

Fascination of chaos

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Abstract. This review introduces most of the concepts used in the study of chaotic phenomena in nonlinear systems and has as its objective to summarize the current understanding of results from the theory of chaotic dynamical systems and to describe the original ideas underlying the study of deterministic chaos. The presentation relies on informal analysis, with abstract mathematical ideas visualized geometrically or by examples from physics. Hyperbolic dynamics, homoclinic trajectories and tangencies, wild hyperbolic sets, and different types of attractors which appear in dynamical systems are considered. The key aspects of ergodic theory are discussed, and the basic statistical properties of chaotic dynamical systems are described. The fundamental difference between stochastic dynamics and deterministic chaos is explained. The review concludes with an investigation of the possibility of studying

complex systems on the basis of the analysis of registered signals, i.e., the generated time series.

Preface

The outstanding 20th-century physicist and Full Member of the Russian Academy of Sciences Vitaly Lazarevich Ginzburg passed away on November 8, 2009. In his work dedicated to the development perspectives of modern physics, the ideas of nonequilibrium processes, solitons, and strange attractors were always given particular attention. V L Ginzburg repeatedly stressed that these ideas are related to both fundamental and practically important problems. This is what he wrote back in 1982 [1]: “*This domain (it can be referred to as nonlinear physics) occupies an increasingly more prominent place, and such is, one can hardly doubt, the tendency of the development of physics nowadays.*” Twenty years later, in 2003, he also mentioned the importance of nonlinear phenomena and chaos in his Nobel lecture. In particular, Vitaly Lazarevich identified the question of entropy growth, irreversibility, and the time arrow, which are also considered in the theory of chaotic dynamical systems, as one of three ‘outstanding’ problems in physics.

This avenue of inquiry has come to the forefront over the last decades. Many new scientific journals have appeared, a

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large number of conferences are held each year, and laboratories or departments of nonlinear dynamics are now created in almost all universities around the world. The successful development of the physics of nonlinear phenomena was also embodied in new approaches to a broad variety of applied problems—from information processing and financial analysis to arrhythmology and neurodynamics.

This review is dedicated to the memory of Academician V L Ginzburg.

1. Introduction

The dynamical approach to the description of systems of very different origins has been known since the time of Newton. This approach underlies the analysis of most classical phenomena in physics and other natural sciences: one first constructs an appropriate mathematical model expressed in terms of dynamical equations and then, in one way or another, studies their solutions which, in principle, can be verified against experimental data. The development of these ideas and the expectation that the state of the model at any instant of time should be unambiguously defined by initial conditions have led researchers to the concept of the dynamical system.

Notwithstanding the fact that any dynamical system is a certain mathematical abstraction, this paradigm proved to be a rather productive tool for describing many real-world phenomena. The greatest success in this area was achieved over the first third of the 20th century when the theory of oscillations of two-dimensional systems was developed. Subsequent efforts of researchers were aimed at studying how to generalize this theory to multidimensional systems. However, despite numerous discoveries in this area, the real degree of complexity in such systems was not appreciated until the 1960s.

The situation changed radically after S Smale's work [2, 3] on the foundations of hyperbolic theory. Research along this line of inquiry uncovered the great diversity in nonlinear system dynamics and led to one of the most significant discoveries of the 20th century—dynamical chaos. U-systems [4–6] (called later the Anosov systems) were introduced, separatrix loop bifurcations causing a complex behavior [7, 8] were described, and billiard models representing simplified models of statistical physics [9, 10] were studied.

At that time, however, these ideas did not enjoy wide recognition because the examples which were built carried a manifestly abstract character, and it was not obvious whether these constructions have some relation to reality. Moreover, a viewpoint was broadly disseminated that chaotic phenomena inherent in physical systems have a transient character and chaos should degenerate to regular motion on a long time lapse.

This viewpoint persisted up to the middle of the 1970s—when there was success in connecting the mathematical ideas of the theory of dynamical systems with a physical model related to fluid dynamics—the celebrated Lorenz system (see Ref. [11]). The systematic study of dynamical chaos has been pursued since then.¹

A classical example of chaos is provided by gambling, which is specifically studied by the probability theory.

Gambling, however, represents a nondeterministic process. Here, an element of randomness is admitted. The theory of chaotic systems utilizes methods of the probability theory, yet it does not make up a part of it. Chaos should be defined as a particular random process observed in dynamical systems which are not affected by noise or any random forces. It is for this reason that the theory of chaos is considered as a part of the theory of dynamical systems.

For systems of statistical mechanics with a large number N of degrees of freedom, which are in equilibrium, the particle configuration does not obey any dynamical laws, and particles have a limit distribution as $N \rightarrow \infty$. Such systems are in a state of spatial disorder. One of the basic achievements of chaos theory is the establishment of the fact that time in dynamics plays the same role as the number of degrees of freedom in statistical mechanics. In other words, the deterministic chaos is described as dynamical disorder.

In conservative systems, whose dynamical chaos nature was covered in-depth in the recent review [13], the phase volume is preserved. This implies that the Liouville theorem holds true. This fundamental property predefines the character of evolution and provides a key for explaining chaos in conservative systems. In dissipative systems, the phase volume contracts because of dissipation. The principal distinction of dissipative systems from the conservative ones is manifested through the occurrence of attracting sets in their phase space, the attractors, which do not exist in conservative systems.

The term 'attractor' can easily be explained through the example of a pendulum in a viscous medium. Assume the pendulum is at the lower point of stable equilibrium (at a stable stationary point). Being slightly perturbed, it will make damped oscillations around its equilibrium state. In this sense, the equilibrium state of the pendulum will be attracting, i.e., serve as an attractor, which apparently has a zero measure. Likewise, one can get an idea of attractors that correspond to periodic (limit cycle) or quasiperiodic (invariant tori) motion. Formalization of these ideas leads to the modern concept of the attractor.

When system's parameters are varied, the attractors change, too. However, they may undergo a qualitative modification at certain values of parameters. For example, a stable focus may turn into a limit cycle. Such parameter values are called bifurcational, while the modification proper is termed the bifurcation. The onset of chaotic behavior in a dynamical system as a result of one sequence of bifurcations or another is customarily referred to as a picture or scenario of chaos development.

Where does the boundary lie between the regular, but complex, dynamics and chaos? The system's stability against small perturbations may provide a criterion. If such stability is absent, the deterministic description loses its sense on large time intervals and it becomes necessary to resort to statistical methods. Such an approach led to diagnosing the chaotic behavior of a system through its sensitive dependence on initial conditions, and also to utilizing Lyapunov exponents and entropy as criteria of dynamical chaos.

The dynamics of dissipative systems are in a certain sense more diverse than those of conservative ones. Such invariant sets as stable and unstable stationary points and limit cycles, multidimensional attracting tori which correspond to a stable quasiperiodic behavior with incommensurate frequencies, the mathematical image of chaotic oscillations—the strange attractor, and some others pertain to dissipative systems.

¹ The history of the origin of the dynamical chaos concept and the development of theoretical views on this phenomenon are detailed in the remarkable monograph [12].

The strange attractor constitutes a certain set of the ‘complex structure’ in the phase space, such that almost all trajectories from some neighborhood of this set are attracted to it, while on the set itself the motion has an exponentially unstable character. This combination of global contraction with local instability requires that the attractor no longer be smooth as, for instance, the torus; it is foliated in a particular way and represents a Cantor set in some section.

The main idea of Ref. [14] (see also Refs [11, 15]), which for the first time introduced the notion of the strange attractor and proved its existence, was that such subsets of phase space are indispensable in paving the way to the solution of the turbulence problem. Although this approach did not succeed to the full extent (see Refs [16, 17]), paper [14] served as a stimulus to the development of the theory of chaotic dynamical systems and its applications.

This review is devoted to the description of chaotic phenomena occurring primarily in dissipative systems. Its outline is as follows. Section 2 introduces the main concepts from the basic theory, gives a definition of chaos, and presents attractors of different types. Sections 3 and 4 deal with hyperbolic sets, Smale’s horseshoe, homoclinic tangencies, and related exotic phenomena— Ω -explosions and wild hyperbolic sets. The statistical properties of dynamical systems, explored in the framework of ergodic theory, are considered in Sections 5–7.

In order to explore the properties of complex systems, including for this experimental methods, an approach based on the analysis of signals produced by the system is widely used. It becomes especially relevant when a mathematical description of the process at hand is practically impossible, but its certain representative quantity is still available for observations. The analysis of such an approach makes up the content of Section 8. Section 9 explains the principal distinction between stochastic dynamics and deterministic chaos.

The concept of deterministic chaos is based on ideas elaborated in such fields of modern mathematics as differential dynamics, measure theory, functional analysis, the theory of singularities, topology, and others. Unfortunately, most concepts from these domains are now excessively formalized and occasionally their physical sense is lost under the veil of abstract reasoning, terminology, and theorems. Recognizing such a state of affairs, this review proposes a basically qualitative presentation of the theory of chaotic dynamical systems, in which the dominant part of formal ideas is accompanied by illustrative examples.

2. Dynamical systems and chaos

Generally speaking, the notion of what is currently called chaos has been known in mathematics from the beginning of the 20th century. Since the time of Henri Poincaré it has become apparent that when the complex behavior of dynamical systems is studied, the common approach consisting in the analytical computation of individual trajectories of differential equations fails to work. For this reason, the main task of the theory lies in exploring stability, studying the role of invariant manifolds, analyzing the geometrical structure of trajectories, searching for invariant measures, computing invariant characteristics, and so on. Although this approach does not allow explicit representation of solutions, it permits one to *qualitatively* describe many important peculiarities of dynamical systems, including their chaotic behavior. The

term *qualitative theory* is therefore frequently used in the literature.

2.1 General concepts

The subject of the study of the qualitative theory is primarily the localized systems described by a set of ordinary differential equations

$$\dot{x} = v(x, a), \tag{1}$$

where $x(t) = \{x_1, x_2, \dots, x_n\}$ is a set of dynamical variables, t is the time, $v = \{v_1, v_2, \dots, v_n\}$ is the vector function of a given smoothness r (i.e., of class C^r), defined in some domain $M \subseteq \mathbb{R}^n$, $v : M \rightarrow \mathbb{R}^n$, and a is a parameter (or a set of parameters). M is commonly referred to as the phase space of system (1).

The function v gives birth to the flow $F^t : M \rightarrow \mathbb{R}^n$, where $F^t(x)$ is a differentiable function defined for values of t from the interval $T \subseteq \mathbb{R}$, such that for all $x \in M$ and $\tau \in T$ the following relationship holds true:

$$\left. \frac{d}{dt} F^t \right|_{t=\tau} = v(F^\tau(x)).$$

Notice that F^t possesses group properties: F^0 is the identity operator, and $F^{t_1+t_2} = F^{t_1}(F^{t_2}) \equiv F^{t_1} \circ F^{t_2}$. Consequently, the flow F^t defines the function v uniquely.

The flow F^t is sometimes referred to as a shift transformation because it transfers the system from the state it occupied at the initial instant to the state at any other instant of time. One can readily understand that F^s at $s = -t$ has a mutually inverse function of the same smoothness class C^r , i.e., the system is reversible in time.

The system of equations (1) can geometrically be interpreted as a vector field in which the vector v is assigned to each point $x \in M$. The solution $x(t) = F^t(x(0))$ then represents a curve tangent to this vector field at each point. Accordingly, the word combinations ‘system of ordinary differential equations’ and ‘vector field’ are frequently used as synonyms. System of equations (1) is called autonomous if the vector field v does not contain explicit dependence on time.

Differential equations satisfy the fundamental theorem of local existence and uniqueness of solutions (see, for example, book [18] for details). Provided the initial state $x(0) = x_0 \in D$ of system (1) is specified, $F^t(x_0) : T \rightarrow \mathbb{R}^n$ defines the phase trajectory, or phase curve, of the differential equation (1). Because of solution uniqueness, there exists a unique phase curve for each initial condition x_0 and any smooth vector field v . This specifically implies that trajectories defined by system (1) cannot intersect.

If time t takes a discrete set of values, the respective dynamical system is referred to as a cascade. For such a system, the following notation is more conventional: $f : M \rightarrow M$,

$$x_{n+1} = f(x_n), \tag{2}$$

which defines a map. In this case, the sequence $x_1 = f(x_0)$, $x_2 = f(f(x_0)) = f^2(x_0) \equiv f \circ f, \dots$, i.e., $\{x_k\}_{k=0}^\infty$ will be its trajectory. The cascade f^n which is C^r -smooth together with its inverse is called the diffeomorphism.

In some cases, the flow F^t admits the existence of global section Π , i.e., a hypersurface of dimension $n - 1$, which is intersected by phase trajectories transversely (without tan-

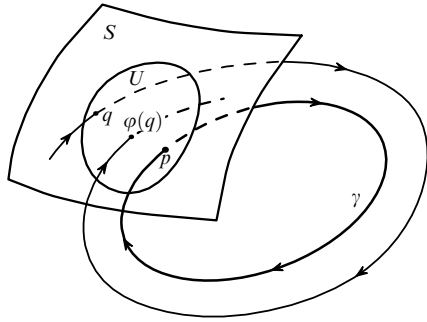


Figure 1. The Poincaré map in the vicinity of the limit cycle.

gency). In this case, the study of the behavior of an original system can be reduced to the analysis of the map $\Phi : \Pi \rightarrow \Pi$, which is called the Poincaré map. The inverse operation, from the Poincaré map to the flow, is called suspension.

One of the key notions pertaining to the dynamical system analysis is that of roughness or structural stability, introduced by A A Andronov and L S Pontryagin [19]. The vector field v of a dynamical system is said to be rough (or structurally stable) if some neighborhood U of field v exists, such that for any vector field v' from this neighborhood the phase portrait does not differ qualitatively from that defined by the field v . Correspondingly, a property of a dynamical system is rough if it is preserved under small perturbations of the system.

A crucial role in the origin of chaos in dissipative dynamical systems belongs to limit cycles — the closed phase trajectories that correspond to a periodic behavior.

Let $\gamma = x(t)$ be a limit cycle, and S be the section surface which is intersected by the cycle at point p . Select some neighborhood $U \subset S$ of this point (Fig. 1) and consider the first return of trajectory that issued out of the point $q \in U$ located close to p . In that case, one may determine the Poincaré map $\Phi : U \rightarrow S$ which maps the point q into $q' = \Phi(q)$ on the surface S .

Obviously, this map can be written in the coordinate representation as $y' = \hat{L}y + g(y)$ in some vicinity of point p , where \hat{L} is the matrix with elements $\partial\varphi_i/\partial y_k|_{y=p}$. The linearized map then takes the form $y' = \hat{L}y$. The matrix \hat{L} is called the monodromy matrix²; its eigenvalues are said to be multipliers. The multipliers can be both real and complex quantities.

