta_\nu'))\sigma_\nu & (\alpha\eta+\beta(1+\delta_\nu''))\tau_\nu\\ (\gamma\eta+\delta(1+\delta_\nu'))\sigma_\nu & (\gamma\eta+\delta(1+\delta_\nu''))\tau_\nu\end{pmatrix},$$

and hence by (7.13)

$$(7.15) \qquad \eta = (\alpha\eta+\beta)(\gamma\eta+\delta)^{-1}\,,$$

so that η is one of the roots of the quadratic equation

$$(7.16) \qquad z\gamma z + z\delta - \alpha z - \beta = 0\,.$$

Of course, by (7.14),

$$\xi \sim \eta \,, \tag{7.17}$$

and consequently, by (2.23)

$$C(\xi) = C(\eta) \,. \tag{7.18}$$

EXAMPLE 1. As a particularly interesting example of a chain (3.19) of Farey simplices containing an irrational quaternion ξ we consider the case, where

$$\mathrm{FS}(p_1^{(n+1)}q_1^{(n+1)^{-1}},\ldots,p_5^{(n+1)}q_5^{(n+1)^{-1}}) = \mathrm{FS}(p_1'^{(n)}q_1'^{(n)^{-1}},\ldots,p_5'^{(n)}q_5'^{(n)^{-1}})$$

for all $n \geqq n_0$, that is, $\mathrm{FS}^{(n+1)}$ is the central Farey simplex in the inner subdivision of $\mathrm{FS}^{(n)}$ for all $n \geqq n_0$. (That such a chain exists is proved later.)

By a simple calculation based on table 1 and following the proof of theorem 1, it is seen that one of the corresponding sequences of Farey matrices is of the form (7.5), where $\mathfrak{M}^{(n)}$ is given by (7.8) and

$$\mathfrak{S}^{(n)} = \begin{pmatrix} \omega - i & i-j \\ -i+j & \omega - j \end{pmatrix} = \mathfrak{S}' \quad \text{or} \quad \mathfrak{S}^{(n)} = \begin{pmatrix} \omega-1-i & i-j \\ -i+j & \omega-1-j \end{pmatrix} = \mathfrak{S}^*, \tag{7.19}$$

$n \geqq n_0$. However, since

$$\mathfrak{S}'\mathfrak{S}^* = \mathfrak{S}^*\mathfrak{S}' = \mathfrak{E}\,,$$

$\mathfrak{E}$ being the unity matrix, it follows from (7.19) and condition (iv) of definition 4 that the sequence $\mathfrak{S}^{(0)}, \mathfrak{S}^{(1)}, \ldots$ is periodic with period 1, hence $\lambda = 1$ and $\mathfrak{S} = \mathfrak{S}'$ or $\mathfrak{S} = \mathfrak{S}^*$ in (7.9).

In both cases ξ is equivalent to one of the roots of the quadratic equation

$$z(-i+j)z + z(\omega - j) - (\omega - i)z - (i-j) = 0 \tag{7.20}$$

by (7.11), (7.16), (7.17) and (7.19). An easy calculation shows that the equation (7.20) has precisely two roots

$$\eta_1 = \tfrac{1}{2} + \tfrac{1}{4}(1+5^{\frac{1}{2}})i + \tfrac{1}{4}(1-5^{\frac{1}{2}})j$$

and

$$\eta_2 = \tfrac{1}{2} + \tfrac{1}{4}(1-5^{\frac{1}{2}})i + \tfrac{1}{4}(1+5^{\frac{1}{2}})j\,,$$

and hence

$$\xi \sim \eta_1 \quad \text{or} \quad \xi \sim \eta_2\,. \tag{7.21}$$

Because of the importance of this result we should like to derive it in a more direct manner based on the relations

$$\mathfrak{S}'^2 = \mathfrak{S}' + \mathfrak{E} \quad \text{and} \quad \mathfrak{S}^{*2} = -\mathfrak{S}^* + \mathfrak{E}\,, \tag{7.22}$$

which follow from (7.19). In fact, by (7.22)

$$\begin{aligned} \mathfrak{S}'^{\nu} &= f_\nu \mathfrak{S}' + f_{\nu-1}\mathfrak{E}\,, \\ \mathfrak{S}^{*\nu} &= (-1)^{\nu-1} f_\nu \mathfrak{S}^* + (-1)^\nu f_{\nu-1}\mathfrak{E}, \end{aligned} \quad \nu \geqq 0\,, \tag{7.23}$$

where the sequence $\{f_\nu\}$, $\nu \geqq 0$, is the Fibonacci sequence defined by

$$f_0 = 0, \quad f_1 = 1, \quad f_\nu = f_{\nu-1} + f_{\nu-2}, \quad \nu \geqq 2\,. \tag{7.24}$$