The eigenvector defining an invariant direction corresponds to each eigenvalue of \hat{L} . One such eigenvector is always aligned with the trajectory. The respective multiplier is therefore equal to one.

When a phase trajectory in the vicinity of the limit cycle makes a single turn (see Fig. 1), the perturbation related to each eigenvector is multiplied by the respective multiplier. Thus, the limit cycle will be stable if all the multipliers save one (bearing the responsibility for the displacement along the trajectory) lie on the complex plane inside the unit circle, i.e., for $|\alpha_j| < 1$. If, however, the multipliers satisfy the condition $|\alpha_j| > 1$, the cycle will be absolutely unstable.

Generally, a part of the multipliers lies inside the circle, while the other part stays outside it. Such limit cycles are called saddle. An inherent feature of a saddle limit cycle is the presence of stable (W^s) and unstable (W^u) manifolds, whose characteristic property is that, if the initial point of trajectory

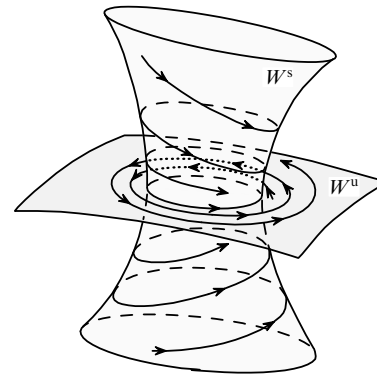


Figure 2. Stable W^s and unstable W^u manifolds of the saddle limit cycle.

$x_0 \in W^s$ ($x_0 \in W^u$), then for $t \rightarrow \infty$ (respectively, for $t \rightarrow -\infty$) the phase trajectories will tend to γ (Fig. 2). The presence of saddle type cycles is a key ingredient of chaotic dynamical systems.

2.2 Attractors

The basic feature of dissipative systems is the contraction of their phase volume: according to dynamical equations, the volume initially occupied by a cloud of phase points decreases with time. Formally, such property is expressed through the inequality $\text{div } v < 0$. In this case, as $t \rightarrow \infty$ all the phase trajectories will converge to some subset $\mathcal{A} \subset M$ of zero (in phase space) volume, which is referred to as the attractor of the dynamical system.

Currently, there are several definitions of the attractor, which seemingly cannot be reduced to each other (see reviews [20–23]). The geometrical definition of attractor is as follows. A compact subset \mathcal{A} of the phase space M is said to be an attractor if it satisfies the following conditions [23]:

- (1) \mathcal{A} is invariant with respect to the flow of the dynamical system;
- (2) there exists a neighborhood U that contracts to \mathcal{A} under the flow action;
- (3) \mathcal{A} cannot be decomposed into two or more nonintersecting invariant subsets.

The invariance of attractor \mathcal{A} with respect to the flow implies that $F^t \mathcal{A} = \mathcal{A}$, i.e., once on the attractor we stay there as $t \rightarrow \infty$. From the second condition it follows that \mathcal{A} is such a subset of the set U , $\mathcal{A} \subset U \subset M$, for which, as $t \rightarrow \infty$, $F^t U \rightarrow \mathcal{A}$ holds. In fact, one is dealing here with the attraction basin of attractor \mathcal{A} , i.e., with the set of initial points $x_0 \in U$, such that the phase trajectories issuing out of these points converge as $t \rightarrow \infty$ to the attractor \mathcal{A} . The third condition serves to exclude the attractors composed of several separate components.

According to this definition, stable equilibrium positions, limit cycles, and tori are attractors; they are called regular, or simple.

2.3 Chaos

There are several ways of introducing the notion of chaos. The most widespread and frequently used definition, proposed in Ref. [25], exploits the property of extreme (exponentially strong) sensitivity of the system to initial conditions or external actions. This seems quite natural because the main manifestation of dynamical chaos is expressed in terms of the exponential divergence of close trajectories.

² The term monodromy originates from *mono* (one) and *drom* (circle), i.e., the monodromy matrix is that obtained through a single cycle run.

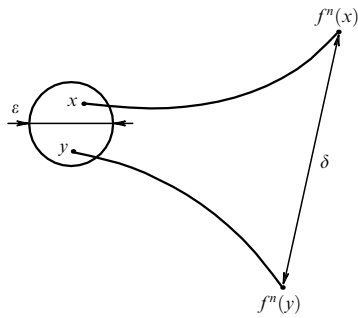


Figure 3. Geometrical interpretation of instability with respect to initial conditions.

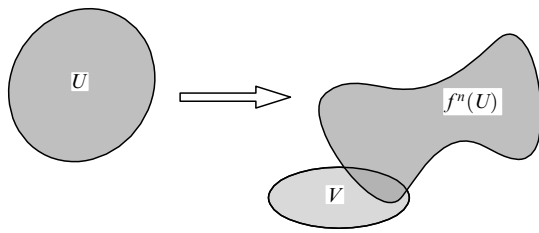


Figure 4. Elucidation of a transitivity property.

And yet, exponential instability alone is insufficient to define the notion of chaos. It is necessary that the condition of transitivity be obeyed and that there exist some regularity called the density of periodic orbits (i.e., cycles). The condition of transitivity is often replaced by a stronger condition of topological mixing.

Let M be a metric space. The map $f : M \rightarrow M$ is called chaotic [25] if (a) f is unstable with respect to initial conditions, (b) f is topologically transitive, and (c) the cycles of map f are dense in the space M .

The map f is said to be unstable with respect to initial conditions if a quantity δ (constant of instability) exists, such that for some point $x \in M$ and $\epsilon > 0$ there exists a point $y \in M$ satisfying the conditions $\text{dist}(x, y) < \epsilon$ and $\text{dist}(f^n(x), f^n(y)) \geq \delta$ at $n \in \mathbb{N}$, where $\text{dist}(\cdot, \cdot)$ denotes the distance. The geometrical interpretation of these relationships is shown in Fig. 3. It is noteworthy that the constant δ is independent of x or ϵ —it is defined only by the properties of the system under consideration.

Further, map f is called transitive if, for any two open sets U and V , an integer number n exists, such that $f^n(U) \cap V \neq \emptyset$. The informal sense of the transitivity property is demonstrated in Fig. 4. Notice that, as known from the theory of metric spaces, the transitivity is equivalent to the existence of a dense trajectory.

Finally, the property of density of periodic trajectories implies that, in any vicinity of any point in the space M , there exist at least one and hence infinitely many periodic trajectories.

Thus, a chaotic system must possess three important properties: (1) unpredictability (exponential instability); (2) indecomposability (transitivity), and (3) some regularity (the density of cycles). However, not long ago it was found [26] that the condition of sensitive dependence on initial conditions is redundant in the definition given above. Consequently, if a map is continuous and possesses the transitivity property, while the cycles are dense, it exhibits sensitive dependence on the initial conditions. Somewhat later it was revealed [27] that in the definition of chaoticity

neither the transitivity nor the density of cycles follows from the other two conditions. Arguably, a transformation defined on a compact set can be referred to as chaotic if it exhibits a sensitive dependence on initial conditions and has dense cycles.

Very recently the authors of paper [28] proposed a definition of the dynamical system chaoticity, which involves the requirement of trajectory complexity in addition to the sensitive dependence on initial conditions. Here, the complexity means the absence of recurrence in a certain sense. It was then shown that, from the definition of system’s chaoticity based on the instability with respect to the initial conditions, transitivity and the density of cycles [25], follow the definition proposed by the authors of Ref. [28].

We remark that one can also approach the definition of chaos from a physical viewpoint by resorting to the notions of entropy and dimension (see Section 9).

3. Hyperbolic dynamics

Hyperbolic sets furnish a good example for gaining insight into the ‘internal structure’ of dynamical systems exhibiting a chaotic behavior. The attractive feature of hyperbolic sets lies also in the fact that studying them allows one to understand the global behavior of trajectories through the analysis of their local properties.

3.1 Hyperbolic sets

The construction of hyperbolic sets is based on a very simple idea: the tangent space Σ of a dynamical system is structured so that it is composed of three subspaces: the stable E^s , unstable E^u , and neutral E^0 . In other words, here one is dealing with such a linear vector space Σ of vectors of infinitesimal perturbations, where arbitrary perturbation vector can be expanded into a linear combination of vectors belonging to the subspaces E^s , E^u , and E^0 . The subspaces E^s and E^u are defined by the fact that close trajectories exponentially converge: those corresponding to E^s as $t \rightarrow \infty$, and those corresponding to E^u as $t \rightarrow -\infty$. In the subspace E^0 , vectors are contracted or stretched more slowly than exponentially. Similar dynamics of trajectories of dynamical systems follows from this behavior of equations in variations.

Consider a system with discrete time which is defined through the map (diffeomorphism) (2), and a compact invariant set $A \subset M$, i.e., $f(A) = A$. For such transformations, the hyperbolicity is introduced as follows [29, 30] (see also Refs [22, 31, 32] and the literature cited therein). Let each iteration f^n be a smooth one in the neighborhood of $x \in M$. Then there exists the differential ∂f_{x_n} of the diffeomorphism f of the maps of tangent space Σ_{x_n} into the tangent space $\Sigma_{f(x_n)}$. The trajectory x_n is called hyperbolic if there exist subspaces $E_{f^k(x)}^s$ and $E_{f^k(x)}^u$ of the tangent space $\Sigma_{f^k(x)}$, $0 \leq k < \infty$ such that $\Sigma_{f^k(x)} = E_{f^k(x)}^s + E_{f^k(x)}^u$ and the following conditions are satisfied:

- (a) $\partial f_{f^k(x)}(E_{f^k(x)}^s) = E_{f^{k+1}(x)}^s$, $\partial f_{f^k(x)}(E_{f^k(x)}^u) = E_{f^{k+1}(x)}^u$;
- (b) $\|\partial f_{f^k(x)}e\| \leq c\|e\|$, $e \in E_{f^k(x)}^s$, $\|\partial f_{f^k(x)}e\| \geq c^{-1}\|e\|$, $e \in E_{f^k(x)}^u$, where $0 < c < 1$ is a constant;
- (c) $\text{dist}(E_{f^k(x)}^s, E_{f^k(x)}^u) \geq \text{const}$, $0 < k < \infty$.

The decomposability of the tangent space Σ implies that any vector v can uniquely be represented as $v = v^s + v^u$, where $v^s \in E^s$ and $v^u \in E^u$. From relations (a) it follows that the subspaces E^u and E^s are invariant, and condition (b)

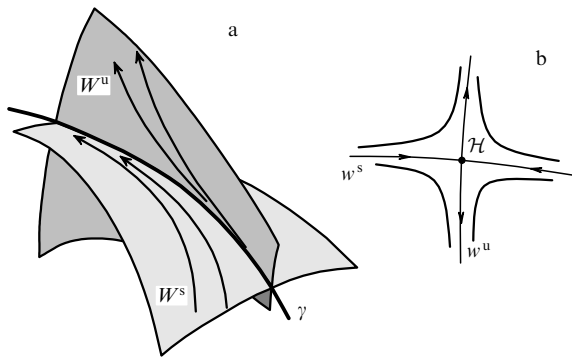


Figure 5. Hyperbolic trajectory γ and hyperbolic fixed point \mathcal{H} .

implies hyperbolicity. Finally, inequality (c) points to the properties of E^u and E^s .

The concept of hyperbolicity is naturally introduced for flows, too.

A set Λ is called hyperbolic if it is closed and is composed of trajectories that satisfy the hyperbolicity conditions. If estimates (b) worsen along the trajectory (with an increase in the number of iterations), i.e., the degree of contraction and extension in subspaces E^u and E^s varies from point to point, then such sets are referred to as nonuniformly hyperbolic. Dynamical systems with uniform hyperbolicity of all trajectories are said to be the Anosov systems [32].

One of key results in this area, which is also important for applications, is the Hadamard–Perron theorem, or the theorem about local manifolds. According to it, a hyperbolic trajectory γ possesses local stable W^s and unstable W^u manifolds (Fig. 5a). The phase curves of a dynamical system exponentially diverge on the manifold W^u , and exponentially converge on the manifold W^s . In a section, such a hyperbolic trajectory corresponds to a hyperbolic (saddle) point \mathcal{H} (Fig. 5b).

Establishing the hyperbolicity of a set under study is, as a rule, very difficult. In applications, the so-called cone condition (see Ref. [31] and references cited therein) turns out to be very convenient to test the hyperbolicity property. In particular, the hyperbolic nature of Smale–Williams and Plykin attractors in physically realizable systems [33–35] (see Section 5.2) was substantiated with its assistance, and the proof of existence of the well-known Lorenz attractor was also found [36, 37] (see Section 5.3).

3.2 The Smale horseshoe

One of the remarkable examples of hyperbolic sets is Smale’s horseshoe. The modern theory of a chaotic dynamical system began by exploring this sample (see Refs [38, 39] about it). At the present time, the horseshoe is well studied; we shall therefore describe only basic elements of its construction using the example of the reversible map of a plane, which can be considered as a Poincaré map for a certain three-dimensional flow.

Consider a unit square on the plane $S = [0, 1] \times [0, 1]$. Let us introduce a map $f: S \rightarrow \mathbb{R}^2$ which acts so that $f(S) \cap S \equiv \Omega_d^1$ forms two rectangular strips, Q_1 and Q_2 , and that the horizontal boundaries AB and DC also transform into horizontal ones (Fig. 6). Two horizontal strips $B_i = f^{-1}(Q_i)$, $i = 1, 2$, are the pre-images of strips Q_1 and Q_2 .

This construction corresponds to a stretching of the original square with a certain coefficient α along the vertical

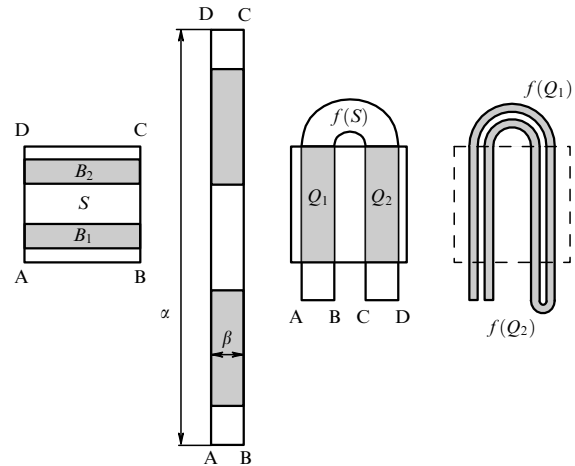


Figure 6. Smale’s horseshoe.

direction, contraction in the horizontal direction with coefficient β , and folding the resulting rectangle into a horseshoe so that the bent part stays outside of S . The next iteration of the map $f(Q_i)$ leads to the appearance of the set $\Omega_d^2 = S \cap f(S) \cap f^2(S)$ composed of four narrower vertical strips (see Fig. 6). The inverse iterations give four horizontal strips $\Omega_r^2 = S \cap f^{-1}(S) \cap f^{-2}(S)$. On completing m iterations, one gets the sets

$$\Omega_d^m = \bigcap_{k=0}^m f^k(S), \quad \Omega_r^m = \bigcap_{k=0}^m f^{-k}(S)$$

composed of 2^m vertical and horizontal strips, respectively. Therefore, the limit sets Ω_d^∞ and Ω_r^∞ will include an uncountable set of lines that represent a product of a Cantor set by an interval. The construction described here can obviously be varied in different ways.

It can readily be seen from the construction of the horseshoe that most points leave the square S in the course of iterations. Moreover, almost all points are not contained in the image $f^k(S)$. Those points p that always stay in S form a nontrivial invariant subset

$$\begin{aligned} \Omega &= \{p \mid f^k(p) \in S, -\infty < k < \infty\} \\ &= \Omega_d^\infty \cap \Omega_r^\infty = \bigcap_{k=-\infty}^{+\infty} f^k(S), \end{aligned}$$

which has a zero measure and complex topological structure. This set is called the Smale horseshoe; however, as remarked by D V Anosov [32], it is not this set that looks like a horseshoe, but rather the set $f(S)$ used in the construction procedure.

As is known, the set Ω contains infinite families of saddle cycles with various periods, some dense trajectory, and an uncountable set of nonperiodic motions. Therefore, the presence of the horseshoe is usually treated as a manifestation of certain chaotic properties by the system.

The Smale horseshoe can coexist with attractors. However, the set itself is not attracting. As an outcome, dynamical systems containing a horseshoe in their phase space may demonstrate a transient (sometimes very long) chaotic behavior which ends with the convergence of trajectory to a certain stable cycle or stationary point. A characteristic example is given by the well-known Duffing equation for a

certain set of parameters [40] (for more details about this system, see Ref. [41]).

From the existence of the horseshoe one can draw an unexpected conclusion which is of utmost importance for physical applications: the presence of horseshoe type complex sets of zero measure in the phase space, which are commonly neglected in the analysis, might have a dramatic effect on the dynamical system behavior (see Section 6). The discussion of these questions by the example of a wide spectrum of phenomena can be found in a recent monograph by G M Zaslavskii [42].

The Smale horseshoe occurs rather often in physical systems, for instance, in the celestial mechanics [43] (see also Ref. [44]), as a result of transversal intersection of manifolds in the Van der Pol [45] and Duffing [40] equations, in a model of a bouncing ball [46], and some others (see Refs [30, 47]). These results revealed all the complexity of the dynamics inherent in the systems which seem very simple at first glance.

4. Homoclinic structures

Invariant sets in the vicinity of homoclinic and heteroclinic trajectories also belong to hyperbolic sets. Such trajectories, discovered by Poincaré when studying the N -body problem (see Ref. [48]), occur in both conservative and dissipative systems. At present, their analysis is an independent field of nonlinear dynamics [49].

4.1 Homoclinic trajectories

In order to get a general idea about the behavior of systems featuring homoclinic and heteroclinic structures, let us consider, for simplicity, a three-dimensional ($n = 3$) dynamical system (1). Assume that this system possesses a saddle limit cycle γ with stable W^s and unstable W^u manifolds. Suppose that γ has a doubly asymptotic trajectory $\Gamma = W^s \cap W^u$, i.e., $\Gamma \rightarrow \gamma$ as $t \rightarrow \pm\infty$. A part of such a structure is displayed in Fig. 7. If the manifolds W^s and W^u intersect along Γ transversely (Fig. 7a), it is said that a rough homoclinic trajectory Γ emerges. When there is tangency of the manifolds along the curve Γ [i.e., the homoclinic tangency (Fig. 7b)], the homoclinic trajectory Γ_0 is called nonrough (structurally unstable).

It is well known that from the existence of a single homoclinic trajectory follows the existence of an infinite number of them (see Ref. [12] for a visual explanation of this phenomenon). A part of a similar structure in the case of transversal intersection is shown in Fig. 8a. In the Poincaré map, the closed hyperbolic trajectory (a saddle cycle) γ corresponds to the hyperbolic point \mathcal{H} . The manifolds of this point are one-dimensional and correspond to the stable

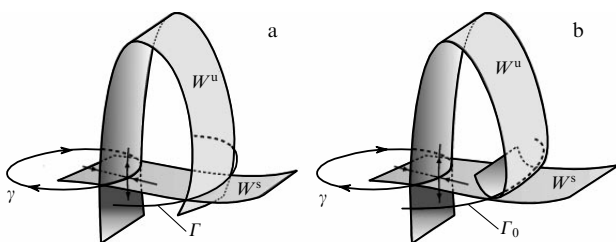


Figure 7. (a) Rough Γ , and (b) nonrough Γ_0 homoclinic trajectories of the saddle cycle γ .

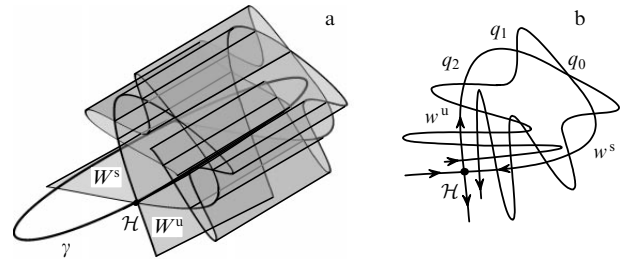


Figure 8. (a) Invariant manifolds in the vicinity of hyperbolic cycle, and (b) a respective Poincaré map (b).

$w^s(\mathcal{H})$ and unstable $w^u(\mathcal{H})$ branches of separatrices (Fig. 8b). The intersection of w^s and w^u implies the presence of a transversal homoclinic point q_0 . Its trajectory is defined as the infinite set of intersections: $\mathcal{Q} = \{q_i\}_{i=-\infty}^{\infty}, q_{i+1} = f(q_i)$.

The homoclinic points are connected by loops which are mapped into each other through f . Since the loops embrace equal areas, and stable and unstable manifolds have no self-intersections, these loops expand more and more with iterations of the map. As a consequence, one obtains a lattice formed by branches of a separatrix in the phase space (Fig. 8b). In the physical literature such regions are called stochastic layers or homoclinic tangles.

Consider a set \mathcal{U} consisting of such intersections and the point \mathcal{H} , and also of neighborhoods U_0, U_1, \dots, U_k which are selected in the following way. Let U_0 be the neighborhood of homoclinic point q , which also includes points $f^k(q)$ except for some finite number of them. The remaining points of the homoclinic lattice are covered by the mutually disjoint neighborhoods U_1, \dots, U_k . It can then be shown [44] (see also Ref. [50]) that for any neighborhood V of the set \mathcal{U} there exists a system of neighborhoods $U_i \subset V, i = 0, 1, 2, \dots$, for which

$$A = \bigcap_{i=-\infty}^{\infty} f^i \left(\bigcup_{j=1}^k U_j \right)$$

is a hyperbolic invariant set. This means that the set A is composed of such hyperbolic points y which stay in the neighborhood V subject to iterations of map $f^k(y), -\infty < k < \infty$. With decreasing V , the set A tends to \mathcal{U} .

Homoclinic points (trajectories) can be considered as a particular case of heteroclinic points (trajectories) which occur at the intersection of stable and unstable branches of separatrices (manifolds) of different hyperbolic points. The results analogous to those obtained for homoclinic points (trajectories) [31, 49] are valid for them, too.

4.2 Horseshoes in homoclinic structures

The Smale horseshoe is closely related to homoclinic and heteroclinic tangles. This relationship is vividly demonstrated by the following construction. Consider a small neighborhood U of hyperbolic point \mathcal{H} . The iterations of map f will ensure that there exist m, n such that $q \in f^k(U)$ for $k \geq m$, and $q \in f^{-l}(U)$ for $l \geq n$. This implies that the domain U is stretched along the unstable direction w^u forming a strip $S = f^m(U)$ which includes the homoclinic point q . On the other hand, a certain neighborhood B around the stable direction w^s and point q is mapped into the domain $U, B = f^{-n}(U)$ (Fig. 9), by inverse iterations. Consequently, the bent horseshoe-like strip $S = f^{m+n}(B)$ intersecting B will be the image of strip B after $n + m$ iterations.

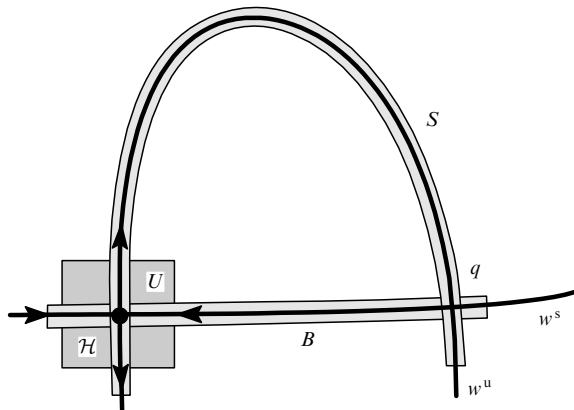


Figure 9. The emergence of a horseshoe from the homoclinic structure.

Subsequent iterations lead to even stronger stretching and new intersections $B \cap f^l(B)$. This structure is analogous to construction of the Smale horseshoe (see Fig. 6), but this time, instead of starting from the square S , one is dealing with the curvilinear rectangle B . We thus arrive at a very important result first obtained by Birkhoff [51, 52] and later rigorously justified by Smale [2]: if a diffeomorphism $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ possesses a fixed hyperbolic point \mathcal{H} and transversal homoclinic point q , then a horseshoe exists in any small neighborhood of \mathcal{H} for a certain iteration of map f . The inverse is also true: if horseshoes are present in dynamical systems, there is also a homoclinic structure.

An important corollary follows from the Smale–Birkhoff result: the presence of a transversal homoclinic point entails positive entropy of the dynamical system (see Section 7.2). The inverse statement for maps of the plane [53] and flows [32] is valid, too. This, among other things, implies that, for the system to behave chaotically, it is sufficient that horseshoes be born (see Section 9). It should, however, be borne in mind that a chaotic set may turn out to be nonattracting.

Among dynamical systems possessing homoclinic structures an important place belongs to those containing a loop of the equilibrium state of the ‘saddle–focus’ type in their phase space (Fig. 10a). The systems with homoclinic loops are nonrough. Having been perturbed, the loops split, which may bring about very complex dynamics.

An analysis of systems with double-asymptotic trajectories of the ‘saddle–focus’ type was first conducted by L P Shil’nikov [7, 49, 54]. Shil’nikov revealed that there are Smale horseshoes in the whole neighborhood of the values of the parameter for which the saddle–focus loop is observed. Moreover, there exist here systems with homoclinic tangencies (see Section 4.3) and systems which have a countable set of stable cycles.³

Consider next a three-dimensional vector field that has a stationary point O at the coordinate origin with a single positive eigenvalue $\rho > 0$ and two complex-conjugate ones, $\lambda_{1,2} = \eta \pm i\zeta$. This means that the stationary point has the structure of a saddle–focus with a two-dimensional locally stable manifold W^s and a one-dimensional unstable manifold W^u . Assume that the separatrices of the saddle form a homoclinic loop Γ (Fig. 10a). The Shil’nikov theorem asserts that if $\rho > |\eta|$, then the perturbed vector field has invariant

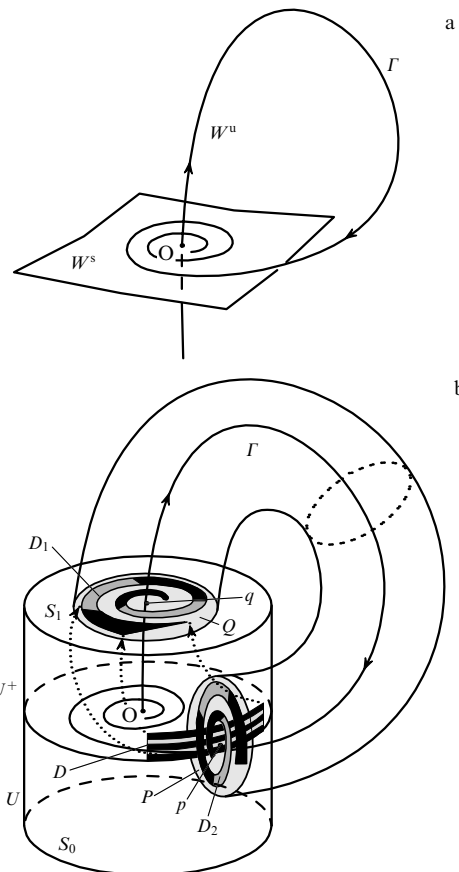


Figure 10. (a) Saddle–focus, and (b) the birth of a horseshoe.

hyperbolic sets, and the Poincaré map has a countable set of horseshoes. In this case, an infinite set of saddle limit cycles exists in any neighborhood of loop Γ .

The emergence of a horseshoe from a saddle–focus loop can easily be understood from the following geometrical construction. Consider the neighborhood U of point O in the form of a cylinder which is defined as the intersection of two surfaces S_0 and S_1 (Fig. 10b). The stable manifold W^s divides this neighborhood into two parts, U^+ and $U \setminus U^+$. We will denote the intersection point of the unstable manifold W^u with the surface S_0 as p , and the intersection point with the surface S_1 as q . If the neighborhood U^+ is sufficiently small, then a map $\psi: S_0 \rightarrow S_1$ exists which establishes correspondence for any point on S_0 with a certain point on S_1 (Fig. 10b). Specifically, this map transforms the domain $D \subset S_0$ into a ‘thick’ logarithmic spiral $D_1 \subset S_1$ with its center at the point q , i.e., $\psi(D) = D_1$. Put differently, the horizontal strips on D are mapped onto the strips lying inside the two spirals belonging to S_1 , which twist around the point q , while the point q itself is the image of the intersection line of D and W^s .

Since the flow from q to p is not degenerate, there exists a diffeomorphism φ of the neighborhood Q of point q into the neighborhood P of point p , $\varphi: Q \rightarrow P$. In this case, the domain D_1 is mapped into some domain D_2 on the surface S_0 and $\varphi(q) = p$. Thus, the map of the first return can be expressed as $\varphi(\psi)$. The action of this map consists in transforming the original domain D into a spiraling strip D_1 which is then mapped into D_2 and superimposed on D in some way (Fig. 10b). As an outcome, $\varphi(\psi(D)) \cap D$ represents a set which is analogous to the Smale horseshoe by construction.

³ As remarked by I M Ovsyannikov, one of L P Shil’nikov disciples, the saddle–focus is inexhaustible, in the same way as an electron.