Hence, according as $\mathfrak{S}^{(n)} = \mathfrak{S}'$ or $\mathfrak{S}^{(n)} = \mathfrak{S}^*$ for $n \geqq n_0$,

$$\begin{aligned} \eta &= \lim_{\nu\to\infty} (f_\nu(\omega - i) + f_{\nu-1})(f_\nu(-i+j))^{-1} \\ &= \lim_{\nu\to\infty} \tfrac{1}{2}(1 + (1 + f_{\nu-1}/f_\nu)i - (f_{\nu-1}/f_\nu)j) = \eta_1 \end{aligned}$$

or

$$\begin{aligned} \eta &= \lim_{\nu\to\infty} ((-1)^{\nu-1} f_\nu(\omega - 1 - i) + (-1)^\nu f_{\nu-1})((-1)^{\nu-1} f_\nu(-i+j))^{-1} \\ &= \lim_{\nu\to\infty} \tfrac{1}{2}(1 - (f_{\nu-1}/f_\nu)i + (1 + f_{\nu-1}/f_\nu)j) = \eta_2 \end{aligned}$$

by (7.12), (7.19), (7.23) and the well-known relation

$$\lim_{\nu\to\infty} f_{\nu-1}/f_\nu = \tfrac{1}{2}(5^{\frac{1}{2}} - 1)$$

for the Fibonacci sequence (7.24).

Of course, by (7.17) this proves once more (7.21) .

In order to calculate $C(\eta_1)$ we consider the sequence of Farey simplices $\widetilde{\mathrm{FS}}^{(1)}\widetilde{\mathrm{FS}},^{(2)},\ldots$, where the corresponding sequence of Farey matrices is

$$\tilde{\mathfrak{F}}^{(n)} = \begin{pmatrix} \tilde{p}_1^{(n)} & \tilde{p}_2^{(n)} \ldots \tilde{p}_5^{(n)} \\ \tilde{q}_1^{(n)} & \tilde{q}_2^{(n)} \ldots \tilde{q}_5^{(n)} \end{pmatrix} = \mathfrak{S}'^n \begin{pmatrix} 1 & 0 & 1 & \omega & \omega - k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{7.25}$$

From (7.19) and (7.23)

$$\mathfrak{S}'^n = \begin{pmatrix} f_n(\omega - i) + f_{n-1} & f_n(i-j) \\ f_n(-i+j) & f_n(\omega - j) + f_{n-1} \end{pmatrix}, \quad n \geqq 1\,, \tag{7.26}$$

and hence by (7.25) and (7.26)

$$-\tilde{q}_1^{(n)^{-1}}\tilde{q}_2^{(n)} = (1-j)^{-1} + \tfrac{1}{2}(j-i)(f_{n-1}/f_n), \quad n \geqq 1\,. \tag{7.27}$$

Since $f_{n-1}/f_n \geqq 0$ for $n \geqq 1$ by (7.24), the j-coordinate of $-\tilde{q}_1^{(n)^{-1}}\tilde{q}_2^{(n)}$ exceeds its i-coordinate, and hence by a result in § 3, $\widetilde{\mathrm{FS}}^{(n)}$, $n \geqq 1$, is a non-degenerate Farey simplex having $\widetilde{\mathrm{FS}}^{(n+1)}$ as central Farey simplex in its inner subdivision.

Further, we consider the sequence of unimodular homographic maps

$$\tilde{\varphi}^{(n)}:\ w = (\tilde{p}_1^{(n)} z + \tilde{p}_2^{(n)})(\tilde{q}_1^{(n)} z + \tilde{q}_2^{(n)})^{-1}, \quad n \geqq 1\,. \tag{7.28}$$

By (7.25) and (7.28)

$$\tilde{\varphi}^{(n)} = \tilde{\varphi}^{(1)^n}, \quad n \geqq 1\,,$$

and hence, since η_1 and η_2 are fixed points under $\varphi^{(1)}$ by the definition of these points,

$$(7.29) \qquad \tilde{\varphi}^{(n)}(\eta_1) = \eta_1 \quad \text{and} \quad \tilde{\varphi}^{(n)}(\eta_2) = \eta_2, \qquad n \geqq 1 .$$