A more complete consideration and rigorous analysis of the construction described here is offered in the monograph [49] (see also the literature cited therein). Other research work [55, 56] provides examples of the systems of differential equations which allow physical interpretation, for which Shil'nikov's results permit direct verification.

4.3 Homoclinic tangencies and wild hyperbolic sets

Dynamical systems with homoclinic tangencies of stable and unstable manifolds of saddle cycles, mentioned in Section 4.1, can manifest a number of new unexpected features and, as a consequence, be characterized by an additional complexity of their trajectories. This hinges on the so-called nonisolated character of tangencies, which is rooted in the fact that systems with homoclinic tangencies are dense in the space of dynamical systems and form whole regions, called the Newhouse domains.

The homoclinic tangencies were first described in Refs [56, 57] (see also Refs [22, 58]), which studied basic bifurcations pertaining to the existence of a nonrough homoclinic curve and leading to a complex behavior.

Let the saddle cycle γ be such that its stable W^s and unstable W^u manifolds have tangency along some homoclinic curve Γ_0 (Fig. 7b). Depending on the geometry and the signs of the cycle multipliers, such a system may allow several different types of homoclinic tangencies. A part of them can be reduced to each other, but irreducible combinations can only be of three distinct classes [58]. For each of them, the structure of the set Δ of trajectories in a small neighborhood of nonrough curve Γ_0 can be qualitatively different.

Consider a Poincaré map in the neighborhood of cycle γ . Assume that $f(x, a)$ is a volume-contracting one-parametric family of smooth diffeomorphisms of plane \mathbb{R}^2 , which has a hyperbolic fixed point \mathcal{H} with eigenvalues ρ and σ , $0 < |\rho| < 1 < |\sigma|$, $|\rho\sigma| \neq 1$. Suppose next that at $a = 0$ the family $f(x, a)$ has a nonrough homoclinic trajectory \mathcal{Q}_0 at points of which the stable $w^s(\mathcal{H})$ and unstable $w^u(\mathcal{H})$ manifolds have a quadratic tangency, i.e., $w^s = x$ in local coordinates, while w^u is expressed as $a - x^2$ (or $a + x^2$).

If the unstable branch $w^u(\mathcal{H})$ approaches the stable one $w^s(\mathcal{H})$ so that the tangency occurs from 'below' (Fig. 11a, b), then such a structure is referred to as a tangency of the first class. In this case, the set Δ includes the homoclinic trajectory \mathcal{Q}_0 and the point \mathcal{H} . Diffeomorphisms with this tangency type are of interest thanks to the fact that in the parameter space they correspond to the boundaries that separate the regions containing systems with a simple behavior of trajectories from those with systems exhibiting chaotic behavior. When passing through such a boundary, complex dynamics emerges immediately, i.e., in an explosive way, which gave the phenomenon its name— Ω -explosion. This dynamics was discovered in Refs [60, 61] (see also Ref. [62]) and were thoroughly explored in Refs [63–66].

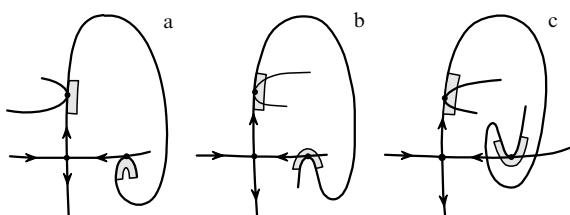


Figure 11. (a, b) Homoclinic tangencies of the first, and (c) second classes.

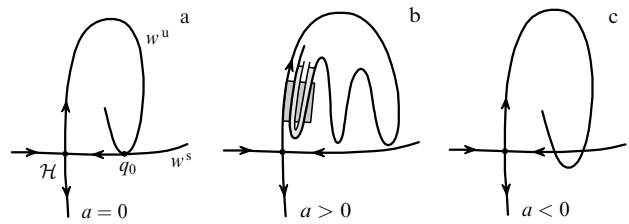


Figure 12. Transformation of manifolds of diffeomorphism $f(x, a)$ in \mathbb{R}^2 .

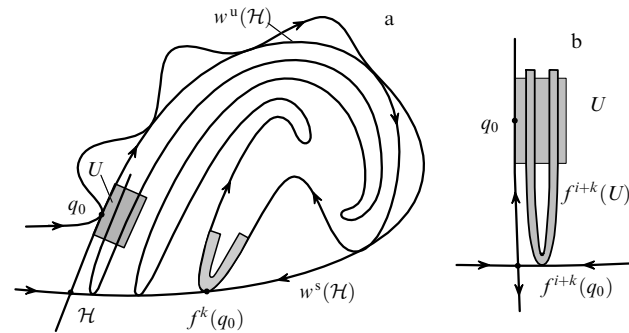


Figure 13. Birth of a horseshoe from a homoclinic tangency.

If the separatrix branch approaches from 'above', as shown in Fig. 11c, the tangency of the second class takes place. The set Δ in such systems has a nonuniform hyperbolic structure, i.e., all trajectories except for \mathcal{Q}_0 are hyperbolic.

When the diffeomorphism $f(x, a)$ is such that segments of the curves $w^u(\mathcal{H})$ and $w^s(\mathcal{H})$ are tangent, as shown in Fig. 12a, then such tangencies belong to the third class. In this case, the set Δ contains nontrivial hyperbolic subsets and, consequently, systems of that type exhibit chaotic dynamics. It turns out that the tangencies of the third class exist in the neighborhood of any system with a homoclinic tangency.

The onset of complex dynamics in such systems follows from the following qualitative analysis [21]. Suppose the homoclinic tangency occur at $a = 0$ (Fig. 12a). Consider a rectangular domain U that contains a point of tangency q_0 of stable $w^s(\mathcal{H})$ and unstable $w^u(\mathcal{H})$ manifolds on its boundary (Fig. 13a). Iterations of map f lead to the result that the point $f^k(q_0)$ belongs to $w^s(\mathcal{H})$ for some k . It is then easy to see that subsequent iterations $f^{i+k}(U)$ for sufficiently large i will lead to an intersection with U and the birth of a horseshoe (Fig. 13b).

As shown in Refs [57, 58], a horseshoe does not occur for some other tangencies prior to the appearance of transversal homoclinic points under the action of perturbations. This finding is clarified in Fig. 11 which shows the evolution of a semineighborhood of homoclinic tangency under the action of map f .

When the family $f(x, a)$ is perturbed, the tangencies may disappear, while intersections of manifolds may appear, causing a qualitative transformation of the diffeomorphism behavior. Namely, if $a > 0$, the tangencies are absent (Fig. 12b), and horseshoes disappear for subsequent map iterations. However, one can readily envisage that for a that is sufficiently small and positive, i.e., just before the instant of tangency, a very complex structure will be present in the system. For $a < 0$, the map has a transversal homoclinic point close to $f^k(q_0)$ and, as a consequence, a horseshoe.

Although the systems considered here exhibit a high degree of complexity, effects of an even more delicate nature may show their worth in the situation above. They are related to the birth of so-called wild hyperbolic sets first described by Newhouse [67–69]. The name is carried by uniformly hyperbolic sets whose stable and unstable manifolds have a quadratic tangency which cannot be removed by small smooth perturbations. More precisely, individual homoclinic tangencies can be eliminated by gently ‘wiggling’ the system, but one cannot get rid of all them in this way because new tangencies inevitably occur.

This is a rather unexpected result. It contradicts, in a certain sense, the common view that even if a tangency takes place at some point, it should disappear subject to perturbations because of structural instability (see Fig. 12). Notably, this naive viewpoint is incorrect. In this case, for general map families $f(x, a)$ there exist intervals where the values of the parameter a , for which $f(x, a)$ has homoclinic tangencies, are dense.

In order to understand in which sense the particular types of tangencies possess ‘stability’, consider a hyperbolic invariant set A on a plane, which is defined through the diffeomorphism $f(x, a)$. From the example of the Smale horseshoe in Section 3.2 it follows that stable and unstable manifolds of this set look like a product of a Cantor set by an interval. Assume that a quadratic tangency of stable $w^s(A)$ and unstable $w^u(A)$ manifolds takes place. In its neighborhood $w^s(A)$ and $w^u(A)$ are the Cantor sets of horizontal intervals and parabolas (Fig. 14). Let us denote the curve passing through the vertices of parabolas as L . The Cantor sets $K_s = w^s(A) \cap L$ and $K_u = w^u(A) \cap L$ exist on it. If there is a point $q_0 = K_s \cap K_u$, it will be the point of tangency for manifolds $w^s(A)$ and $w^u(A)$.

In order to establish whether K_s and K_u intersect, the author of Ref. [67] used the metric characteristics of a Cantor set—its ‘thickness’ $d(K)$. This quantity for a Cantor set on an interval characterizes the ratio of the length of parts removed in the process of construction to that of the remaining parts. As proven in Ref. [66], if K_1 and K_2 are the Cantor sets satisfying the inequality $d(K_1)d(K_2) > 1$, then (under some additional conditions) $K_1 \cap K_2 \neq \emptyset$. This implies that the problem of proving the existence of tangencies that do not disappear subject to perturbations can be reduced to constructing Cantor sets of

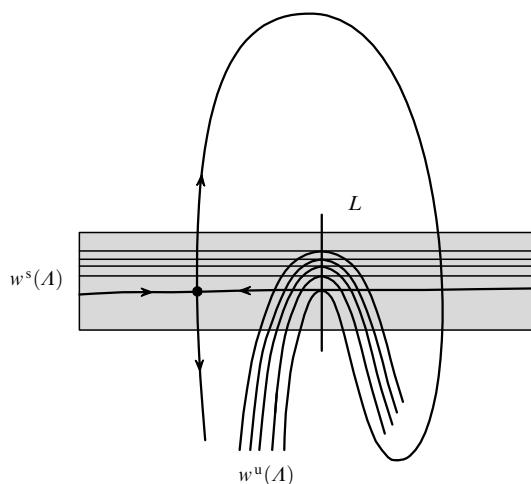


Figure 14. Tangency of manifolds of a Cantor set.

finite thicknesses. It is precisely the existence of such sets that was proven in Ref. [67].

Later on, when adapting this approach to dynamical systems, it was proven in Ref. [69] that open domains exist in the space of smooth dynamical systems, domains in which systems with homoclinic tangencies are dense. They are called Newhouse domains. The invariant hyperbolic sets near A , which contain tangencies, are said to be wild hyperbolic sets.

The dynamics of systems with homoclinic tangencies are extremely complex. Indeed, it was proven in Ref. [68] (see also Ref. [70]) that systems possessing infinitely many stable cycles are dense in Newhouse domains. A countable set of saddle, stable, and absolutely unstable cycles was discovered in systems from Newhouse domains with heteroclinic tangencies [71]. Moreover, such systems may simultaneously have a countable set of stable and unstable invariant tori which coexist with a countable set of saddle, stable, and absolutely unstable cycles [72].

In the research performed in Ref. [73] it was found that both homoclinic tangencies and cycles of an arbitrarily high degeneracy order may appear in systems with homoclinic tangencies subject to smooth perturbations. The degeneracy is linked to the fact that one of the multipliers becomes equal to $+1$ (or -1), and some or all Lyapunov values come to zero. One of the ensuing fundamental implications, which had an essential impact on nonlinear dynamics as a whole, is the impossibility of getting a full qualitative description of models with complex behavior in the framework of a finite-parametric family of dynamical systems [70, 74, 75]. In addition, a number of dynamical properties that seemed exotic earlier are, on the contrary, typical for systems with homoclinic tangencies [76, 77].

Thus, nonrough homoclinic trajectories are never found to be isolated. Moreover, if homoclinic tangencies occur in deterministic systems, they prove to be a source of additional dynamical complexity and entail essential transformations of the behavior of systems.

Notwithstanding their abstract character, homoclinic tangencies and Newhouse domains have been found in many systems, including physical ones. For example, they are encountered in families of two-dimensional maps that demonstrate transition to chaos through period doubling [73], in systems of Lorenz [78] and Duffing [22] types, in the destruction of quasiperiodic motion [79, 80], in systems containing the homoclinic loop of the ‘saddle–focus’ type [54, 81, 82] (see Fig. 10), and in some others.

The most complete information on homoclinic tangencies, Newhouse domains, and related phenomena is presented in the paper collection [59] and monograph [83].

4.4 Symbolic dynamics

The construction of the horseshoe proposed by Smale to prove the hyperbolicity in the neighborhood of rough homoclinic trajectory turns out to be insufficiently universal. In particular, it cannot be adapted to a broad class of Hamiltonian systems. Moreover, using this approach one does not succeed in substantiating the Birkhoff hypothesis [52] (known as the Poincaré–Birkhoff problem) which asserts that a two-dimensional area-preserving diffeomorphism contains a countable set of periodic orbits in any neighborhood of the closure of a homoclinic trajectory.

In due time Birkhoff conjectured that, in order to prove this statement, one may apply methods of so-called symbolic

dynamics, where, instead of system's trajectory, one considers the sequence corresponding to the phase space regions visited by this trajectory. In this case, each trajectory is coded with the help of some finite or countable alphabet by an infinite sequence. In other words, the original trajectory turns out to be associated with a shift in space of the sequences obtained. Despite its somewhat uncommon character, this approach proved to be rather productive and facilitated proofs of a number of important conjectures related, for example, to the hyperbolicity of dynamical systems, the complexity of dynamics in the three-body problem, and some others [44].

Shil'nikov [84] succeeded in obtaining the full solution to the Poincaré–Birkhoff problem. In his proof he made use of precisely the possibility of coding trajectories from the neighborhood of the homoclinic curve. It is hard to imagine how this complex dynamics could be described analytically using traditional approaches.

The simplest example of a symbolic system is furnished by the so-called Bernoulli topological scheme in two symbols. Consider some set \mathcal{X} whose elements are sequences $\mathbf{a} = \{a_i\}_{i=-\infty}^{\infty}$ infinite to both sides and composed of only two symbols, 0 and 1. If one introduces metrics $d(\mathbf{a}, \mathbf{b}) = \sum_{m=-\infty}^{\infty} \delta_m 2^{-|m|}$, where $\delta_m = 0$ at $a_m = b_m$ and $\delta_m = 1$ for $a_m \neq b_m$, then \mathcal{X} will be a compact. Let us now define the map $g: \mathcal{X} \rightarrow \mathcal{X}$ as a shift by one symbol, i.e., $g(\mathbf{a}) = \mathbf{b}$, where $b_n = a_{n+1}$. The map constructed in this way, which can be called the shift map, will satisfy the definition of chaoticity (see Section 2.3).

Now it is possible to prove that a one-to-one correspondence φ exist between the Smale horseshoe Ω and the set \mathcal{X} , such that the sequence $\mathbf{b} = \varphi(f(p))$ is obtained from the sequence $\mathbf{a} = \varphi(p)$ through a one-symbol shift: $b_k = a_{k+1}$. Indeed, $\varphi(p) = \{a_k\}_{k=-\infty}^{\infty}$, where $f^k(p) \in B_{a_k}$. This implies that the point p lies in Ω only if the relationship $f \in B_1 \cup B_2$ is valid for all k (see Fig. 6). Since $f^{k+1}(p) = f^k(f(p))$, then $\varphi(f(p))$ will follow from $\varphi(x)$ with the help of the superscript shift.

Thus, we get that $\varphi \circ f|_{\Omega} = g \circ \varphi$. Hence, a topological conjugation exists between $f|_{\Omega}$ and g . In other words, from the relationship $f|_{\Omega} = \varphi^{-1} \circ g \circ \varphi$ one arrives at $f^k|_{\Omega} = \varphi^{-1} \circ g^k \circ \varphi$. Consequently, there exists a transformation φ of trajectories of f on Ω to trajectories of g on \mathcal{X} .

More elaborate constructions are applied in order to analyze a system with homoclinic and heteroclinic trajectories with the help of the symbolic approach (see Refs [22, 31, 44]). The monograph [85] presents the basic principles of symbolic dynamics.

5. Chaotic attractors of dynamical systems

The constructions described above point to the existence of hyperbolic sets and to the complex behavior of dynamical systems under rather general assumptions. In the general case, however, these results does not imply that typical trajectories are asymptotically chaotic, because attractors will always present in the phase space of systems given the dissipation processes. If these are the regular attractors mentioned in Section 2.2, the complex dynamics pertaining to the presence of above-described hyperbolic sets will be a transient process and in the long run the behavior of such systems will be stationary, periodic, or quasiperiodic. The chaotic behavior of dissipative systems is assured by the presence of a nontrivial subset in their phase space—the

strange attractor. The possibility of its existence was first proven in Ref. [14].

5.1 Strange and chaotic attractors

An attractor of a dynamical system is called strange if it is different from a finite union of smooth submanifolds of the space M [11, 23]. It should be remembered that the (smooth) submanifold of space M is such a subset W in the space M' ($M' \subseteq M$), which locally resembles a part of space M' and has a unique tangent at each point. In this way, one emphasizes in the definition of strange attractor precisely its nonsmooth structure: it is a Cantor set (i.e., a fractal) in a certain section. This property and the exponential instability of trajectories on the attractor motivated labeling it 'strange' in Ref. [14]: the birth of such a set in smooth dynamical systems was indeed conceived of as quite extraordinary.⁴

Strange attractors possess a certain degree of hyperbolicity, but this hyperbolicity has another form than the uniform hyperbolicity described in Section 3. Such attractors are indeed sets of a complex structure, yet they cannot be studied in detail in the framework of hyperbolic theory (this issue is considered at length, for instance, in the review [87]). Nevertheless, the examples of just hyperbolic attractors of dynamical systems (see Section 5.2), which correspond to uniform hyperbolicity, are well known.

Analytical results pertaining to the existence of strange attractors are relatively rare. For example, it is agreed that chaotic systems with a countable number of strange attractors can be dense in the Newhouse domains in the case of saddle–focus [88]. Moreover, it can be anticipated that, in the neighborhood of a family of diffeomorphisms, which has a homoclinic tangency of the stable and unstable manifolds of a hyperbolic point, subsets of systems can exist which have no stable cycles but possess strange attractors.

Many statements have been proven for a family of diffeomorphisms of arbitrary dimensions [89–92]. The main result obtained here reads as follows. Let f_a be a family of diffeomorphisms of manifold M , $\dim M \geq 2$, that has a homoclinic tangency at $a = \tilde{a}$. Then, a set $A \subset \mathbb{R}$ exists such that f_a possesses a strange attractor for each $a \in A$ and $A \cap [\tilde{a} - \varepsilon, \tilde{a} + \varepsilon]$ has a positive Lebesgue measure for all $\varepsilon > 0$.

Nowadays, the notion of a 'strange attractor' has gained a collective sense—by using it one tries to emphasize the chaotic character of the system under study. It is commonly assumed that a dynamical system possesses a strange attractor if in its phase space there is a limit set composed of chaotic trajectories. The chaotic behavior in this case can be ensured by various criteria: the homoclinicity, presence of positive Lyapunov exponents, continuous character of spectrum, bifurcations of period doubling, and so on (see Ref. [28] and references cited therein). Therefore, this notion is more a paradigm than a characteristic of a mathematical object.

It is, in addition, known that there exist strange nonchaotic attractors [93], i.e., attractors which exhibit a fractal structure and yet the system incorporating them does not manifest chaos in any sense. Therefore, the broad usage of the notion 'strange attractor' turns out to be not always justified.

⁴ According to one of the authors of this term, D Ruelle, the 'strange' attractors look actually strange [86].

If the structure of the attractor in a chaotic system is not known, as not infrequently is the case in numerical and experimental research, it is more relevant to term such attractors chaotic, accenting just the complex motion of the trajectories. This does not contradict the fact that chaotic attractors can be hyperbolic attracting sets, possess a transversal homoclinic trajectory, and so on.

Strict mathematical results have been obtained primarily for attractors having hyperbolicity properties. They include hyperbolic attractors and quasihyperbolic type attractors which are sometimes referred to as stochastic.

5.2 Hyperbolic attractors

The set A is said to be a hyperbolic attractor of a dynamical system if A is a closed topologically transitive hyperbolic set and there exists a neighborhood $U \supset A$, such that $A = \bigcap_{n \geq 0} f^n(U)$. A hyperbolic attractor constitutes a structurally stable (rough) set. This means that small perturbations cannot lead to qualitative modifications in both the attractor and the behavior of the system as a whole. Dynamical systems with a hyperbolic type of attractor serve as models of structurally stable systems with most distinct chaotic properties [21, 50, 87].

The hyperbolic attractors discovered thus far are few in number; therefore, each example carries the name of the author who constructed it. Well-known attractors of that type include the Smale–Williams solenoid (see, for example, Refs [3, 21, 30, 31, 87]) and the Plykin attractor [22, 31, 87, 94, 95].

Attractors that, by their properties, are close to hyperbolic type include the Belykh attractor [29, 96, 97] and the Lozi one [21, 29, 98, 99]. In order to describe such attractors one uses a widened notion of hyperbolicity, which goes outside the class of smooth invertible maps, because such attractors exist in dynamical systems with singularities [97, 100–102].

Note that neither Lorenz [13, 103] nor Hénon [13, 104] attractors belong to the hyperbolic type: the Lorenz attractor is nonrough (see Section 5.3), and transitivity and sensitive dependence on the initial conditions have been proven for the Hénon attractor [105] (see also Ref. [89]).

Of all the hyperbolic attractors the construction of the Smale–Williams solenoid is topologically the simplest. The attractor is obtained through mapping a toroidal region into itself in the space of three or more dimensions.

Let D be a solid torus, i.e., a filled two-dimensional torus. Let us cut and unbend it, so as to obtain a cylinder. We then stretch it strongly and compress it along the radius. The result will be a structure that resembles a thick cord with a circular cross section. Fold then the cord into a two-turn hoop, glue its ends, and place it into the original solid torus D . As a result, one gets the domain D' lying in D (Fig. 15a). At the next step, such a transformation is applied to the domain D' , thereby transforming D' into the domain

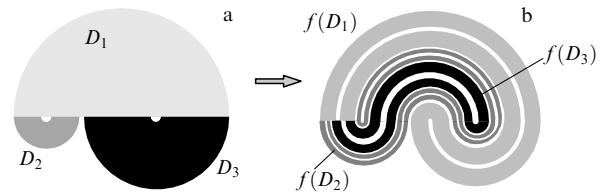


Figure 16. Construction of a Plykin attractor.

D'' lying in D' , and so on.⁵ It should be noted that in the course of construction the cross-section area contracts more than twofold, i.e., the total volume diminishes. In a section, the construction represents a hierarchy of structures, shown in Fig. 15b.

The above-described construction can formally be written as follows. The toroidal domain is a three-dimensional structure $\mathbf{T} = S^1 \times D^2$, where S^1 is a unit circle, and D^2 is a unit disk in \mathbb{R}^2 . Then, the transformation

$$f: \mathbf{T} \rightarrow \mathbf{T}, \quad f(x, y, \varphi) = \left(\frac{1}{k}x + \frac{1}{2}\cos\varphi, \frac{1}{k}y + \frac{1}{2}\sin\varphi, 2\varphi \right), \quad (3)$$

where the parameter $k > 2$ stands for the degree of torus contraction in the direction of its ‘thickness’, defines the solenoid as a subset of $\mathbf{T} \subset \mathbb{R}^3$. Consider a section of domain \mathbf{T} by a two-dimensional plane. We then obtain two disks with radius $1/k$. It is obvious therewith that $f^2(\mathbf{T}) \subset f(\mathbf{T})$.

When the map (3) is subjected to iterations, the picture is repeated on smaller and smaller scales (Fig. 15b). As a result, the set $f^m(\mathbf{T})$ in a section will be composed of 2^m disks. As $m \rightarrow \infty$, one gets the Smale–Williams attractor, which is hyperbolic. It is constructed so that stretching in one direction and compression in the other one are uniform.

Another well-known hyperbolic attractor was constructed in Ref. [94]. It is obtained through applying a special mapping to the domain D which is a union of three domains, D_1 , D_2 , and D_3 , each representing a semicircle from which a smaller semicircle is cut concentrically, with the sum of the diameters of semicircles D_2 and D_3 being smaller than the diameter of D_1 . Two semicircles, D_2 and D_3 , are glued along the diameter to the larger semicircle D_1 (Fig. 16a). The map $f: D \rightarrow D$ transforms every radius of the partition of domain D into the radius of the same partition with the help of a certain contraction. As a result, one gets the pattern shown in Fig. 16b. By generalizing this construction, it is possible to build a variety of hyperbolic attractors with different topological structures (see Ref. [95]).

For a long period of time it was believed that the Smale–Williams, the Plykin, and other hyperbolic attractors are artificial mathematical constructions, and that there are no physically realizable systems which may exhibit them. However, not long ago the authors of Refs [33, 34] proposed dynamical systems that contain a set with properties resembling those of a Smale–Williams type hyperbolic attractor in their phase space. Moreover, the properties of such an attractor have been studied *experimentally* [106]. In other studies, the possibility of the Plykin attractor occurring in a modified Lorenz model [107] and a neuron model [108] was demonstrated. The recent Ref. [35] suggested an example of a nonautonomous system which in a section has

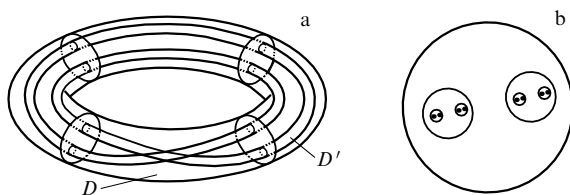


Figure 15. Construction of a Smale–Williams attractor.

⁵ Such an object is sometimes called the Vietoris–van Dantzig solenoid.

a Plykin type attractor on a plane (see also Ref. [109] and references cited therein).

It is necessary, however, to remark that many mathematicians do not accept the existence of hyperbolic attractors in physically realizable systems.

5.3 Stochastic and other attractors

Apparently, the stochastic attractor proposed by Ya G Sinai [110] can serve as an adequate mathematical image of the developed chaotic behavior observed in physical systems. In this case, however, the definition ‘stochastic’ is not associated with the presence of random external perturbation or noise in the system. The term is borrowed from probability theory and relates to the existence of an invariant measure (see Section 7.5).

Any hyperbolic limit set constitutes a stochastic attractor. For such an attractor, the properties of stretching and contraction on, respectively, unstable and stable manifolds are structurally stable, but the geometric picture and the degree of stretching and contraction might be different at various points of the attractor. At the same time, such an attractor does not contain stable trajectories, and they cannot appear there through small perturbations of the system.

The formal definition of a stochastic attractor relies on the notion of invariant measure and the mixing property. It should be reminded that the attractor \mathcal{A} is an invariant closed set which has such a neighborhood U that $U_t = F^t U \subset U$ and $\bigcap_t U_t = \mathcal{A}$. The attractor \mathcal{A} is stochastic if for any absolutely continuous invariant measure μ on U its shift μ_t converges as $t \rightarrow \infty$ to the limit invariant measure ν which is independent of μ , and the dynamical system $(\mathcal{A}, \nu, \{F^t\})$ possesses a mixing property [110].

The well-known example of a stochastic attractor is Lorenz’s attractor, which appears in the famous Lorenz system for $b = 8/3$, $\sigma = 10$, and $r = 28$ [97], namely

$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = rx - xz - y, \quad \dot{z} = xy - bz,$$

describing, under certain assumptions, convective motions in fluid [103], disk dynamos (see Ref. [11]), and some models of lasers [111], as well. The proof of existence of the Lorenz attractor relies on fairly elaborate constructions based on combinations of rigorous mathematical manipulations with numerical analysis. It was proven in Refs [36, 37] that the Lorenz attractor is stable, i.e., it is preserved under small perturbations of parameters, and has a unique SRB measure (see Section 8.5). These properties imply, in particular, that the Lorenz attractor is a physically observable object.

However, by no means do all chaotic attractors observed in models of real physical systems possess such ‘nice’ properties. This happens because the chaotic attractors may contain, within themselves, an infinite set of saddle cycles (see Section 2.3). In this case, the unstable manifold of a cycle also belongs to the attractor. Then, bifurcations pertaining to the tangency with stable and unstable manifolds bring about the appearance of an infinite set of stable periodic trajectories, as described in detail in Section 4.3. Such attractors belong to the quasistochastic type (i.e., they are so-called quasiattractors) [112, 113]. Weak perturbations of systems with a quasistochastic attractor entail complex qualitative transformations in both the system dynamics and the structure of the attractor itself (see Ref. [114] for more details on the physical aspects).

The following example is illuminating. Analytical results of bifurcation theory show that in the Lorenz system with parameters that are arbitrarily close to the values of $b = 8/3$, $\sigma = 10.2$, $r = 30.2$ stable limit cycles exist [115, 116]. Yet, thus far no numerical analysis has succeeded in revealing them. Moreover, there always exists such a parameter region where, according to a rigorous theory, stable cycles can occur, and yet they escape detection by numerical means, whatever the accuracy of numerical simulations.

It should be noted in closing that the dynamics of dissipative systems with typical chaotic attractors are in a certain sense analogous to the dynamics of Hamiltonian systems in which, in addition to stochastic layers, invariant Kolmogorov–Arnold–Moser (KAM) tori exist [13].