Now let L be the line perpendicular to the regular tetrahedron spanned by 0, 1, ω, $\omega-k$ and passing through its centre. Then L contains η_1, η_2 and all the points $-\tilde{q}_1^{(n)^{-1}}q_2^{(n)}$, $n \geqq 1$. Further the sphere through 0, 1, ω, $\omega-k$ and η_1 is cut by L (apart from η_1) in the point

$$(1-j)^{-1} + \tfrac{1}{2}(j-i)(\tfrac{3}{10}5^{\frac{1}{2}}-\tfrac{1}{2}) .$$

By these geometric properties of the line L together with (7.25), (7.27), (7.28) and (7.29),

$$(7.30) \qquad \eta_1 \in \operatorname{int}\widetilde{\mathrm{FS}}^{(n)}, \quad n \geqq 2 ,$$

since

$$f_{n-1}/f_n \geqq f_2/f_3 = \tfrac{1}{2} > \tfrac{3}{10}5^{\frac{1}{2}}-\tfrac{1}{2}, \qquad n \geqq 2,$$

by well-known inequalities for the Fibonacci sequence (7.24). Also, by the same argument

$$\eta_1 \notin \widetilde{\mathrm{FS}}^{(1)} ,$$

since

$$f_0/f_1 = 0 < \tfrac{3}{10}5^{\frac{1}{2}} - \tfrac{1}{2} ,$$

and consequently the sequence $\widetilde{\mathrm{FS}}^{(n)}$, $n \geqq 1$, is not a chain of Farey simplices containing η_1.

However, it is evident that the sequence of Farey simplices

$$(7.31) \quad \mathrm{FS}(1+i,i,\omega-j-k,\omega-j,(\omega+1-i)(-i+j)^{-1}), \ \widetilde{\mathrm{FS}}^{(2)}, \ \widetilde{\mathrm{FS}}^{(3)}, \ \ldots$$

actually is a chain of Farey simplices containing η_1. Also by (7.30) η_1 is clearly an interior point in each Farey simplex in (7.31). Hence by corollary 1 of theorem 4, (7.31) is the only chain of Farey simplices containing η_1.

A straightforward calculation using (7.24), (7.25) and (7.26) and the well-known relation

$$\lim_{n\to\infty} \{|\tfrac{1}{2}(5^{\frac{1}{2}}-1)f_n - f_{n-1}|f_n\} = 5^{-\frac{1}{2}}$$

shows that

$$(7.32) \qquad \lim_{n\to\infty} (|\tilde{q}_l^{(n)}|\,|\eta_1\tilde{q}_l^{(n)}-\tilde{p}_l^{(n)}|)^{-1} = (\tfrac{5}{2})^{\frac{1}{2}}, \quad 1 \leqq l \leqq 5 ,$$

and

$$(7.33) \qquad \lim_{n\to\infty} \cos\tilde{\alpha}_{l,m}^{(n)} = -\tfrac{1}{4}, \quad l \neq m, \ 1 \leqq l \leqq 5, \ 1 \leqq m \leqq 5 ,$$

where $\tilde{\alpha}_{l,m}^{(n)}$ is the angle $\tilde{p}_l^{(n)}\tilde{q}_l^{(n)^{-1}}, \eta_1, \tilde{p}_m^{(n)}\tilde{q}_m^{(n)^{-1}}$. Further, it follows immediately from the linear norm relations (4.2) that

(7.34) $$\lim_{n\to\infty} N(\tilde{q}_m^{(n)})/N(\tilde{q}_l^{(n)}) = 1, \quad 1 \leqq l \leqq 5,\ 1 \leqq m \leqq 5 .$$

Finally, since η_1 and the chain (7.31) of Farey simplices satisfy the conditions in the corollary of theorem 5,

(7.35) $$C(\eta_1) = (\tfrac{5}{2})^{\frac{1}{2}}$$

by (7.4) and (7.32).