6. Chaos in physical systems

Physical interest in chaotic phenomena goes back a long way and was prompted by the need to describe systems with a very large number of degrees of freedom (for more details see review [13]). However, it turned out after the discovery of exponential instability that practically any typical nonlinear system with more than one degree of freedom may exhibit chaotic features. In these circumstances, one of the main questions is that about the measure of chaotic trajectories.

It is also known that in nonlinear systems chaotic trajectories always neighbor the set of coexisting regular (not chaotic) orbits, and that the measure of such regular trajectories might be zero. As follows from the material presented above, answering the question about the role of these regular trajectories in dynamics may be far from simple.

6.1 Chaos and strange attractors

When one talks about the physical manifestations of chaos, such a system behavior is commonly borne in mind when its chaotic trajectories look indistinguishable from a certain random process. This assumes a certain correspondence between solutions of some stochastic equation and deterministic equations. And yet the dynamics of chaotic systems are not fully random (see Section 9). Despite some paradoxicality in these statements, the research over the last forty years indicates that the approach to studying chaos, developed in the framework of the theory of dynamical systems, has fully proven its worth.

The basic concept in exploring chaotic systems proves to be that of roughness, or structural stability [19]. Historically, the rough chaotic attractors were constructed first. They were all designed on geometrical premises, without resorting to model equations describing some real process. From a physical viewpoint, the rough properties are the most important, as they are preserved under small perturbations and are therefore observable. In contrast, the specific feature of structurally unstable systems lies in the fact that they can be qualitatively modified under the action of small perturbations. Generally speaking, because of measurement errors, factors not accounted for, rounding off, and so on, the derived equations will be therefore adequate only if their qualitative characteristics are preserved under perturbations. From a mathematical perspective, structural stability was thus considered a necessary condition to be adhered to when designing model systems.

However, the onset of chaoticity is underlain by exponential instability. Admittedly, for a system to exhibit a chaotic behavior (i.e., to have a sensitive dependence on initial

conditions) only the instability is needed, which shows up in the definition of hyperbolicity. It is by no means necessary that this instability be the same for all trajectories. Moreover, the number of unstable directions may vary from trajectory to trajectory. The Lorenz attractor, for example, has such a structure. Attractors of that type are not destroyed subject to small perturbations, but their geometric structure may, in general, undergo some changes. In another situation, because of the homoclinic tangencies emerging in typical systems, the chaotic attractors will by necessity contain stable cycles of a large period with very small attraction basins. This implies that such attractors are structurally unstable and that their detailed structure is unobservable. Nevertheless, the property of exponential instability is of utmost importance, as it allows the chaotic behavior in a corresponding system to be, in principle, examined experimentally.

Thus, when describing chaotic systems one relies on a qualitative approach which, as remarked in the monograph [22], is in a certain sense analogous to the question as to whether the length of a rod is a rational or irrational number (one is only interested in a qualitative property of the rod).

This is what V I Arnold [117] wrote about one of the chaotic attractors (the Hénon attractor [104]) discovered by numerical simulations in a two-dimensional map: “*The convergence to a set that locally has the form of the product of Cantor set and interval that is observed in numerical experiment does not allow a description in the framework of accepted definitions of hyperbolicity⁶ (it cannot even be ruled out that attraction basins of long cycles are interspersed in this set). For that reason, mathematicians do not accept that the Hénon set is hyperbolic. And yet, from the experimentalist’s viewpoint, the motion of a phase point under the action of map iterations bears a manifestly stochastic character.*”

Consequently, in the context of dynamical chaos, the approach based on the notion of structural stability needs to be revisited. Namely, when constructing a dynamical system which describes some physical process it is necessary that only essential (from one viewpoint or another) properties be preserved subject to small perturbations. This is qualitatively different from the conventional concept that demands preservation of all qualitative characteristics in perturbed systems. For each concrete case, apparently, one should specify precisely which perturbations are admissible and which properties are essential for the system studied.

6.2 Unstable sets and trajectory trapping

It is well known that in a typical case the Hamiltonian systems are nonintegrable and nonergodic [118] but manifest complex dynamics in the sense that the hierarchy of KAM islands of stability coexist in their phase space with a chaotic layer formed by one or several ergodic components (see Refs [13, 22, 42]). This implies, in particular, that, as a whole, the regular and chaotic dynamics are tightly interlaced. This type of behavior of Hamiltonian systems, described in many studies, was also examined experimentally (see Ref. [42] and references cited therein).

The divided phase space leads to a fairly unexpected stickiness effect or trajectory trapping effect [119–121], which is manifested through the fact that a chaotic trajectory, having visited a small neighborhood of KAM islands, may stay there sufficiently long.

When a phase point resides in a chaotic region far from the set of KAM tori, it moves randomly. However, if it finds itself in a domain in the vicinity of such a set, it gets trapped by its neighborhood. The phase point can travel in the vicinity of KAM tori for an arbitrarily long, but finite interval of time during which the system behaves itself in a regular way. In this case, the exponential decay of correlations characteristic of the system before being trapped in the basin around KAM tori will be replaced by a power-law type of decay [122]. At the expiration of the time interval, the system ‘re-establishes’ its chaotic properties. Then, the whole sequence is repeated.

Thus, a Hamiltonian system with a divided phase space exhibits, as a whole, an intermittent dynamical regime: following chaotic behavior over a long time interval almost regular dynamics are observed. Such a process, apparently, has implications for the statistical properties of Hamiltonian systems.

The phenomenon of stickiness was discovered in exploring numerous problems emerging in the analysis of a number of physical phenomena. Among them are the kinetic description of many-particle systems, problems of plasma physics, the problem of particle dynamics in a potential well, and some others (see Refs [42, 120] and references cited therein). However, it is not known in a general case whether stickiness will be observed in systems which comprise a finite or infinite number of KAM islands [121].

Recently, Ref. [123] has proposed the construction of curious and highly illustrative families of Hamiltonian systems containing clearly separated chaotic and regular components in the phase space. In the simplest case, their dynamics are governed by a single KAM island and a single ergodic chaotic component.

Such a behavior is demonstrated by billiards with a boundary shaped like a mushroom, which is composed of semicircle S and rectangular base Π (Fig. 17). In the configuration space of such a billiard table, the boundary between the ergodic and regular components is furnished by those trajectories in the semicircle S that are tangent to the circle c of radius r . As is apparent from Fig. 17, the regular domain consist here of trajectories that do not intersect this circle, staying inside the semicircle S and never visiting the base Π . However, the chaotic orbits are bound to get there.

Based on the example of such mushroom billiards it was recently discovered [124] that the existence of a hierarchy of stability islands is not necessarily required in order to observe dynamical stickiness of chaotic trajectories. Moreover, in the billiard table considered, this process unfolds owing to the presence of an infinite set of zero measure of unstable periodic

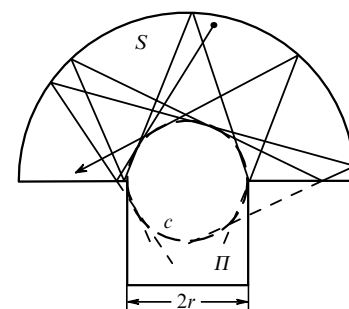


Figure 17. Billiard table with a mushroom-shaped boundary [122].

⁶ It was proven later [105] that the Hénon attractor is chaotic in the sense of the definition given in Section 2.3. (Remark by AL.)

trajectories in the chaotic domain. In other words, sets of zero measure may essentially influence the behavior of the whole system.

This result seems highly counterintuitive since from the viewpoint of dynamics it was always assumed that these sets should not be accounted for and that their presence can be neglected. Certainly, because of zero measure, they asymptotically have no effect on the ergodicity of a dynamical system. Nevertheless, the stickiness of chaotic trajectories in the neighborhood of the regular domain boundary occurs just because of their presence.

There are many other systems (see Ref. [42] and references cited therein) containing sets of zero measure which cannot be discarded when dynamics are analyzed. One of them is the Smale horseshoe in the Duffing system at some parameter values. The appearance of such a set entails an exponential instability with respect to initial conditions; however, almost all trajectories leave this set as time progresses and are attracted to regular attractors. It is noteworthy that the period of such a transient chaotic process can be however as long as you like.

The example with a horseshoe in the so-called capture problem (see Refs [43, 44]) is illuminating, in which a binary star captures the third one approaching from infinity.⁷ It was proven in Ref. [43] that the presence of the horseshoe makes the capture feasible. One more example is furnished by the billiards of Sinai [10] and Bunimovich [126] ('stadium' type billiards). In such billiards, all the trajectories, excluding a set of zero measure, are chaotic. And still these sets strongly influence the dynamics and long-term motion characteristics [42].

Thus, all the complexity exhibited by dynamical systems may consist in the existence of sets which are commonly neglected. Hence, it follows that the general approach to the description of chaotic systems calls for a new vision.

7. Ergodic aspects of dynamical chaos

Getting a full description of the chaotic system dynamics presents (at least at the current stage) an unsolvable problem. However, another approach exists suggesting that one inquire only about typical properties of a dynamical system. This viewpoint, which ignores the treatment of exclusive (rare) trajectories, is adopted in the ergodic theory or metric theory of dynamical systems, where the concept of measure is used as the fundamental idea.

The existence of invariant measures for dynamical systems, at least of one invariant measure, is ensured by existence theorems. One of them is the Krylov–Bogoliubov theorem, according to which at least one invariant measure always exists for dynamical systems with a compact phase space. In other cases, the existence of invariant measures follows from dynamical laws.

The ergodic theory is now enjoying a period of rapid development thanks to the possibility of using it in various applications (see, for example, Refs [29, 127] and references cited therein). With the help of characteristics considered in this theory, such as the dimension, entropy, Lyapunov exponents, and some others, one can obtain more complete information on the properties of systems under study,

and assess the depth of chaoticity.

Some important properties of dynamical systems considered by the ergodic theory, for example, the fulfillment of the central limit theorem, exponential decay of correlations, and the Bernoulli and K-properties, lie at the heart of the modern idea of deterministic chaos. In Sections 7.1–7.5, we shall describe those concepts which are especially useful in applications.

7.1 Lyapunov exponents

As is known, the dynamics of system (1) in the vicinity of a stationary point can be studied by analyzing the eigenvalues of a linearization matrix. To analyze motions in the vicinity of periodic motion, one employs multipliers. Generally, the (characteristic) Lyapunov exponents are used to explore the behavior in the vicinity of an arbitrary trajectory.

Geometrically, the Lyapunov exponents characterize the degree of stretching and contraction in the system along, respectively, stable and unstable directions (see Fig. 5). Let $x(t)$ be a typical phase trajectory of system (1), and $x_1(t)$ a trajectory close to it, i.e., $x_1(t) = x(t) + \xi(t)$. Consider the function

$$\Xi(\xi(0)) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left| \frac{\xi(t)}{\xi(0)} \right|,$$

which is defined for the vectors $\xi(0)$ of an initial displacement such that $|\xi(0)| = \varepsilon$, where $\varepsilon \rightarrow 0$. Then, depending on the direction of vector $\xi(0)$, the function $\Xi(\xi(0))$ will assume a finite set of values $\{\lambda_i\}$, $i = 1, 2, \dots, n$. These values are said to be Lyapunov exponents (see, for example, Refs [41, 128, 129] and references cited therein). The set of all possible Lyapunov exponents is called the Lyapunov spectrum.

The theory of Lyapunov exponents got its strict foundation after the proof of the well-known multiplicative ergodic theorem [130–132], which ensures their existence for almost any $x \in M$.

The Lyapunov exponents are universal invariant characteristics allowing one to judge certain properties of dynamical systems. As follows from the geometrical representation, in particular, the Lyapunov exponents serve as a measure of chaoticity: if some of them are positive, the behavior of a dynamical system is chaotic.

The number of characteristic exponents is equal to the dimension n of the phase space of the dynamical system, and they can be sorted in a descending order: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. For systems of differential equations (1), one of the Lyapunov exponents, corresponding to the displacement along the trajectory which does not end in a stationary point, is always equal to zero. For Hamiltonian systems, the exponents exhibit the following symmetry: $\lambda_j = -\lambda_{2k-j+1}$, where k is the number of degrees of freedom in the system. Consequently, at least two exponents are zero for Hamiltonian systems. In the case of fully integrable systems, all the exponents take zero values.

The sum of all Lyapunov exponents equals the average value of the divergence of the vector field governing the system dynamics:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \operatorname{div} \dot{\xi} \, dt = \sum_{j=1}^n \lambda_j.$$

⁷ This well-known problem pertains to the question of final motion in the three-body problem (for information see the fundamental monograph [125]).

As is well known, the relative variation of the phase volume is determined by the sign of the divergence. Consequently, it is readily seen that $\sum_j \lambda_j = 0$ for Hamiltonian systems, while for dissipative ones $\sum_j \lambda_j < 0$.

The Lyapunov exponents also carry information on the dimension of the strange attractor of a dynamical system (see Section 7.3). The quantity

$$D_L = j + \sum_{i=1}^j \frac{\lambda_i}{|\lambda_{j+1}|},$$

where j is determined from the conditions $\lambda_1 + \lambda_2 + \dots + \lambda_j > 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_{j+1} < 0$, is called the Lyapunov dimension, or Kaplan–Yorke dimension, from the names of authors who proposed at their time the hypothesis on the connection of fractal dimension and characteristic exponents. The value of D_L provides the upper bound for the Hausdorff dimension of the attractor [133, 134].

Since the Lyapunov exponents determine the degree of instability, they turn out to be connected in a natural way to the entropy of the dynamical system.

7.2 Entropy

Each dynamical system can be characterized by a certain quantity (sometimes infinite) which is called entropy. The related circumstance, difficult to anticipate, is that if the entropy is positive, the system also possesses some other, rather strong, statistical properties. Moreover, for two-dimensional diffeomorphisms such dynamical complexity is ensured by the existence of horseshoes in the phase space [31] (see Section 3).

The entropy is in a certain sense the measure of a system’s order (or disorder). At present, in describing this property in one context or another, one uses a large number of characteristics also referred to as ‘entropy’. For instance, the entropy H in the information theory is introduced for a system that can be in states x_i with probabilities $p_i = p(x_i)$ via Shannon’s formula

$$H = - \sum_i p_i \log p_i. \tag{4}$$

In a similar way, one may define the metric entropy of a dynamical system, or the Kolmogorov–Sinai entropy. This concept enabled strict formulation of the absolute criterion of chaoticity as the motion with positive metric entropy, which is Lyapunov unstable.

From the relations for the entropy it follows that this quantity serves as a measure of the exponential divergence (or convergence) of trajectories of a dynamical system. This fact on its own was known long ago, but the entropy approach offered the possibility of exploring the behavior of complex systems in a new way [135].

Suppose that the dynamical system is defined through the map $x_{n+1} = f(x_n)$ on a compact set A . Let the partition of set A into the finite number of measurable subsets A_i be defined. We shall denote the set of points mapped into A_i by the transformation f^k as $f^{-k}(A_i)$. Without writing out all the partitions created by such an inverse map, we present the general case: at the m th step, the partition into $A_{i_1 i_2 \dots i_m}^{(m)} = A_{i_1} \cap f^{-1}(A_{i_2}) \cap f^{-2}(A_{i_3}) \cap \dots \cap f^{-(m-1)}(A_{i_m})$ consist of points from the set A_{i_1} , which at the next $m - 1$ steps will subsequently fall in $A_{i_2}, A_{i_3}, \dots, A_{i_m}$.

Let us calculate the entropy for each partition using the formula which is analogous to Shannon’s:

$$H^{(m)} = - \sum_{i_1 i_2 \dots i_m} p(A_{i_1 i_2 \dots i_m}) \log p(A_{i_1 i_2 \dots i_m}).$$

With the notation $\varepsilon = \max_i \text{diam } A_i$, the entropy of a dynamical system will be expressed as

$$K = \lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} (H^{(m+1)} - H^{(m)}) = \lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} \frac{1}{m} H^{(m)}. \tag{5}$$

In other words, the entropy is defined as the asymptotic increment of uncertainty for the partitioning of an infinitely small diameter. Notice that this definition is never used in practice. Instead, the common approach is that the entropy is estimated with the help of Lyapunov exponents [see Eqn (8)] or one calculates a quantity K_2 approximating it, which is introduced in the following manner.

Consider the Rényi entropy

$$H_q = \frac{1}{1-q} \log \left(\sum_i p_i^q \right). \tag{6}$$

It is easy to show that Shannon’s entropy (4) is a particular case of the Rényi entropy for $q \rightarrow 1$. Using this quantity, it is also possible to introduce the generalized dimension D_q (see Section 7.3) and quantity K_q , which is called the generalized entropy and characterizes the degree of disorder of a dynamical system in the most general way. Specifically, if one substitutes H_q for H in the definition of entropy K [see Eqn (5)], the result is

$$K_q = \lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} (H_q^{(m+1)} - H_q^{(m)}) = \lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} \frac{1}{m} H_q^{(m)}.$$

One of the most important properties pertaining to the quantity K_q can be written as a sequence of inequalities

$$K = K_1 \leq K_2 \leq \dots \leq K_{q-1} \leq K_q \leq \dots, \tag{7}$$

where K is the Kolmogorov–Sinai entropy. Correspondingly, the Kolmogorov–Sinai entropy can be estimated from above if $K_i, i = 2, 3, \dots$ are found (see Section 7.3). In turn, the Kolmogorov–Sinai entropy is expressed through the Lyapunov exponents in the following way [132]:

$$K_1 = \int_M \sum_{\lambda_i \geq 0} \lambda_i(x) d\mu. \tag{8}$$

In a number of cases, expression (8) can be simplified. If f is a differentiable map of a finite-dimensional manifold, and μ is the ergodic probability measure for the dynamical system, then $K_1 \leq \sum_{\lambda_i > 0} \lambda_i$ [128, 132]. The equality in the last expression takes place when only the chaotic component of motion is considered, i.e., if μ is the Sinai–Ruelle–Bowen measure [128] (see Section 7.5).

The value of entropy does not depend on the partitioning of the phase space. Moreover, if two dynamical systems have equal entropies, their statistical laws of motion coincide, too [136, 137].

7.3 Dimensional characteristics

The generalized dimension D_q is defined as

$$D_q = \lim_{\varepsilon \rightarrow 0} - \frac{H_q(\varepsilon)}{\ln \varepsilon},$$

where ε is the characteristic size of the cells of phase space subdivision. Let us explain the sense of generalized dimension D_q for various q . If $q = 0$, then $p_i^q = 1$. The quantity obtained is called the fractal dimension, or the set capacity. In some cases, D_0 is also called the Hausdorff dimension. This is related to the fact that, although there are exceptions, the values of the capacity and Hausdorff dimension are very close for typical attractors (which are encountered in numerical simulations and physical experiments).

If $q = 1$, the numerator and denominator in relation (6) become zero, since $\sum p_i = 1$. However, it can easily be shown then that

$$D_1 = \lim_{\varepsilon \rightarrow 0} \frac{\sum p_i \ln p_i}{\ln \varepsilon}.$$

Since the relationship $\sum p_i \ln p_i$ defines the entropy, the quantity D_1 describes how the amount of information about the set under consideration increases when $\varepsilon \rightarrow 0$. The quantity D_1 is therefore called the information dimension.

If $q = 2$, then one finds

$$D_2 = \lim_{\varepsilon \rightarrow 0} \frac{\ln \sum p_i^2}{\ln \varepsilon},$$

where p_i^2 is the probability that there are at least two points residing in the i th cell. Consequently, the value of D_2 is determined through binary correlations in the probability distribution, and therefore D_2 is called the correlation dimension.

For $q = 3, 4, \dots$ the dimensions D_3, D_4, \dots are linked with correlations of a higher order. They give additional information about the structure of the sets under consideration. Moreover, the generalized dimension D_q for $q \rightarrow \infty$ was defined [138].

Notice that the entropy and dimension are in a certain sense the independent invariants. For instance, in the case of quasiperiodic dynamics with a large number of incommensurable frequencies (i.e., a torus of high dimension), the entropy equals zero. But, for example, for saw-tooth one-dimensional map $x_{n+1} = lx_n \pmod{1}$, where $l \gg 1$, the entropy can be fairly large. It is therefore necessary to be in a position to calculate both these characteristics, when exploring real systems.

7.4 Estimates of entropy and dimension

In order to evaluate the generalized quantities, the concept of generalized correlation integral is introduced as follows:

$$C_q(\varepsilon) = \left(\sum_i p_i^q \right)^{1/(q-1)}.$$

Because $-\ln C_q^{(m)}(\varepsilon) = H_q^{(m)}(\varepsilon)$, C_q is just a redesignation of H_q . Therefore, the connections with other generalized quantities are preserved. The generalized correlation integral is convenient because it allows an essential simplification in the case of $q = 2$:

$$C_2(\varepsilon) \equiv C(\varepsilon) = \frac{1}{N^2} \sum_i S_i(\varepsilon),$$

where S_i is the number of points within the cell i , and N is the number of sequence elements. The quantity C_2 is referred to as the correlation integral. The algorithm of computing it is

well developed (see Refs [41, 139, 140] and references cited therein).

Let us return to the method of determining the generalized dimension D_q and entropy K_q . The relationship $-\ln C_q^{(m)}(\varepsilon) = H_q^{(m)}(\varepsilon)$ mentioned above allows the generalized quantities to be expressed in terms of the generalized correlation integral:

$$D_q = \lim_{\varepsilon \rightarrow 0} -\frac{H_q(\varepsilon)}{\ln \varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\ln C_q^{(m)}(\varepsilon)}{\ln \varepsilon},$$

$$K_q = \lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} m^{-1} H_q^{(m)} = \lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} -\frac{\ln C_q^{(m)}(\varepsilon)}{m}.$$

Based on the above-given formulas, one may obtain that for small ε and sufficiently large m the relation $\ln C_q^{(m)}(\varepsilon) \simeq D_q \ln \varepsilon - K_q w + \text{const}$ holds true, where $w = (m - 1)\tau$ is the so-called reconstruction window width. Hence, in addition to the estimate of D_q one may get, by fixing the scale ε and exploring the dependence on w , an estimate of the generalized entropy K_q of the dynamical system.

It was proposed in Refs [141, 142] that there exists another scheme of computing the K_2 -entropy, which relies on the representation of K_q as the limit of difference $H_q^{(m+1)} - H_q^{(m)}$ for $\varepsilon \rightarrow 0$ and $m \rightarrow \infty$:

$$K_2 \sim \lim_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} K_2^{(m)}(\varepsilon), \quad K_2^{(m)} = \frac{1}{k\tau} \ln \frac{C^{(m)}(\varepsilon)}{C^{(m+k)}(\varepsilon)}.$$

Here, $C^{(m)}(\varepsilon)$ is the correlation integral $C_2^{(m)}(\varepsilon)$. The dependence of $K_2^{(m)}$ on the embedding dimension m can be approximated with the least square procedure through the following relationship:

$$K_2^{(m)}(\varepsilon) = K_2(\varepsilon) + \frac{b}{m^c},$$

where b and c are some positive values.

Thus, in the case of small fixed ε the entropy K_2 is defined as the limit of $K_2^{(m)}$ for $m \rightarrow \infty$.

7.5 Statistical properties of dynamical systems

A dynamical system is chaotic if its behavior possesses certain properties characteristic of random processes which are the subject of probability theory. In this section, we present the hierarchy of chaotic properties of dynamical systems [143, 144]. Each subsequent property will be stronger than the preceding one.

I. The existence of an invariant measure [30, 31, 145, 146]. Sufficiently general theorems on the existence of invariant measure were proven for dynamical systems with a compact phase space. For Hamiltonian systems, the existence theorem follows from the Liouville theorem. For dissipative systems, the invariant measure frequently turns out to be on the attractor.

The important part of this avenue of research is the proof of the existence of so-called physically observable measures, or SRB (Sinai–Ruelle–Bowen) measures introduced in the 1970s [147–150] (see also Ref. [151]). The presence of the SRB measure in the system signifies that there exists such an invariant measure defined on the attractor that almost all trajectories localized in the vicinity of the attractor turn out to be uniformly distributed with respect to it. In this case, the time average of any continuous function will coincide with its space average, which is just

defined by the invariant measure. Hence follows the name—the physically observable measure.

It is well known that strange attractors do not yield to analytical description in an overwhelming majority of cases—they can be studied only numerically. The importance of the SRB measure consists in the fact that, from its existence, it follows that images of strange (chaotic) attractors obtained with the help of computers correspond to the actual attractors.

A hyperbolic attractor has an SRB measure. For nonuniform hyperbolic attractors, such measures were also constructed for the family of Hénon maps [152], Lorenz equations [36, 37], and other dynamical systems (see Ref. [151]).

One of the important corollaries that follows from the existence of invariant measure in conservative systems is the famous Poincaré return theorem [127], according to which the system returns to the neighborhood of its initial state infinitely many times, excluding the set of initial conditions of zero measure. In its day, this result prompted the formulation of the Zermelo paradox in statistical mechanics [153], which was proposed as an objection against Boltzmann's H-theorem for the gas kinetic equation [154]. A gas of particles, which can be considered as a compact Hamiltonian system, should return to the domain where it was initially. For example, if the gas occupied half of some volume, then after a certain period of time all its particles would gather there again. Such an event contradicts Boltzmann's H-theorem, which argues that a system composed of a large number of colliding particles irreversibly tends to the equilibrium state.

The solution to this paradox given by Boltzmann [155] relies on two basic issues: the extremely small probability of returns, and coarse-graining of the distribution function. For systems of interest in statistical mechanics, the return time is extremely long, so that the probability of returns is negligibly small. The procedure of coarse-graining of the particle distribution function does not make provision for such rare events, which makes the relaxation to equilibrium irreversible.

Although such an explanation seems rather plausible, it remains unclear how the coarse-graining occurs in the original system because the process of coarse-graining is ambiguous, while the system itself remains unchanged. Discussions of the questions raised can be found in Refs [13, 42, 156, 157].

II. If the measure is already constructed, then it is natural to inquire about ergodic properties of the system with respect to this measure [144, 146]. According to the Birkhoff–Khinchin ergodic theorem (see, for example, Refs [29, 127]), the existence of the invariant measure ensures the possibility of time averaging almost everywhere, i.e., the existence of time-averaged quantities.

III. Mixing [13, 29–31, 127, 144, 146]. The transformation $F^t : M \rightarrow M$ is called mixing if, for any two functions h and g , the relation

$$\lim_{t \rightarrow \infty} \int_M h(F^t x) g(x) d\mu = \int_M h(x) d\mu \int_M g(x) d\mu$$

holds true, i.e., past a sufficiently long time interval the functions $h(F^t x)$ and $g(x)$ will be statistically independent. The expression $\int h(F^t x) g(x) d\mu$ is called a time correlation function. The existence of mixing in the system entails

irreversibility and the unpredictability of dynamics (see Ref. [13]).

IV. K-property (quasiregular, as proposed by A N Kolmogorov, or the Kolmogorov property, according to modern terminology) [29, 127, 135]. If a dynamical system represents a K-system, its entropy is positive. Such a division of dynamical systems into K-systems and deterministic transformations with zero entropy formalizes the very important idea about stochastic and deterministic systems considered in statistical physics (see also Section 9).

V. The Bernoulli property [126, 135]. The behavior of a dynamical system looks the more stochastic, the better it is described by a sequence of independent random variables. If coding a dynamical system into the regular stationary process (see Section 4.4) represents such a sequence, the system is called the Bernoulli system.

VI. Fulfillment of the conditions of the central limit theorem [85, 144].

Let $f : M \rightarrow M$ be some map of the space M into itself. For a given function g , consider time fluctuations (around its average), namely

$$\frac{1}{n} \sum_{k=1}^n g(f^k(x)) - \bar{g},$$

where $\bar{g} = \int_M g(x) d\mu(x)$. The fulfillment of the central limit theorem means that there exists $\sigma = \sigma(g)$ such that

$$\lim_{n \rightarrow \infty} \mu \left\{ x : \sigma \sqrt{n} \left[\frac{1}{n} \sum_{k=1}^n g(f^k(x)) - \bar{g} \right] < a \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a \exp\left(-\frac{u^2}{2}\right) du.$$

The sense of this relationship is that the sequence of values $g(f^k(x))$ generated by the dynamical system f is described by the standard normal distribution function, while time fluctuations do not tend to any limit but have the Gauss limiting distribution.

This property is inherent in hyperbolic dynamical systems and scattering billiards.

VII. Exponential decay of correlations [145]. In applications, it is not only establishing the presence of mixing (see III) that is important, but also estimation of the rate of convergence to the limit. If for a function g with zero mean, $\bar{g} = 0$, there exist numbers $C > 0$ and $0 < q < 1$ such that

$$\left| \int_M g(f^k(x)) g(x) d\mu \right| \leq C(g) q^{|k|},$$

then g possesses the property of exponential decay of correlations.

8. Reconstruction of dynamical systems

As already mentioned, notwithstanding the intensive development of the dynamical system's theory and the availability of fairly powerful theoretical apparatuses, getting an adequate description of chaotic processes presents a far from simple problem, if it is solvable at all. Moreover, the majority of complex systems considered in natural sciences do not yield to treatment in the framework of model (1) since the derivation of equations from the first principles is not always possible. Because of this, the study of systems, especially in

experimental research, frequently proceeds through processing the recorded signals generated by the system. For example, in arrhythmology one uses electrocardiograms as such signals, and in seismology the records of Earth crust displacements, in meteorology the observational data, in financial analyses the rates of securities, and so on [158–162]. Commonly, such a signal is called the observable, and the method of signal processing is known as the reconstruction of dynamical systems [139, 163].