Analogously, η_2 is an interior point in each Farey simplex in the chain

(7.36)
$$\mathrm{FS}(1+j, j, \omega-i-k, \omega-i, (\omega+1-j-k)(i-j)^{-1}),\quad \widehat{\mathrm{FS}}^{(2)},\quad \widehat{\mathrm{FS}}^{(3)},\quad \ldots ,$$

where the Farey matrix $\hat{\mathfrak{F}}^{(n)}$ corresponding to $\widehat{\mathrm{FS}}^{(n)}$, $n \geqq 2$, is

(7.37) $$\hat{\mathfrak{F}}^{(n)} = \begin{pmatrix} \hat{p}_1^{(n)} & \hat{p}_2^{(n)} & \ldots & \hat{p}_5^{(n)} \\ \hat{q}_1^{(n)} & \hat{q}_2^{(n)} & \ldots & \hat{q}_5^{(n)} \end{pmatrix} = \mathfrak{S}^{*n} \begin{pmatrix} 1 & 0 & 1 & \omega & \omega-k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The point η_2 and the chain (7.36) of Farey simplices satisfy the conditions in the corollary of theorem 5, and hence by (7.4) and the analogue of (7.32)

(7.38) $$C(\eta_2) = (\tfrac{5}{2})^{\frac{1}{2}} .$$

Combining (7.21), (7.35) and (7.38) we obtain the important result

(7.39) $$C(\xi) = (\tfrac{5}{2})^{\frac{1}{2}} .$$

Finally, we shall prove that the two quaternions η_1 and η_2 despite their analogous approximation properties are non-equivalent. Indeed, assume on the contrary that ψ is a unimodular homographic map such that $\psi(\eta_1) = \eta_2$. Then it follows by the conformal property of ψ and the Farey matrix-preserving property of the corresponding unimodular linear map Ψ together with (7.33) and (7.34) that the vertices of $\widetilde{\mathrm{FS}}^{(n)}$ for sufficiently large n are mapped by ψ onto the vertices of a non-degenerate Farey simplex of norm greater than 6, and having $\eta_2 = \psi(\eta_1)$ as an interior point. However, every non-degenerate Farey simplex containing η_2 is in the unique chain (7.36) of Farey simplices containing η_2 by theorem 4. Hence for suitable integers $n, r, n \geqq 2, r \geqq 2$, the unimodular homographic map

$$\varphi = \tilde{\varphi}^{(r)} \psi \tilde{\varphi}^{(n)}$$

permutes the five points $(\infty, 0, 1, \omega, \omega-k)$, and $\varphi(\eta_1) = \eta_2$ by (7.25), (7.31), (7.36), (7.37), (7.28), (7.29) and the relation $\mathfrak{S}' \mathfrak{S}^* = \mathfrak{E}$.

Hence, in order to prove the non-equivalence of η_1 and η_2 it suffices to establish the non-existence of a unimodular homographic map φ with such properties.

It is easily seen that, among the unimodular homographic maps permuting the five points $(\infty, 0, 1, \omega, \omega-k)$,

$$\varphi_1 : w = \omega z \omega + 1, \qquad \varphi_2 : w = (\omega-k)z(\omega-k)+1,$$
$$\varphi_3 : w = (-z+1)^{-1}$$

are the only ones that give rise to the permutations given by the cycles

$$(0 1 \omega), \quad (0 1 \omega-k), \quad (0 1 \infty),$$

respectively. Since these three cycles generate the alternating group $\mathfrak{A}_5$ of the five symbols $(\infty, 0, 1, \omega, \omega-k)$, the multiplicative group of unimodular homographic maps inducing an even permutation of $(\infty, 0, 1, \omega, \omega-k)$ is isomorphic to $\mathfrak{A}_5$. Hence, since

$$\varphi_l(\eta_1) = \eta_1 \quad \text{and} \quad \varphi_l(\eta_2) = \eta_2, \qquad 1 \leqq l \leqq 3,$$

there is no unimodular homographic map φ inducing an even permutation of $(\infty, 0, 1, \omega, \omega-k)$ and satisfying $\varphi(\eta_1)=\eta_2$.

Finally, we claim that there is no unimodular homographic map φ inducing an odd permutation of $(\infty, 0, 1, \omega, \omega-k)$. From the results above it is enough to prove the non-existence of a unimodular homographic map φ with

$$(7.40) \qquad \varphi(\infty) = \infty, \quad \varphi(0) = 0, \quad \varphi(1) = 1, \quad \varphi(\omega) = \omega-k,$$
$$\varphi(\omega-k) = \omega.$$

By the three first conditions of (7.40) and the definition of a unimodular homographic map,

$$\varphi : w = \varepsilon^{-1} z \varepsilon,$$

where ε is a unit in H. But then the last two conditions of (7.40) cannot be complied with since ω and $\omega-k$ are in different conjugacy classes in the group of units in H (cf. § 2).

This finishes the proof of the non-equivalence of η_1 and η_2.

8. The approximation spectrum of quaternions.

We are now prepared to prove the properties of the approximation spectrum of quaternions already announced in the introduction using the theory of Farey simplices developped.

THEOREM 6. *Let ξ be an irrational quaternion with the approximation constant*

$$C(\xi) < (2.51)^{\frac{1}{2}} = 1.5842\ldots.$$

Then ξ has the approximation properties described in example 1 *in* § 7, *especially ξ is equivalent to one of the quaternions*

$$\tfrac{1}{2}+\tfrac{1}{4}(1+5^{\frac{1}{2}})i+\tfrac{1}{4}(1-5^{\frac{1}{2}})j \quad \text{or} \quad \tfrac{1}{2}+\tfrac{1}{4}(1-5^{\frac{1}{2}})i+\tfrac{1}{4}(1+5^{\frac{1}{2}})j\,,$$

and

$$C(\xi) = (\tfrac{5}{2})^{\frac{1}{2}} = 1.5812\ldots .$$

PROOF. Let $\mathrm{FS}(p_1^{(n)}q_1^{(n)^{-1}},\ldots,p_5^{(n)}q_5^{(n)^{-1}})$, $N_1^{(n)} \leqq \ldots \leqq N_5^{(n)}$, $n \geqq 0$, be a chain of Farey simplices containing ξ. Since by assumption $C(\xi) < (2.51)^{\frac{1}{2}}$, there exists a positive integer n_0 such that

$$N_5^{(n)}/N_1^{(n)} \leqq 1.538, \qquad n \geqq n_0, \tag{8.1}$$

by the definition of $C(\xi)$ and lemma 10 with $\delta = 1/200$.

For a Farey simplex $\mathrm{FS}(p_1q_1^{-1},\ldots,p_5q_5^{-1})$ with $N_1 \leqq \ldots \leqq N_5$ and $N_5/N_1 \leqq 1.538$, we have $N_1' \geqq \ldots \geqq N_5'$ by (4.2), and hence by (4.2) and (5.1)

$$\begin{aligned} N_m'/N_l &\geqq N_5'/N_5 = (N_5'+N_5)/N_5 - 1 \\ &= \tfrac{1}{5}(N'+N)/N_5 - 1 \\ &= (N+(N^2-4N^{(2)})^{\frac{1}{2}})/(2N_5) - 1 \\ &\geqq \left(N+\left(N^2-4\times\frac{4+1.538^2}{(4+1.538)^2}N^2\right)^{\frac{1}{2}}\right)\Big/\left(2\times\frac{1.538}{4+1.538}N\right) - 1 \\ &= 1.54\ldots > 1.538 \end{aligned}$$

for $1 \leqq l \leqq 5$ and $1 \leqq m \leqq 5$. From this result and (8.1)

$$N_m'^{(n)}/N_l^{(n)} > 1.538, \quad 1 \leqq l \leqq 5,\ 1 \leqq m \leqq 5,\ n \geqq n_0\,. \tag{8.2}$$

Now recall that by definition 4 and theorem 3, the $p_r^{(n+1)}q_r^{(n+1)^{-1}}$ are a selection of the $p_l^{(n)}q_l^{(n)^{-1}}$ and the $p_m'^{(n)}q_m'^{(n)^{-1}}$. Then (8.2) and the fact that (8.1) is valid for $n+1$ instead of n show that only one selection is possible, i.e.

$$\mathrm{FS}(p_1^{(n+1)}q_1^{(n+1)^{-1}},\ldots,p_5^{(n+1)}q_5^{(n+1)^{-1}}) = \mathrm{FS}(p_1'^{(n)}q_1'^{(n)^{-1}},\ldots,p_5'^{(n)}q_5'^{(n)^{-1}})$$

for all $n \geqq n_0$.

This proves theorem 6.

Although the constant $(2.51)^{\frac{1}{2}}$ in theorem 6 may be raised somewhat by minor modifications of the proof of theorem 6, it is obvious that the determination of the second minimum of the approximation spectrum of quaternions is more involved. However, it is a reasonable conjecture that the second minimum corresponds to one or several equivalence classes of quaternions having periodic chains of Farey simplices.

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