The observable comprises the sequence of values taken by some variable (or variables) recorded continuously or in some time intervals. Often the notion of ‘time series’ is used instead of ‘observable’. It is clear that the availability of only a time series instead of a full solution of equations strongly limits our knowledge of the system being studied. This imposes strong limitation on feasible reconstructions.

A scalar time series $\{x_i\}_{i=1}^N$ is an array of N numbers representing the values of some dynamical variable $x(t)$ measured with a constant time step $t_i = t_0 + (i - 1)\tau$: $x_i = x(t_i)$, $i = 1, \dots, N$.

The contemporary analysis of time series is based on the idea [163] that one can get a satisfactory geometrical picture of a strange attractor if, instead of original variables showing up in system (1), so-called delay vectors $z_i = \{x_i, x_{i+1}, \dots, x_{i+m-1}\}$ are utilized. This approach to the analysis of time series was first mathematically verified in the work by F Takens [165].

8.1 Elements of the Takens theory

Let the dynamical system $F^t(x)$ with the phase space M be defined. The quantities forming the time series are the values of some function of state, $x(t)$, of this dynamical system on the manifold W^d from M : $x_i = h(x(t_i)) = h(F^{t_i}(x_0))$. Consider the construction of delay vectors given the time step τ . Obviously, the equalities $x_{i+1} = F^\tau(x_i)$, $x_{i+2} = F^{2\tau}(x_i), \dots, x_{i+m-1} = F^{(m-1)\tau}(x_i)$ hold true for state vectors $x(t_i) = x_i$. Hence, the components of delay vectors z satisfy the following relations

$$\begin{aligned} x_i &= h(x_i) \equiv \Phi_0(x_i), \\ x_{i+1} &= h(x_{i+1}) = h(F^\tau(x_i)) \equiv \Phi_1(x_i), \\ x_{i+2} &= h(x_{i+2}) = h(F^{2\tau}(x_i)) \equiv \Phi_2(x_i), \\ &\dots\dots\dots, \\ x_{i+m-1} &= h(x_{i+m-1}) = h(F^{(m-1)\tau}(x_i)) \equiv \Phi_{m-1}(x_i). \end{aligned}$$

Since all components of vector $z_i = \{x_i, x_{i+1}, \dots, x_{i+m-1}\}$ can be connected to one and the same state x_i of the dynamical system, a vector function A that maps vectors $x_i \in W^d$ into vectors of m -dimensional Euclidean space \mathbb{R}^m exists:

$$z_i = A(x_i), \quad x_i \in W^d, \quad z_i \in \mathbb{R}^m.$$

This reasoning makes up the essence of the Takens theorem [164] which asserts that the typical property of map A for $m \geq 2d + 1$ is the embedding of W^d into \mathbb{R}^m . In other words, the image W^d in \mathbb{R}^m [$V^d = A(W^d)$] does not have self-intersections. In this case, the function A is differentiable and possesses the inverse differentiable function A^{-1} defined on V^d : $W^d = A^{-1}(V^d)$. Then, a dynamical system can be defined on V^d because $x_i = A^{-1}(z_i)$ and $x_{i+1} = F^\tau(x_i)$:

$$z_{i+1} = A(x_{i+1}) = A(F^\tau(A^{-1}(z_i))) \equiv \Psi(z_i), \quad z_i \in V^d,$$

where $\Psi : V^d \rightarrow V^d$. It is easy to see that if the last component in this expression is kept, one obtains a component-wise variant of the time series presentation: $x_i = G(x_{i-1}, \dots, x_{i-m})$. This relation can be used to solve the forecast problem, since one only needs to know m .

Thus, we have two transformations, Φ and Ψ , viz.

$$\begin{aligned} x_{i+1} &= F^\tau(x_i) \equiv \Phi(x_i), \quad x \in W^d, \\ z_{i+1} &= \Psi(z_i), \quad z \in V^d. \end{aligned}$$

Since $z = A(x)$, these are different representations of the same map. Consequently, the characteristics invariant with respect to nondegenerate substitution should coincide for both systems.

Therefore, the attractor dimension, the entropy of a dynamical system, and some other invariant characteristics can be retrieved from the measured values.

In the applied analysis of time series one distinguishes two main tasks: that of identification, and that of forecast.

8.2 Identification problem

The identification problem in the analysis of observables aims at studying the question of the parameters of the system that has generated the given time series—the correlation dimension, entropy, Lyapunov exponents, and others (see Section 7).

Since the entropy is the measure characterizing the degree of disorder in the system, it turns out to be related in a natural way to the notion of predictability. This notion may imply (a) how the initial information about the system is lost with time as a consequence of initial error growth, and (b) how the difference between the real and perturbed trajectories grows in time.

The entropy K is related to the first treatment—it determines the time of predictable behavior of a dynamical system:

$$T_K \sim \frac{1}{K} \log \frac{1}{\varepsilon},$$

where ε is the relative error of data on its state. It should be noted that the dependence on ε is only a logarithmic one. The second treatment is related to the notion of Lyapunov exponents. In this case, one can also introduce the predictability horizon:

$$T_\lambda \sim \frac{1}{\lambda_{\max}} \log \frac{1}{\varepsilon},$$

where λ_{\max} is the maximum Lyapunov exponent. Both estimates agree to an order of magnitude. In practice, however, the predictability time is estimated from the simplified formulas $T_K \sim K^{-1}$ and $T_\lambda \sim \lambda_{\max}^{-1}$.

8.3 Forecast problem

Apparently, the forecast problem is one of the oldest in the analysis of time series. It came into being long before the inception of the theory of dynamical systems and was related to the forecast problem in the theory of stochastic processes.

The forecast problem aims to predict, based on observational data, the future values of measured characteristics of the object under study, i.e., make a forecast for a certain time interval. At the current stage, several different forecast methods are being developed and rigorously justified. However, they all fall into two main categories: local and

global. This subdivision takes into account the domain of definition of the parameters of the approximating function which evaluates the next value of the time series in a recurrent way, based on the several preceding ones [139].

In the historical context, the global methods were developed first. It was proposed to use autoregression, running mean, and other tools based on statistical analysis. Later on, new practical techniques were suggested in the framework of nonlinear dynamics:

- singular spectral analysis (SSA) [166, 167];
- local approximation (LA) [168];
- the SSA-LA combination [169].

The SSA method has a rigorous mathematical foundation and is largely used to determine principal components of time series, and also for noise filtration [166, 170]. Original forecast algorithms that are based on this method also exist [167].

The local methods of forecast, based on LA, have an advantage over the other methods in tasks related to the forecast of irregular series [171, 172]. The application of LA does not require *a priori* information about the system generating the time series, so that there is no need to construct a specific model which would describe the dynamics of the series under study (see Ref. [41]).

With the progress in nonlinear dynamics it turned out that the forecast problem is essentially more involved and frequently does not fit to the schemes proposed by the theory. In particular, the length of the series being analyzed should be sufficiently large, while the noise component should be small enough. One of effective approaches, which allows one to overcome these difficulties and explore noisy and relatively short series, is based on the SSA-LA method [169].

These important topics, however, involve a different range of problems, namely, that of reconstruction of dynamical systems by their time series. Their description is, therefore, outside the scope of the current review. The main results pertaining to the questions touched on in this section are elaborated in studies [41, 139, 158, 173–176] (see also the references cited therein).

9. Randomness and chaos

The statistical properties of deterministic systems (see Section 7.5) reflect the stochastic character of their dynamics. For such systems, however, their future is uniquely determined by their evolution operator and given initial conditions. A question naturally arises: in which sense may a dynamical system possess stochastic (or statistical) properties?

Sometimes the view is expressed that chaos and randomness are the collective concepts characterizing one and the same phenomenon of unpredictability. However, such a view is in principle incorrect, because chaotic and stochastic systems have deep distinctions. The Takens theory (see Section 8.1) allows one to tell apart a dynamical process from a purely stochastic, i.e., nondeterministic, one. This aspect proves to be very important for applications because it enables the nature of a signal obtained experimentally to be established.

9.1 Finite-dimensional and infinite-dimensional observables

The sequence of measurements $\hat{x} = \{x_n\}_{n=0}^N$ is called deterministically generated if the following conditions take place [165, 177, 178]:

(a) there exist a finite-dimensional dynamical system f , point y_0 , and function ϕ such that for all $n = 0, 1, 2, \dots$ the equality $\phi(f^n(y_0)) = x_n$ holds;

(b) the distance $\text{dist}(f^k x, f^k x') \leq \text{const exp}(\lambda k) \text{dist}(x, x')$, i.e., the maximum Lyapunov exponent (see Section 7.1) is bounded for f ;

(c) the function ϕ is Lipschitz continuous, i.e., $|\phi(x) - \phi(x')| \leq L|x - x'|$ for a certain L called the Lipschitz constant.

Let us introduce the space B of all observables as the set $\hat{x} = \{x_0, x_1, x_2, \dots\}$, $\sum_{n=0}^{\infty} |x_n|/2^n < \infty$. With a proper norm, the space B will be a complete normalized linear space. We define a dynamical system in B by the shift map $\hat{x} \mapsto f\hat{x}$, where $f\hat{x} = (x_1, x_2, x_3, \dots)$. We then obtain a universal dynamical system generating any bounded sequence of observations.

Consider the limit set $A(\hat{x})$ and limit capacity $D_0(A)$ of the observable. They can easily be introduced if one considers an arbitrary sequence \hat{x} as an initial state of the universal dynamical system in the space B . If $D_0(A) < \infty$, then a finite-dimensional dynamical system corresponds to the given observable. If, in addition, the entropy and maximum Lyapunov exponent are bounded, the process under consideration is deterministically generated.

Thus, the analysis of the observed signal can answer the principal question of the finite dimensionality and deterministic character of the process under study. However, this does not exhaust the significance of this approach.

9.2 To the definition of chaos and randomness

Based on the concepts of entropy and dimension, the chaotic dynamics can be defined differently with respect to Section 2.3. Such an approach was proposed by Takens in Ref. [165] (see also Ref. [179]). In a certain sense this definition is physically better grounded because it relies on more common notions.

Consider the sequence $\{x_n\}_{n \geq 0}$ which is obtained through measurements of some stationary process or represents the values of the dynamical variable of system (1), recorded at certain discrete instants of time n . For this sequence, the entropy may take one of three admissible values: $H = 0$, $0 < H < \infty$, or $H \rightarrow \infty$.

If the system is a dynamical one with a compact space of states, the value $H \rightarrow \infty$ for it should be excluded. This case takes place if the system evolves in a stochastic (random) way. When $0 < H < \infty$, the system is referred to as possessing chaotic behavior. Consequently, the dynamic system is chaotic if there exists some subset Ω in its state space such that any trajectory that starts in Ω has a positive entropy. Finally, the system dynamics for $H = 0$ are regular.

Thus, this definition of chaos differs from the others through the fact that it additionally considers stochastic dynamics. However, it includes therewith two common elements: stationary, periodic, and quasiperiodic dynamics are not chaotic; the system with chaos shows sensitive dependence on its initial conditions. This latter statement follows from formula (8) which connects the entropy and exponential instability (see Section 2.3).

10. Conclusions

The development of the theory of dynamical systems in the second half of the 20th century led to extremely important consequences, not only in theoretical physics and mathe-

matics, but also in natural sciences as a whole — the discovery of dynamical chaos and issues connected to it. It turned out that numerous nonlinear systems, despite their fully deterministic character, i.e., the absence of noise, stochastic perturbations, and so on, can demonstrate behavior similar to that of random processes. By this means, the emerging statistics of system's behavior are exclusively defined by the specifics of dynamics. Although premises of this ideology were formulated more than one hundred years ago in the work of A M Lyapunov and H Poincaré, the main discoveries and comprehension of this phenomenon took place in the 1960s–1980s.

It is worth noting that the classical analytical tools, such as series of perturbation theory, asymptotic methods, and some others turned out to be useless for studying chaotic systems. For instance, the Smale horseshoe, which in a certain sense prompted the modern research of chaotic phenomena, cannot be described by relations in the mathematical style of the 19th century. In order to create and analyze such constructions, essentially new methods had to be designed. Their development led to important discoveries in the theory of dynamical systems.

The main goal of this review is the description of various approaches that assist the analysis of nonlinear chaotic dynamical systems. Understandably enough, many lines of inquiry were left out of our consideration. In particular, we barely touched such broad topics as the chaos onset scenarios, dynamics of one-dimensional maps, spacetime chaos and turbulence, holomorphic dynamics and fractal sets, and others. However, some of them are described in the papers, manuals, and monographs listed in the References.

The main achievements of the theory of chaotic dynamical systems can briefly be summarized as follows.

It was proven that even very simple systems (such, as for example, the Lorenz system) may exhibit random properties. This drastically changed the idea of randomness, which was previously conceived of as occurring exclusively in systems with a large number of degrees of freedom.

Based on the analysis of billiards significant progress was reached in understanding the roots of randomness in a gas of hard spheres and, as a consequence, in substantiating the Boltzmann ergodic hypothesis.

With the help of the theory of chaotic dynamical systems, a partial success was in getting solution to the problem of the origin of irreversibility in reversible, deterministic equations of motion.

It was proven that chaos may evolve along universal routes, independent of the nature of the system. This outstanding discovery, backed also by experimental research, has led to the development of the renormalization group method in the theory of dynamical systems.

It was found that randomness may derive from both internal properties and outside factors. Moreover, exploring time series of observable quantities one can always distinguish the random, stochastic behavior of systems from the deterministic chaos, and in this way judge on the finite dimensionality of the process under study.

The methods developed for the time series analysis made it possible to use in practice the results of the theory of dynamical systems in order to calculate such characteristics as the entropy, Lyapunov exponents, and dimension. This enables, based on only experimental data, defining the predictability horizon for the process under study and, in some cases, predicting the subsequent system's evolution.

This line of inquiry is continuously gaining in popularity in such applied sciences as financial analysis and medicine.

Finally, one cannot avoid mentioning the esthetic attractiveness of the results obtained. Thanks to books published by B Mandelbrot [180], and H-O Peitgen and P Richter [181], the computer images of fractal sets, strange attractors, and their attraction basins have uncovered the artistic side of the chaos theory. As noted by D Ruelle, this is the field of research where new harmonies will be found [182].

The world of chaos and the theory of dynamical systems is fascinating indeed, and, having once uncovered it, one stays forever enchanted by its beauty.

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